COL 8184 : Algorithms for Fair Representation

Lecture 3

Apportionment Paradoxes (Contd.) & Axioms

Jan 12, 2026 | Rohit Vaish


U.S. Constitution enacted
1789

Hamilton's method rejected,
Jefferson's method adopted

U.S. Constitution enacted

1789

1792

HAMILTON'S METHOD

(aka method of largest remainder)

* Assign each state i its lower quota $\lfloor q_i \rfloor$.

* Assign remaining seats one at a time to the state with the largest remainder $r_i = q_i - \lfloor q_i \rfloor$.

JEFFERSON'S METHOD

* Let D be a divisor such that

$$\left\lfloor \frac{p_1}{D} \right\rfloor + \left\lfloor \frac{p_2}{D} \right\rfloor + \cdots + \left\lfloor \frac{p_n}{D} \right\rfloor = h .$$

* Assign $s_i = \left\lfloor \frac{p_i}{D} \right\rfloor$ to state i.


Hamilton's method rejected,
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Large-state bias
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Adams' method
rejected by Congress

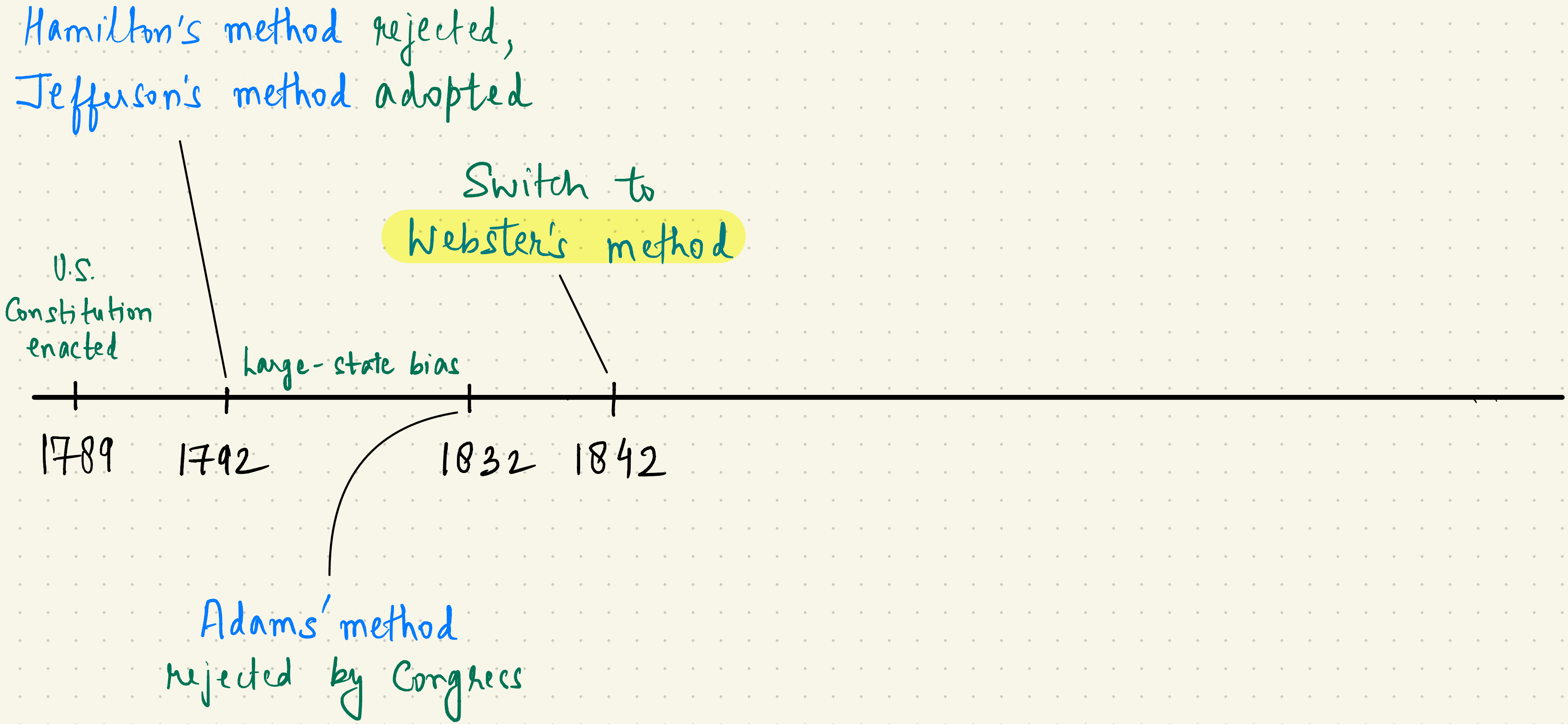
ADAMS' METHOD

* Let D be a divisor such that

$$\left\lceil \frac{p_1}{D} \right\rceil + \left\lceil \frac{p_2}{D} \right\rceil + \cdots + \left\lceil \frac{p_n}{D} \right\rceil = h.$$

* Assign $s_i = \left\lceil \frac{p_i}{D} \right\rceil$ to state i.

A round-up version of Jefferson's method.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Switch to
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WEBSTER'S METHOD

