COL 8184 : ALGORITHMS FOR FAIR REPRESENTATION

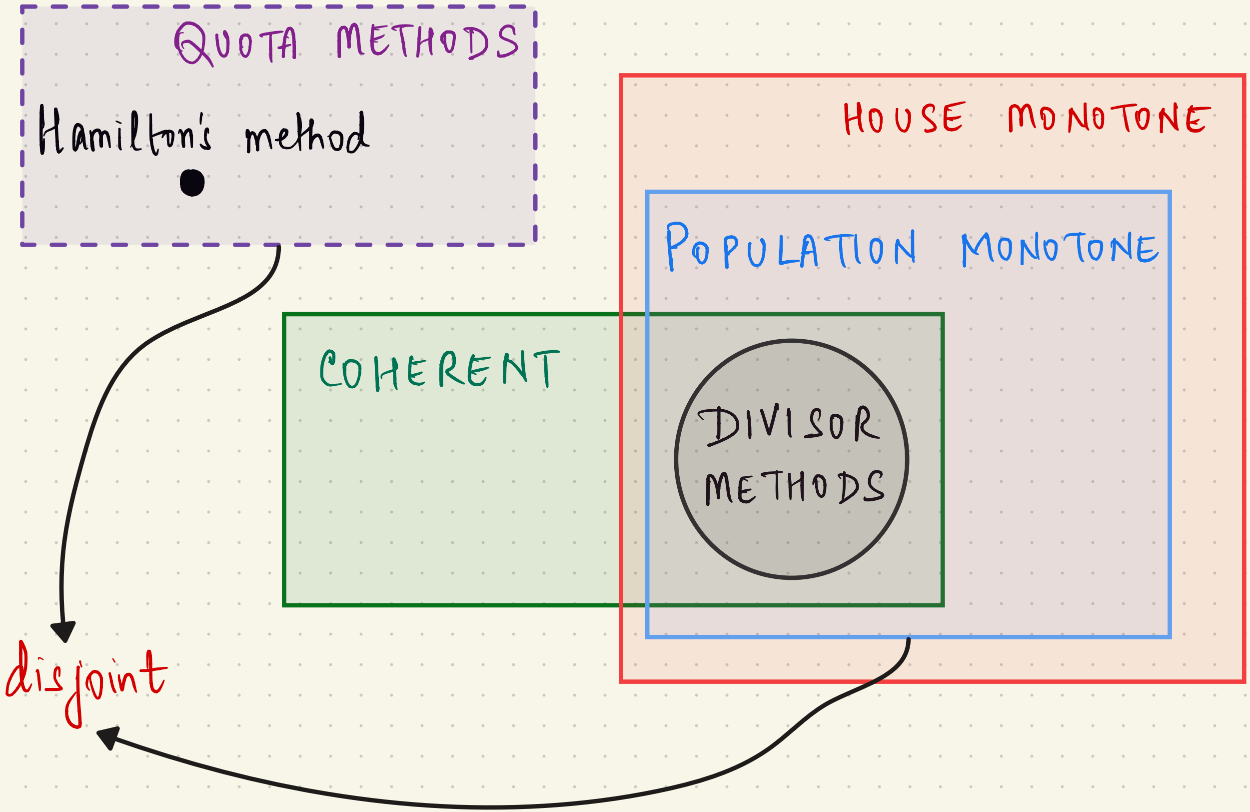
LECTURE 9

AXIOMS FOR RANDOMIZED APPORTIONMENT

FEB 12, 2026 | ROHIT VAISH

BALINSKI - YOUNG IMPOSSIBILITY

Theorem : Population monotonicity is incompatible with quota criterion.


QUOTA METHODS
Hamilton's method
HOUSE MONOTONE
POPULATION MONOTONE
COHERENT
DIVISOR METHODS
disjoint

CIRCUMVENTING BALINSKI-YOUNG IMPOSSIBILITY

1. Beyond the worst case

   Is quota almost always compatible with population monotonicity?

2. Beyond population monotonicity

   Is quota always compatible with house monotonicity?

3. Beyond deterministic methods

   Do randomized apportionment methods satisfy quota and population monotonicity?

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FREQUENCY OF QUOTA VIOLATIONS

	Adams	Dean	H-H	Webster	Jefferson
Expected number of violations per 1000 samples	1000	15.40	2.86	0.61	1000

