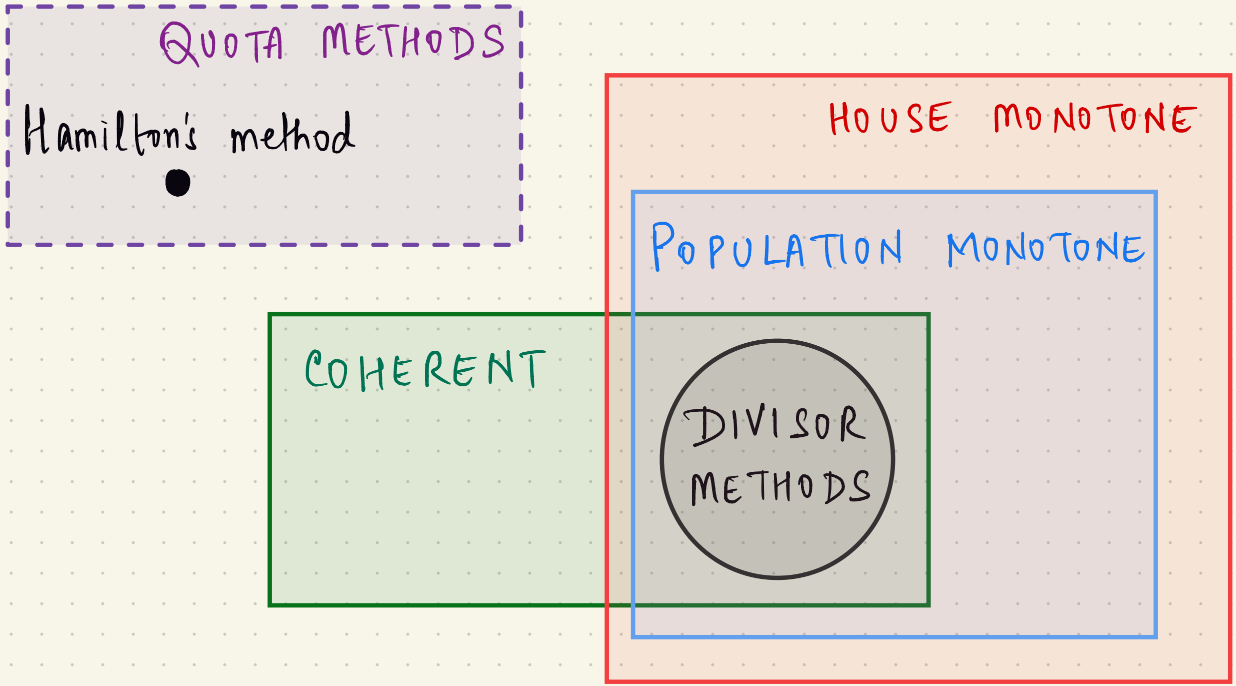
COL 8184 : ALGORITHMS FOR FAIR REPRESENTATION

