COL 8184 : ALGORITHMS FOR FAIR REPRESENTATION

LECTURE 6

OPTIMIZATION IN APPORTIONMENT (CONTD.) & BALINSKI-YOUNG IMPOSSIBILITY RESULT

JAN 29, 2026 | ROHIT VAISH

DIVISOR METHODS

Jefferson's

$$f(x) = \lfloor x \rfloor$$

$$1,\ 2,\ 3,\ \cdots,\ k$$

Webster's

$$f(x) = [x]$$

arithmetic mean

$$\frac{1}{2},\ \frac{3}{2},\ \frac{5}{2},\ \cdots,\ k - \frac{1}{2}$$

Huntington-Hill's

$$f(x) = \begin{cases} \lceil x \rceil & \text{if } x \geq \sqrt{\lfloor x \rfloor \lceil x \rceil} \\ \lfloor x \rfloor & \text{o/w} \end{cases}$$

geometric mean

$$0,\ \sqrt{2},\ \sqrt{6},\ \cdots,\ \sqrt{(k-1)\cdot k}$$

Dean's

$$f(x) = \begin{cases} \lceil x \rceil & \text{if } x \geq \dfrac{2}{\frac{1}{\lfloor x \rfloor} + \frac{1}{\lceil x \rceil}} \\ \lfloor x \rfloor & \text{o/w} \end{cases}$$

harmonic mean

$$0,\ \frac{4}{3},\ \frac{12}{5},\ \cdots,\ \frac{2k(k-1)}{2k-1}$$

Adams'

$$f(x) = \lceil x \rceil$$

$$0,\ 1,\ 2,\ \cdots,\ k-1$$

PROPERTIES OF DIVISOR METHODS

Divisor methods

+ avoid Alabama paradox (house monotonicity)

+ avoid population paradox (population monotonicity)

+ avoid new state paradox (coherence)

− fail quota criterion

WHEN IS AN APPORTIONMENT METHOD "FAIR"?

Hope : Each citizen's vote carries equal power.

$$\frac{s_i}{p_i} = \frac{s_j}{p_j} \quad \text{for every pair of states } i,j$$

Reality : Almost never the case 🙁

WHEN IS AN APPORTIONMENT METHOD "FAIR"?

State i shouldn't get many more seats per capita than state j.

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minimize absolute difference

$$\left| \frac{s_i}{p_i} - \frac{s_j}{p_j} \right|$$

minimize relative difference

$$\frac{\left| s_i/p_i - s_j/p_j \right|}{\min\{ s_i/p_i , s_j/p_j \}}$$

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16 TYPES OF ABSOLUTE INEQUALITY

$\left\lvert \frac{s_i}{p_i} - \frac{s_j}{p_j} \right\rvert$	$\left\lvert \frac{s_i}{p_i s_j} - \frac{1}{p_j} \right\rvert$	$\left\lvert \frac{1}{p_i} - \frac{s_j}{s_i p_j} \right\rvert$	$\left\lvert \frac{1}{s_j p_i} - \frac{1}{s_i p_j} \right\rvert$
$\left\lvert \frac{s_i p_j}{p_i} - s_j \right\rvert$	$\left\lvert \frac{s_i p_j}{s_j p_i} - 1 \right\rvert$	$\left\lvert \frac{p_j}{p_i} - \frac{s_j}{s_i} \right\rvert$	$\left\lvert \frac{p_j}{s_j p_i} - \frac{1}{s_i} \right\rvert$
$\left\lvert s_i - \frac{s_j p_i}{p_j} \right\rvert$	$\left\lvert \frac{s_i}{s_j} - \frac{p_i}{p_j} \right\rvert$	$\left\lvert 1 - \frac{s_j p_i}{s_i p_j} \right\rvert$	$\left\lvert \frac{1}{s_j} - \frac{p_i}{s_i p_j} \right\rvert$
$\left\lvert s_i p_j - s_j p_i \right\rvert$	$\left\lvert \frac{s_i p_j}{s_j} - p_i \right\rvert$	$\left\lvert p_j - \frac{s_j p_i}{s_i} \right\rvert$	$\left\lvert \frac{p_j}{s_j} - \frac{p_i}{s_i} \right\rvert$

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What does it mean to minimize the inequality?

STABILITY

No reallocation of a seat from one state to another can reduce the inequality between them.

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e.g., for the measure $\left|\frac{s_i}{p_i} - \frac{s_j}{p_j}\right|$, stability requires that

$$\left|\frac{s_i}{p_i} - \frac{s_j}{p_j}\right| < \left|\frac{s_i - 1}{p_i} - \frac{s_j + 1}{p_j}\right|.$$

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🤔 Does a given measure of inequality admit a stable apportionment method?

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Challenge : Improving inequality between states i and j may worsen the inequality w.r.t. other states.

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$\left\lvert \frac{s_i}{p_i} - \frac{s_j}{p_j} \right\rvert$	$\left\lvert \frac{s_i}{p_i s_j} - \frac{1}{p_j} \right\rvert$	$\left\lvert \frac{1}{p_i} - \frac{s_j}{s_i p_j} \right\rvert$	$\left\lvert \frac{1}{s_j p_i} - \frac{1}{s_i p_j} \right\rvert$
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Unstable

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ABSOLUTE V/S RELATIVE

		Absolute	Relative (wrt smaller)
State A	10	5	50%
State B	15		

		Absolute	Relative (wrt smaller)
State A	250,000	5	0.002 %
State B	250,005		

16 TYPES OF RELATIVE INEQUALITY

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All 16 relative inequality measures are equivalent.

Huntington-Hill's method is stable for all 16 measures

THE APPORTIONMENT OF REPRESENTATIVES IN CONGRESS*

BY

E. V. HUNTINGTON

Transactions of the American Mathematical Society

Vol. 30, No. 1, pages 85-110, 1928

GEOMETRIC MEAN IS ALL YOU NEED

Theorem: A divisor method minimizes relative disparity if and only if it rounds at geometric mean.