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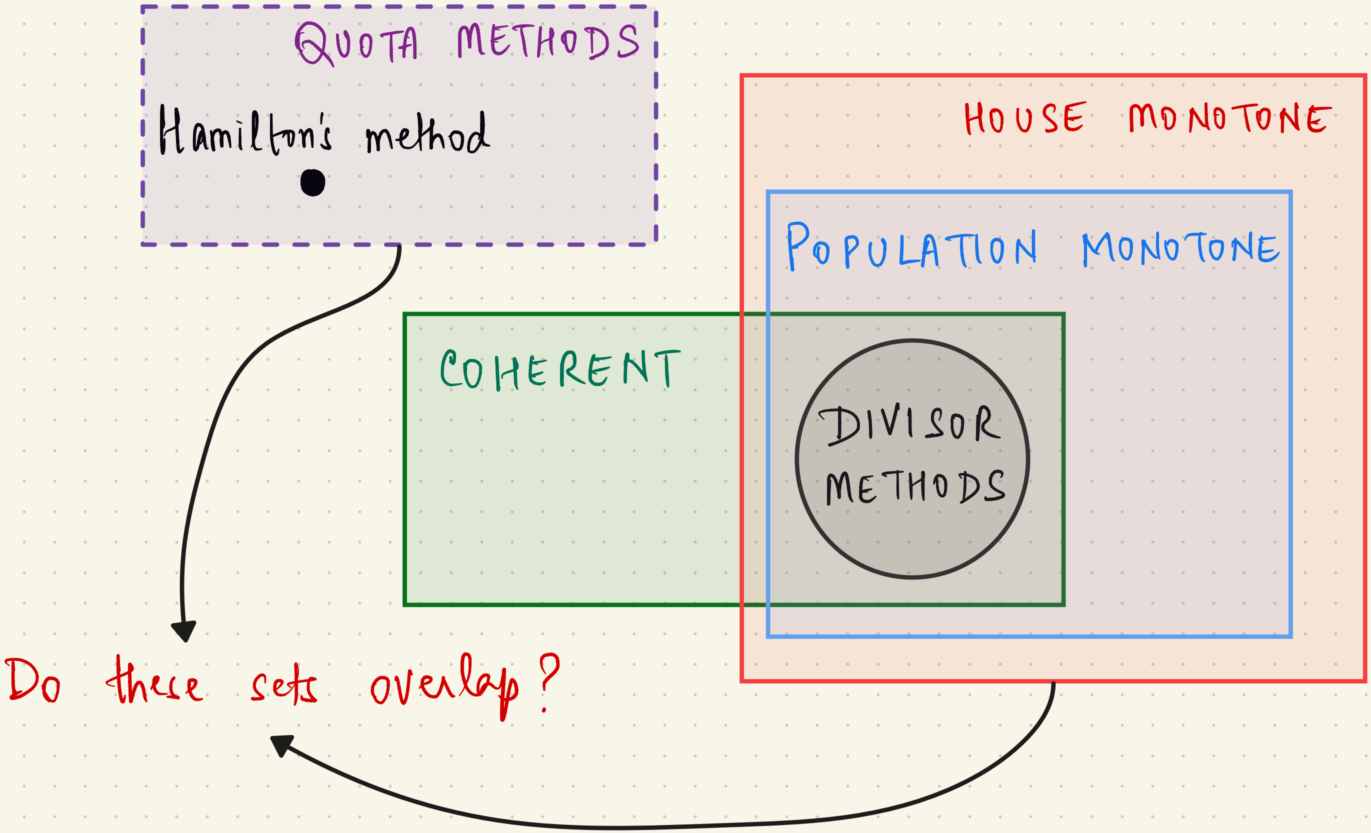
QUOTA METHODS
Hamilton's method
HOUSE MONOTONE
POPULATION MONOTONE
COHERENT
DIVISOR METHODS
Do these sets overlap?

BALINSKI - YOUNG QUOTA METHOD

Recall the table definition of divisor methods.

* Write the entries p_i/k for each state i and each $k \in \{1, \dots, h\}$

  (priority score)

* Select the h largest entries sequentially,

  but skip them if selecting an entry violates upper quota.

PROPERTIES OF BALINSKI-YOUNG QUOTA METHOD

* Upper quota : By construction

* House monotonicity : $h \uparrow$ $D \downarrow$ no state loses a seat

* Lower quota : Last quiz!

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GRIMMETT'S METHOD

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* Consider a unif. random permutation of the states (wolog, identity).

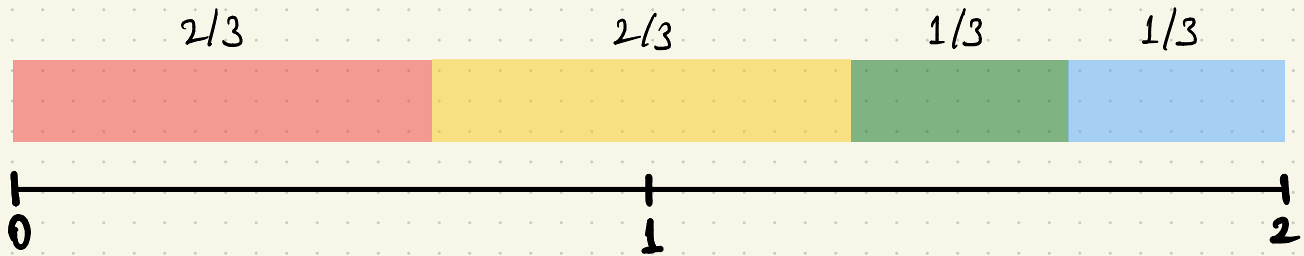
* Provisionally assign $\lfloor q_i \rfloor$ seats to state i. Let $r_i := q_i - \lfloor q_i \rfloor$.
  i.e., the residue

* Draw $U \sim \text{Unif}[0,1]$. Let $Q_i := U + \sum_{j=1}^{i-1} r_j$.