LECTURE 8

CIRCUMVENTING BALINSKI - YOUNG IMPOSSIBILITY

FEB 05, 2026 | ROHIT VAISH

PICTURE SO FAR



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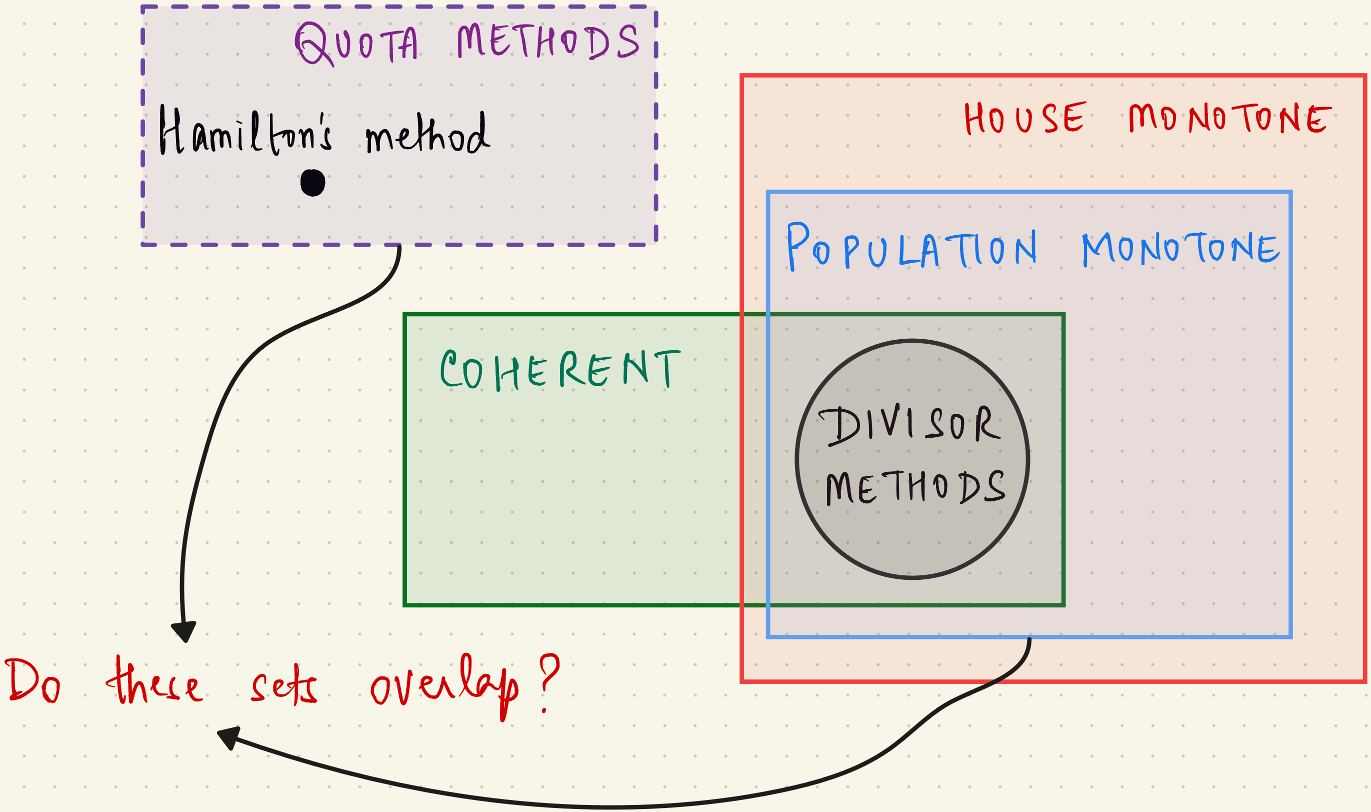
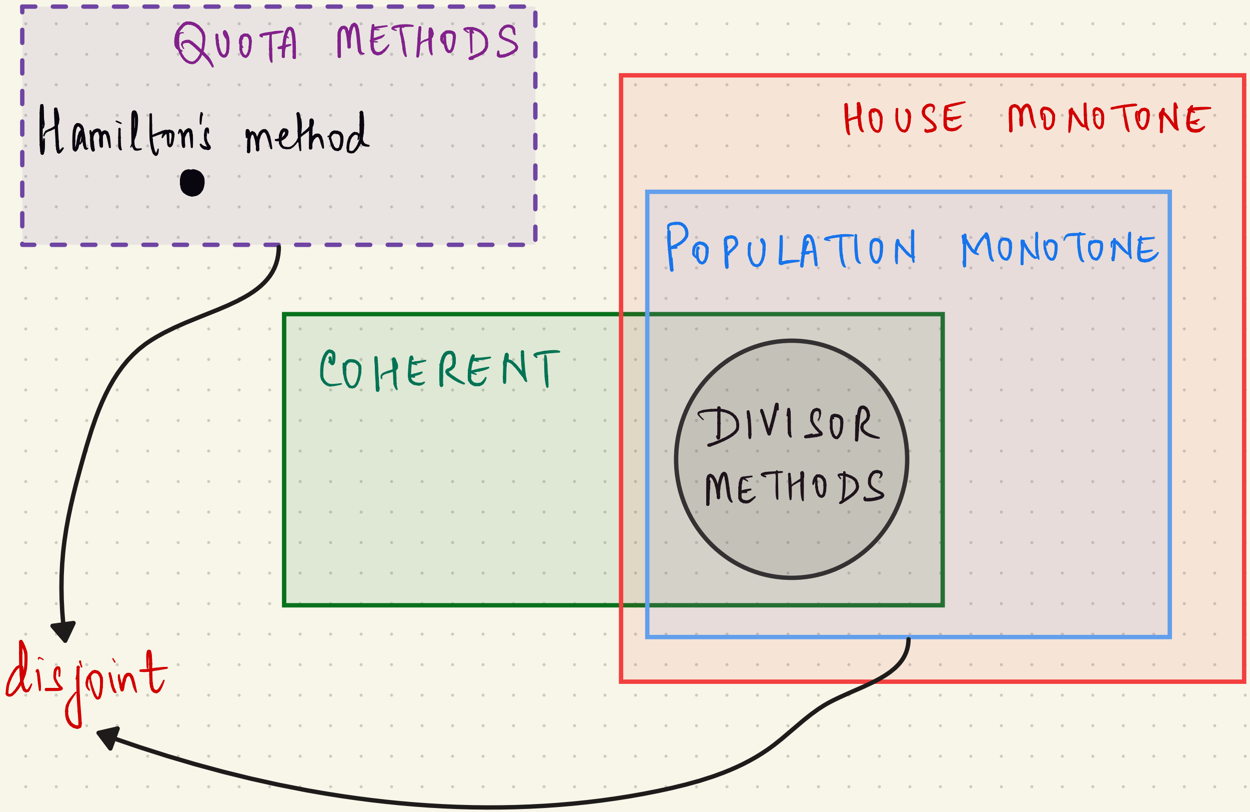


BALINSKI - YOUNG IMPOSSIBILITY

Theorem: Population monotonicity is incompatible with quota criterion.


QUOTA METHODS
Hamilton's method
HOUSE MONOTONE
POPULATION MONOTONE
COHERENT
DIVISOR METHODS
disjoint

NON-UNIFORMITY OF QUOTA CONSTRAINT

State A $\quad q_A = 1.5 \quad \pm 0.5 \Rightarrow$ 33% fluctuation

State B $\quad q_B = 41.5 \quad \pm 0.5 \Rightarrow$ 1.2% fluctuation

CIRCUMVENTING BALINSKI-YOUNG IMPOSSIBILITY

1. Beyond the worst case

   Is quota almost always compatible with population monotonicity?

2. Beyond population monotonicity

   Is quota always compatible with house monotonicity?

3. Beyond deterministic methods

   Do randomized apportionment methods satisfy quota and population monotonicity?

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* 50 states, 435 seats

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  (as per the 1970 apportionment)

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* Evaluate the probability of quota violation.

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$\frac{p_i}{D}$ generated uniformly at random between $\max\{0.5, d_{s_i}\}$ and d_{s_i+1}.
s.t. $\sum p_i$ = population in 1970.

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avoid wastage of seats on small states

* Evaluate the probability of quota violation.

FREQUENCY OF QUOTA VIOLATIONS

	Adams	Dean	H-H	Webster	Jefferson
Expected number of violations per 1000 samples					

FREQUENCY OF QUOTA VIOLATIONS

	Adams	Dean	H-H	Webster	Jefferson
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Expected number of violations per 1000 samples	1000	15.40	2.86	0.61	1000

Frequency of Quota Violations

	Adams	Dean	H-H	Webster	Jefferson
Expected number of violations per 1000 samples	1000	15.40	2.86	0.61	1000