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Find a divisor D such that

$$f\left(\frac{p_1}{D}\right) + f\left(\frac{p_2}{D}\right) + \cdots + f\left(\frac{p_n}{D}\right) = h$$

where

$$f(x) = \begin{cases} \lceil x \rceil & \text{if } x \geq \sqrt{\lfloor x \rfloor \lceil x \rceil} \\ \lfloor x \rfloor & \text{o/w} \end{cases}$$

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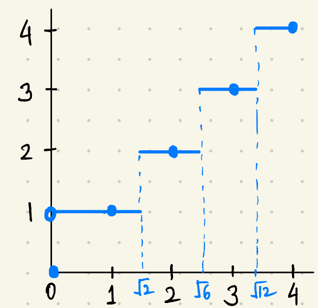

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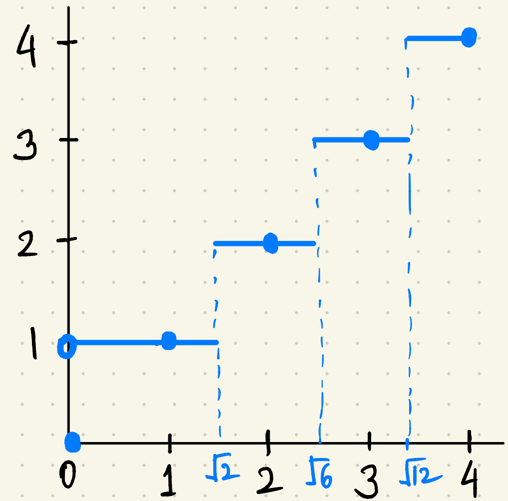

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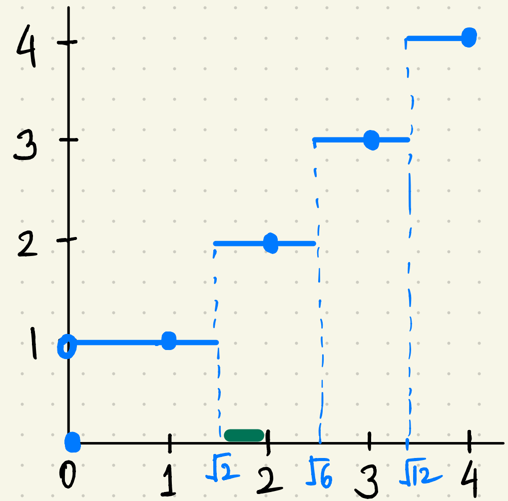
$(\Leftarrow)$

fraction x is rounded up to 2 if $\sqrt{1.2} \leq x \leq 2$



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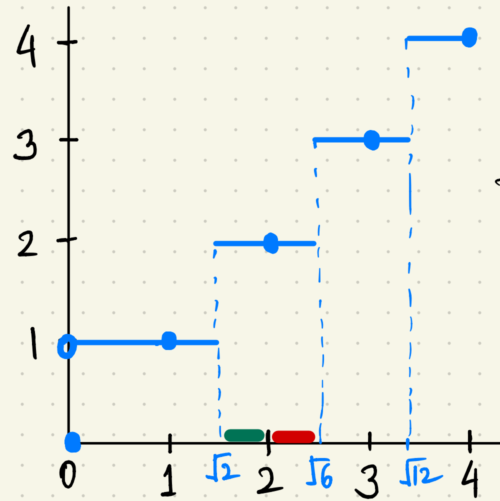
Proof:

($\Longleftarrow$)

fraction x is rounded up to 2 if $\sqrt{1\cdot 2} \leq x \leq 2$



fraction x is rounded down to 2 if $2 \leq x < \sqrt{2\cdot 3}$

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Proof: Let D be the divisor.

$(\Leftarrow)$

GEOMETRIC MEAN IS ALL YOU NEED

Theorem: A divisor method minimizes relative disparity if and only if it rounds at geometric mean.

Proof ($\Leftarrow$) **:** Let D be the divisor. Then, for every state i,

$\frac{p_i}{D}$ is rounded up to s_i when $\sqrt{(s_i - 1)\, s_i} \leq \frac{p_i}{D} \leq s_i$, and

$\frac{p_i}{D}$ is rounded down to s_i when $s_i \leq \frac{p_i}{D} < \sqrt{s_i\,(s_i + 1)}$.

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$$\sqrt{(s_i - 1)\, s_i} \leq \frac{p_i}{D} < \sqrt{s_i\,(s_i + 1)}$$

or,

$$\frac{(s_i - 1)\, s_i}{p_i^2} \leq \frac{1}{D^2} < \frac{s_i\,(s_i + 1)}{p_i^2} .$$

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Or,
$$\frac{(s_i - 1)\, s_i}{p_i^2} \leq \frac{1}{D^2} < \frac{s_i\,(s_i + 1)}{p_i^2}.$$

Then, for every pair of states i and j,

$$\frac{(s_i - 1)\, s_i}{p_i^2} < \frac{s_j\,(s_j + 1)}{p_j^2}.$$

☺

GEOMETRIC MEAN IS ALL YOU NEED

Theorem: A divisor method minimizes relative disparity if and only if it rounds at geometric mean.

Proof ($\Leftarrow$) : Suppose, for contradiction, that there exist states i and j such that $p_i/s_i \leq p_j/s_j$ and $\dfrac{p_i/s_i}{p_j/s_j} \leqslant \dfrac{p_j/(s_j+1)}{p_i/(s_i-1)}$.

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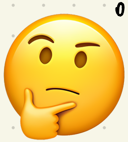
Why does "not minimizing relative disparity" imply this?



THOUGHT BUBBLE

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Minimizing relative disparity requires that for every pair of states i and j, the following holds:

$$\frac{\left| s_i/p_i - s_j/p_j \right|}{\min\left\{ s_i/p_i , s_j/p_j \right\}} < \frac{\left| \frac{s_i - 1}{p_i} - \frac{s_j + 1}{p_j} \right|}{\min\left\{ \frac{s_i - 1}{p_i} , \frac{s_j + 1}{p_j} \right\}}$$

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Not Minimizing relative disparity requires that for ~~every~~ Some pair of states i and j, the following holds:

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If state i is under-represented, i.e., $s_i/p_i < s_j/p_j$, then transferring a seat from state i to state j will necessarily make the relative disparity worse.

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$\Longrightarrow$ State i must be over-represented, i.e., $s_i/p_i \geq s_j/p_j$.

