COL 8184 : ALGORITHMS FOR FAIR REPRESENTATION

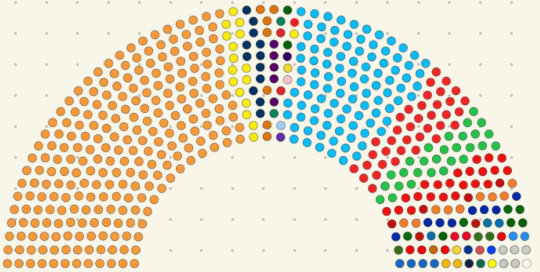
LECTURE 2

APPORTIONMENT METHODS & PARADOXES

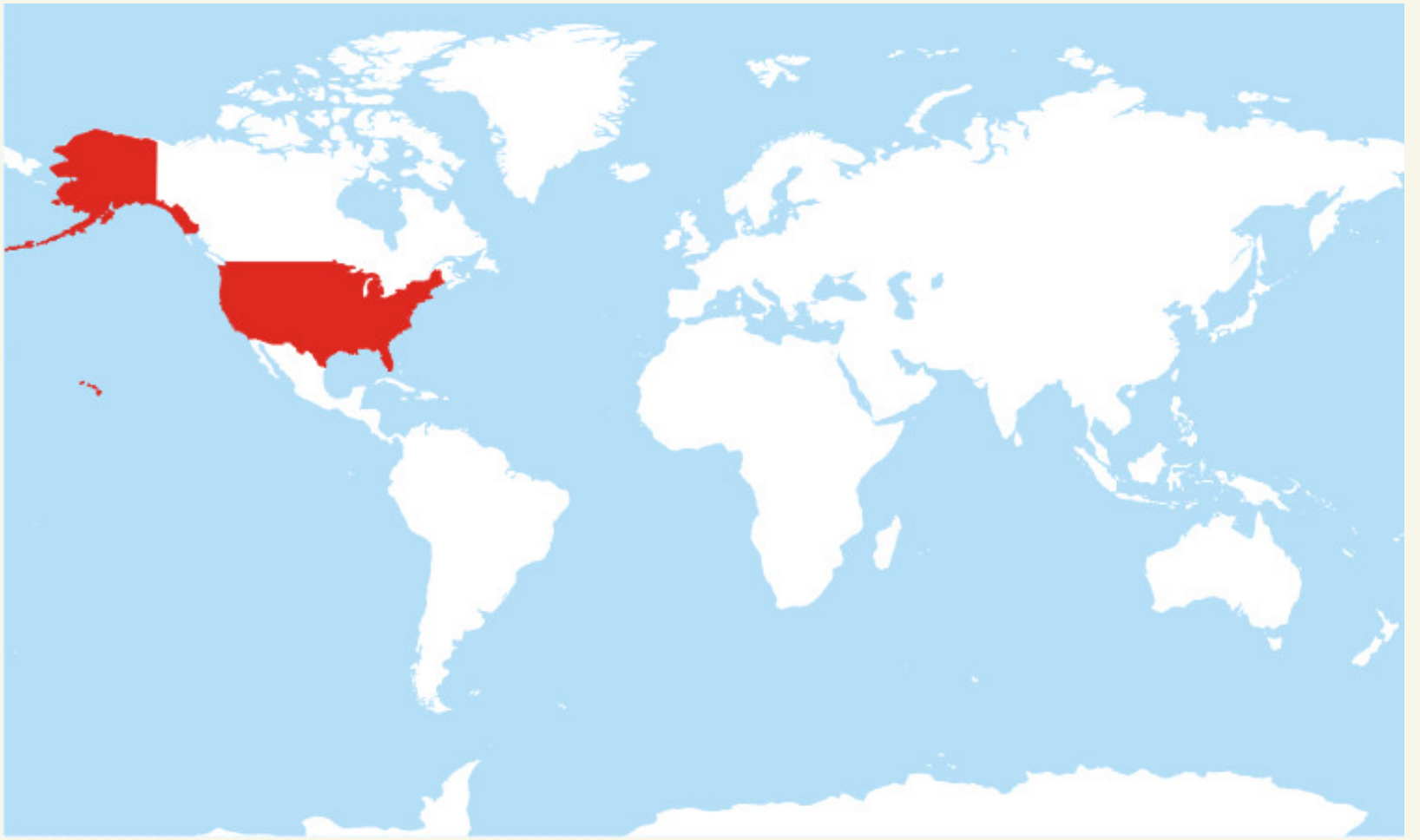
JAN 08, 2026 | ROHIT VAISH

COURSE WEBPAGE

TEAMS CHANNEL



APPORTIONMENT



We the People of the United States, in order to form a more perfect Union, establish Justice, insure domestic Tranquility, provide for the common defence, promote the general Welfare, and secure the Blessings of Liberty to ourselves and our Posterity, do ordain and establish this Constitution for the United States of America.

Article. 1.

Section. 1. All legislative Powers herein granted shall be vested in a Congress of the United States, which shall consist of a Senate and House of Representatives.

Section. 2. The House of Representatives shall be composed of Members chosen every second Year by the People of the several States, and the Electors in each State shall have the Qualifications requisite for Electors of the most numerous Branch of the State Legislature.

No Person shall be a Representative who shall not have attained to the Age of twenty five Years, and been seven Years a Citizen of the United States, and who shall not, when elected, be an Inhabitant of that State in which he shall be chosen.

Representatives and direct Taxes shall be apportioned among the several States which may be included within this Union, according to their respective Numbers, which shall be determined by adding to the whole Number of free Persons, including those bound to Service for a Term of Years, and excluding Indians not taxed, three fifths of all other Persons. The actual Enumeration shall be made within three Years after the first Meeting of the Congress of the United States, and within every subsequent Term of ten Years, in such Manner as they shall by Law direct. The Number of Representatives shall not exceed one for every thirty Thousand, but each State shall have at Least one Representative; and until such enumeration shall be made, the State of New Hampshire shall be entitled to chuse three, Massachusetts eight, Rhode-Island and Providence Plantations one, Connecticut five, New York six, New Jersey four, Pennsylvania eight, Delaware one, Maryland six, Virginia ten, North Carolina five, South Carolina five, and Georgia three.

US Constitution, Article 1, Section 2, Clause 3

Came into force on March 4th, 1789

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(1) Proportionality is the ideal

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(2) Basis of proportionality: Representative population

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(4) Upper and lower bounds on no. of representatives

Missing: Exact no. of representatives ("how many") & Method of apportionment ("how")

THE MODEL

* n states with populations $p_1, p_2, \cdots, p_n$. $(p_i > 0)$
* House size $h \geq 0$

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Goal: Assign seats $s_1, s_2, \dots, s_n \in \mathbb{N}_{\geq 0}$ such that

$$s_1 + s_2 + \dots + s_n = h.$$

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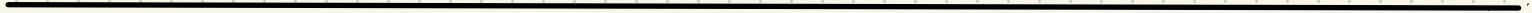
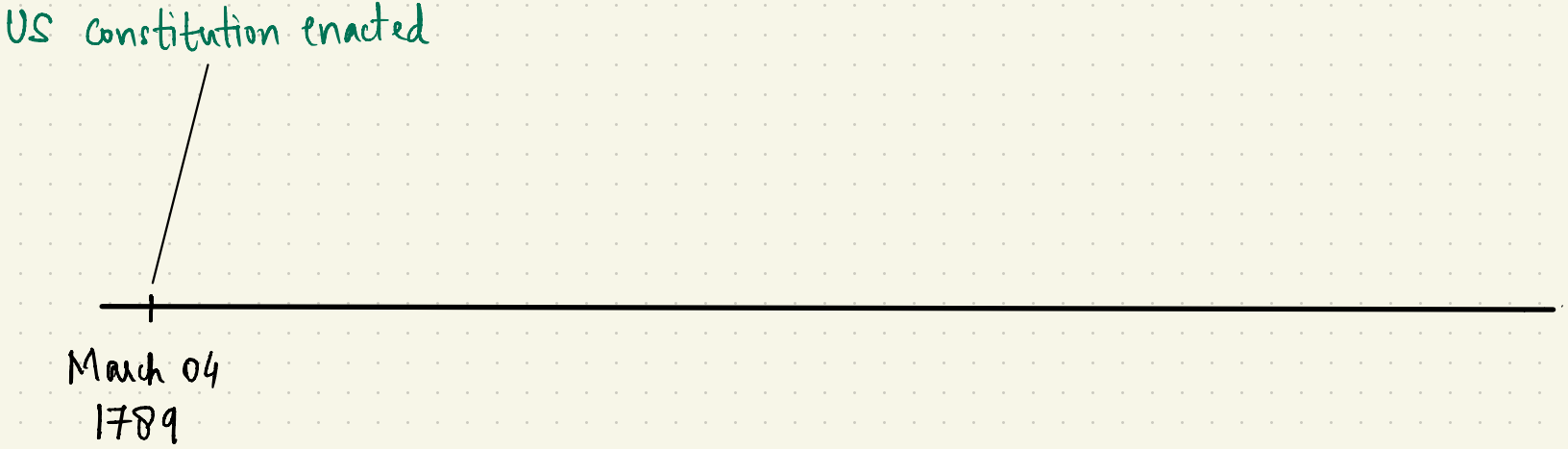
$$s_1 + s_2 + \dots + s_n = h. \qquad \text{(apportionment method)}$$