$\approx$ 1 in every 1600 apportionments

$\approx$ once every 16,000 years

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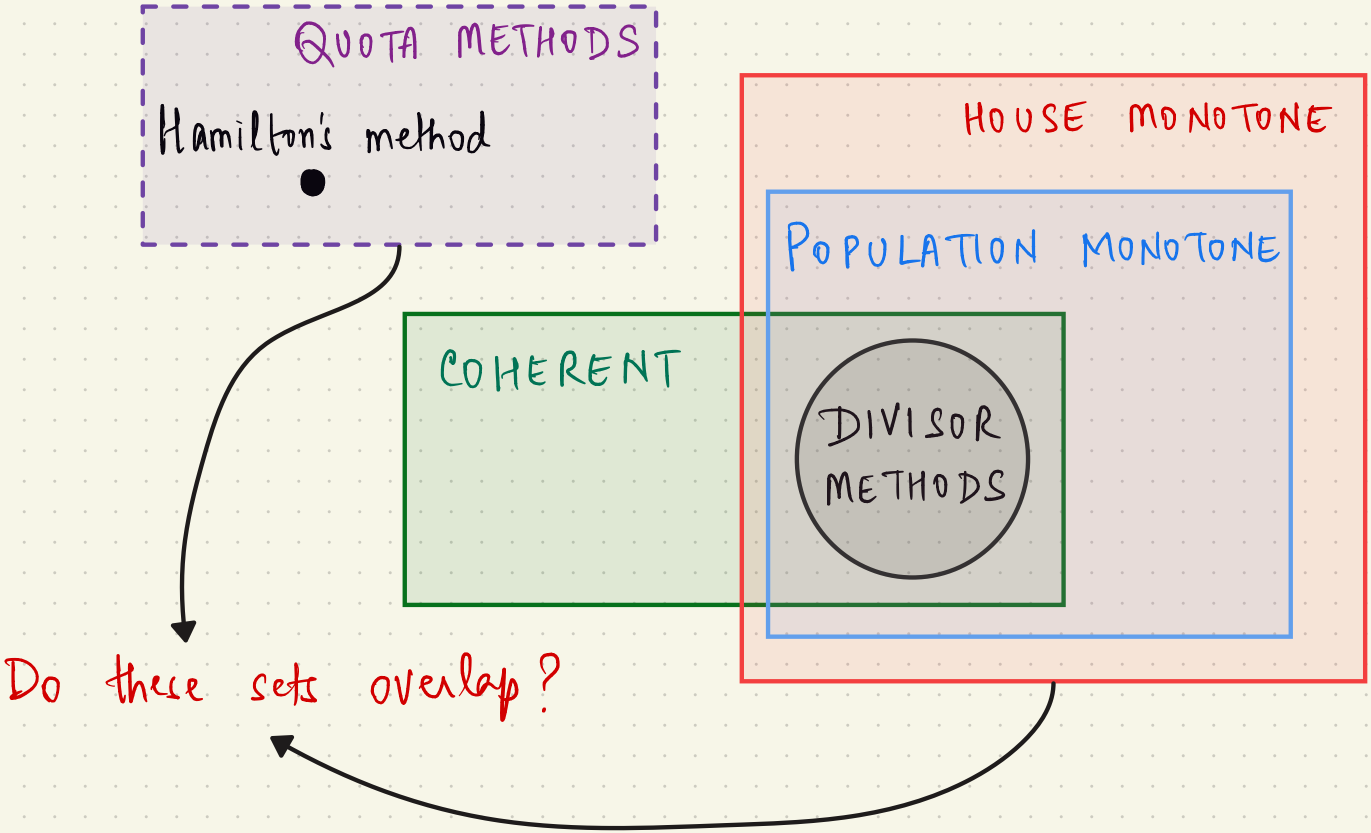
QUOTA METHODS
Hamilton's method
HOUSE MONOTONE
POPULATION MONOTONE
COHERENT
DIVISOR METHODS
Do these sets overlap?

BALINSKI - YOUNG QUOTA METHOD

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Recall the table definition of divisor methods.

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* Write the entries p_i/k for each state i and each $k \in \{1, \dots, h\}$

  (priority score)

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Recall the table definition of divisor methods.

* Write the entries p_i/k for each state i and each $k \in \{1, \dots, h\}$
  (priority score)

* Select the h largest entries sequentially,
  but skip them if selecting an entry violates upper quota.

BALINSKI - YOUNG QUOTA METHOD

House size $h = 10$

Jefferson example from Lecture 2

State	Population
A	15
B	14
C	13
D	58
	100

BALINSKI - YOUNG QUOTA METHOD

House size $h = 10$

Jefferson example from Lecture 2

State	Population	$P_i/8$	$P_i/7$	$P_i/6$	$P_i/5$	$P_i/4$	$P_i/3$	$P_i/2$	$P_i/1$
A	15	-	-	-	-	3.75	5	7.5	15
B	14	-	-	-	-	-	4.66	7	14
C	13	-	-	-	-	-	4.33	6.5	13
D	58	7.25	8.28	9.67	11.6	14.5	19.3	29	58
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Standard Jefferson violates quota

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C	13	-	-	-	-	-	4.33	6.5	13
D	58 → 6	7.25	~~8.28~~	9.67	11.6	14.5	19.3	29	58
	100								

Balinski-Young Quota Method satisfies quota

PROPERTIES OF BALINSKI-YOUNG QUOTA METHOD

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PROPERTIES OF BALINSKI-YOUNG QUOTA METHOD

* Upper quota : By construction

* House monotonicity : $h \uparrow \quad D \downarrow$ no state loses a seat

* Lower quota : 🤔

QUIZ

QUIZ

Prove that the Balinski-Young Quota Method satisfies lower quota.

Theorem : Balinski-Young Quota Method satisfies lower quota.

Proof : Suppose lower quota is violated for some state, say state 1, on the problem instance $(h; p_1, \cdots, p_n)$.

Define the set E of "excess" states as

$$E := \left\{ i \in N \ : \ s_i > \frac{p_i h}{P} \right\}$$

Note that $E \neq \emptyset$.

Theorem : Balinski-Young Quota Method satisfies lower quota.

Proof : Consider the timeline of the assignment of h seats:

$$\underset{1}{\underline{\qquad}} \quad \underset{2}{\underline{\qquad}} \quad - - - \quad \underset{h-1}{\underline{\qquad}} \quad \underset{h}{\underline{\qquad}} \, .$$

For each state i, let h^i be the time at which state i received its last (i.e., s_i^{th}) seat.

Let $j \in E$ be the last state in E to receive a seat.

That is, $j \in \underset{\ell \in E}{\operatorname{argmax}} \, h_\ell$.

Theorem : Balinski-Young Quota Method satisfies lower quota.

Proof : Since state 1 did not meet its lower quota :

$$s_1 < \lfloor q_1 \rfloor \implies s_1 + 1 \leq \frac{p_1}{P} \cdot h \quad \text{——— } (1)$$

Recall that $j \in E$. So, $s_j > \frac{p_j}{P} \cdot h$. ——— (2)

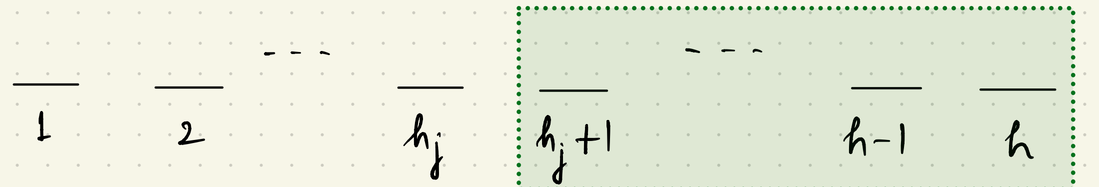
From (1) and (2), $\frac{p_1}{s_1+1} > \frac{p_j}{s_j}$. (State 1 has higher priority)

$\implies$ state j could not have received the h^{th} seat

$\implies h_j < h$.