* Let D be a divisor such that

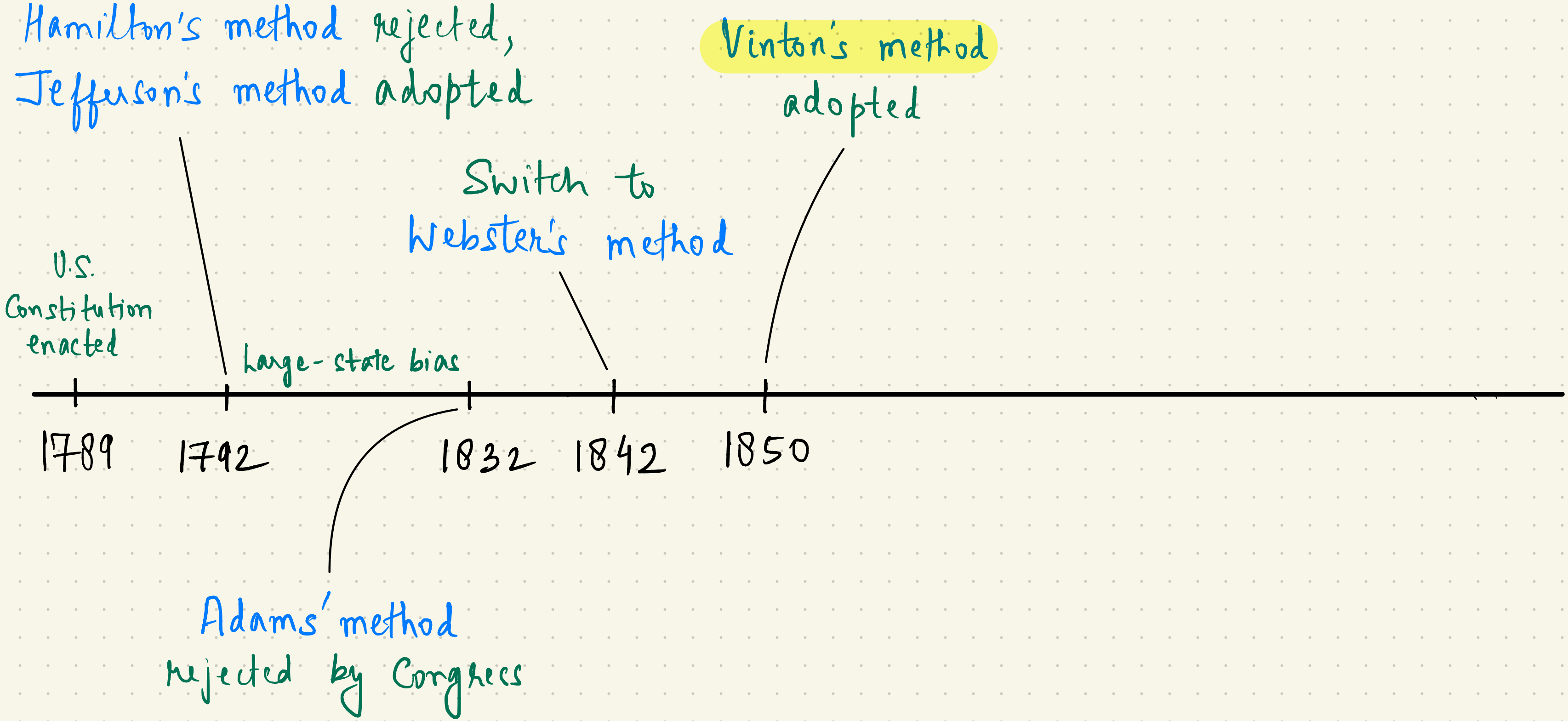
$$\left[\frac{p_1}{D}\right] + \left[\frac{p_2}{D}\right] + \cdots + \left[\frac{p_n}{D}\right] = h.$$

* Assign $s_i = \left[\frac{p_i}{D}\right]$ to state i.

$$[x] = \begin{cases} \lceil x \rceil & \text{if } \operatorname{frac}(x) \geq 0.5 \\ \lfloor x \rfloor & \text{o/w} \end{cases}$$

WEBSTER IS "UNBIASED"