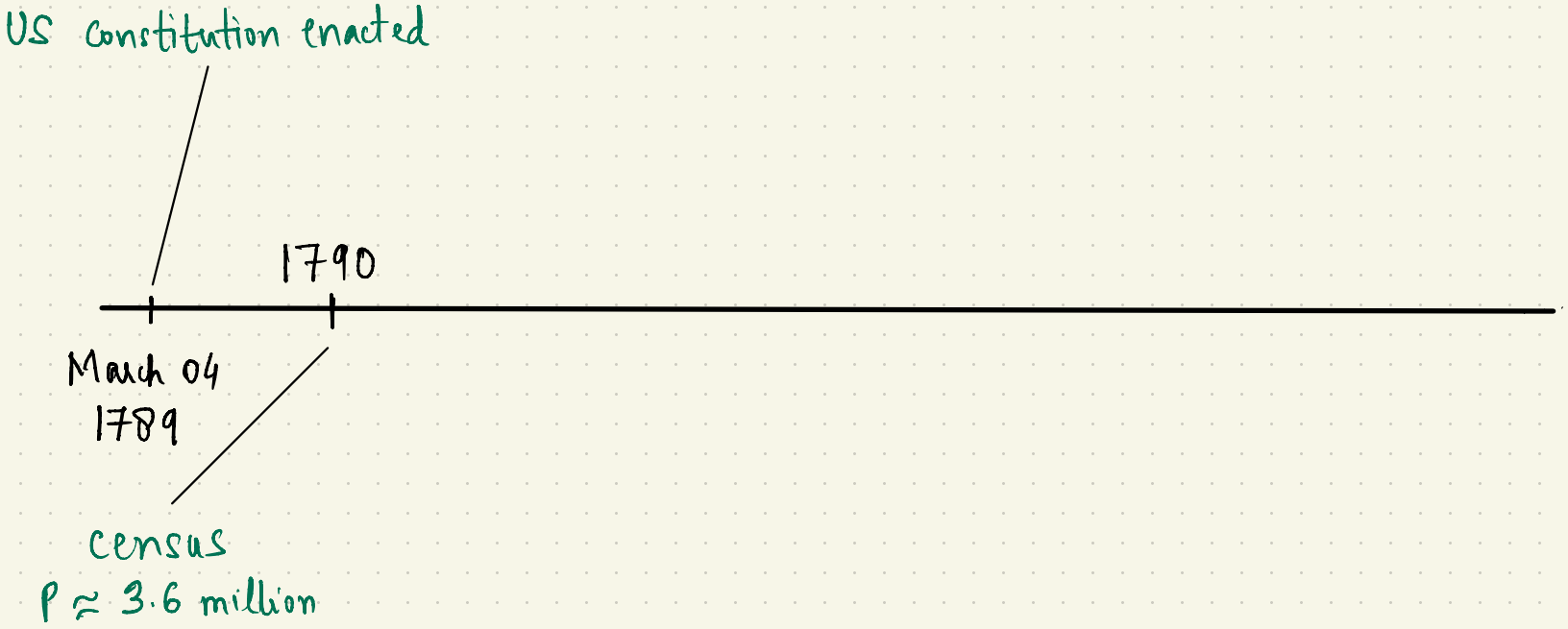
* Total population $P = p_1 + p_2 + \dots + p_n$

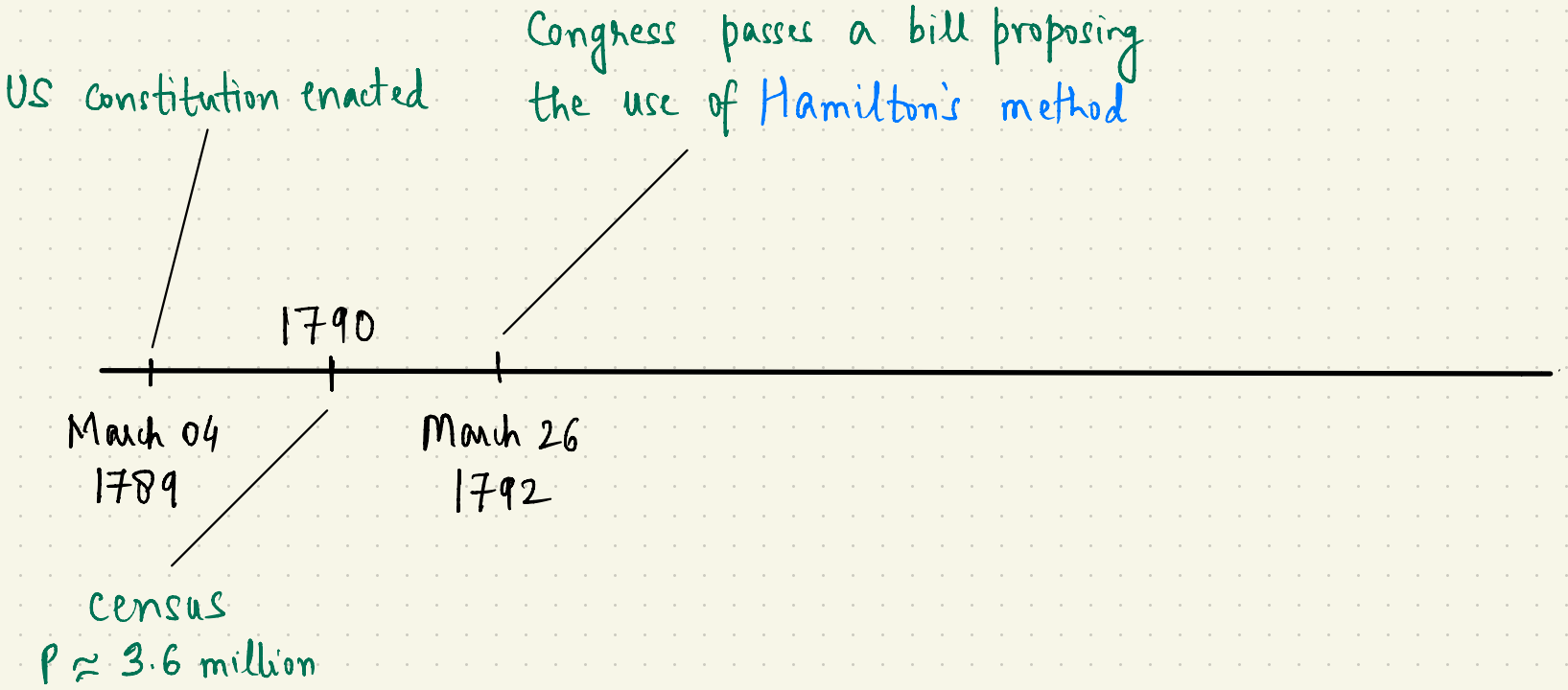
* Standard quota of state i: $q_i = \frac{p_i}{P} \times h$ (entitlement)

* Upper quota: $\lceil q_i \rceil$, Lower quota $= \lfloor q_i \rfloor$




US Constitution enacted
March 04
1789


US constitution enacted
1790
March 04
1789
census
P ≈ 3.6 million


US constitution enacted
Congress passes a bill proposing the use of Hamilton's method
1790
March 04
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March 26
1792
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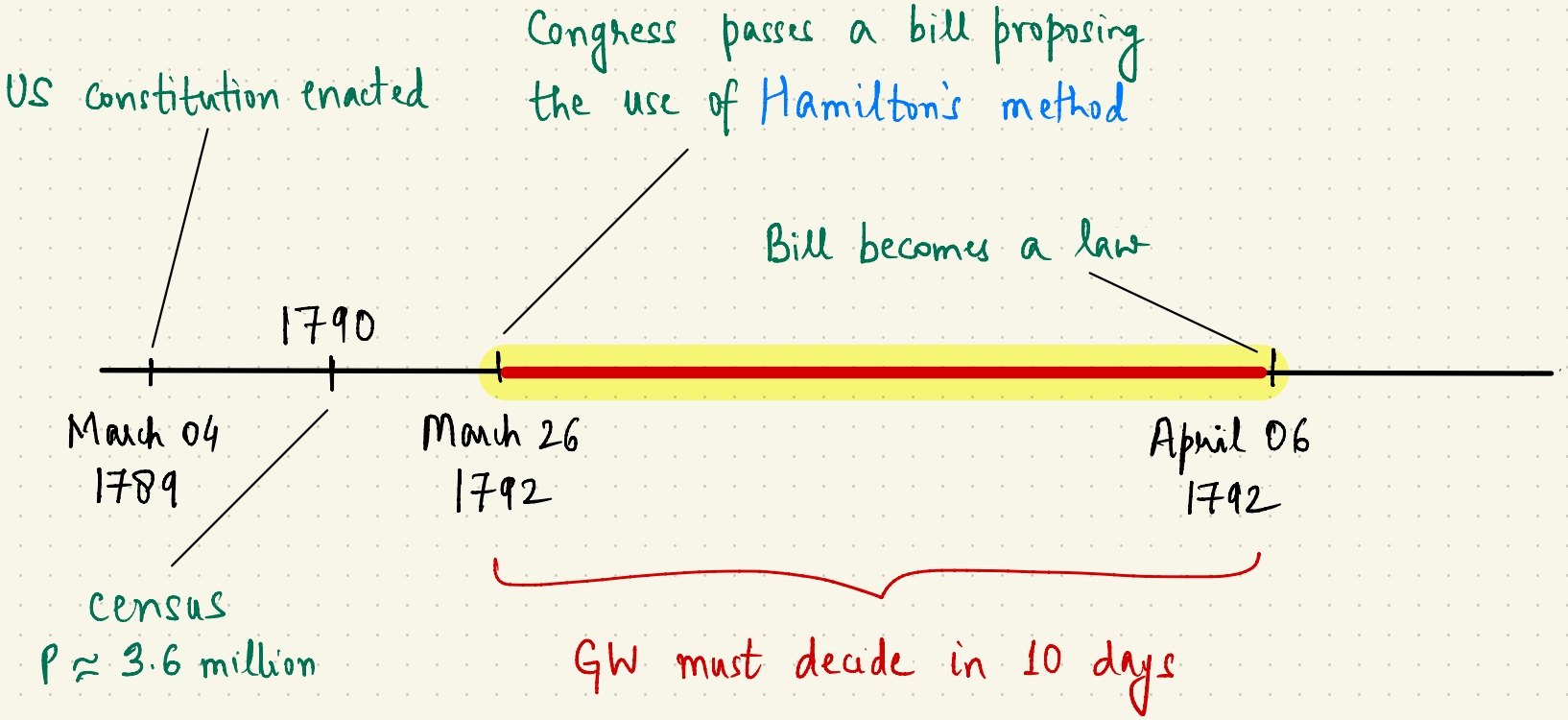
HAMILTON'S METHOD

(aka method of largest remainder)

* Assign each state i its lower quota $\lfloor q_i \rfloor$.