Theorem : Balinski-Young Quota Method satisfies lower quota.

Proof : Let T be the set of states that receive seats during $h_j+1, \ldots, h$.



Since $h_j < h$, $T \neq \emptyset$.

By definition of T, $T \cap E = \emptyset$.

Thus, for every $i \in T$, $s_i \leq \frac{p_i h}{P}$. —— (3)

Theorem : Balinski–Young Quota Method satisfies lower quota.

Proof : For any state $i \in N$, let s_i^t denote the seats received by i up to and including time t. Thus, $s_i^h = s_i$.

For every $i \in T$, observe that $s_i \geq s_i^{h_j} + 1$.

Recall from (3): For every $i \in T$, $s_i \leq \frac{p_i h}{P}$.

Therefore, for every $i \in T$, $\frac{P}{h} \leq \frac{p_i}{s_i} \leq \frac{p_i}{s_i^{h_j}+1}$. —— (4)

Theorem : Balinski-Young Quota Method satisfies lower quota.

Proof : Recall that $j \in E \implies s_j > \frac{p_j}{p} \cdot h$

and $s_j^{h_j} = s_j$.

Combining these observations with (4), we get that for every $i \in T$:

$$\frac{p_j}{s_j^{h_j}} = \underbrace{\frac{p_j}{s_j} < \frac{p}{h}}_{j \in E} \underbrace{\leq \frac{p_i}{s_i} \leq \frac{p_i}{s_i^{h_i}+1}}_{\text{from (4)}} \quad \text{—— (5)}$$

Theorem : Balinski-Young Quota Method satisfies lower quota.

Proof : From 5, we have that for every $i \in T$:

$$\frac{p_j}{s_j^{h_j}} < \frac{p_i}{s_i^{h_j}+1}.$$

Furthermore,

$$\frac{p_i}{s_i^{h_j}+1} < \frac{p_i}{s_i^{h_j}}.$$

Thus, every state in T (nonempty) had a higher priority for receiving the h_j^{th} seat than state j. Contradiction! ∎

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* Draw $U \sim \text{Unif}[0,1]$. Let $Q_i := U + \sum_{j=1}^{i-1} r_j$.

* For each state i, allocate an extra seat to state i if $[Q_i, Q_{i+1})$ contains an integer.

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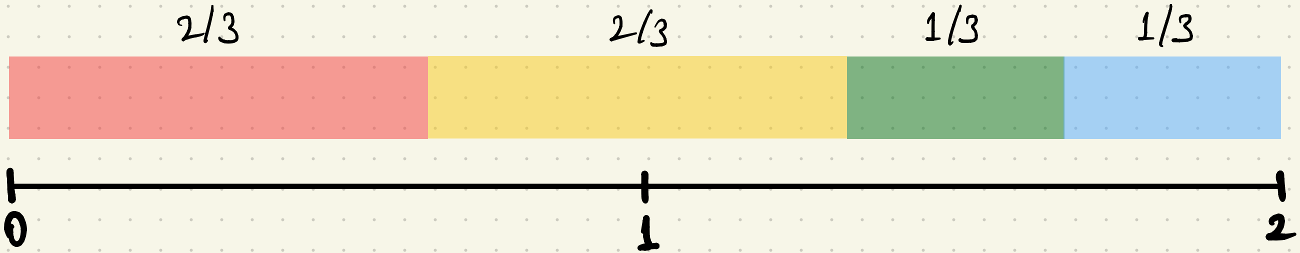

Note : Sum of residues $r_1 + r_2 + \cdots + r_n$

$= $ No. of leftover seats $h - (\lfloor q_1 \rfloor + \lfloor q_2 \rfloor + \cdots + \lfloor q_n \rfloor)$.

$=$ an integer!

Grimmett's Method

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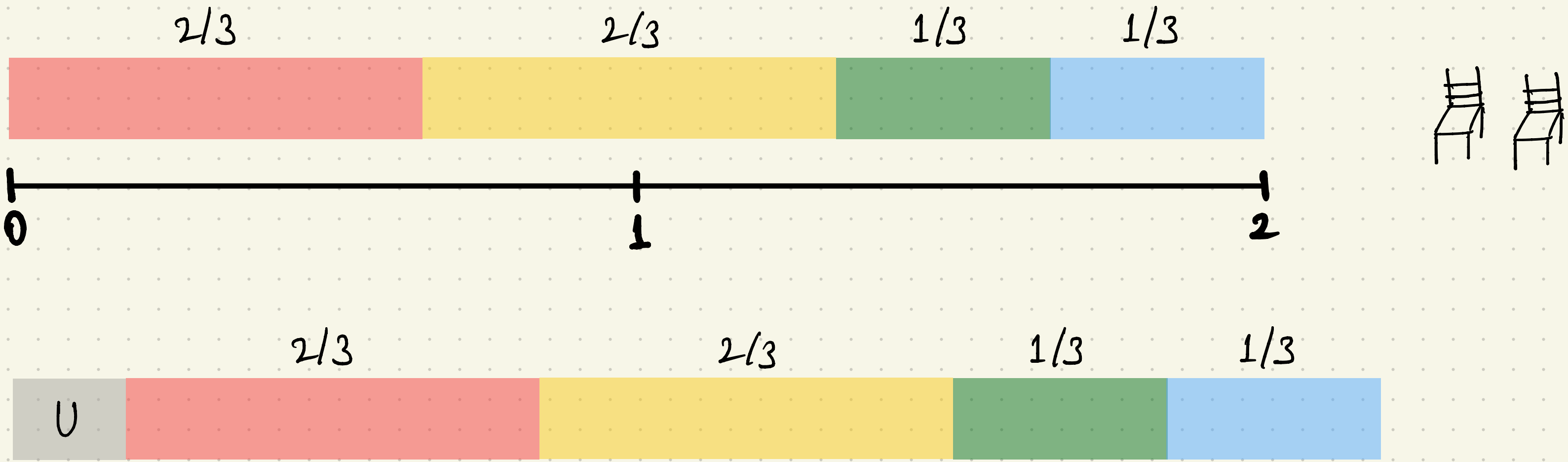