State	Population	Quota	Adams Appt.	Webster	Jefferson Appt. (Polk Bill)
New York	1,918,578	38.593	37	39	40
Pennsylvania	1,348,072	27.117	26	27	28
Kentucky	621,832	12.509	12	12	13
Vermont	280,657	5.646	6	6	5
Louisiana	171,904	3.458	4	3	3
Illinois	157,147	3.161	4	3	3
Missouri	130,419	2.623	3	3	2
Mississippi	110,358	2.220	3	2	2
Delaware	75,432	1.517	2	2	1
U.S. Total	11,931,000	240	240		240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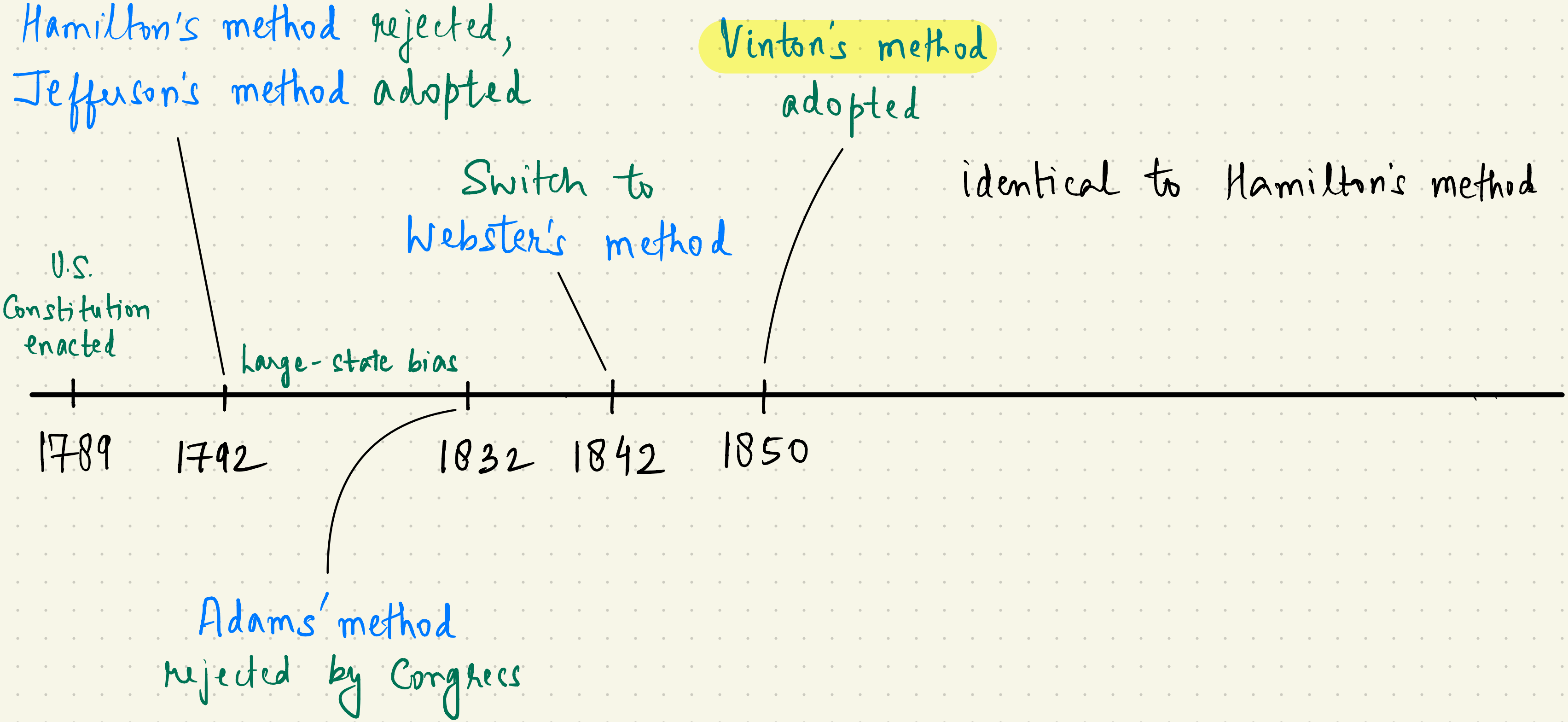
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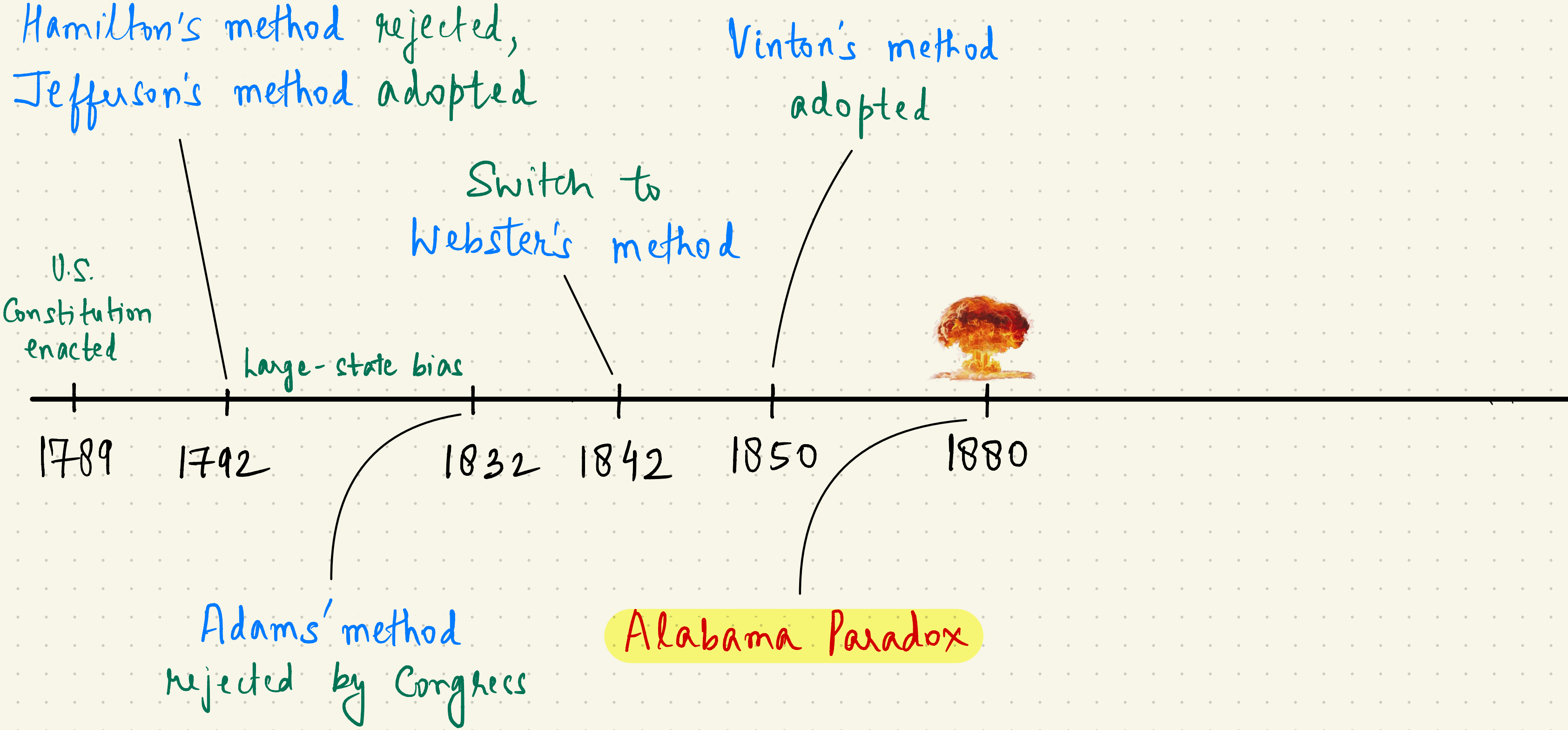
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Adams' method rejected by Congress
Alabama Paradox

* Chief Clerk of Census Office

* Using 1880 Census results, Calculated Hamilton/Vinton outcomes for all house sizes between 275 and 350.

C.W. SEATON

299 seats are to be allocated. Total population is ~~49,713,370~~ 49,371,340 and the appropriate divisor is 165,120.

	Alabama	*Texas*	*Illinois*
Population	1,262,505	1,591,749	3,077,871
"Raw" allocation	7.646	9.640	18.640
Seats in first round	7	9	18
Fractional part	0.646	0.640	0.640
Additional seats	1	0	0
Total seats	8	9	18

Now 300 seats are to be allocated. The appropriate divisor is 164,580.

	Alabama	*Texas*	*Illinois*
Population	1,262,505	1,591,749	3,077,871
"Raw" allocation	7.671	9.672	18.701
Seats in first round	7	9	18
Fractional part	0.671	0.672	0.701
Additional seats	0	1	1
Total seats	7	10	19

Alabama loses one seat; Texas and Illinois each gain a seat.

House size grows but Alabama loses a seat!