* Assign remaining seats one at a time to the state with the largest remainder $r_i = q_i - \lfloor q_i \rfloor$.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Congress passes a bill proposing the use of Hamilton's method
Bill becomes a law
1790
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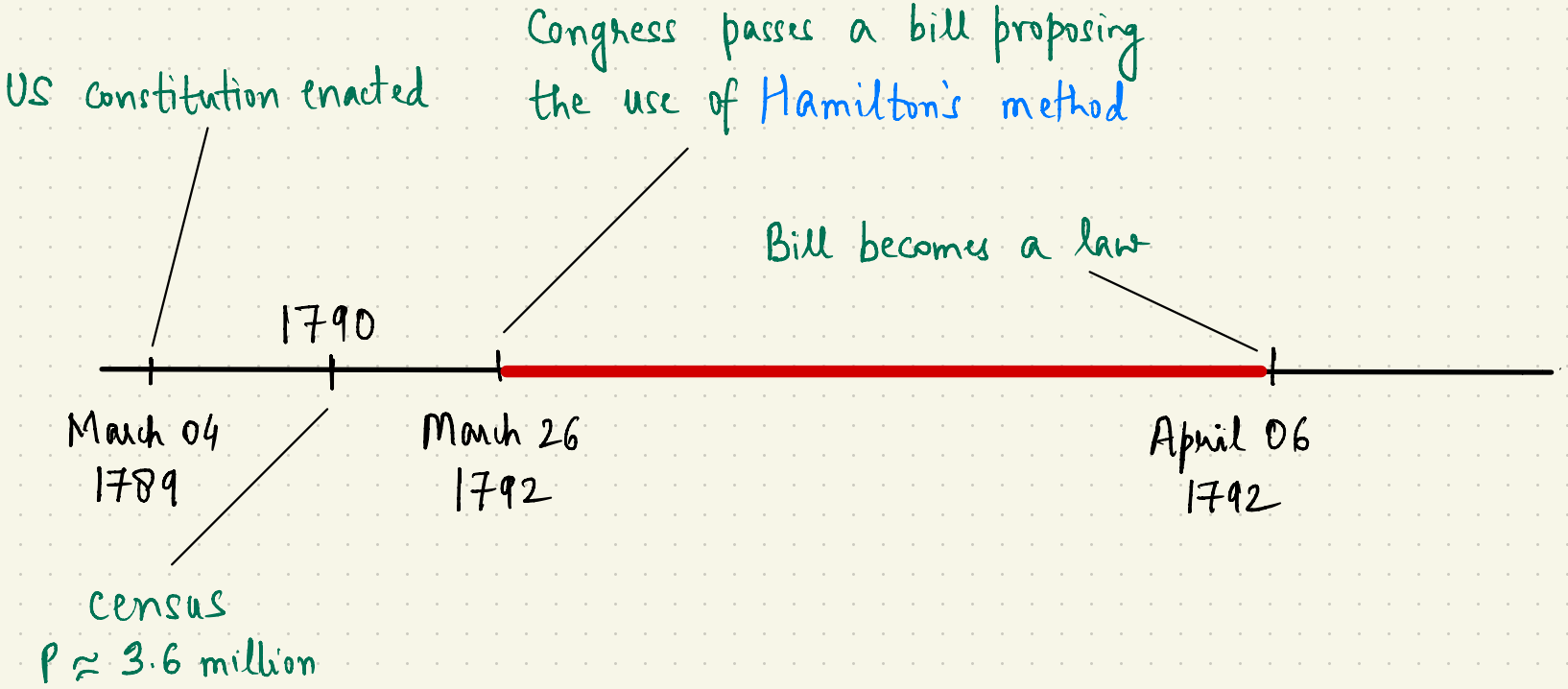
March 26 1792
April 06 1792
census
P ≈ 3.6 million
GW must decide in 10 days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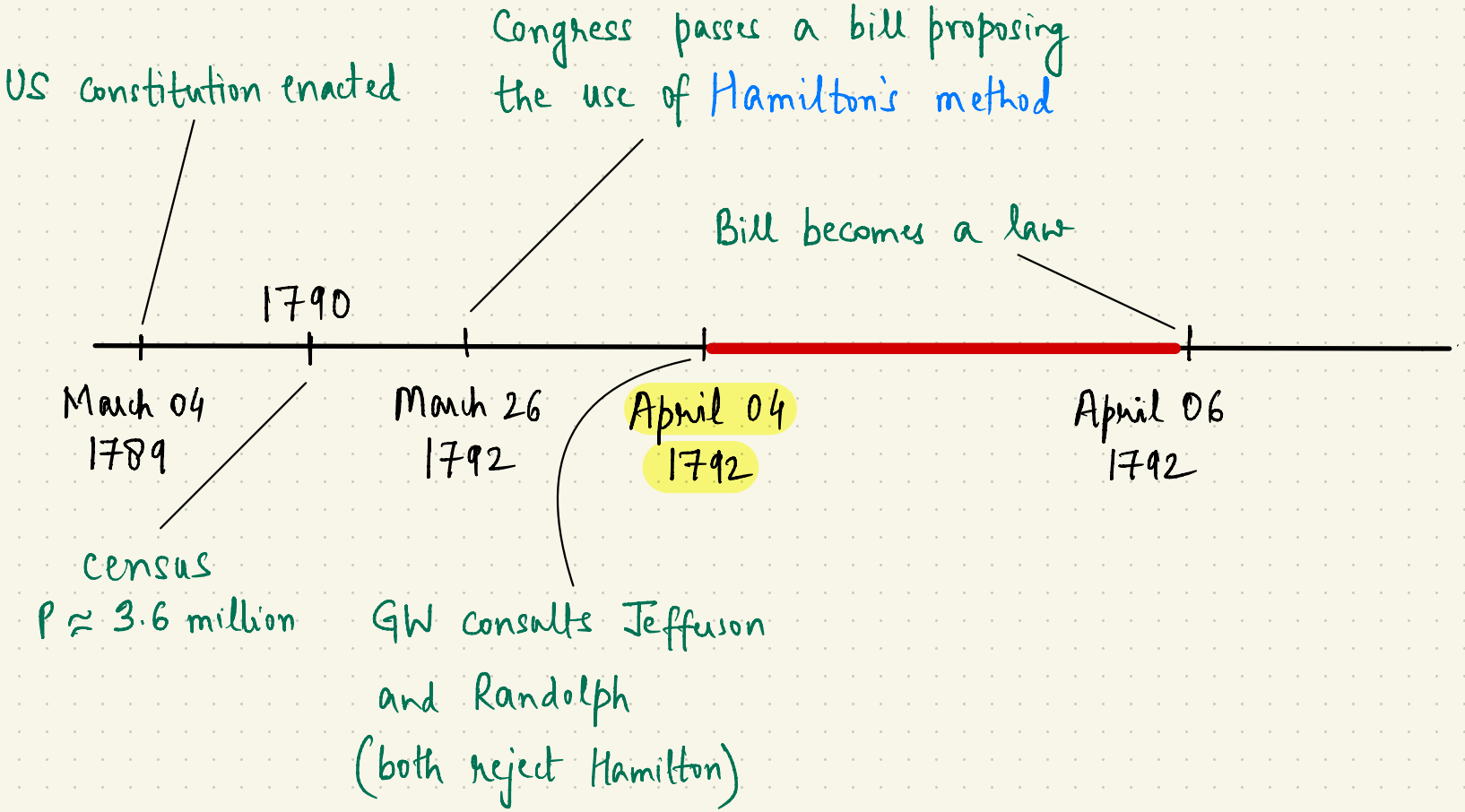
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GW consults Jefferson and Randolph (both reject Hamilton)

Census of 1790. 120 seats to be allocated, 30,000 citizens per representative

State	Population	Seats allocated		
		"Raw"	*Initial*	*Final*
Connecticut*	236,841	7.895	7	8
Delaware*	55,540	1.851	1	2
Georgia	70,835	2.361	2	2
Kentucky	68,705	2.290	2	2
Maryland	278,514	9.284	9	9
Massachusetts*	475,327	15.844	15	16
New Hampshire*	141,822	4.727	4	5
New Jersey*	179,570	5.986	5	6
New York	331,589	11.053	11	11
North Carolina*	353,523	11.784	11	12
Pennsylvania	432,879	14.419	14	14
Rhode Island	68,446	2.282	2	2
South Carolina*	206,236	6.875	6	7
Vermont*	85,533	2.851	2	3
Virginia	630,560	21.019	21	21
Total	3,615,920	120.531	112	120

*States receiving an additional seat after the initial allocation.