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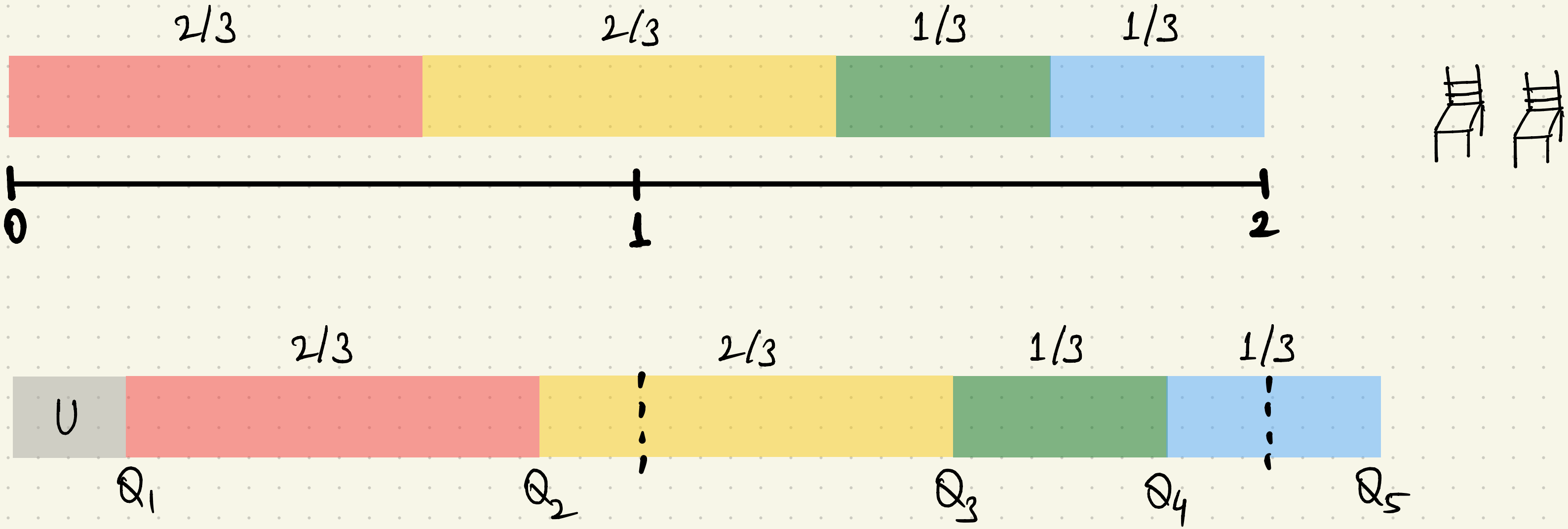

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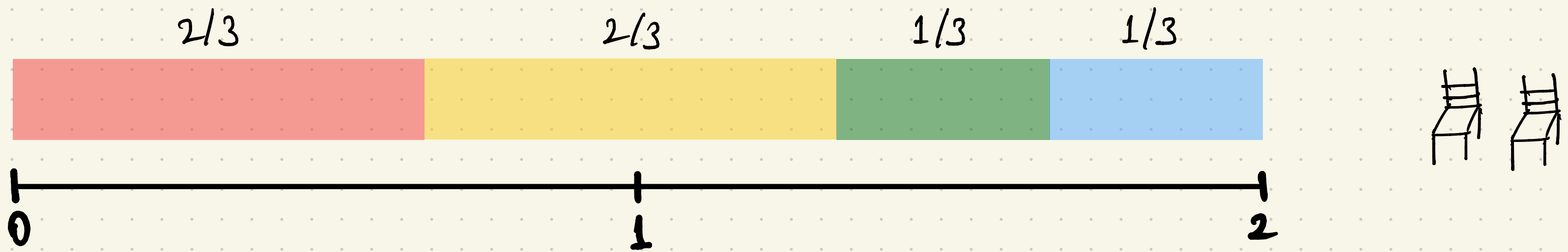