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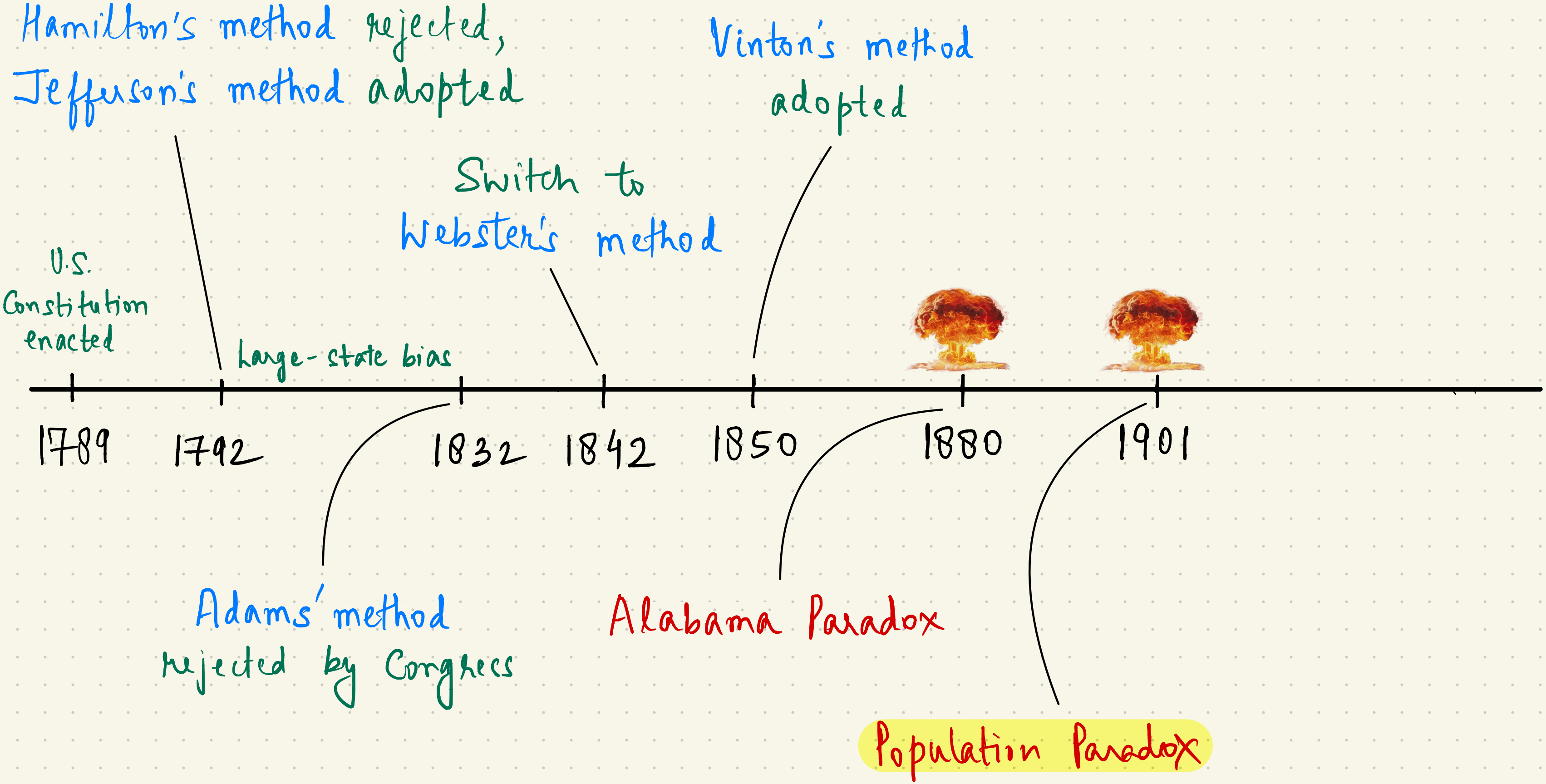
Alabama loses one seat; Texas and Illinois each gain a seat.

Why? Larger states (Texas, Illinois) outpace Alabama.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Population Paradox

POPULATION PARADOX

	1900	*Seats*		*1901*	*Seats*	
	Population	*raw*	*rounded*	*Population*	*raw*	*rounded*
Virginia	1,854,184	9.599*	10	1,873,951	9.509	9
Maine	694,466	3.595	3	699,114	3.548*	4
Total	74,562,608		386	76,069,522		386

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Virginia lost a seat to Maine even though Virginia's population grew more rapidly [+1.06% (V) and +0.7% (M)]

Why? US population grew by 2.02%, faster than either state.

POPULATION PARADOX

House size $h = 10$

State	Population	Quota q_i	Seats
A	145	1.45	2
B	340	3.40	3
C	515	5.15	5
	1000		

State	Population	Quota q_i	Seats
A	147 ↑	1.55	1
B	338 ↓	3.56	4
C	465 ↓	4.89	5
	950 ↓		