* For each state i, allocate an extra seat to state i if $[Q_i, Q_{i+1})$ contains an integer.

Grimmett's method

GRIMMETT'S METHOD

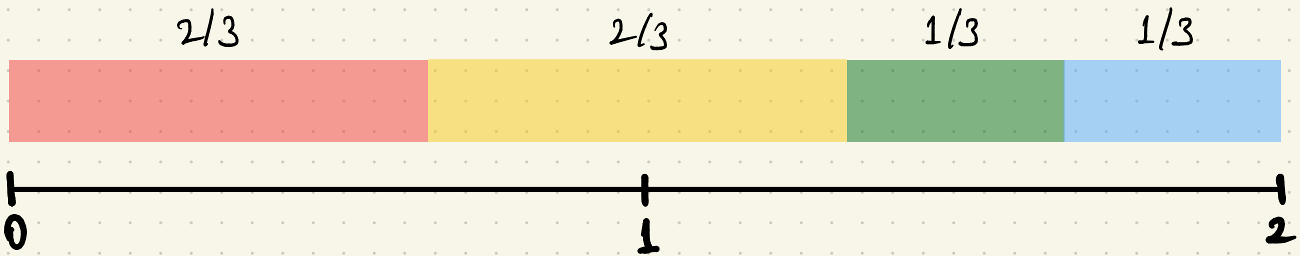


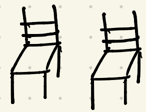
Note: Sum of residues $r_1 + r_2 + \cdots + r_n$

$= $ No. of leftover seats $h - (\lfloor q_1 \rfloor + \lfloor q_2 \rfloor + \cdots + \lfloor q_n \rfloor)$.

$= $ an integer!

GRIMMETT'S METHOD





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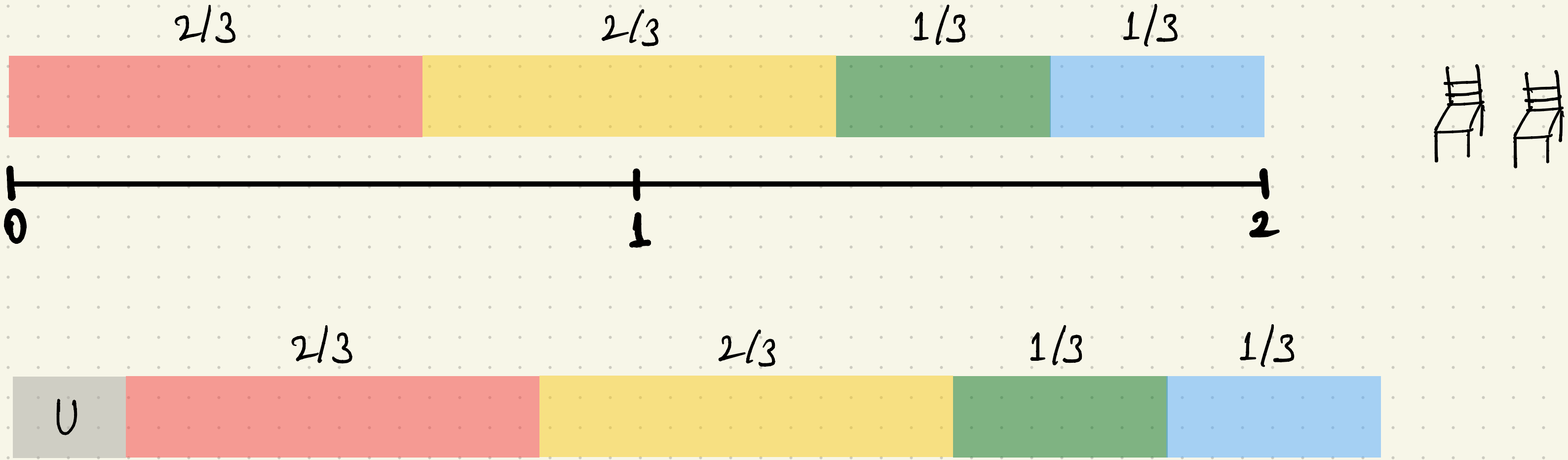

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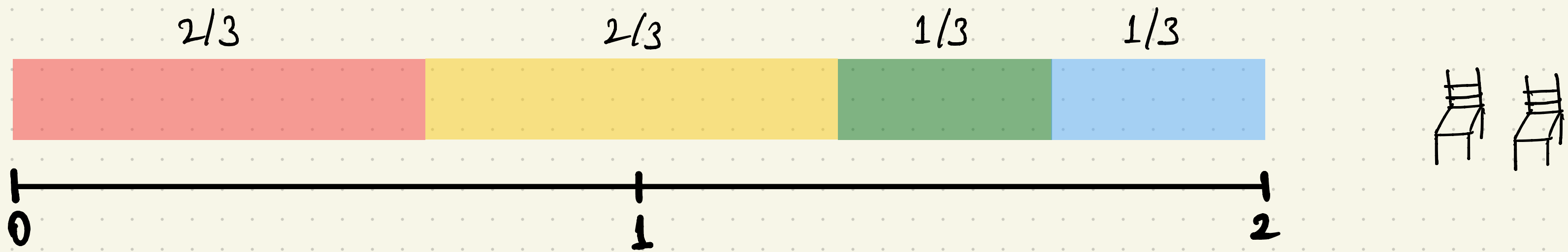