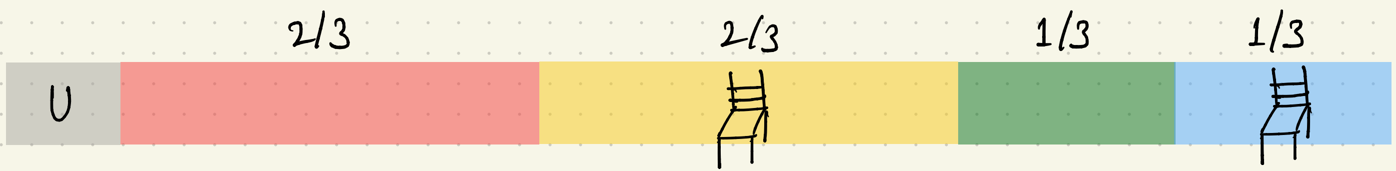

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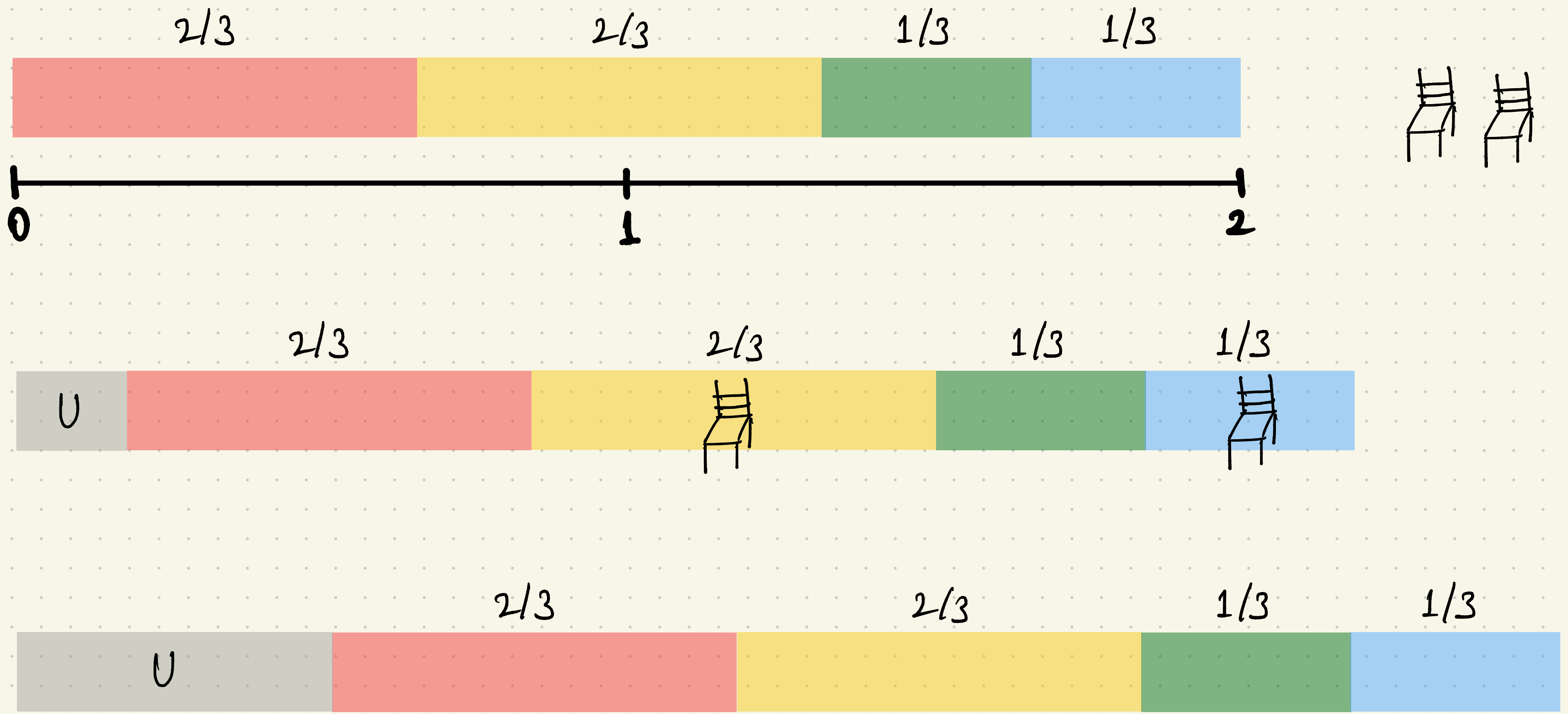

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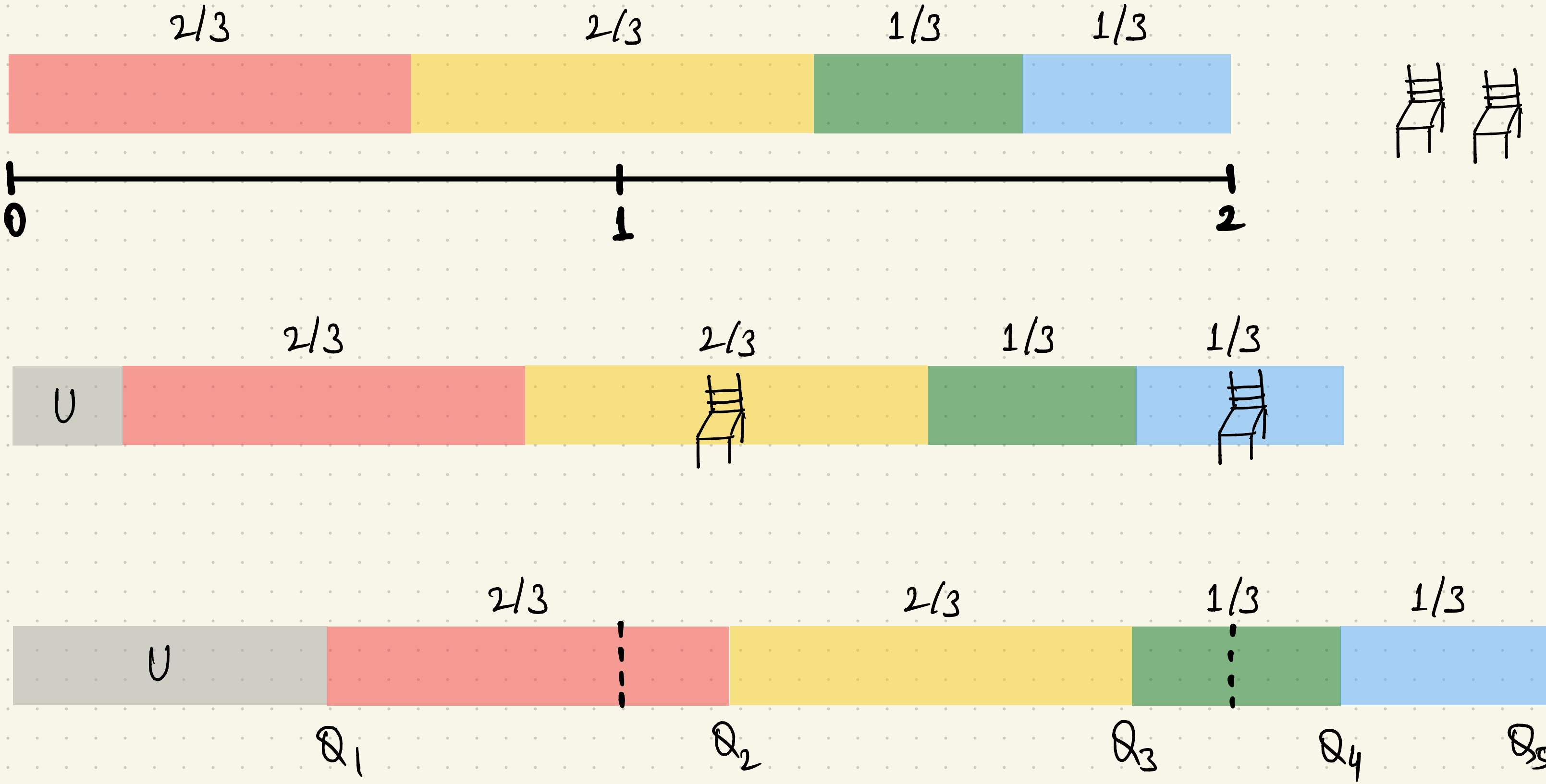