U.S. Constitution enacted

1789

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Large-state bias

Adams' method rejected by Congress

1832

Switch to Webster's method

1842

Vinton's method adopted

1850

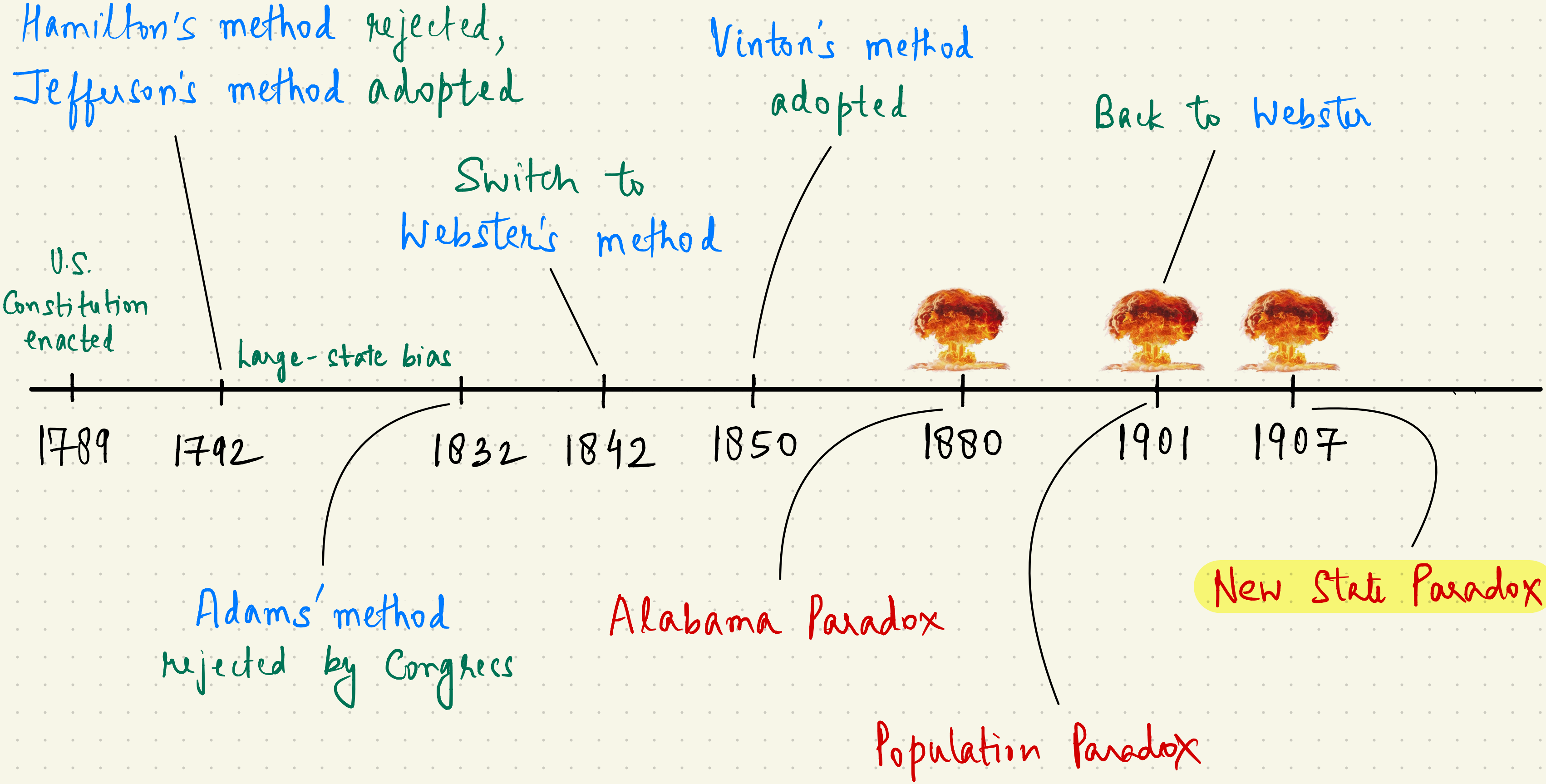
1880

Alabama Paradox

Back to Webster

1901

Population Paradox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New State Paradox

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* Before Oklahoma joined :
  US population $\approx$ 74 mil , House size = 386 $\Rightarrow$ 0.19 mil / seat

* Oklahoma's population $\approx$ 1 mil

* After Oklahoma joined :
  US population $\approx$ 75 mil , House size = 391

* Hope : Seat assignments of other states won't be affected.

NEW STATE PARADOX

	Population before incl. of Oklahoma	Seats		Population after incl. of Oklahoma	Seats	
		raw	rounded		raw	rounded
New York	7,264,183	37.606*	38	7,264,183	37.589	37
Maine	694,466	3.595	3	694,466	3.594*	4
Oklahoma	—	—	—	1,000,000	5.175	5
Total	74,562,608		386	75,562,608		391

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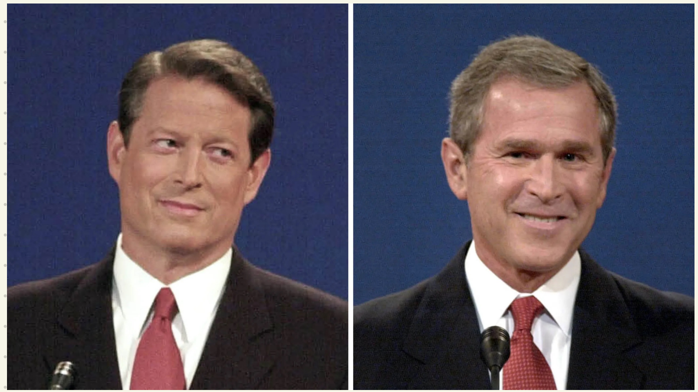
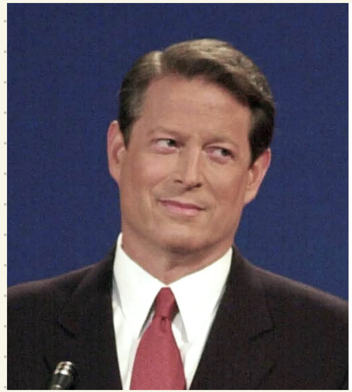
Why? Larger state's (NY) quota decreases faster.

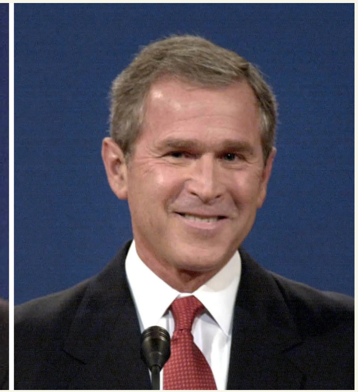


Winner under

Jefferson's method

266 - 271





Winner under

Huntington - Hill method

271 - 266

Method	*Hamilton*	*Jefferson*	*Adams*	*Webster*
Bias toward:	large states	large states	small states	Neither
Paradox				
Alabama	Yes	No	No	No
Population	Yes	No	No	No
New State	Yes	No	No	No

Divisor methods

Method	*Hamilton*	*Jefferson*	*Adams*	*Webster*
Bias toward:	large states	large states	small states	Neither
Paradox				
Alabama	Yes	No	No	No
Population	Yes	No	No	No
New State	Yes	No	No	No

RECALL JEFFERSON'S METHOD

* Let D be a divisor such that

$$\left\lfloor \frac{p_1}{D} \right\rfloor + \left\lfloor \frac{p_2}{D} \right\rfloor + \cdots + \left\lfloor \frac{p_n}{D} \right\rfloor = h.$$

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* Assign $s_i = \left\lfloor \frac{p_i}{D} \right\rfloor$ to state i.