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Claim : $\frac{s_i - 1}{p_i} < \frac{s_j + 1}{p_j}$

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Proof : For any divisor method:

p_i/D rounded to $s_i \Rightarrow s_i - 1 < p_i/D$

p_j/D rounded to $s_j \Rightarrow p_j/D < s_j + 1.$

THOUGHT BUBBLE

Not minimizing relative disparity requires that for some pair of states i and j such that $s_i/p_i \geq s_j/p_j$, the following holds

$$\frac{s_i/p_i - s_j/p_j}{s_j/p_j} \geq \frac{\left| \frac{s_i - 1}{p_i} - \frac{s_j + 1}{p_j} \right|}{\boxed{\min\left\{ \frac{s_i - 1}{p_i}, \frac{s_j + 1}{p_j} \right\}}}$$

Claim : $\dfrac{s_i - 1}{p_i} < \dfrac{s_j + 1}{p_j}$

Proof : For any divisor method:

p_i/D rounded to $s_i \Rightarrow s_i - 1 < p_i/D$

p_j/D rounded to $s_j \Rightarrow p_j/D < s_j + 1$.

Combine these to get:

$$(s_i - 1)/p_i < \frac{1}{D} < (s_j + 1)/p_j \qquad \blacksquare$$

THOUGHT BUBBLE

Not minimizing relative disparity requires that for some pair of states i and j such that $s_i/p_i \geq s_j/p_j$, the following holds

$$\frac{s_i/p_i - s_j/p_j}{s_j/p_j} \geq \frac{-\left(\frac{s_i-1}{p_i} - \frac{s_j+1}{p_j}\right)}{\frac{s_i-1}{p_i}}$$

THOUGHT BUBBLE

Not minimizing relative disparity requires that for some pair of states i and j such that $s_i/p_i \geq s_j/p_j$, the following holds

$$\frac{s_i/p_i - s_j/p_j}{s_j/p_j} \geq \frac{-\left(\frac{s_i-1}{p_i} - \frac{s_j+1}{p_j}\right)}{\frac{s_i-1}{p_i}}$$

$$\Longleftrightarrow \qquad \frac{p_j/(s_j+1)}{p_i/(s_i-1)} \geq \frac{p_i/s_i}{p_j/s_j}$$

GEOMETRIC MEAN IS ALL YOU NEED

Theorem: A divisor method minimizes relative disparity if and only if it rounds at geometric mean.

Proof $(\Leftarrow)$: Suppose, for contradiction, that there exist states i and j such that $p_i/s_i \leq p_j/s_j$ and $\boxed{\dfrac{p_i/s_i}{p_j/s_j} \leqslant \dfrac{p_j/(s_j+1)}{p_i/(s_i-1)}}$.

GEOMETRIC MEAN IS ALL YOU NEED

Theorem: A divisor method minimizes relative disparity if and only if it rounds at geometric mean.

Proof: Suppose, for contradiction, that there exist states i and j such that $p_i/s_i \leq p_j/s_j$ and $\dfrac{p_i/s_i}{p_j/s_j} \ll \dfrac{p_j/(s_j+1)}{p_i/(s_i-1)}$.

($\Leftarrow$)

Then, $\dfrac{s_j\,(s_j+1)}{p_j^2} \ll \dfrac{(s_i-1)\,s_i}{p_i^2}$, contradicting ☺.

GEOMETRIC MEAN IS ALL YOU NEED

Theorem: A divisor method minimizes relative disparity if and only if it rounds at geometric mean.

Proof ($\Leftarrow$): Suppose, for contradiction, that there exist states i and j such that $p_i/s_i \le p_j/s_j$ and $\dfrac{p_i/s_i}{p_j/s_j} < \dfrac{p_j/(s_j+1)}{p_i/(s_i-1)}$.

Then, $$\frac{s_j(s_j+1)}{p_j^2} < \frac{(s_i-1)\,s_i}{p_i^2},$$ contradicting ☺.

Therefore, round at GM $\Longrightarrow$ min. relative disparity. $\blacksquare$

GEOMETRIC MEAN IS ALL YOU NEED

Theorem: A divisor method minimizes relative disparity if and only if it rounds at geometric mean.

Proof:

($\Rightarrow$)

GEOMETRIC MEAN IS ALL YOU NEED

Theorem: A divisor method minimizes relative disparity if and only if it rounds at geometric mean.

Proof: Suppose, for every pair of states i and j s.t. $p_i/s_i \le p_j/s_j$,

($\Longrightarrow$) it holds that

$$\boxed{\frac{p_i/s_i}{p_j/s_j} > \frac{p_j/(s_j+1)}{p_i/(s_i-1)}}.$$

Verify!

GEOMETRIC MEAN IS ALL YOU NEED

Theorem: A divisor method minimizes relative disparity if and only if it rounds at geometric mean.

Proof: Suppose, for every pair of states i and j s.t. $p_i/s_i \leq p_j/s_j$,

($\Rightarrow$) it holds that $\dfrac{p_i/s_i}{p_j/s_j} > \dfrac{p_j/(s_j+1)}{p_i/(s_i-1)}$.

Then, $\dfrac{s_j(s_j+1)}{p_j^2} > \dfrac{(s_i-1)s_i}{p_i^2}$ for every pair of states i and j s.t. $p_i/s_i \leq p_j/s_j$.

GEOMETRIC MEAN IS ALL YOU NEED

Theorem: A divisor method minimizes relative disparity if and only if it rounds at geometric mean.

Proof: ($\Rightarrow$) Suppose, for every pair of states i and j s.t. $p_i/s_i \leq p_j/s_j$, it holds that

$$\frac{p_i/s_i}{p_j/s_j} > \frac{p_j/(s_j+1)}{p_i/(s_i-1)}.$$

Then,

$$\underbrace{\frac{s_j(s_j+1)}{p_j^2}}_{\substack{\text{smallest} \\ \text{for } j^* \text{ (say)}}} > \underbrace{\frac{(s_i-1)s_i}{p_i^2}}_{\substack{\text{largest} \\ \text{for } i^* \text{ (say)}}}$$

for every pair of states i and j s.t. $p_i/s_i \leq p_j/s_j$.

GEOMETRIC MEAN IS ALL YOU NEED

Theorem: A divisor method minimizes relative disparity if and only if it rounds at geometric mean.