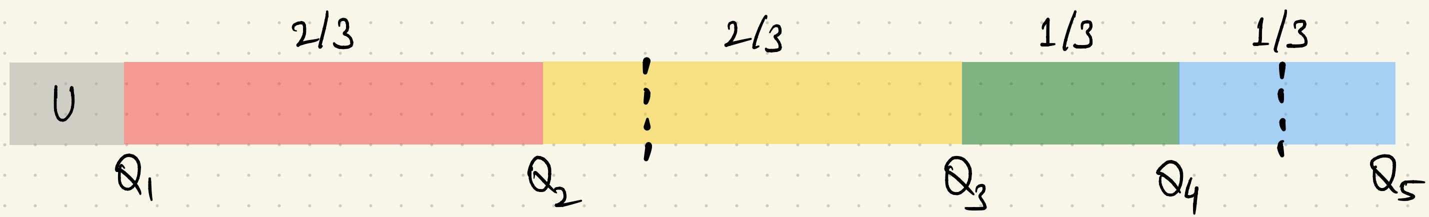

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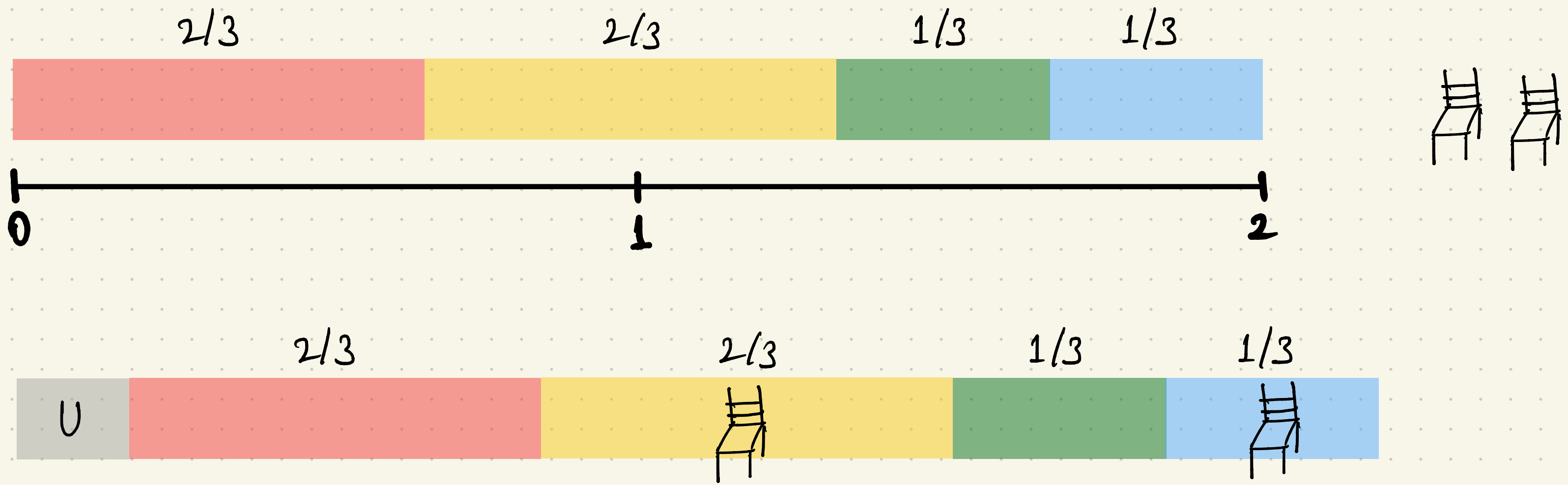