🤔 Does a Jefferson divisor always exist?

🤔 Do all Jefferson divisors give the same seat assignment?

🤔 Can Jefferson outcome be efficiently computed?

JEFFERSON'S METHOD : "TABLE" DEFINITION

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Given the populations $p_1, p_2, \cdots, p_n$ and house size h.

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$$\iff \frac{p_i}{D} \geq 1$$

$$\iff D \leq p_i$$

JEFFERSON'S METHOD : "TABLE" DEFINITION

Given the populations $p_1, p_2, \cdots, p_n$ and house size h.

State i gets at least k seats $\iff \left\lfloor \frac{p_i}{D} \right\rfloor \geq k$

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$$\iff D \leq \frac{p_i}{k}$$

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State i gets at least k seats $\iff D \leq \frac{p_i}{k}$

$\frac{p_i}{h}$	$\frac{p_i}{h-1}$	- - -	$\frac{p_i}{3}$	$\frac{p_i}{2}$	$\frac{p_i}{1}$

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$\frac{p_i}{h}$	$\frac{p_i}{h-1}$	- - -	$\frac{p_i}{3}$	$\frac{p_i}{2}$	$\frac{p_i}{1}$

For any D, we can determine no. of state i's seats by counting how many of the entries are $\geq D$.

JEFFERSON'S METHOD : "TABLE" DEFINITION

State i gets at least k seats $\iff D \leq \frac{p_i}{k}$

$\frac{p_i}{h}$	$\frac{p_i}{h-1}$	- - -	$\frac{p_i}{3}$	$\frac{p_i}{2}$	$\frac{p_i}{1}$

e.g., if $\frac{p_i}{3} < D \leq \frac{p_i}{2}$, then state i gets two seats

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Algorithm to compute Jefferson's outcome :

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Algorithm to compute Jefferson's outcome :

1. Draw the table for all states (of size $n \times h$).

JEFFERSON'S METHOD : "TABLE" DEFINITION

State i gets at least k seats $\Longleftrightarrow D \leq \frac{p_i}{k}$

$\frac{p_i}{h}$	$\frac{p_i}{h-1}$	- - -	$\frac{p_i}{3}$	$\frac{p_i}{2}$	$\frac{p_i}{1}$

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Algorithm to compute Jefferson's outcome :

1. Draw the table for all states (of size $n \times h$).
2. Select the h largest entries in the table.
3. # seats for state i = # selected entries from row i.

"TABLE" DEFINITION : EXAMPLE

House size $h = 10$

State	Population
A	15
B	32
C	53
	100

"TABLE" DEFINITION : EXAMPLE

House size $h = 10$

State	Population	$\frac{p_i}{10}$	$\frac{p_i}{9}$	$\frac{p_i}{8}$	$\frac{p_i}{7}$	$\frac{p_i}{6}$	$\frac{p_i}{5}$	$\frac{p_i}{4}$	$\frac{p_i}{3}$	$\frac{p_i}{2}$	$\frac{p_i}{1}$
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B	32	3.2	-	-	4.57	5.33	6.4	8	10.67	16	32
C	53	5.3	-	-	7.57	8.83	10.6	13.2	17.67	26.5	53
	100										

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State	Population	$\frac{p_i}{10}$	$\frac{p_i}{9}$	$\frac{p_i}{8}$	$\frac{p_i}{7}$	$\frac{p_i}{6}$	$\frac{p_i}{5}$	$\frac{p_i}{4}$	$\frac{p_i}{3}$	$\frac{p_i}{2}$	$\frac{p_i}{1}$
A	15	1.5	-	-	2.14	2.5	3	3.75	5	7.5	15
B	32	3.2	-	-	4.57	5.33	6.4	8	10.67	16	32
C	53	5.3	-	-	7.57	8.83	10.6	13.2	17.67	26.5	53
	100										

10 largest entries?

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State	Population	$\frac{p_i}{10}$	$\frac{p_i}{9}$	$\frac{p_i}{8}$	$\frac{p_i}{7}$	$\frac{p_i}{6}$	$\frac{p_i}{5}$	$\frac{p_i}{4}$	$\frac{p_i}{3}$	$\frac{p_i}{2}$	$\frac{p_i}{1}$
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State	Population	$\frac{p_i}{10}$	$\frac{p_i}{9}$	$\frac{p_i}{8}$	$\frac{p_i}{7}$	$\frac{p_i}{6}$	$\frac{p_i}{5}$	$\frac{p_i}{4}$	$\frac{p_i}{3}$	$\frac{p_i}{2}$	$\frac{p_i}{1}$
A	15	1.5	-	-	2.14	2.5	3	3.75	5	7.5	15
B	32	3.2	-	-	4.57	5.33	6.4	8	10.67	16	32
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	100										

10 largest entries ?

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State	Population	$\frac{p_i}{10}$	$\frac{p_i}{9}$	$\frac{p_i}{8}$	$\frac{p_i}{7}$	$\frac{p_i}{6}$	$\frac{p_i}{5}$	$\frac{p_i}{4}$	$\frac{p_i}{3}$	$\frac{p_i}{2}$	$\frac{p_i}{1}$
A	15	1.5	-	-	2.14	2.5	3	3.75	5	7.5	15
B	32	3.2	-	-	4.57	5.33	6.4	8	10.67	16	32
C	53	5.3	-	-	7.57	8.83	10.6	13.2	17.67	26.5	53
	100										

10 largest entries? Done!