Proof: Then, there must exist $D > 0$ such that
$(\Rightarrow)$

$$\frac{s_{j^*}(s_{j^*}+1)}{p_{j^*}^2} > \frac{1}{D^2} \geq \frac{(s_{i^*}-1)\, s_{i^*}}{p_{i^*}^2} .$$

GEOMETRIC MEAN IS ALL YOU NEED

Theorem: A divisor method minimizes relative disparity if and only if it rounds at geometric mean.

Proof: Then, there must exist $D > 0$ such that
$(\Rightarrow)$

$$\frac{s_{j^*}(s_{j^*}+1)}{p_{j^*}^2} > \frac{1}{D^2} \geq \frac{(s_{i^*}-1)\, s_{i^*}}{p_{i^*}^2}.$$

For this choice of D, rounding p_i/D at geometric mean necessarily results in $(s_1, s_2, \ldots, s_n)$. [Exercise] ∎

QUIZ

QUIZ

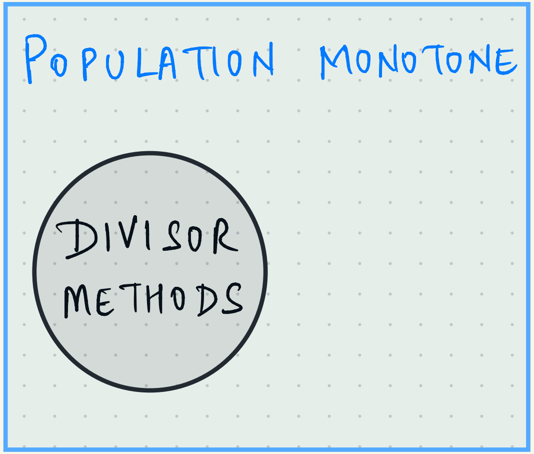
Prove that Webster's method is stable under the inequality measure $\left|\frac{s_i}{p_i} - \frac{s_j}{p_j}\right|$.