States gaining a seat (*) have fewer than 30,000 people / representative

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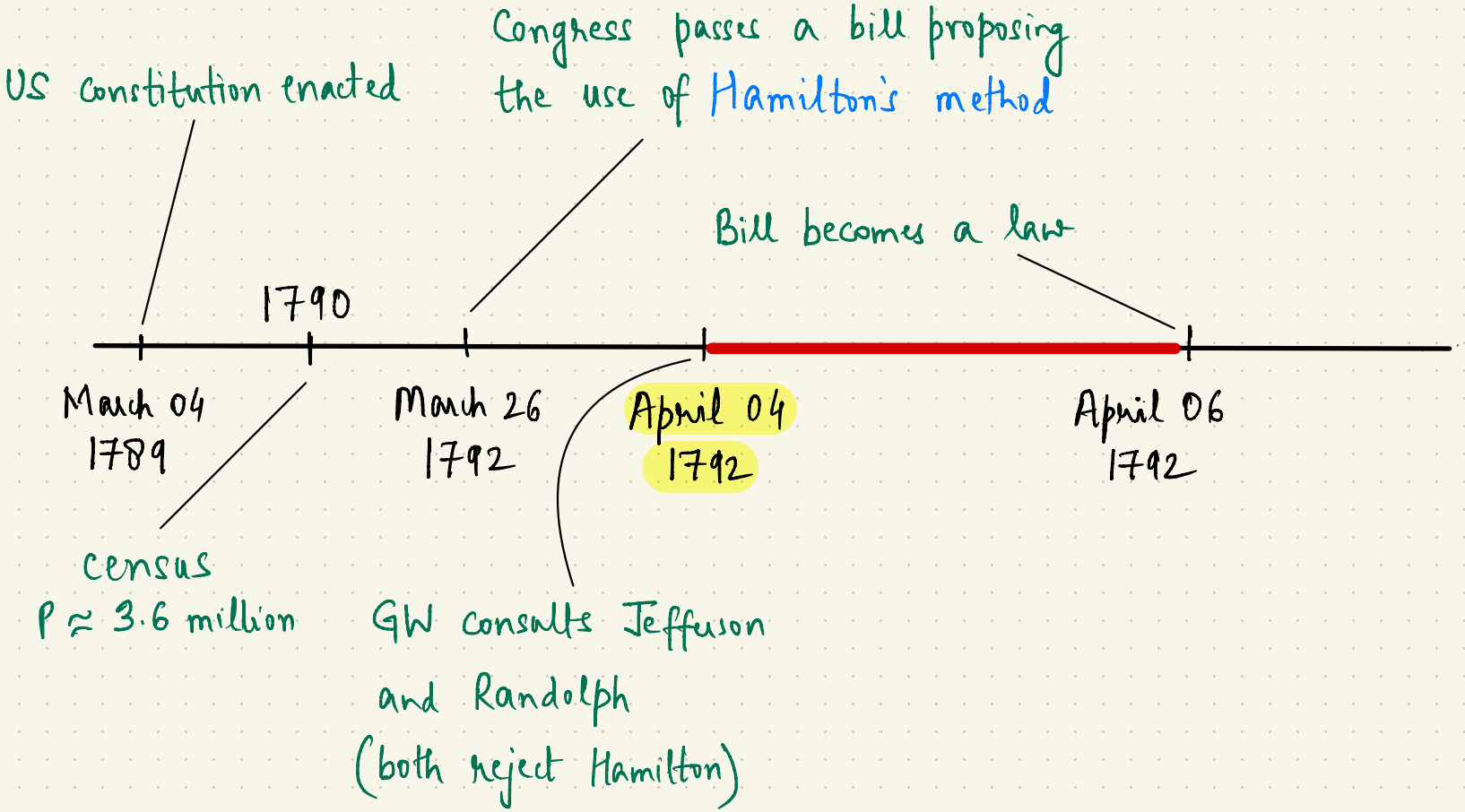
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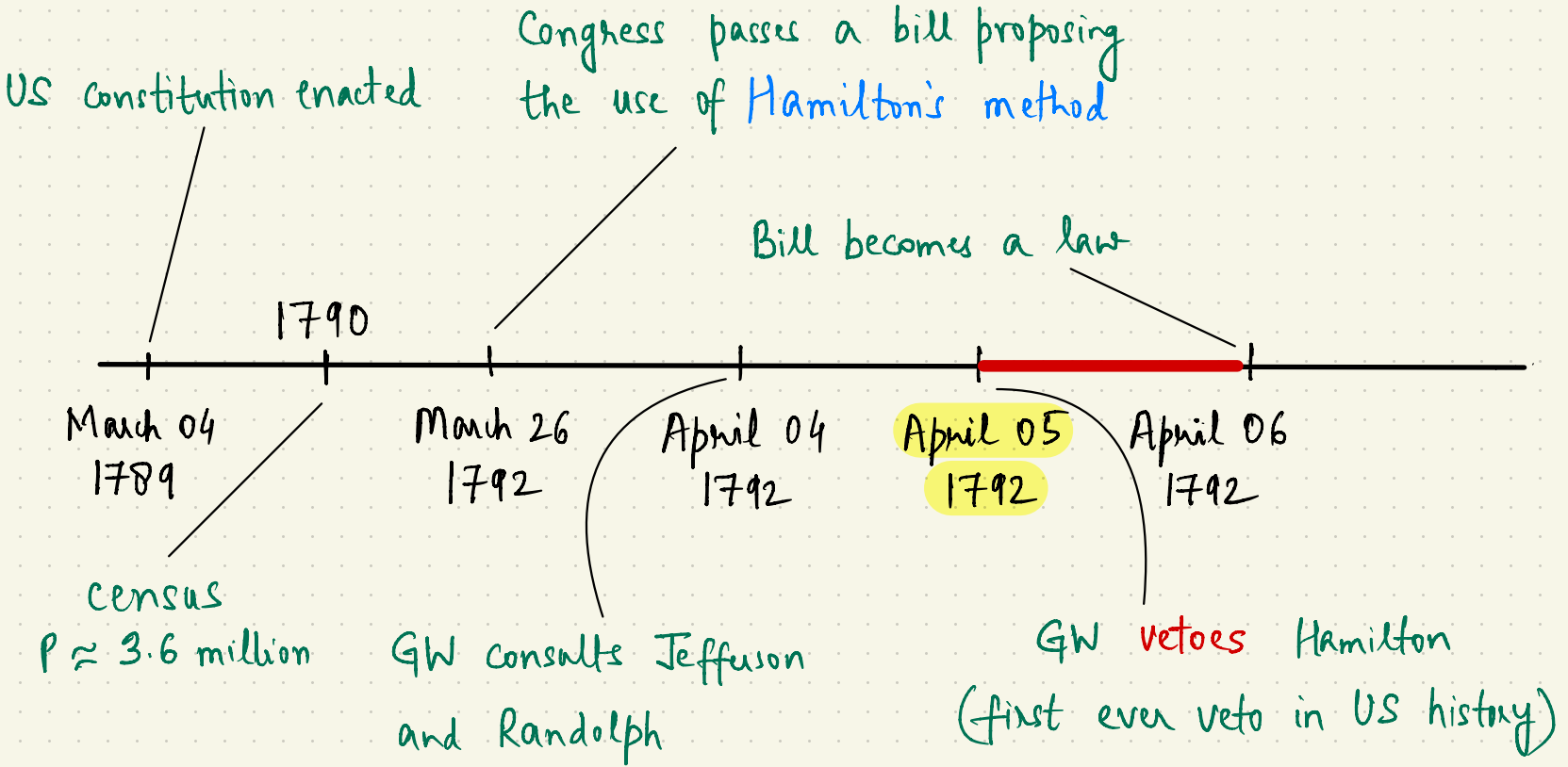
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Gentlemen of the House of Representatives

I have maturely considered the Act passed by the two Houses, intitled, "An Act for an apportionment of Representatives among the several States according to the first enumeration," and I return it to your House, wherein it originated, with the following objections.

First—The Constitution has prescribed that representatives shall be apportioned among the several States according to their respective numbers: and there is no one proportion or divisor which, applied to the respective numbers of the States will yield the number and allotment of representatives proposed by the Bill.

Second—The Constitution has also provided that the number of Representatives shall not exceed one for every thirty thousand; which restriction is, by the context, and by fair and obvious construction, to be applied to the seperate and respective numbers of the States: and the bill has allotted to eight of the States, more than one for thirty thousand.

George Washington.

Differences in no. of people represented per seat across states (i.e., method is "unfair")

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violation of upper bound in the constitution

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Washington interpreted this as a "per state" requirement

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		"Raw"	*Initial*	*Final*
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Georgia	70,835	2.361	2	2
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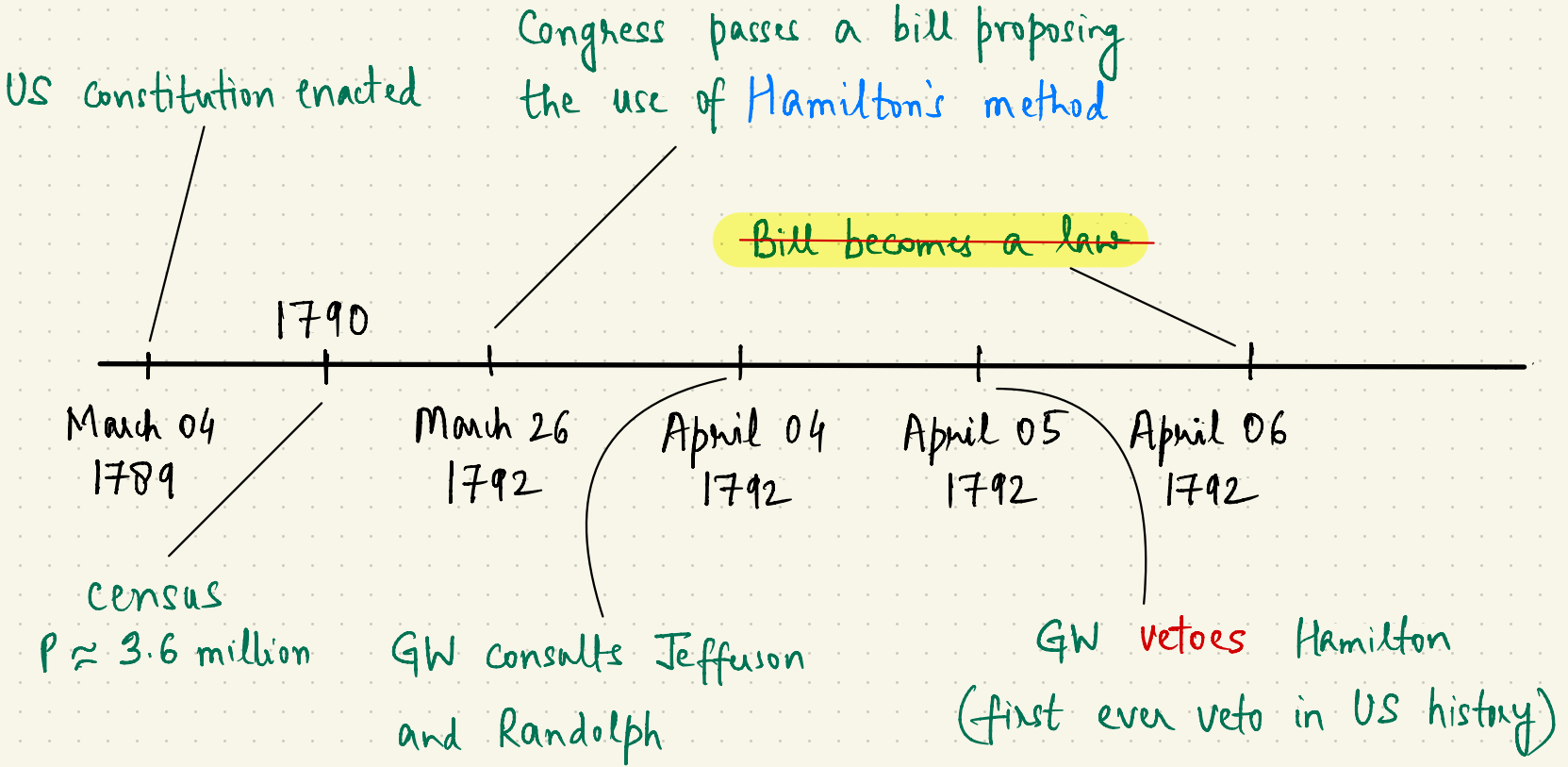
Jefferson and Washington were from →