"TABLE" DEFINITION : EXAMPLE

House size $h = 10$

State	Population	$\frac{p_i}{10}$	$\frac{p_i}{9}$	$\frac{p_i}{8}$	$\frac{p_i}{7}$	$\frac{p_i}{6}$	$\frac{p_i}{5}$	$\frac{p_i}{4}$	$\frac{p_i}{3}$	$\frac{p_i}{2}$	$\frac{p_i}{1}$
A	15	1.5	-	-	2.14	2.5	3	3.75	5	7.5	15
B	32	3.2	-	-	4.57	5.33	6.4	8	10.67	16	32
C	53	5.3	-	-	7.57	8.83	10.6	13.2	17.67	26.5	53
	100										

A : 1 , B : 3 , C : 6

"TABLE" DEFINITION : EXAMPLE

House size $h = 10$

State	Population	$\frac{p_i}{10}$	$\frac{p_i}{9}$	$\frac{p_i}{8}$	$\frac{p_i}{7}$	$\frac{p_i}{6}$	$\frac{p_i}{5}$	$\frac{p_i}{4}$	$\frac{p_i}{3}$	$\frac{p_i}{2}$	$\frac{p_i}{1}$
A	15	1.5	-	-	2.14	2.5	3	3.75	5	7.5	15
B	32	3.2	-	-	4.57	5.33	6.4	8	10.67	16	32
C	53	5.3	-	-	7.57	8.83	10.6	13.2	17.67	26.5	53
	100										

$\rightarrow$ Divisor D

A : 1 , B : 3 , C : 6

"TABLE" DEFINITION : EXAMPLE

House size $h = 10$

		$D = 10$		$D = 8$		$D = 8.5$	
State	Population	$\frac{p_i}{D}$	$\lfloor \frac{p_i}{D} \rfloor$	$\frac{p_i}{D}$	$\lfloor \frac{p_i}{D} \rfloor$	$\frac{p_i}{D}$	$\lfloor \frac{p_i}{D} \rfloor$
A	15	1.5	1	1.87	1	1.76	1
B	32	3.2	3	4	4	3.76	3
C	53	5.3	5	6.62	6	6.24	6
	100		9		11		10

JEFFERSON AVOIDS ALABAMA PARADOX

JEFFERSON AVOIDS ALABAMA PARADOX

$\frac{p_i}{10}$	$\frac{p_i}{9}$	$\frac{p_i}{8}$	$\frac{p_i}{7}$	$\frac{p_i}{6}$	$\frac{p_i}{5}$	$\frac{p_i}{4}$	$\frac{p_i}{3}$	$\frac{p_i}{2}$	$\frac{p_i}{1}$
1.5	-	-	2.14	2.5	3	3.75	5	7.5	15
3.2	-	-	4.57	5.33	6.4	8	10.67	16	32
5.3	-	-	7.57	8.83	10.6	13.2	17.67	26.5	53

JEFFERSON AVOIDS ALABAMA PARADOX

$\frac{p_i}{10}$	$\frac{p_i}{9}$	$\frac{p_i}{8}$	$\frac{p_i}{7}$	$\frac{p_i}{6}$	$\frac{p_i}{5}$	$\frac{p_i}{4}$	$\frac{p_i}{3}$	$\frac{p_i}{2}$	$\frac{p_i}{1}$
1.5	-	-	2.14	2.5	3	3.75	5	7.5	15
3.2	-	-	4.57	5.33	6.4	8	10.67	16	32
5.3	-	-	7.57	8.83	10.6	13.2	17.67	26.5	53

When $h \to h+1$, we just select an additional entry.

JEFFERSON AVOIDS ALABAMA PARADOX

$\frac{p_i}{11}$	$\frac{p_i}{10}$	$\frac{p_i}{9}$	$\frac{p_i}{8}$	$\frac{p_i}{7}$	$\frac{p_i}{6}$	$\frac{p_i}{5}$	$\frac{p_i}{4}$	$\frac{p_i}{3}$	$\frac{p_i}{2}$	$\frac{p_i}{1}$
-	1.5	-	-	2.14	2.5	3	3.75	5	7.5	15
-	3.2	-	-	4.57	5.33	6.4	8	10.67	16	32
-	5.3	-	-	7.57	8.83	10.6	13.2	17.67	26.5	53

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JEFFERSON AVOIDS ALABAMA PARADOX

$\frac{p_i}{11}$	$\frac{p_i}{10}$	$\frac{p_i}{9}$	$\frac{p_i}{8}$	$\frac{p_i}{7}$	$\frac{p_i}{6}$	$\frac{p_i}{5}$	$\frac{p_i}{4}$	$\frac{p_i}{3}$	$\frac{p_i}{2}$	$\frac{p_i}{1}$
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-	5.3	-	-	7.57	8.83	10.6	13.2	17.67	26.5	53

No change

When $h \to h+1$, we just select an additional entry.

JEFFERSON AVOIDS ALABAMA PARADOX

$\frac{p_i}{11}$	$\frac{p_i}{10}$	$\frac{p_i}{9}$	$\frac{p_i}{8}$	$\frac{p_i}{7}$	$\frac{p_i}{6}$	$\frac{p_i}{5}$	$\frac{p_i}{4}$	$\frac{p_i}{3}$	$\frac{p_i}{2}$	$\frac{p_i}{1}$
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JEFFERSON AVOIDS ALABAMA PARADOX

$\frac{p_i}{11}$	$\frac{p_i}{10}$	$\frac{p_i}{9}$	$\frac{p_i}{8}$	$\frac{p_i}{7}$	$\frac{p_i}{6}$	$\frac{p_i}{5}$	$\frac{p_i}{4}$	$\frac{p_i}{3}$	$\frac{p_i}{2}$	$\frac{p_i}{1}$
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-	5.3	-	-	7.57	8.83	10.6	13.2	17.67	26.5	53

When $h \to h+1$, we just select an additional entry.

$\Rightarrow$ Jefferson's method is house monotone.