PICTURE SO FAR

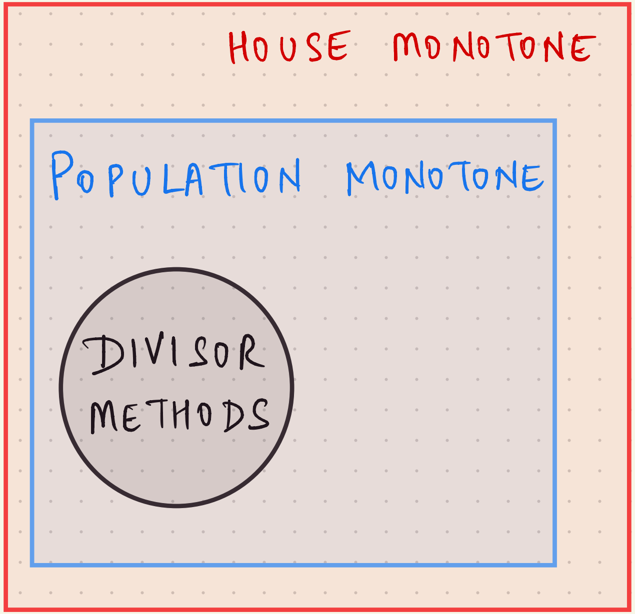
PICTURE SO FAR

DIVISOR METHODS

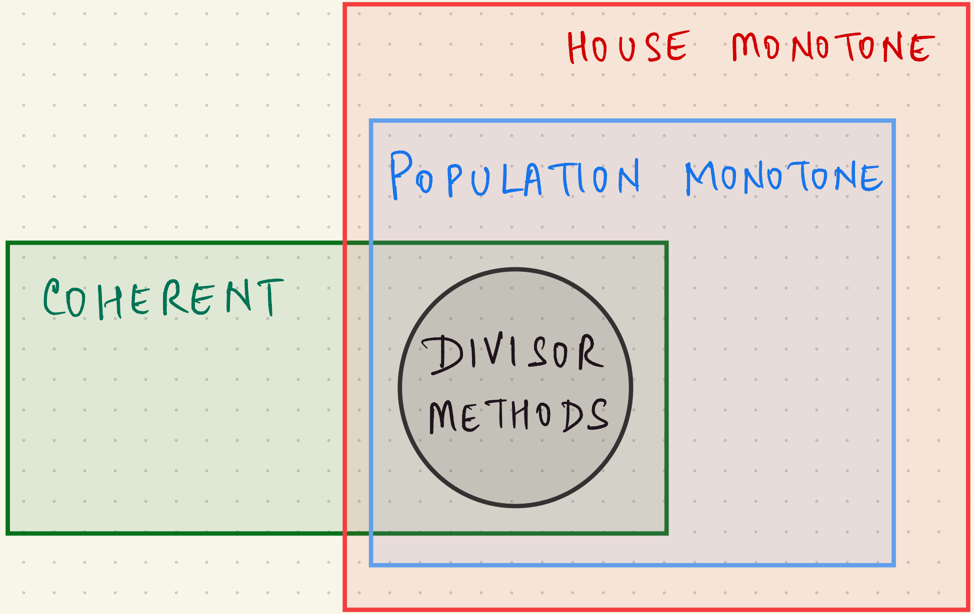
PICTURE SO FAR



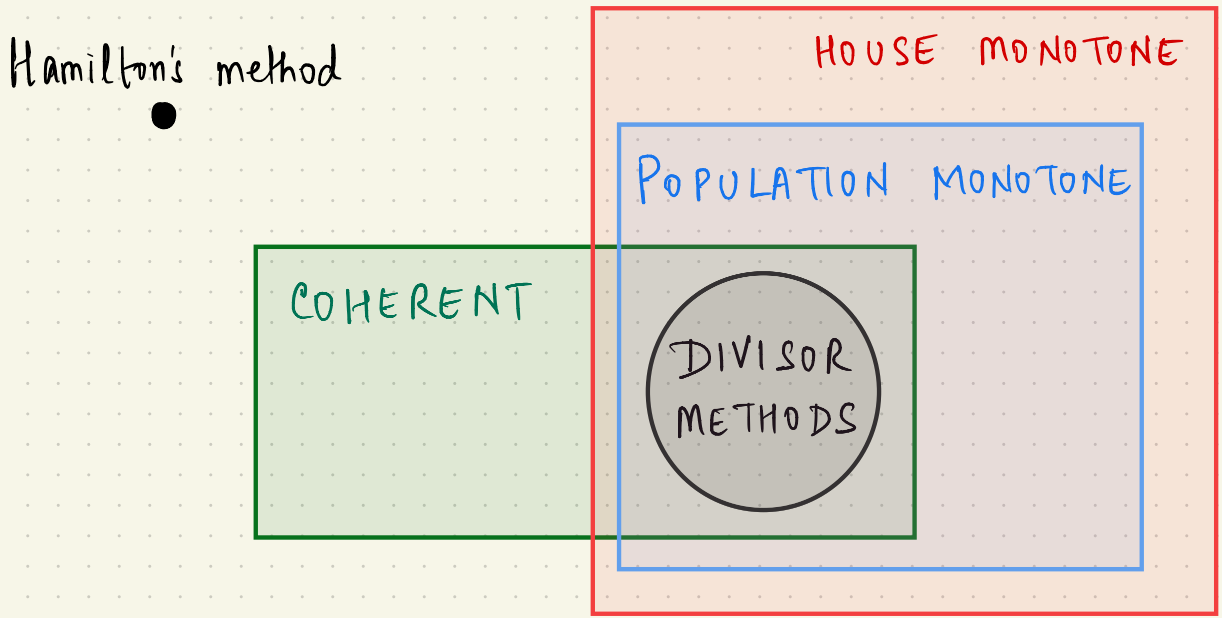
PICTURE SO FAR



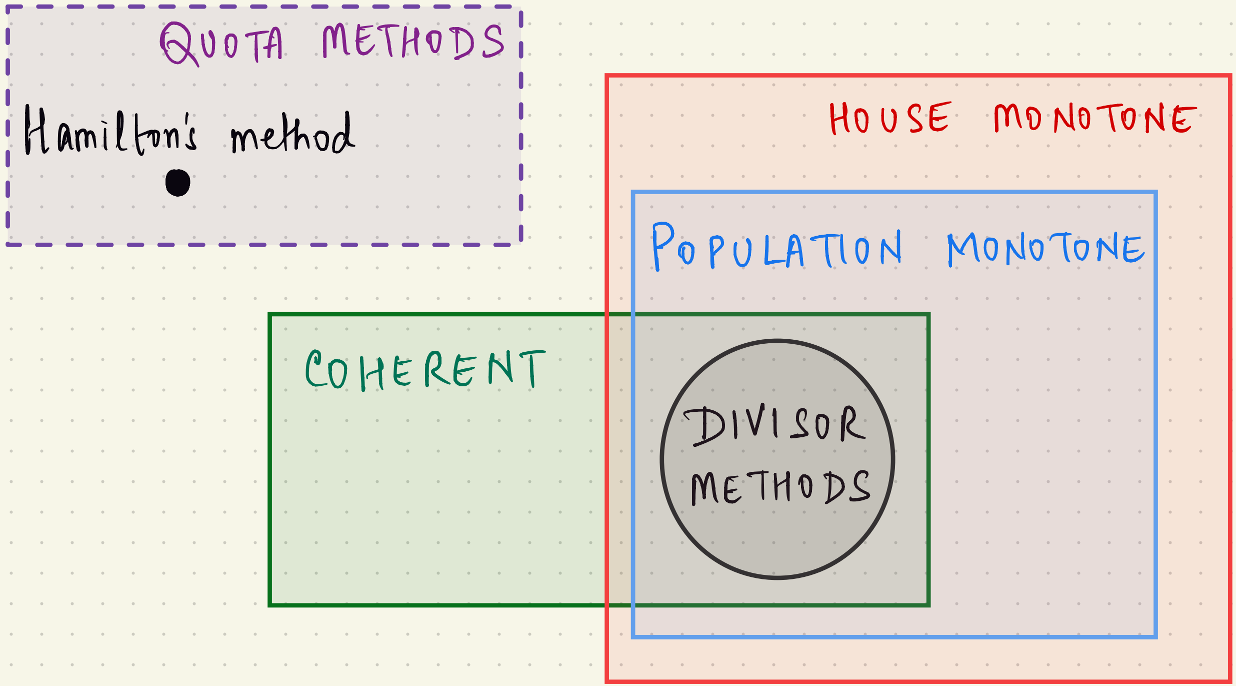
PICTURE SO FAR



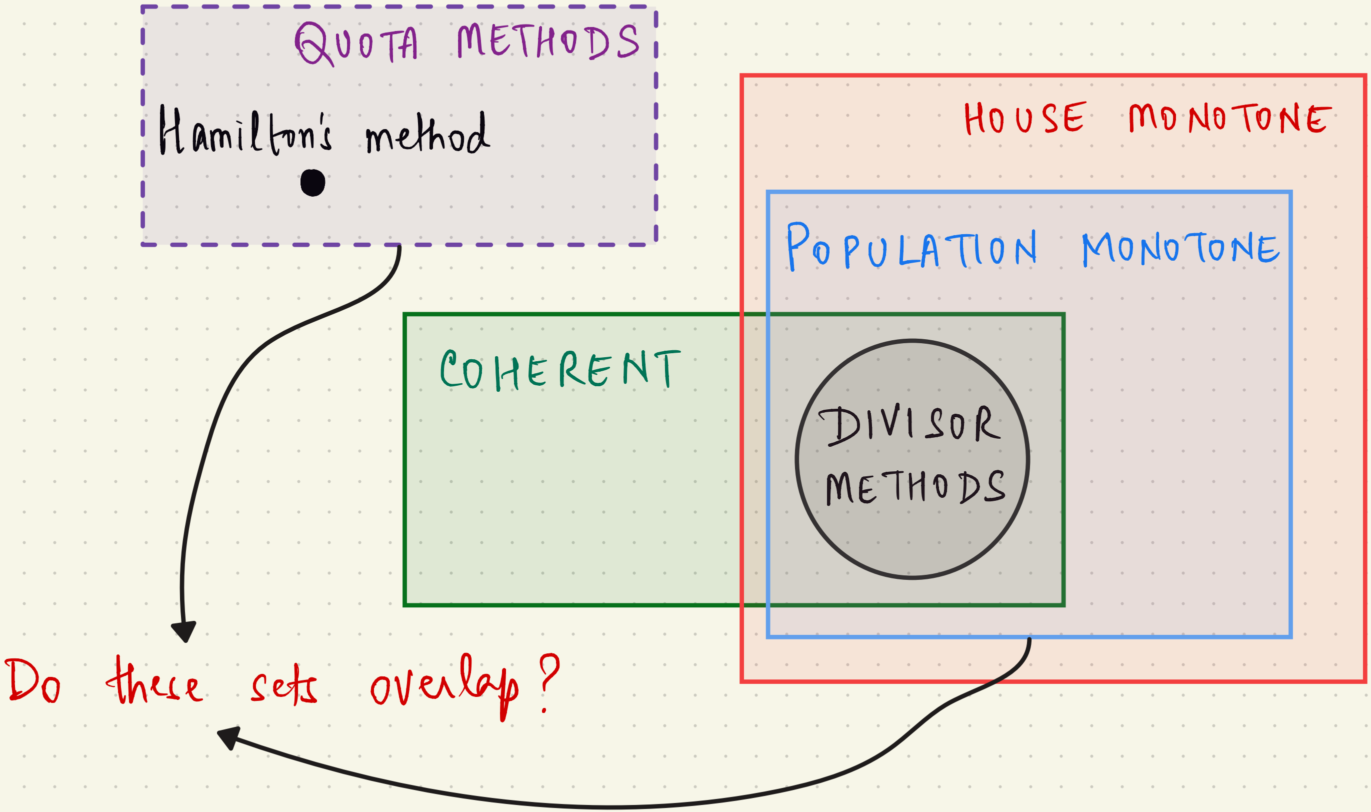
PICTURE SO FAR



PICTURE SO FAR



PICTURE SO FAR



MICHEL LOUIS BALINSKI

* Faculty of Mathematics at CUNY from 1965-1977

* Leading expert in operations research especially, integer programming.

* Notable contributions to diameter of polytopes, primal-dual algorithms, stable matchings, and voting theory.

1933 - 2019

HOBART PEYTON YOUNG

* Faculty of Mathematics at CUNY from 1971 - 1976
* Notable contributions to evolutionary game theory and voting theory
* The famous Kemeny - Young method is named after him.