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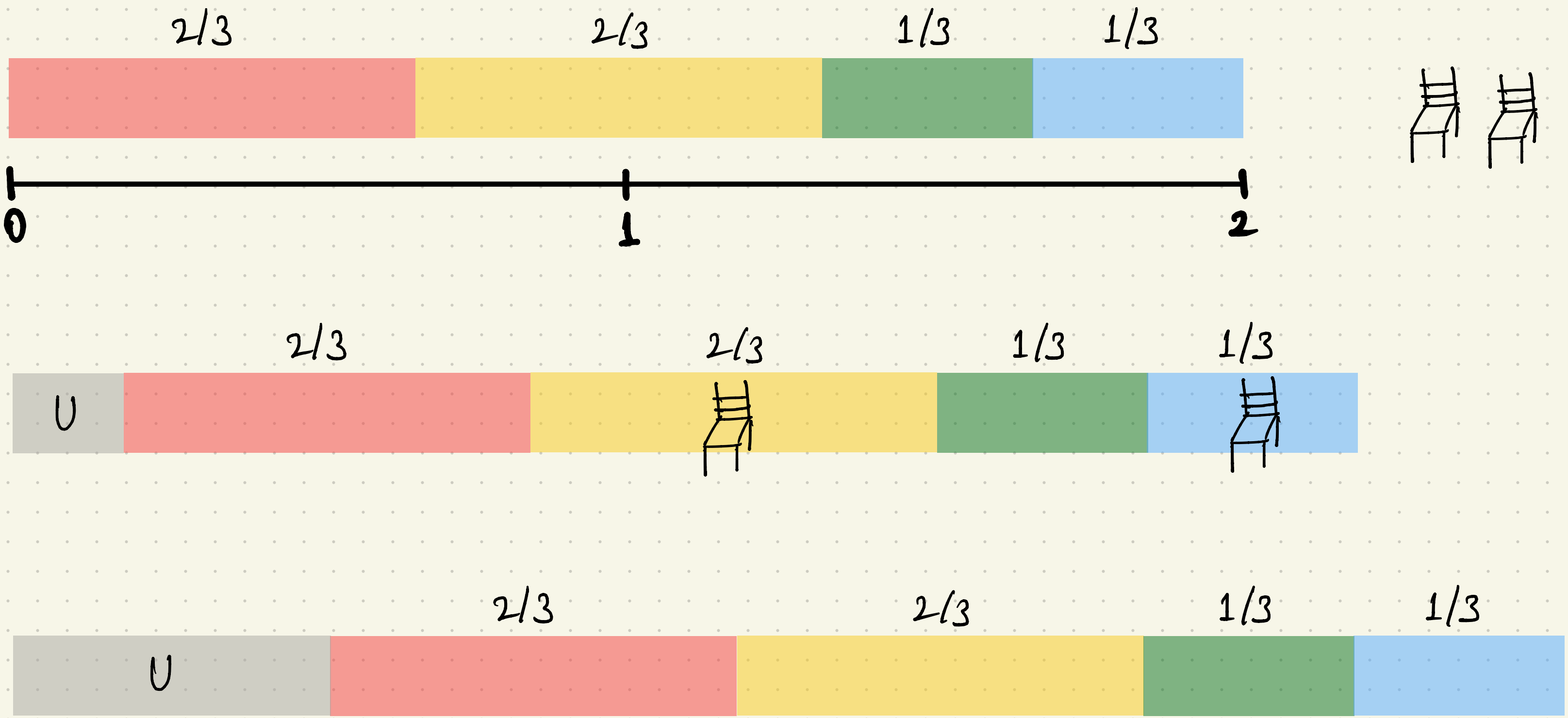

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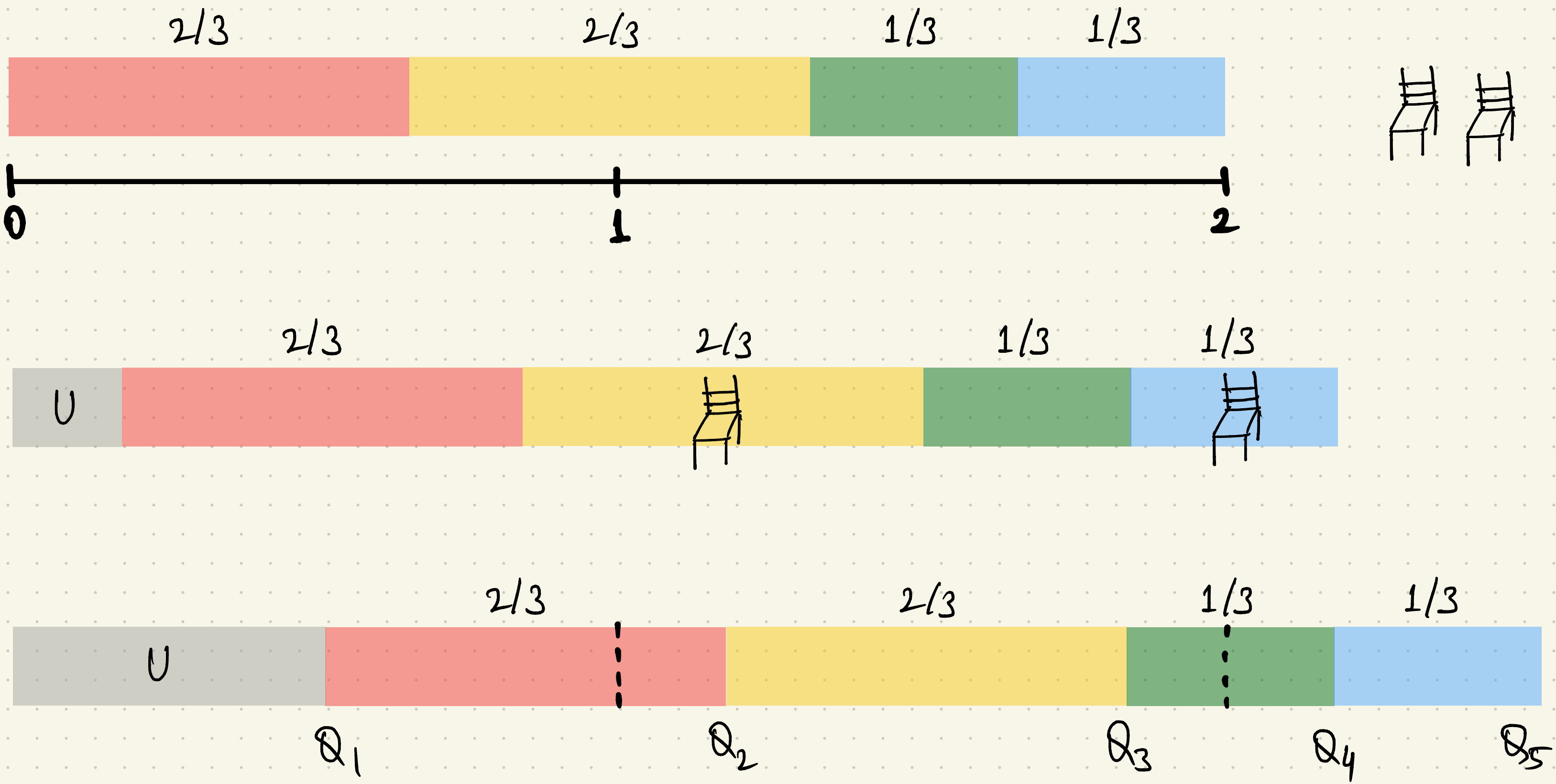