RECAP : THE MODEL

* n states with populations $p_1, p_2, \cdots, p_n$. $(p_i > 0)$
* House size $h \geq 0$

Goal : Assign seats $s_1, s_2, \cdots, s_n \in \mathbb{N}_{\geq 0}$ such that

$$s_1 + s_2 + \cdots + s_n = h.$$

(apportionment method)

* Total population $P = p_1 + p_2 + \cdots + p_n$
* Standard quota of state i : $q_i = \frac{p_i}{P} \times h$ (entitlement)
* Upper quota : $\lceil q_i \rceil$, lower quota $= \lfloor q_i \rfloor$

HOUSE MONOTONICITY

HOUSE MONOTONICITY

An apportionment method is house monotone if, for any two problem instances $I = (h; p_1, p_2, \dots p_n)$ and $I' = (h+1; p_1, p_2, \dots, p_n)$ with seat assignments $I \longmapsto (s_1, s_2, \dots, s_n)$ and $I' \longmapsto (s_1', s_2', \dots, s_n')$, we have

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$$s_i' \geq s_i \quad \text{for every state } i.$$

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$$s_i' \geq s_i \quad \text{for every state } i.$$

Failure of house monotonicity is Alabama paradox.

HOUSE MONOTONICITY

Theorem : Jefferson's method is house monotone

We have seen a "proof by picture" using Table definition.

HOUSE MONOTONICITY

Theorem : Jefferson's method is house monotone

Proof : For problem instance $I = (h;\ p_1, p_2, \ldots, p_n)$, suppose Jefferson's divisor is D. Then, $s_i = \left\lfloor \frac{p_i}{D} \right\rfloor$.

House Monotonicity

Theorem : Jefferson's method is house monotone

Proof : For problem instance $I = (h;\ p_1, p_2, \dots, p_n)$, suppose Jefferson's divisor is D. Then, $s_i = \left\lfloor \frac{p_i}{D} \right\rfloor$.

Note that $\left\lfloor \frac{p_1}{D} \right\rfloor + \cdots + \left\lfloor \frac{p_n}{D} \right\rfloor = h < h+1$.

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Note that $\left\lfloor \frac{p_1}{D} \right\rfloor + \cdots + \left\lfloor \frac{p_n}{D} \right\rfloor = h < h+1$.

Consider the $I' = (h+1;\ p_1, p_2, \ldots, p_n)$ with divisor D'.

Note that $D' < D$.

House Monotonicity

Theorem : Jefferson's method is house monotone

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Note that $\left\lfloor \frac{p_1}{D} \right\rfloor + \cdots + \left\lfloor \frac{p_n}{D} \right\rfloor = h < h+1$.

Consider the $I' = (h+1; p_1, p_2, \dots, p_n)$ with divisor D'.

Note that $D' < D$.

Then, for every state i, $s_i' = \left\lfloor \frac{p_i}{D'} \right\rfloor \geq \left\lfloor \frac{p_i}{D} \right\rfloor = s_i$. $\blacksquare$

POPULATION MONOTONICITY

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An apportionment method is population monotone if,

for any two problem instances $I = (h; p_1, \cdots, p_n)$ and

$I' = (h'; p_1', \cdots, p_n')$ with seat assignments $I \longmapsto (s_1, s_2, \cdots, s_n)$

and $I' \longmapsto (s_1', s_2', \cdots, s_n')$,

Population Monotonicity

An apportionment method is population monotone if, for any two problem instances $I = (h; p_1, \cdots, p_n)$ and $I' = (h'; p_1', \cdots, p_n')$ with seat assignments $I \longmapsto (s_1, s_2, \cdots, s_n)$ and $I' \longmapsto (s_1', s_2', \cdots, s_n')$,

$$s_i < s_i' \text{ and } s_j > s_j' \Rightarrow p_i < p_i' \text{ or } p_j > p_j' .$$

POPULATION MONOTONICITY

An apportionment method is population monotone if, for any two problem instances $I = (h; p_1, \cdots, p_n)$ and $I' = (h'; p_1', \cdots, p_n')$ with seat assignments $I \longmapsto (s_1, s_2, \cdots, s_n)$ and $I' \longmapsto (s_1', s_2', \cdots, s_n')$,

$$\underbrace{s_i < s_i'}_{\substack{i \text{ gets} \\ \text{more seats}}} \text{ and } \underbrace{s_j > s_j'}_{\substack{j \text{ gets} \\ \text{fewer seats}}} \Rightarrow \underbrace{p_i < p_i'}_{i \text{ grows}} \text{ or } \underbrace{p_j > p_j'}_{j \text{ shrinks}}.$$

POPULATION MONOTONICITY

An apportionment method is population monotone if, for any two problem instances $I = (h; p_1, \dots, p_n)$ and $I' = (h'; p_1', \dots, p_n')$ with seat assignments $I \longmapsto (s_1, s_2, \dots, s_n)$ and $I' \longmapsto (s_1', s_2', \dots, s_n')$,

$$s_i < s_i' \text{ and } s_j > s_j' \Rightarrow p_i < p_i' \text{ or } p_j > p_j'.$$

Note: This definition prevents the paradox in the three-state example, but not in the Virginia and Maine example.

POPULATION MONOTONICITY

Theorem : Jefferson's method is population monotone.

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Proof : Let D and D' be the divisors for I and I', respectively.

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Then, $s_i < s_i' \Rightarrow \left\lfloor \frac{p_i}{D} \right\rfloor < \left\lfloor \frac{p_i'}{D'} \right\rfloor \Rightarrow \frac{p_i}{D} < \frac{p_i'}{D'}$.

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Similarly, $s_j > s_j' \Rightarrow \frac{p_j}{D} > \frac{p_j'}{D'}$.

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Similarly, $s_j > s_j' \Rightarrow \frac{p_j}{D} > \frac{p_j'}{D'}$.

Rearrangement gives $p_i' > \frac{D'}{D} \cdot p_i$ and $p_j' < \frac{D'}{D} \cdot p_j$.

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If $\frac{D'}{D} \geq 1$, then $p_i < p_i'$.

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Similarly, $s_j > s_j' \Rightarrow \frac{p_j}{D} > \frac{p_j'}{D'}$.

Rearrangement gives $p_i' > \frac{D'}{D} \cdot p_i$ and $p_j' < \frac{D'}{D} \cdot p_j$.

If $\frac{D'}{D} \geq 1$, then $p_i < p_i'$. If $\frac{D'}{D} < 1$, then $p_j > p_j'$. $\blacksquare$

QUIZ

QUIZ

Prove that Population monotonicity implies house monotonicity.

NEXT LECTURE

Divisor Methods