US constitution enacted
Congress passes a bill proposing the use of Hamilton's method
Bill becomes a law
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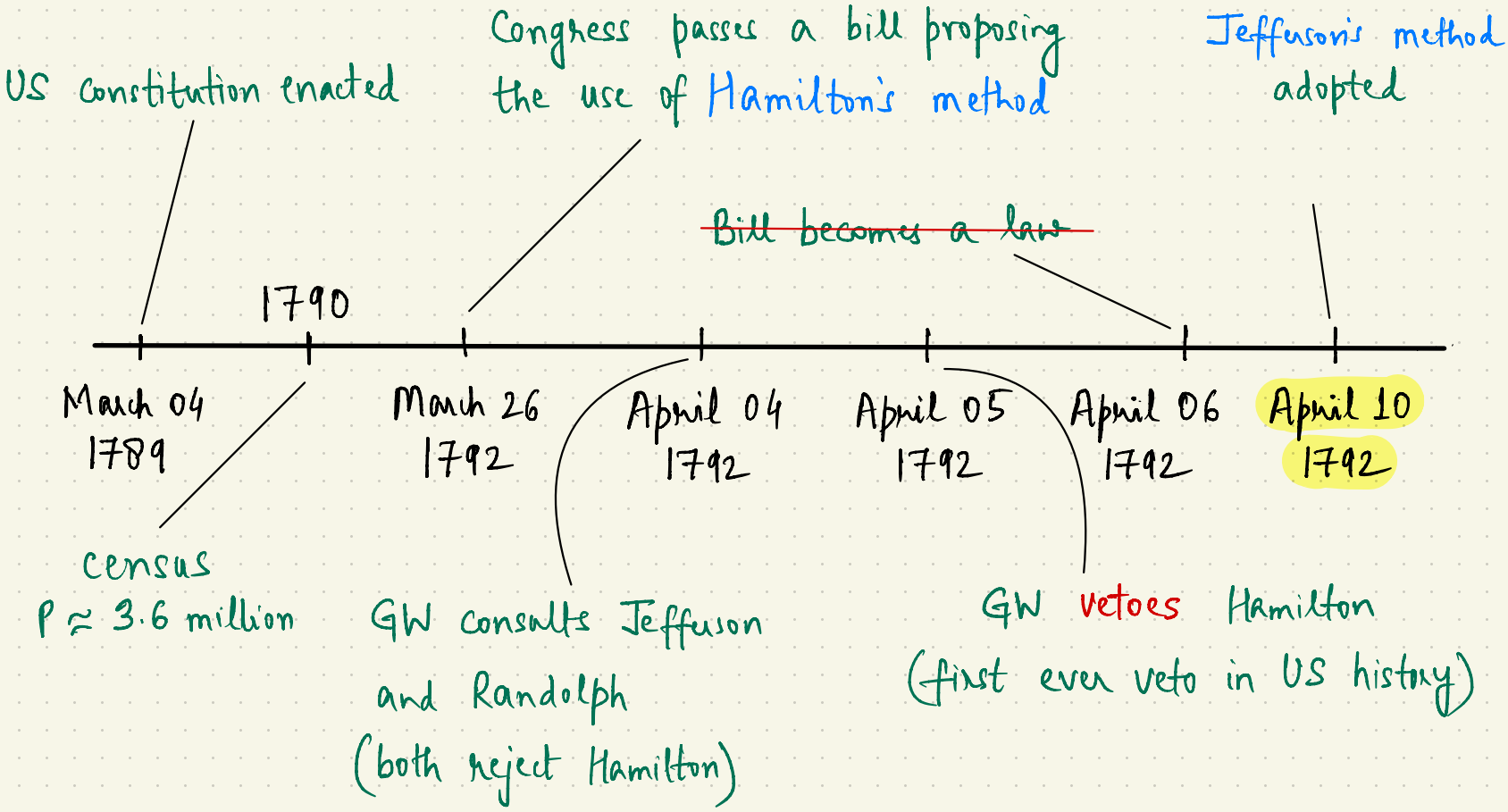
census
P ≈ 3.6 million
GW consults Jefferson and Randolph (both reject Hamilton)
GW vetoes Hamilton (first ever veto in US history)


US constitution enacted
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Jefferson's method adopted
Bill becomes a law
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Modified quota $\hat{q}_i = \frac{p_i}{D}$

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  " " $>$ ",

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* Calculate $\left\lfloor \frac{p_i}{D} \right\rfloor$ for each state i

* If $\sum_i \left\lfloor \frac{p_i}{D} \right\rfloor < h$, decrease D and repeat

  " " $>$ ", increase D and repeat

  " " $=$ ", assign $s_i = \left\lfloor \frac{p_i}{D} \right\rfloor$ to each state s_i.

JEFFERSON'S METHOD

* Let D be a divisor such that

$$\left\lfloor \frac{p_1}{D} \right\rfloor + \left\lfloor \frac{p_2}{D} \right\rfloor + \cdots + \left\lfloor \frac{p_n}{D} \right\rfloor = h.$$

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🤔 Does a Jefferson divisor always exist?

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🤔 Does a Jefferson divisor always exist?

🤔 Do all Jefferson divisors give the same seat assignment?

🤔 Can Jefferson outcome be efficiently computed?

Jefferson's Method Example

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House size $h = 10$

State	Population
A	15
B	32
C	53
	100

Jefferson's Method Example

House size $h = 10$

$D = 10$

State	Population
A	15
B	32
C	53
	100

?

Jefferson's Method Example

House size $h = 10$

$D = 10$

State	Population	$\frac{p_i}{D}$	$\lfloor \frac{p_i}{D} \rfloor$
A	15	1.5	1
B	32	3.2	3
C	53	5.3	5
	100		9

Jefferson's Method Example

House size $h = 10$

$D = 10$

State	Population	$\frac{p_i}{D}$	$\lfloor \frac{p_i}{D} \rfloor$
A	15	1.5	1
B	32	3.2	3
C	53	5.3	5
	100		$9 < h$

$\Rightarrow$ Lower the divisor

Jefferson's Method Example

House size $h = 10$

State	Population	$D = 10$ $\frac{p_i}{D}$	$D = 10$ $\lfloor \frac{p_i}{D} \rfloor$	$D = 8$ $\frac{p_i}{D}$	$D = 8$ $\lfloor \frac{p_i}{D} \rfloor$
A	15	1.5	1	1.87	1
B	32	3.2	3	4	4
C	53	5.3	5	6.62	6
	100		9		11

Jefferson's Method Example

House size $h = 10$

State	Population	$D = 10$ $\frac{p_i}{D}$	$D = 10$ $\lfloor \frac{p_i}{D} \rfloor$	$D = 8$ $\frac{p_i}{D}$	$D = 8$ $\lfloor \frac{p_i}{D} \rfloor$
A	15	1.5	1	1.87	1
B	32	3.2	3	4	4
C	53	5.3	5	6.62	6
	100		9		11 > h

$\Rightarrow$ Increase the divisor

Jefferson's Method Example

House size $h = 10$

State	Population	$D = 10$ $\frac{p_i}{D}$	$D = 10$ $\lfloor \frac{p_i}{D} \rfloor$	$D = 8$ $\frac{p_i}{D}$	$D = 8$ $\lfloor \frac{p_i}{D} \rfloor$	$D = 8.5$ $\frac{p_i}{D}$	$D = 8.5$ $\lfloor \frac{p_i}{D} \rfloor$
A	15	1.5	1	1.87	1	1.76	1
B	32	3.2	3	4	4	3.76	3
C	53	5.3	5	6.62	6	6.24	6
	100		9		11		10

Done!

Quiz

QUIZ

House size $h = 10$

State	Population	Hamilton	Jefferson
A	15		
B	14	?	?
C	13		
D	58		
	100		

QUIZ

House size $h = 10$

State	Population	Hamilton	Jefferson
A	15		
B	14	?	?
C	13		
D	58		
	100		

Notice anything strange?

QUIZ

House size $h = 10$

State	Population	Hamilton	Jefferson
A	15	2	1
B	14	1	1
C	13	1	1
D	58	6	7
	100		

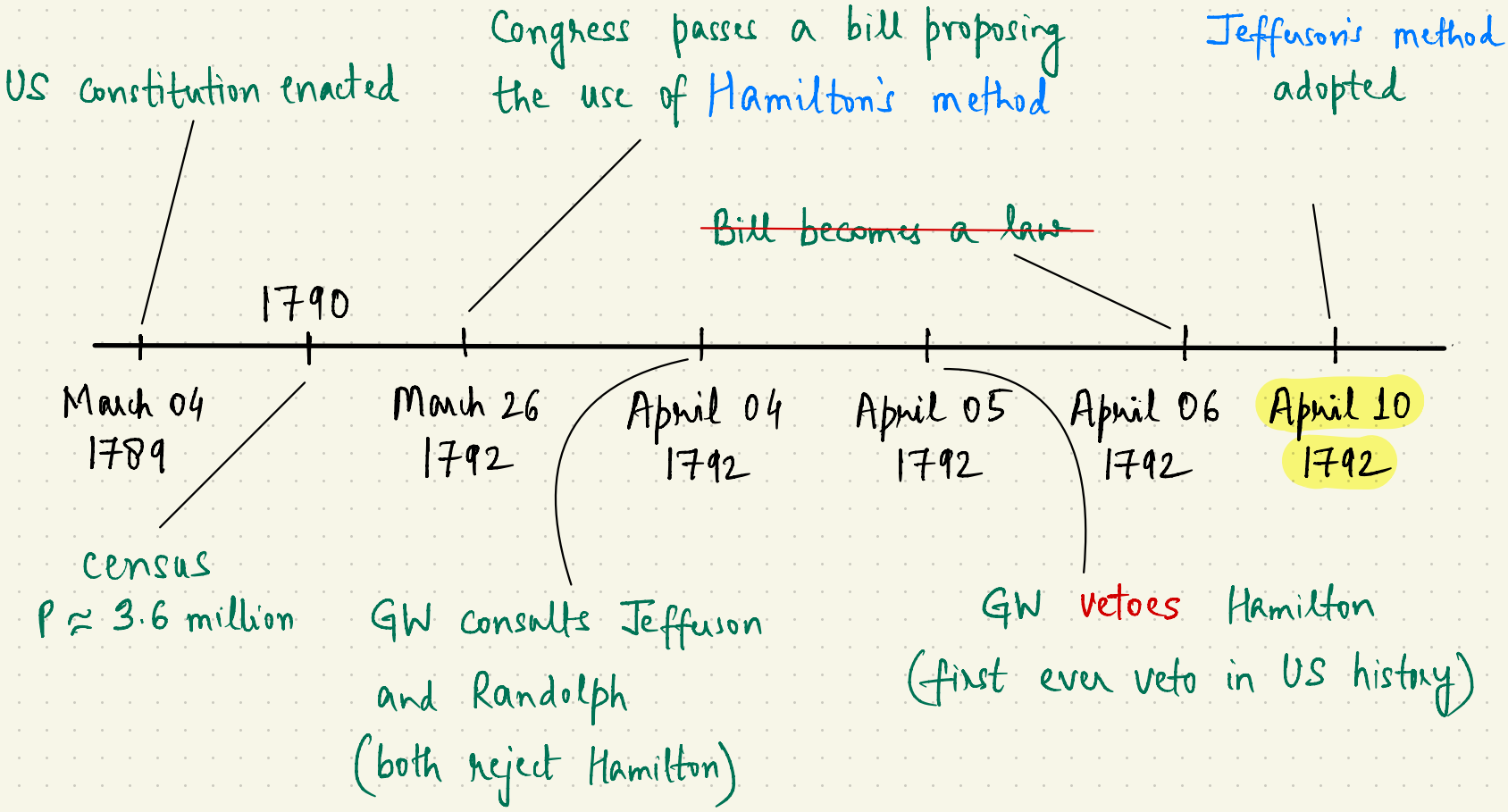
(divisor $D = 8$)

Quiz

House size $h = 10$

State	Population	Hamilton	Jefferson
A	15	2	1
B	14	1	1
C	13	1	1
D	58	6	$7 > \lceil \frac{58}{10} \rceil$
	100		