1945 - Present

Fair Representation

Meeting the Ideal of One Man, One Vote

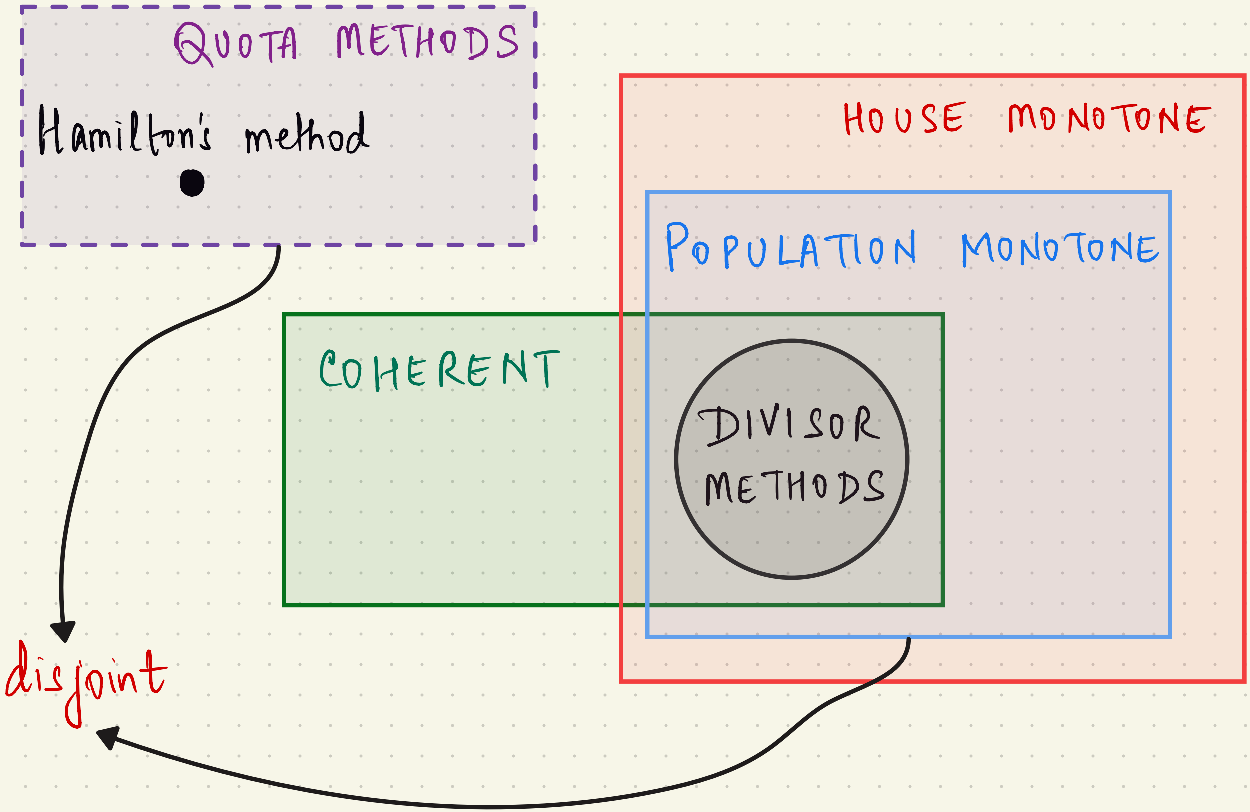
SECOND EDITION

MICHEL L. BALINSKI AND
H. PEYTON YOUNG

BALINSKI - YOUNG IMPOSSIBILITY

Theorem : Population monotonicity is incompatible with quota criterion.

PICTURE SO FAR



BALINSKI - YOUNG IMPOSSIBILITY

Theorem : Population monotonicity is incompatible with quota criterion.

Corollary : No divisor method satisfies the quota criterion.

BALINSKI - YOUNG IMPOSSIBILITY

Theorem : Population monotonicity is incompatible with quota criterion.

Proof :

In This Apportionment Lottery, the House Always Wins

Paul Gölz[1], Dominik Peters[2], and Ariel D. Procaccia[3]

BALINSKI - YOUNG IMPOSSIBILITY

Theorem : Population monotonicity is incompatible with quota criterion.