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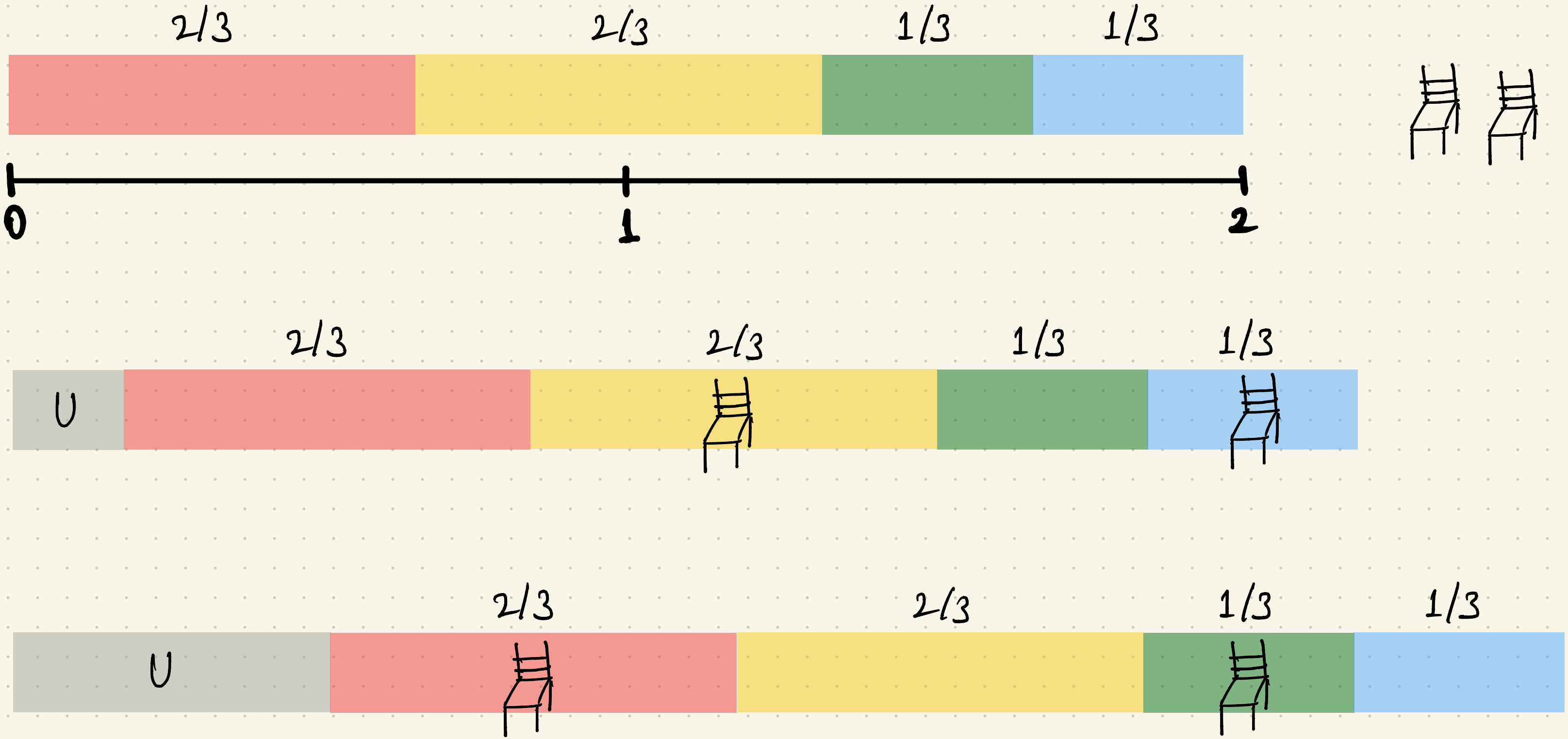