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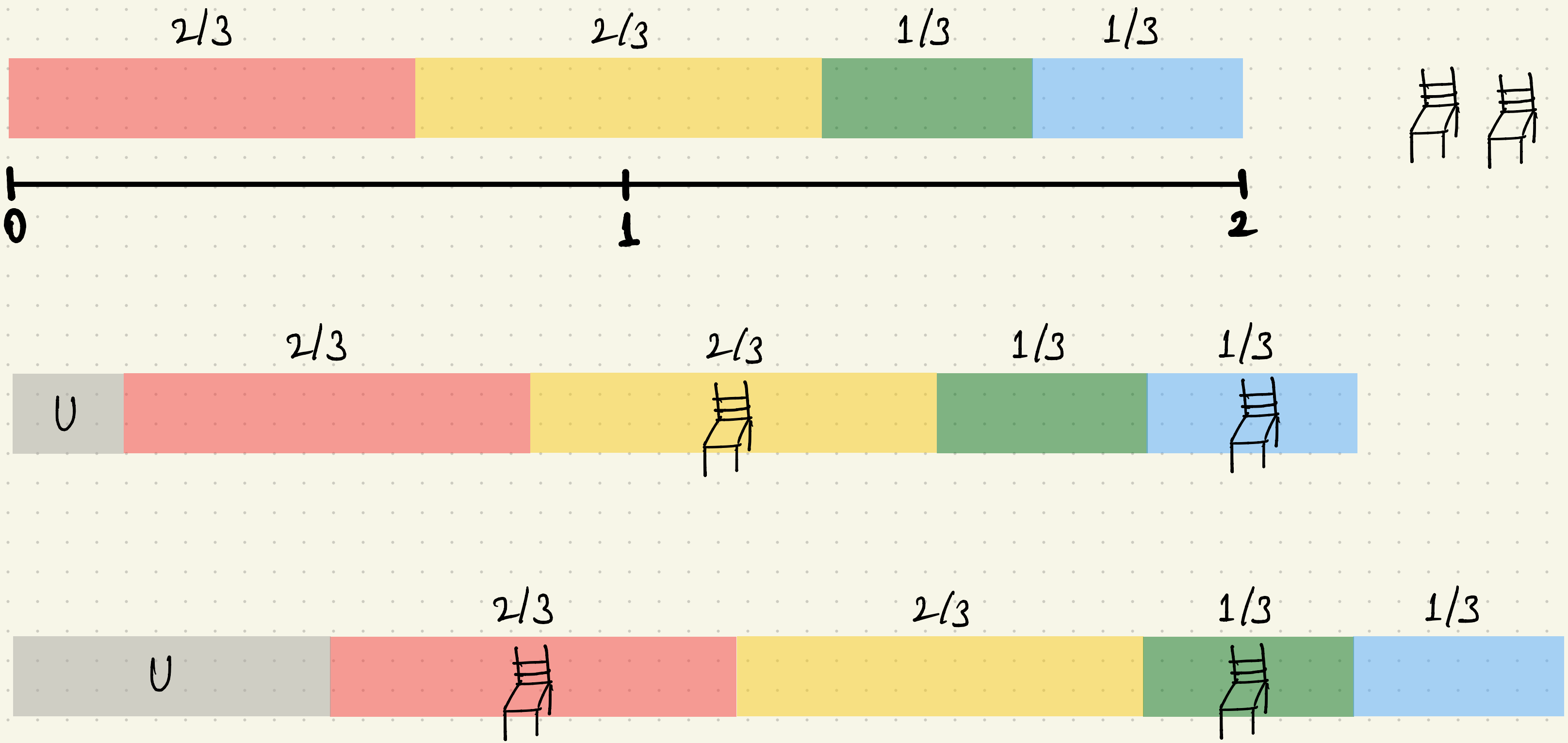

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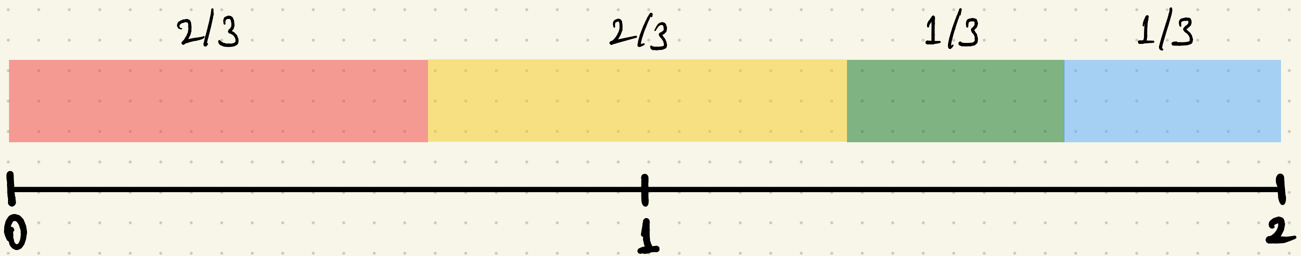

$$\Pr\left(\square \text{ gets an extra seat}\right) = ?$$

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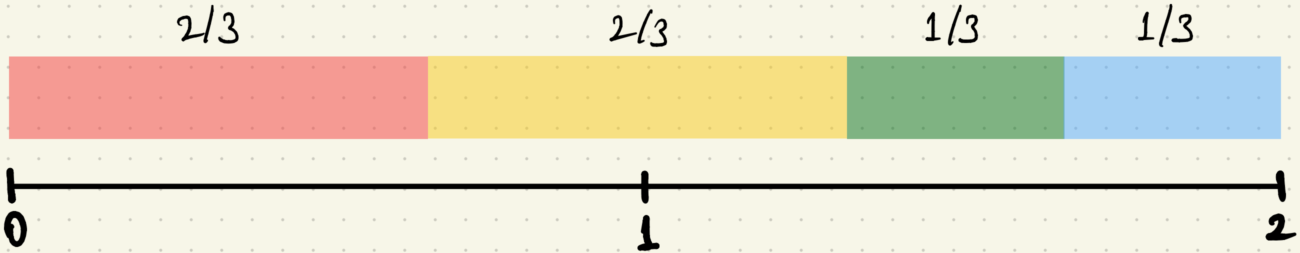
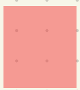


$\Pr\big($  gets an extra seat $\big) = \Pr\left(U \in \left(\frac{1}{3}, 1\right]\right)$