Jefferson overshoots upper quota


US constitution enacted
Congress passes a bill proposing the use of Hamilton's method
Jefferson's method adopted
Bill becomes a law
1790
March 04 1789
March 26 1792
April 04 1792
April 05 1792
April 06 1792
April 10 1792
census
P ≈ 3.6 million
GW consults Jefferson and Randolph (both reject Hamilton)
GW vetoes Hamilton (first ever veto in US history)

Jefferson's method adopted

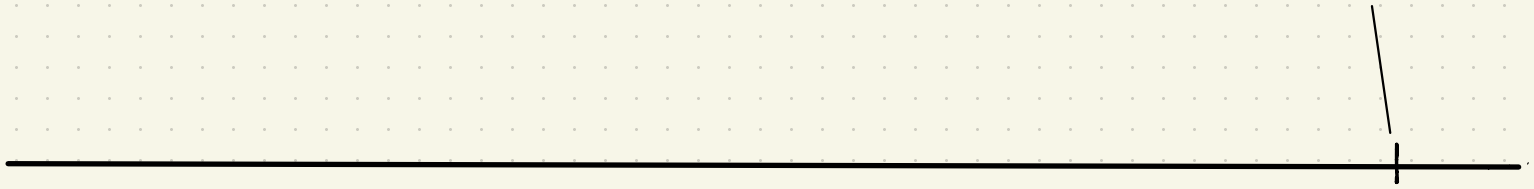
Jefferson Divisor

House Size

April 10 1792

30,000 $\Longrightarrow$ 112

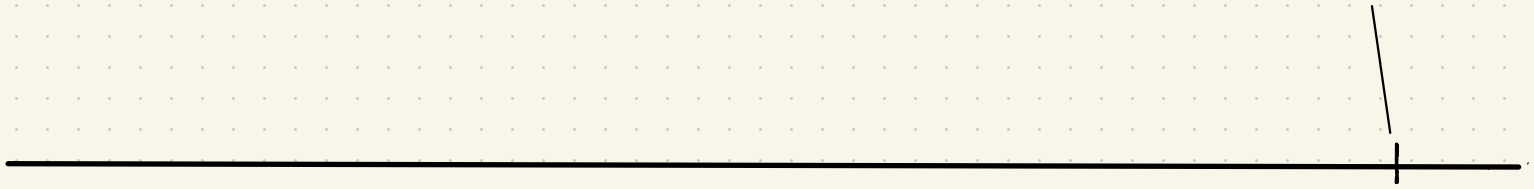
Jefferson's method adopted



Jefferson Divisor		House Size
30,000		112
28,500	⇐	120

April 10
1792

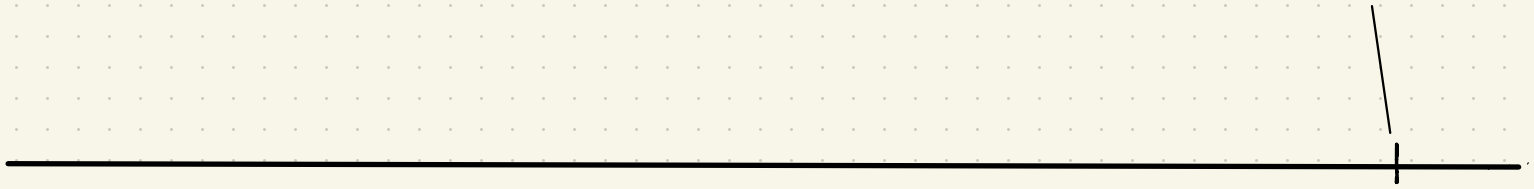
Jefferson's method adopted



April 10
1792

Jefferson Divisor	House Size
30,000	112
☹ 28,500	120
33,000	105

Jefferson's method adopted

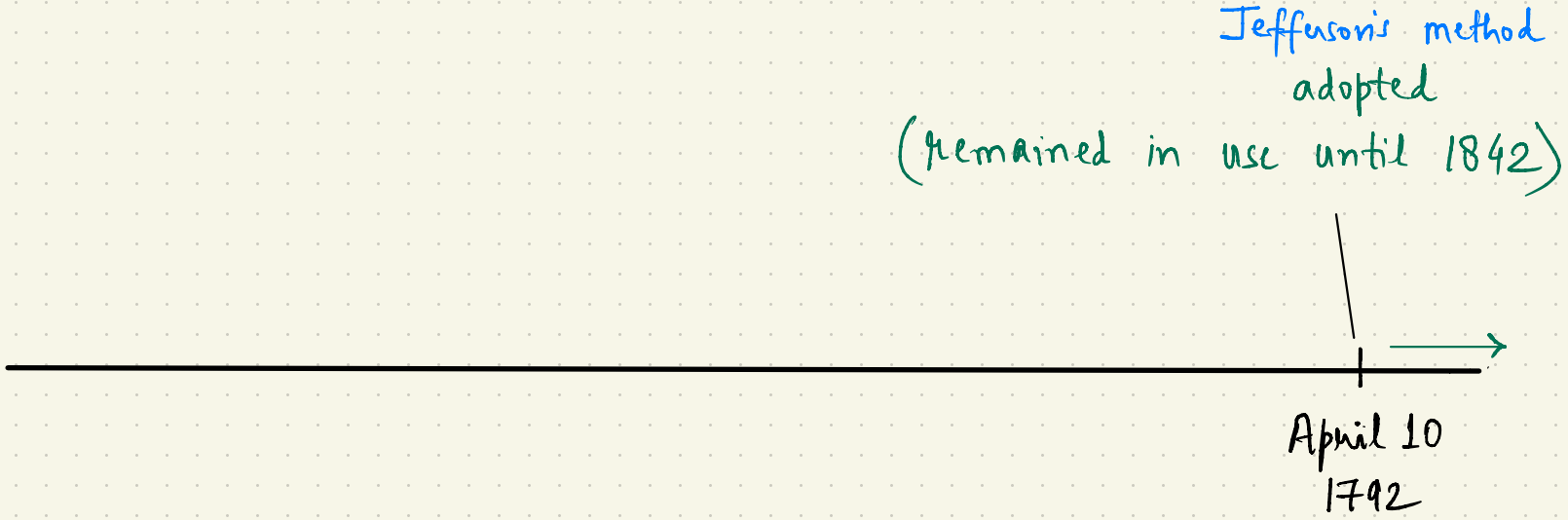


Jefferson Divisor	House Size	April 10 1792
30,000	112	
☹ 28,500	120	
33,000	105	→

Virginia's seats

Jefferson	19 / 105
Hamilton	18 / 105

😐


Jefferson's method
adopted
(remained in use until 1842)
April 10
1792

Jefferson's method adopted
(remained in use until 1842)

April 10
1792


Jefferson's method
adopted
April 10
1792

No. of states : 15 → 24

Population : 3.6 mil → 12 mil

House size : 104 → 240

JEFFERSON'S LARGE STATE BIAS

JEFFERSON'S LARGE STATE BIAS

House size $h = 10$.

State	Population	$q_i = p_i/10$	Hamilton
A	18	1.8	2
B	82	8.2	8
	100		

JEFFERSON'S LARGE STATE BIAS

House size $h = 10$.

State	Population	$q_i = p_i/10$	Hamilton	$p_i/q.1$	Jefferson
A	18	1.8	2	1.98	1
B	82	8.2	8	9.02	9
	100				

JEFFERSON'S LARGE STATE BIAS

House size $h = 10$.

State	Population	$q_i = p_i/10$	Hamilton	$p_i/9.1$	Jefferson
A	18	1.8	2	1.98	1
B	82	8.2	8	9.02	9
	100				

A : 18 citizens per seat

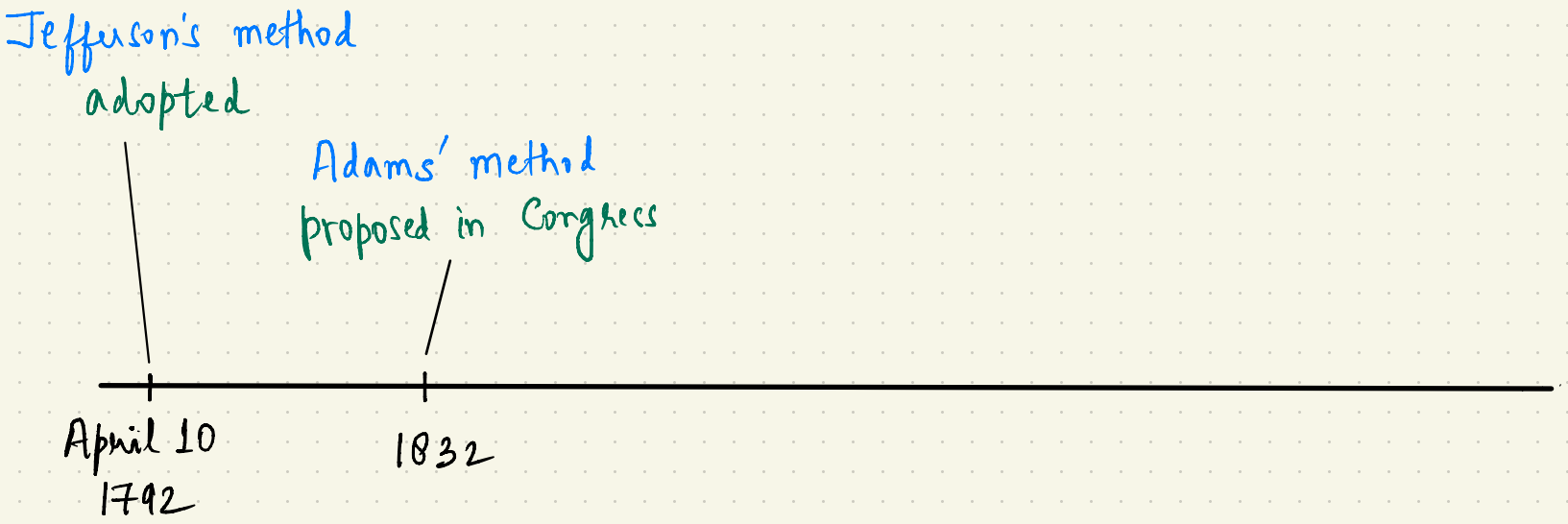
B : 9.1 citizens per seat

JEFFERSON'S LARGE STATE BIAS

House size $h = 10$.

State	Population	$q_i = p_i/10$	Hamilton	$p_i/q.1$	Jefferson
A	18	1.8	2	1.98	1
B	82	8.2	8	9.02	9
	100				

Quiz problem : Jefferson can push large states beyond upper quota.