Proof :

$h = 10$

	profile A		profile B		profile C	
State	p_i^A	q_i^A	p_i^B	q_i^B	p_i^C	q_i^C
1	824	8.24	824	6.99	824	9.02
2	44	0.44	44	0.37	1	0.01
3	44	0.44	44	0.37	1	0.01
4	44	0.44	44	0.37	44	0.48
5	44	0.44	222	1.88	44	0.48

RECALL : POPULATION MONOTONICITY

An apportionment method is population monotone if, for any two problem instances $I = (h; p_1, \cdots, p_n)$ and $I' = (h'; p_1', \cdots, p_n')$ with seat assignments $I \longmapsto (s_1, s_2, \cdots, s_n)$ and $I' \longmapsto (s_1', s_2', \cdots, s_n')$,

$$\underbrace{s_i < s_i'}_{\text{i gets more seats}} \text{ and } \underbrace{s_j > s_j'}_{\text{j gets fewer seats}} \Rightarrow \underbrace{p_i < p_i'}_{\text{i grows}} \text{ or } \underbrace{p_j > p_j'}_{\text{j shrinks}}.$$

BALINSKI - YOUNG IMPOSSIBILITY

Theorem : Population monotonicity is incompatible with quota criterion.

Proof :

$h = 10$	profile A		profile B		profile C	
State	p_i^A	q_i^A	p_i^B	q_i^B	p_i^C	q_i^C
1	824	8.24	824	6.99	824	9.02
2	44	0.44	44	0.37	1	0.01
3	44	0.44	44	0.37	1	0.01
4	44	0.44	44	0.37	44	0.48
5	44	0.44	222	1.88	44	0.48

BALINSKI - YOUNG IMPOSSIBILITY

Theorem : Population monotonicity is incompatible with quota criterion.

Proof :

On profile A, state 1 should receive 8 or 9 seats (by quota).

$h=10$	profile A		profile B		profile C	
State	p_i^A	q_i^A	p_i^B	q_i^B	p_i^C	q_i^C
1	824	8.24	824	6.99	824	9.02
2	44	0.44	44	0.37	1	0.01
3	44	0.44	44	0.37	1	0.01
4	44	0.44	44	0.37	44	0.48
5	44	0.44	222	1.88	44	0.48

BALINSKI - YOUNG IMPOSSIBILITY

Theorem : Population monotonicity is incompatible with quota criterion.

Proof :

$h=10$

State	profile A p_i^A	q_i^A	profile B p_i^B	q_i^B	profile C p_i^C	q_i^C
1	824	8.24	824	6.99	824	9.02
2	44	0.44	44	0.37	1	0.01
3	44	0.44	44	0.37	1	0.01
4	44	0.44	44	0.37	44	0.48
5	44	0.44	222	1.88	44	0.48

On profile A, state 1 should receive 8 or 9 seats (by quota).

* 8 seats : pop. mon. violated for profile C

Balinski - Young Impossibility

Theorem : Population monotonicity is incompatible with quota criterion.

Proof :

$h = 10$

	profile A		profile B		profile C	
State	p_i^A	q_i^A	p_i^B	q_i^B	p_i^C	q_i^C
1	824	8.24	824	6.99	824	9.02
2	44	0.44	44	0.37	1	0.01
3	44	0.44	44	0.37	1	0.01
4	44	0.44	44	0.37	44	0.48
5	44	0.44	222	1.88	44	0.48

On profile A, state 1 should receive 8 or 9 seats (by quota).

* 8 seats : pop. mon. violated for profile C

* 9 seats : pop. mon. violated for profile B

BALINSKI - YOUNG IMPOSSIBILITY

Theorem : Population monotonicity is incompatible with quota criterion.

Proof :

Suppose state 1 gets 9 seats in profile A.

$h = 10$

	profile A		profile B		profile C	
State	p_i^A	q_i^A	p_i^B	q_i^B	p_i^C	q_i^C
1	824	8.24 → 9	824	6.99	824	9.02
2	44	0.44	44	0.37	1	0.01
3	44	0.44	44	0.37	1	0.01
4	44	0.44	44	0.37	44	0.48
5	44	0.44	222	1.88	44	0.48

BALINSKI - YOUNG IMPOSSIBILITY

Theorem : Population monotonicity is incompatible with quota criterion.

Proof :

Suppose state 1 gets 9 seats in profile A.

WOLOG, seat assignment is (9, 0, 0, 0, 1).

$h = 10$

	profile A		profile B		profile C	
State	p_i^A	q_i^A	p_i^B	q_i^B	p_i^C	q_i^C
1	824	8.24 → 9	824	6.99	824	9.02
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BALINSKI - YOUNG IMPOSSIBILITY

Theorem : Population monotonicity is incompatible with quota criterion.

Proof :

$h = 10$	profile A		profile B		profile C	
State	p_i^A	q_i^A	p_i^B	q_i^B	p_i^C	q_i^C
1	824	8.24 → 9	824	6.99	824	9.02
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3	44	0.44	44	0.37	1	0.01
4	44	0.44	44	0.37	44	0.48
5	44	0.44	222	1.88	44	0.48

Suppose state 1 gets 9 seats in profile A.

WOLOG, seat assignment is (9, 0, 0, 0, 1).

On profile B, quota constraint implies:

state 1 : ≤ 7 seats

state 5 : ≤ 2 seats

BALINSKI - YOUNG IMPOSSIBILITY

Theorem : Population monotonicity is incompatible with quota criterion.

Proof :

$h = 10$

	profile A		profile B		profile C	
State	p_i^A	q_i^A	p_i^B	q_i^B	p_i^C	q_i^C
1	824	$\overset{9}{8.24}$	824	6.99	824	9.02
2	44	0.44	44	0.37	1	0.01
3	44	0.44	44	0.37	1	0.01
4	44	0.44	44	0.37	44	0.48
5	44	0.44	222	1.88	44	0.48

Suppose state 1 gets 9 seats in profile A.
WOLOG, seat assignment is (9, 0, 0, 0, 1).

On profile B, quota constraint implies:

state 1 : ≤ 7 seats

state 5 : ≤ 2 seats

$\implies \geq 1$ seat to state 2, 3, or 4.

BALINSKI - YOUNG IMPOSSIBILITY

Theorem: Population monotonicity is incompatible with quota criterion.