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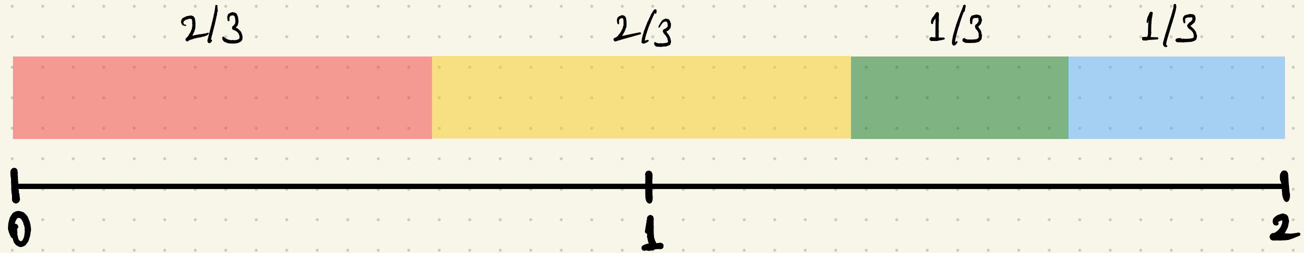
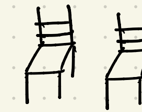
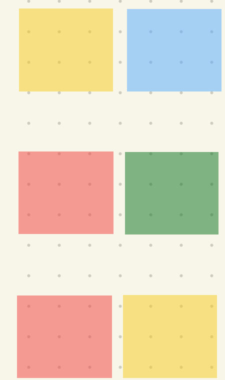




Resulting probability distribution is



w.p. $1/3$

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Quiz

QUIZ

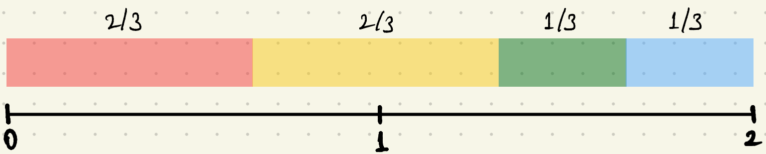
Prove or disprove:

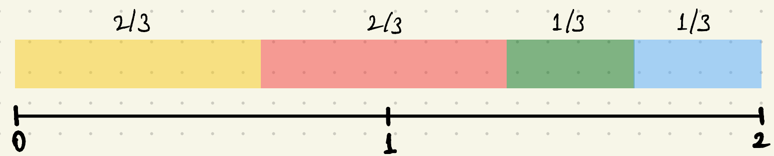
For any two distinct permutations sampled in step I of Grimmett's method, the resulting probability distribution over seat assignments is the same.

Quiz

Different permutations result in different distributions.

switch  and 





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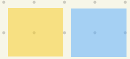
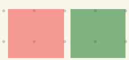
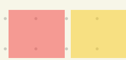
 w.p. $\frac{1}{3}$

 w.p. $\frac{1}{3}$

 w.p. $\frac{1}{3}$

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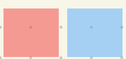 w.p. $\frac{1}{3}$

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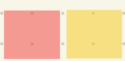 w.p. $\frac{1}{3}$

Properties of Grimmett's method

PROPERTIES OF GRIMMETT'S METHOD

Property 1 : Each state is rounded up with probability equal to its residue.

PROPERTIES OF GRIMMETT'S METHOD

Property 2 : Each state receives its quota in expectation.

(ex-ante proportionality)

i.e., $\mathbb{E}[s_i] = q_i$

PROPERTIES OF GRIMMETT'S METHOD

Theorem: Under Grimmett's method, each state receives its quota in expectation.

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Proof : State i receives $\lceil q_i \rceil$ seats with probability r_i

and $\lfloor q_i \rfloor$ " " " $(1-r_i)$.

By Property 1

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$$= q_i \qquad \blacksquare$$

PROPERTIES OF GRIMMETT'S METHOD

Property 3 : Population monotonicity is satisfied in expectation.

(ex-ante population monotonicity)

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Pop. Mon. : $s_i < s_i'$ and $s_j > s_j'$ $\Rightarrow$ $p_i < p_i'$ or $p_j > p_j'$.

Exp. Pop. Mon. : $\mathbb{E}[s_i] < \mathbb{E}[s_i']$ and $\mathbb{E}[s_j] > \mathbb{E}[s_j']$ $\Rightarrow$ $p_i < p_i'$ or $p_j > p_j'$

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by ex-ante proportionality : $\mathbb{E}[s_i] = q_i < q_i' = \mathbb{E}[s_i']$ and $\mathbb{E}[s_j] = q_j > q_j' = \mathbb{E}[s_j']$

PROPERTIES OF GRIMMETT'S METHOD

Property 4 : Quota criterion is satisfied ex-post.