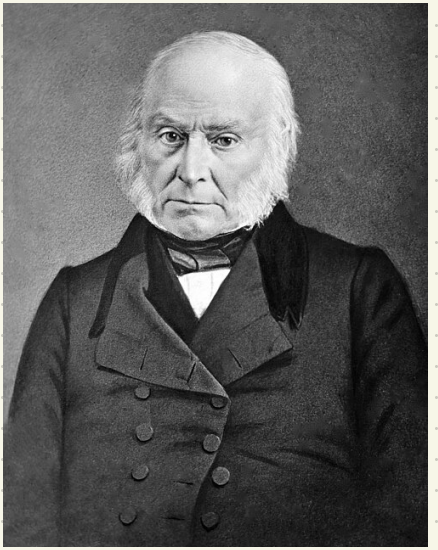

Jefferson's method adopted
Adams' method proposed in Congress
April 10 1792
1832

John Quincy Adams

* Sixth US President (1825-29)

  [Not to be confused with John Adams, his father and the second US President]

* Returned to Congress after his Presidential tenure



1767 - 1848

ADAMS' METHOD

* Let D be a divisor such that

$$\left\lceil \frac{p_1}{D} \right\rceil + \left\lceil \frac{p_2}{D} \right\rceil + \cdots + \left\lceil \frac{p_n}{D} \right\rceil = h.$$

* Assign $s_i = \left\lceil \frac{p_i}{D} \right\rceil$ to state i.

A round-up version of Jefferson's method.

ADAMS' SMALL STATE BIAS

ADAMS' SMALL STATE BIAS

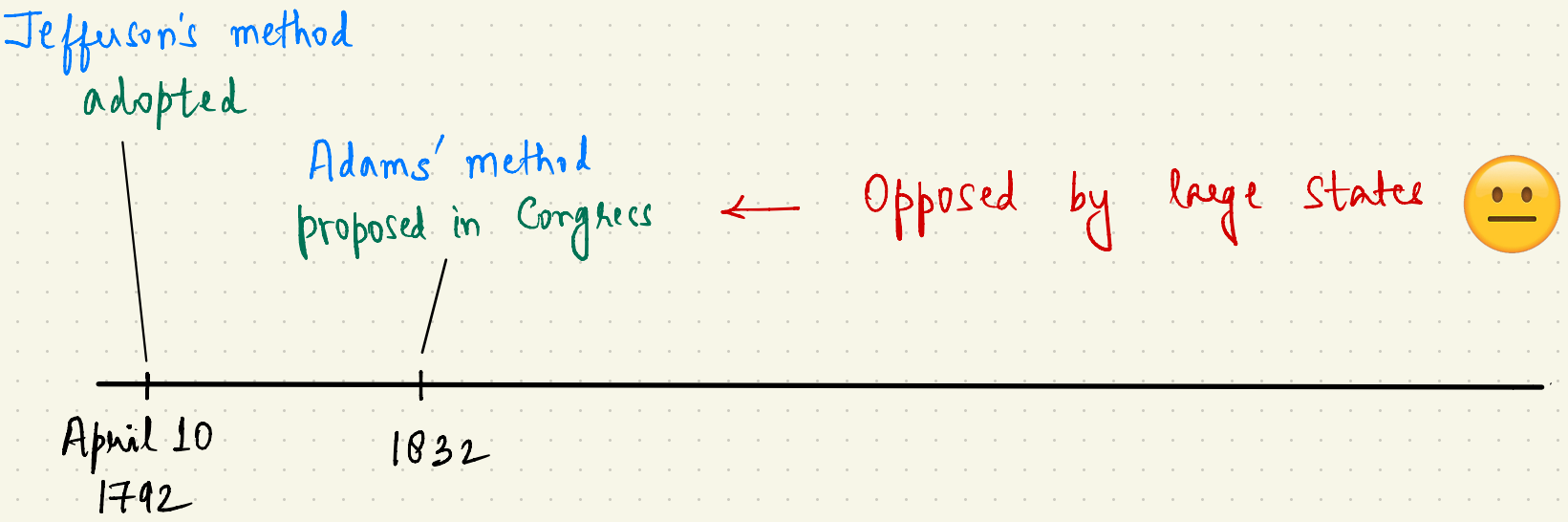
House size $h = 10$.

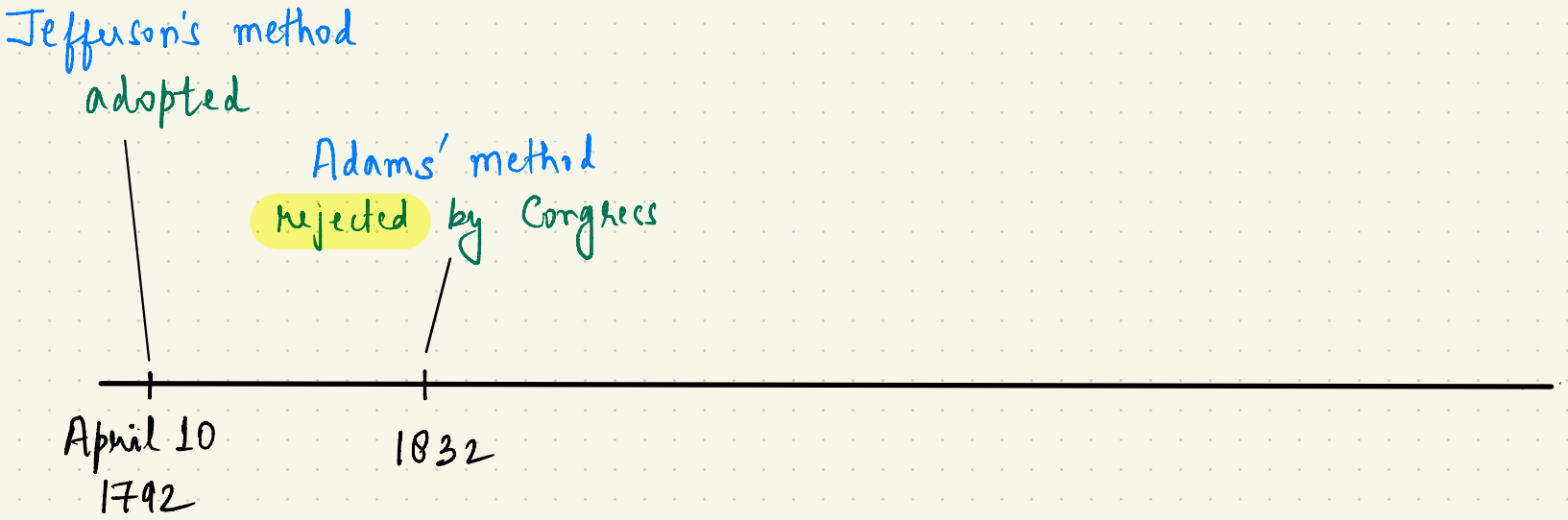
State	Population	$D = 10$		$D = 10.3$	
		$p_i/10$	$\lceil p_i/10 \rceil$	$p_i/10.3$	$\lceil p_i/10.3 \rceil$
A	18	1.8	2	1.75	2
B	82	8.2	9	7.96	8
	100		11		10

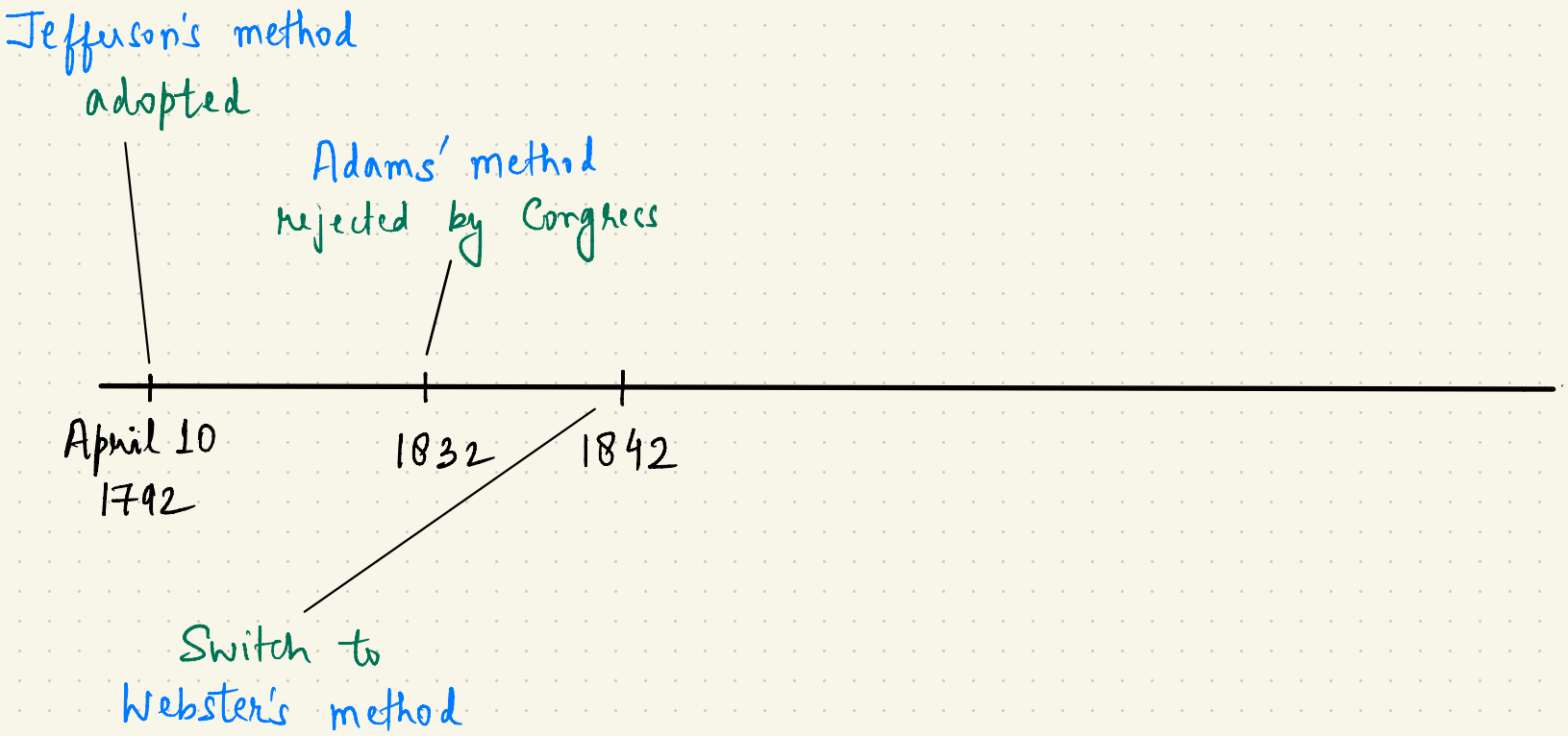
A: 9 citizens per seat

B: 10.25 citizens per seat

State	*Population*	*Quota*	*Adams Appt.*	*Jefferson Appt. (Polk Bill)*
New York	1,918,578	38.593	37	40
Pennsylvania	1,348,072	27.117	26	28
Kentucky	621,832	12.509	12	13
Vermont	280,657	5.646	6	5
Louisiana	171,904	3.458	4	3
Illinois	157,147	3.161	4	3
Missouri	130,419	2.623	3	2
Mississippi	110,358	2.220	3	2
Delaware	75,432	1.517	2	1
U.S. Total	11,931,000	240	240	240