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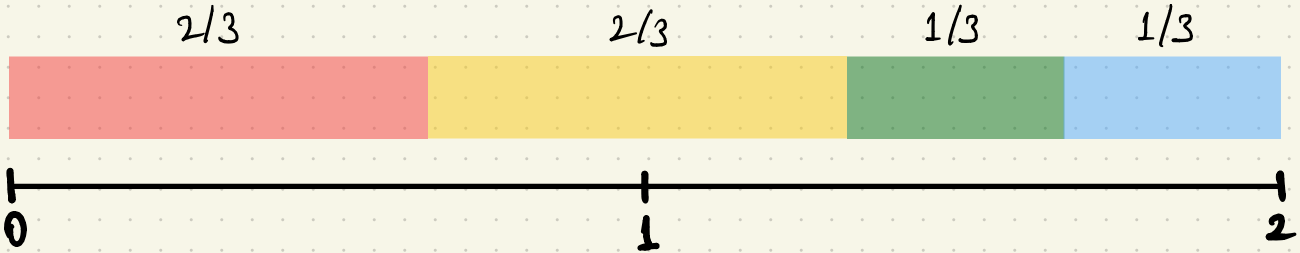
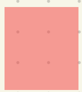
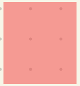


$\mathbb{P}\Big($  gets an extra seat$\Big) = \mathbb{P}\Big(U \in \big(\tfrac{1}{3}, 1\big]\Big)$

$=$ length of 

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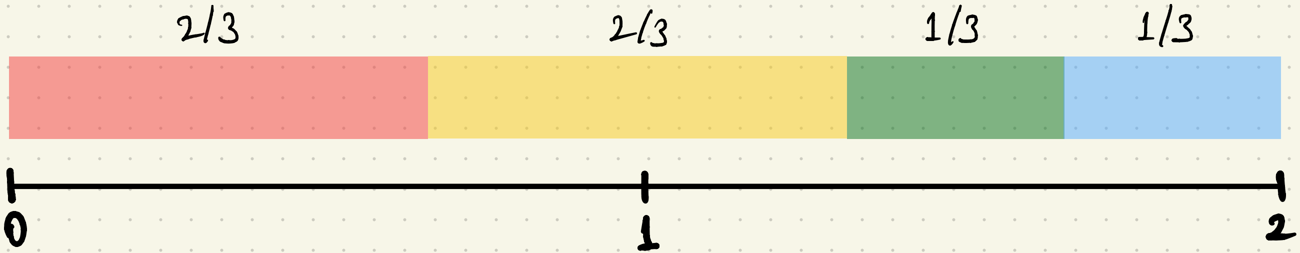

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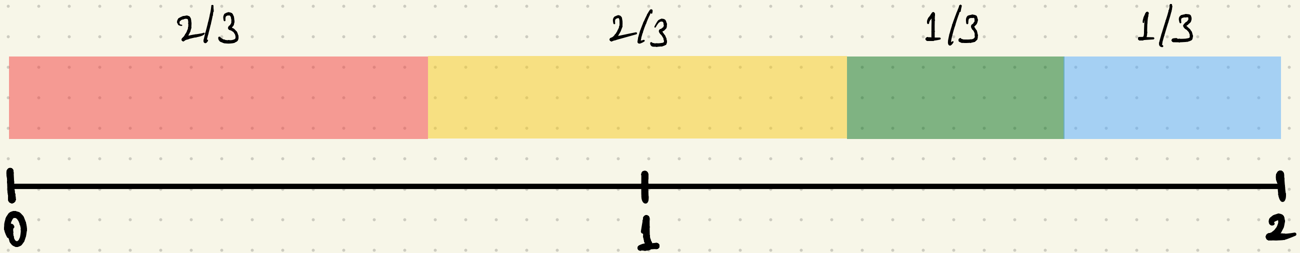

$$\Pr\left(\square \text{ gets an extra seat}\right) = \Pr\left(U \in \left[0, \tfrac{1}{3}\right) \cup \left(\tfrac{2}{3}, 1\right]\right)$$

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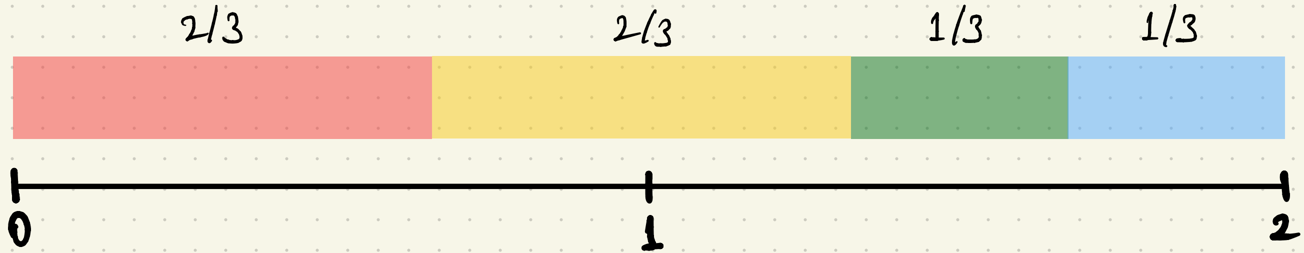

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$$= \text{length of } \square$$

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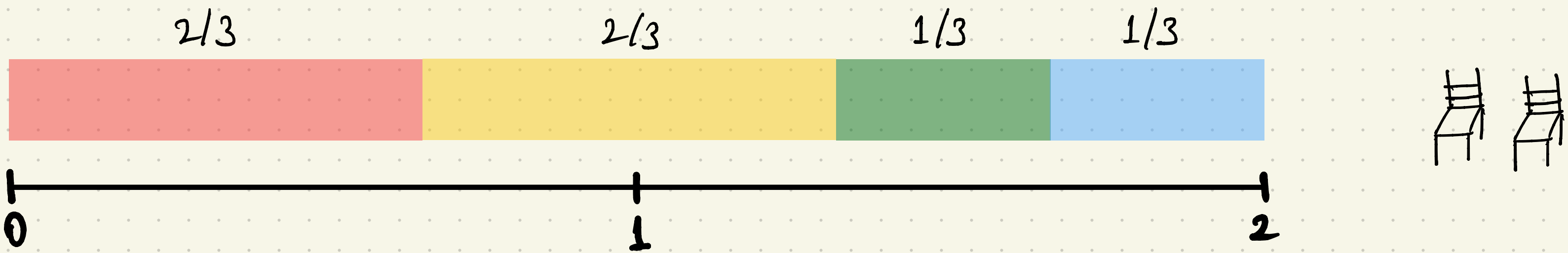
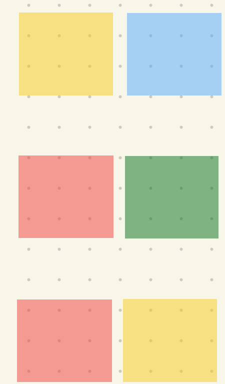


Resulting probability distribution is



w.p. $1/3$

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NEXT LECTURE

Randomized Apportionment (Contd.)