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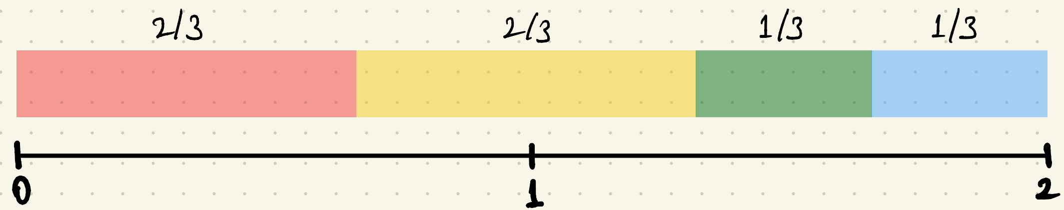

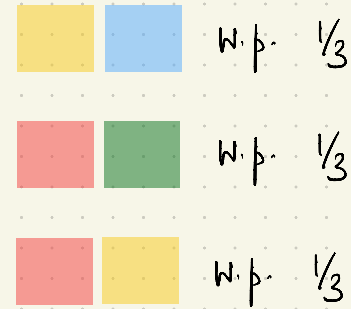

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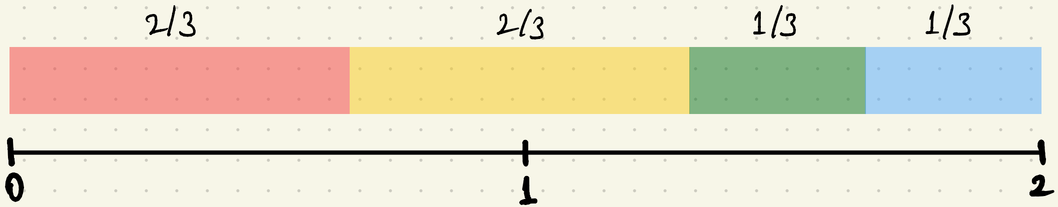
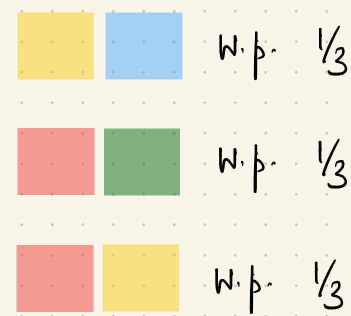


ex-post ⇒

each of these outcomes satisfies quota



PROPERTIES OF GRIMMETT'S METHOD

Theorem : Grimmett's method satisfies :

* ex-ante proportionality
* ex-ante population monotonicity
* ex-post quota

LANDSCAPE

LANDSCAPE

Ex-post proportionality

Ex-ante proportionality

Ex-post quota

Ex-ante quota

LANDSCAPE

Ex-post proportionality $\rightarrow$ $s_i = q_i$ for each deterministic seat assignment in the support of the distribution

Ex-ante proportionality

Ex-post quota

$\rightarrow$ $\mathbb{E}[s_i] = q_i$

Ex-ante quota

LANDSCAPE

Ex-post proportionality

$s_i \in \{\lfloor q_i \rfloor, \lceil q_i \rceil\}$ for each deterministic seat assignment in the support of the distribution

Ex-ante proportionality

Ex-post quota

Ex-ante quota

$\mathbb{E}[s_i] \in [\lfloor q_i \rfloor, \lceil q_i \rceil]$

LANDSCAPE

$s_i = q_i$

Ex-post proportionality

Ex-ante proportionality

$\mathbb{E}[s_i] = q_i$

Ex-post quota

$s_i \in \{\lfloor q_i \rfloor, \lceil q_i \rceil\}$

Ex-ante quota

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LANDSCAPE

$$\begin{tikzcd}
 & \begin{array}{c} S_i = q_i \\ \text{Ex-post proportionality} \end{array} \arrow[dl] \arrow[dr] & \\
\begin{array}{c} \text{Ex-ante proportionality} \\ \mathbb{E}[S_i] = q_i \end{array} \arrow[dr] & & \begin{array}{c} \text{Ex-post quota} \\ S_i \in \{\lfloor q_i \rfloor, \lceil q_i \rceil\} \end{array} \arrow[dl] \\
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\end{tikzcd}$$

LANDSCAPE

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& \text{Ex-ante quota} &
\end{tikzcd}
$$

may fail

always exist

LANDSCAPE

$$
\begin{tikzcd}
 & \text{Ex-post proportionality} \arrow[dl] \arrow[dr] & \\
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\end{tikzcd}
$$

Ex-post pop. mon.

Ex-ante pop. mon.

Ex-post house mon.

LANDSCAPE

$$
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\end{tikzcd}
$$

$$
\begin{tikzcd}
\text{Ex-post pop. mon.} \arrow[d] \arrow[r, dotted, "🤔"] & \text{Ex-ante pop. mon.} \\
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\end{tikzcd}
$$

LANDSCAPE

Ex-post proportionality

Ex-post pop. mon.

Ex-ante proportionality

Ex-post quota

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LANDSCAPE

Balinski - Young Impossibility

Ex-post proportionality

Ex-post pop. mon.

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Ex-ante pop. mon.

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Ex-ante quota

LANDSCAPE

Ex-post proportionality

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by Grimmett's method

LANDSCAPE

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$$

Possible ?

GRIMMETT'S METHOD FAILS HOUSE MONOTONICITY

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Identity permutation and $U > \frac{2}{3}$ $\Longrightarrow$

Grimmett returns the seat allocation $(1, 0, 1, 0)$.

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"toxic" assignment

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NEXT LECTURE

The Careful Sliding Method