Proof:

$h = 10$

	profile A		profile B		profile C	
State	p_i^A	q_i^A	p_i^B	q_i^B	p_i^C	q_i^C
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2	44	0.44	44	0.37	1	0.01
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5	44	0.44	222	1.88	44	0.48

Suppose state 1 gets 9 seats in profile A.

WOLOG, seat assignment is $(9, 0, 0, 0, 1)$.

On profile B, quota constraint implies:

state 1 : ≤ 7 seats

state 5 : ≤ 2 seats

$\implies \geq 1$ seat to state 2, 3, or 4.

Overall: $A \to B$

State 1 loses a seat, state 2 (say) gains a seat without change in population

BALINSKI - YOUNG IMPOSSIBILITY

Theorem : Population monotonicity is incompatible with quota criterion.

Proof :

$h = 10$

	profile A		profile B		profile C	
State	p_i^A	q_i^A	p_i^B	q_i^B	p_i^C	q_i^C
1	824	8.24 → 9	824	6.99	824	9.02
2	44	0.44	44	0.37	1	0.01
3	44	0.44	44	0.37	1	0.01
4	44	0.44	44	0.37	44	0.48
5	44	0.44	222	1.88	44	0.48

Suppose state 1 gets 9 seats in profile A.

WOLOG, seat assignment is (9, 0, 0, 0, 1).

On profile B, quota constraint implies:

state 1 : ≤ 7 seats

state 5 : ≤ 2 seats

$\Rightarrow \geq 1$ seat to state 2, 3, or 4.

Overall : State 1 loses a seat, state 2 (say) gains a seat
$A \to B$

without change in population $\Rightarrow$ Violates population monotonicity.

BALINSKI - YOUNG IMPOSSIBILITY

Theorem : Population monotonicity is incompatible with quota criterion.

Proof :

Suppose state 1 gets 8 seats in profile A.

$h = 10$

State	profile A p_i^A	q_i^A	profile B p_i^B	q_i^B	profile C p_i^C	q_i^C
1	824	8.24 (8)	824	6.99	824	9.02
2	44	0.44	44	0.37	1	0.01
3	44	0.44	44	0.37	1	0.01
4	44	0.44	44	0.37	44	0.48
5	44	0.44	222	1.88	44	0.48

BALINSKI - YOUNG IMPOSSIBILITY

Theorem : Population monotonicity is incompatible with quota criterion.

Proof :

Suppose state 1 gets 8 seats in profile A.

WOLOG, seat assignment is (8, 0, 0, 1, 1).

$h = 10$

State	profile A p_i^A	q_i^A	profile B p_i^B	q_i^B	profile C p_i^C	q_i^C
1	824	8.24 (8)	824	6.99	824	9.02
2	44	0.44	44	0.37	1	0.01
3	44	0.44	44	0.37	1	0.01
4	44	0.44	44	0.37	44	0.48
5	44	0.44	222	1.88	44	0.48

BALINSKI - YOUNG IMPOSSIBILITY

Theorem : Population monotonicity is incompatible with quota criterion.

Proof :

$h = 10$

State	profile A p_i^A	q_i^A	profile B p_i^B	q_i^B	profile C p_i^C	q_i^C
1	824	8.24 → 8	824	6.99	824	9.02
2	44	0.44	44	0.37	1	0.01
3	44	0.44	44	0.37	1	0.01
4	44	0.44	44	0.37	44	0.48
5	44	0.44	222	1.88	44	0.48

Suppose state 1 gets 8 seats in profile A.

WOLOG, seat assignment is (8, 0, 0, 1, 1).

On profile C, quota constraint implies:

state 1 : ≥ 9 seats

BALINSKI - YOUNG IMPOSSIBILITY

Theorem : Population monotonicity is incompatible with quota criterion.

Proof :

$h = 10$

State	profile A p_i^A	q_i^A	profile B p_i^B	q_i^B	profile C p_i^C	q_i^C
1	824	8.24 → 8	824	6.99	824	9.02
2	44	0.44	44	0.37	1	0.01
3	44	0.44	44	0.37	1	0.01
4	44	0.44	44	0.37	44	0.48
5	44	0.44	222	1.88	44	0.48

Suppose state 1 gets 8 seats in profile A.

WOLOG, seat assignment is $(8, 0, 0, 1, 1)$.

On profile C, quota constraint implies:

state 1 : ≥ 9 seats

$\Rightarrow$ at least one of states 4 or 5 receives 0 seats

BALINSKI - YOUNG IMPOSSIBILITY

Theorem : Population monotonicity is incompatible with quota criterion.

Proof :

$h = 10$

State	profile A p_i^A	q_i^A	profile B p_i^B	q_i^B	profile C p_i^C	q_i^C
1	824	8.24 — 8	824	6.99	824	9.02
2	44	0.44	44	0.37	1	0.01
3	44	0.44	44	0.37	1	0.01
4	44	0.44	44	0.37	44	0.48
5	44	0.44	222	1.88	44	0.48

Suppose state 1 gets 8 seats in profile A.
WOLOG, seat assignment is (8, 0, 0, 1, 1).

On profile C, quota constraint implies:

state 1 : ≥ 9 seats

$\Rightarrow$ at least one of states 4 or 5 receives 0 seats

Overall : $A \rightarrow C$ State 1 gains a seat, state 4 (say) loses a seat without change in population

BALINSKI - YOUNG IMPOSSIBILITY

Theorem: Population monotonicity is incompatible with quota criterion.

Proof :

$h=10$	profile A		profile B		profile C	
State	p_i^A	q_i^A	p_i^B	q_i^B	p_i^C	q_i^C
1	824	8.24	824	6.99	824	9.02
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4	44	0.44	44	0.37	44	0.48
5	44	0.44	222	1.88	44	0.48

8

Suppose state 1 gets 8 seats in profile A.
WOLOG, seat assignment is $(8, 0, 0, 1, 1)$.

On profile C, quota constraint implies:

state 1 : ≥ 9 seats

$\Rightarrow$ at least one of states 4 or 5 receives 0 seats

Overall : State 1 gains a seat, state 4 (say) loses a seat
$A \to C$
without change in population $\Rightarrow$ Violates population monotonicity. ∎

NEXT LECTURE

Circumventing Balinski - Young Impossibility