Jefferson's method adopted
Adams' method proposed in Congress
← Opposed by large states
April 10 1792
1832

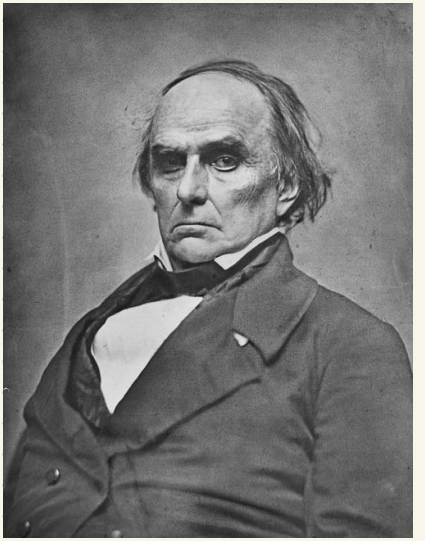

Jefferson's method adopted
Adams' method rejected by Congress
April 10 1792
1832


Jefferson's method adopted
Adams' method rejected by Congress
April 10 1792
1832
1842
Switch to Webster's method

DANIEL WEBSTER

* Lawyer and Senator

* Spellbinding orator



1782 – 1852

WEBSTER'S METHOD

* Let D be a divisor such that

$$\left[\frac{p_1}{D}\right] + \left[\frac{p_2}{D}\right] + \cdots + \left[\frac{p_n}{D}\right] = h .$$

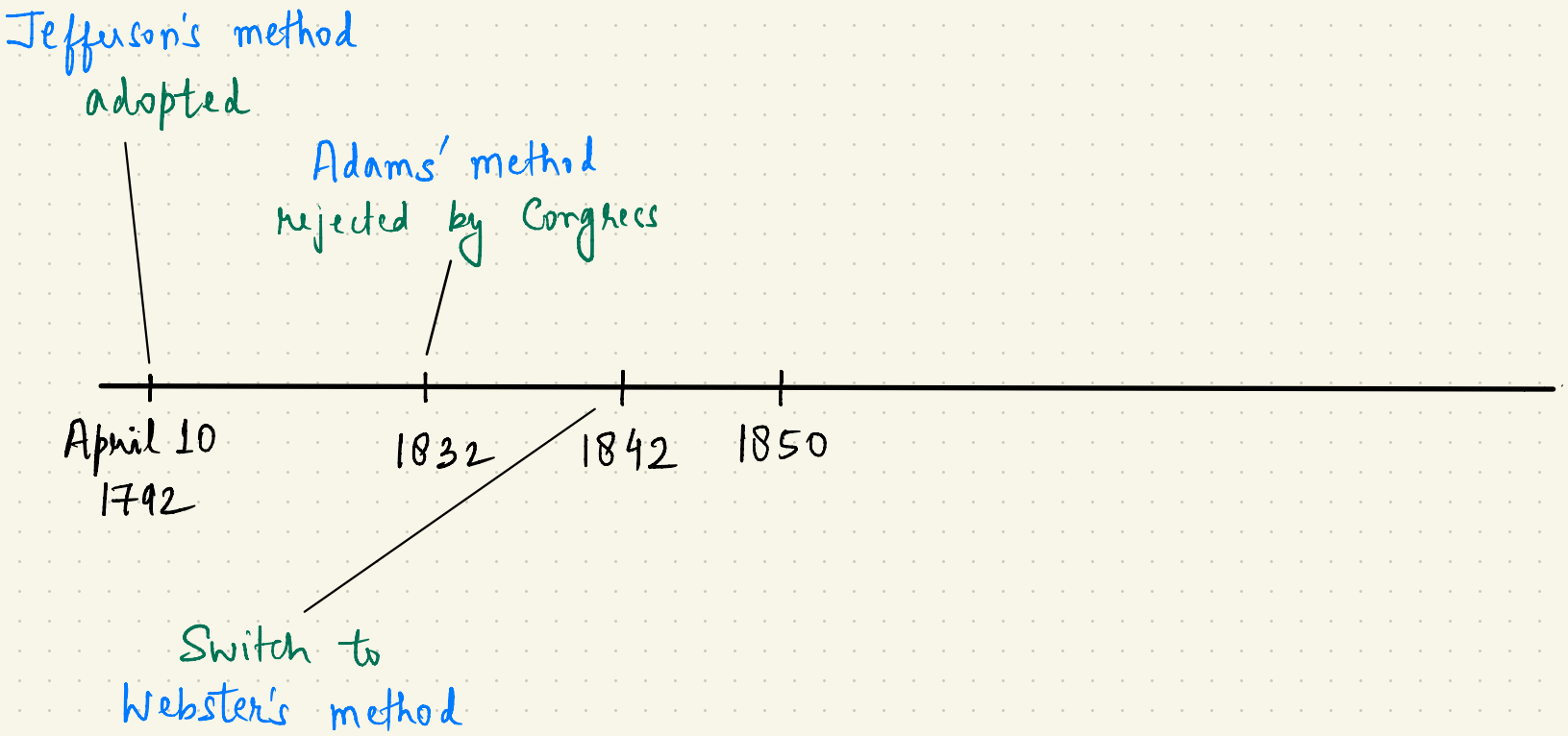
* Assign $s_i = \left[\frac{p_i}{D}\right]$ to state i.

$$[x] = \begin{cases} \lceil x \rceil & \text{if } \text{frac}(x) \geq 0.5 \\ \lfloor x \rfloor & \text{o/w} \end{cases}$$

WEBSTER IS "UNBIASED"

Webster

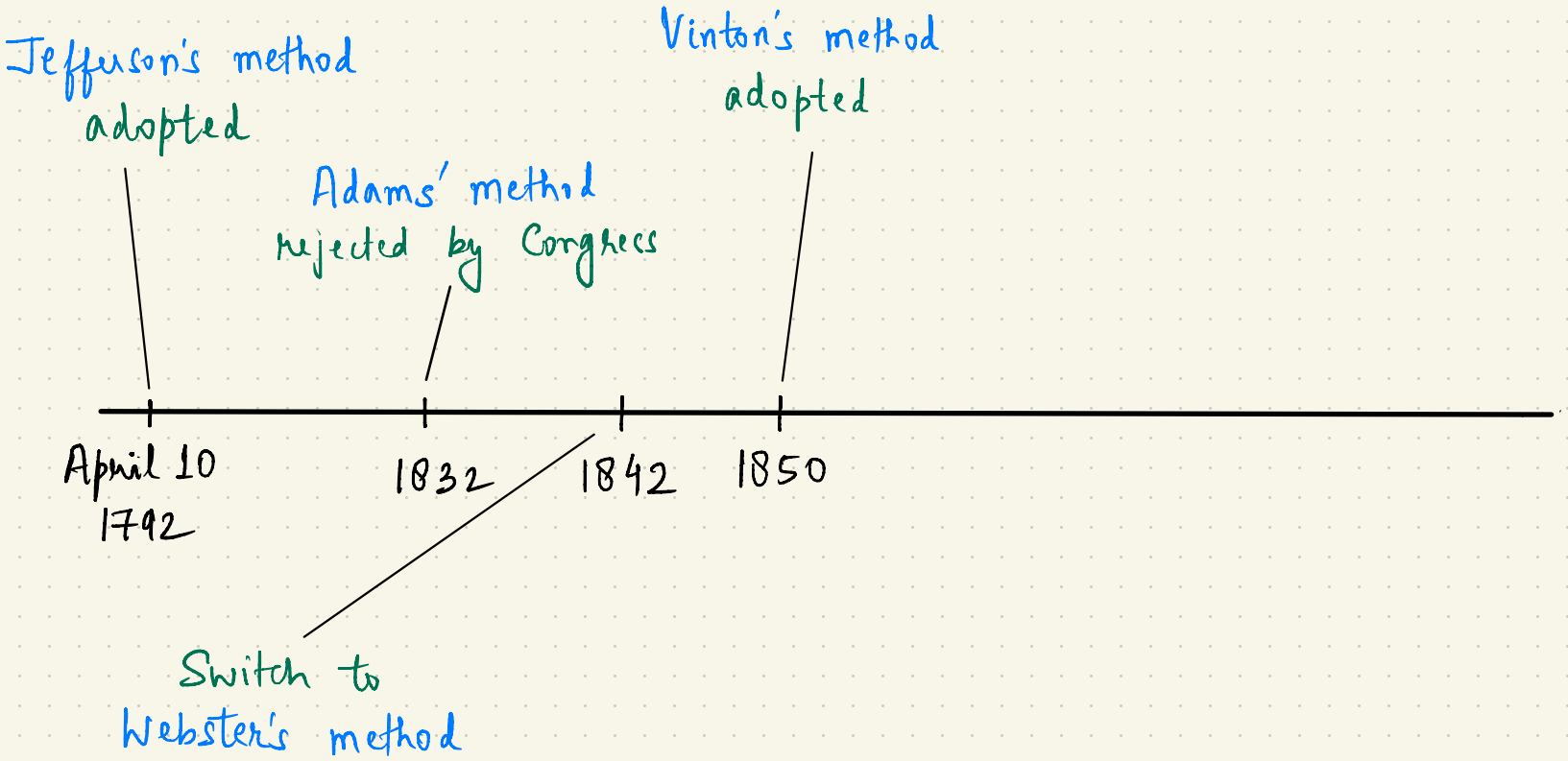
State	Population	Quota	Adams Appt.		Jefferson Appt. (Polk Bill)
New York	1,918,578	38.593	37	39	40
Pennsylvania	1,348,072	27.117	26	27	28
Kentucky	621,832	12.509	12	12	13
Vermont	280,657	5.646	6	6	5
Louisiana	171,904	3.458	4	3	3
Illinois	157,147	3.161	4	3	3
Missouri	130,419	2.623	3	3	2
Mississippi	110,358	2.220	3	2	2
Delaware	75,432	1.517	2	2	1
U.S. Total	11,931,000	240	240		240

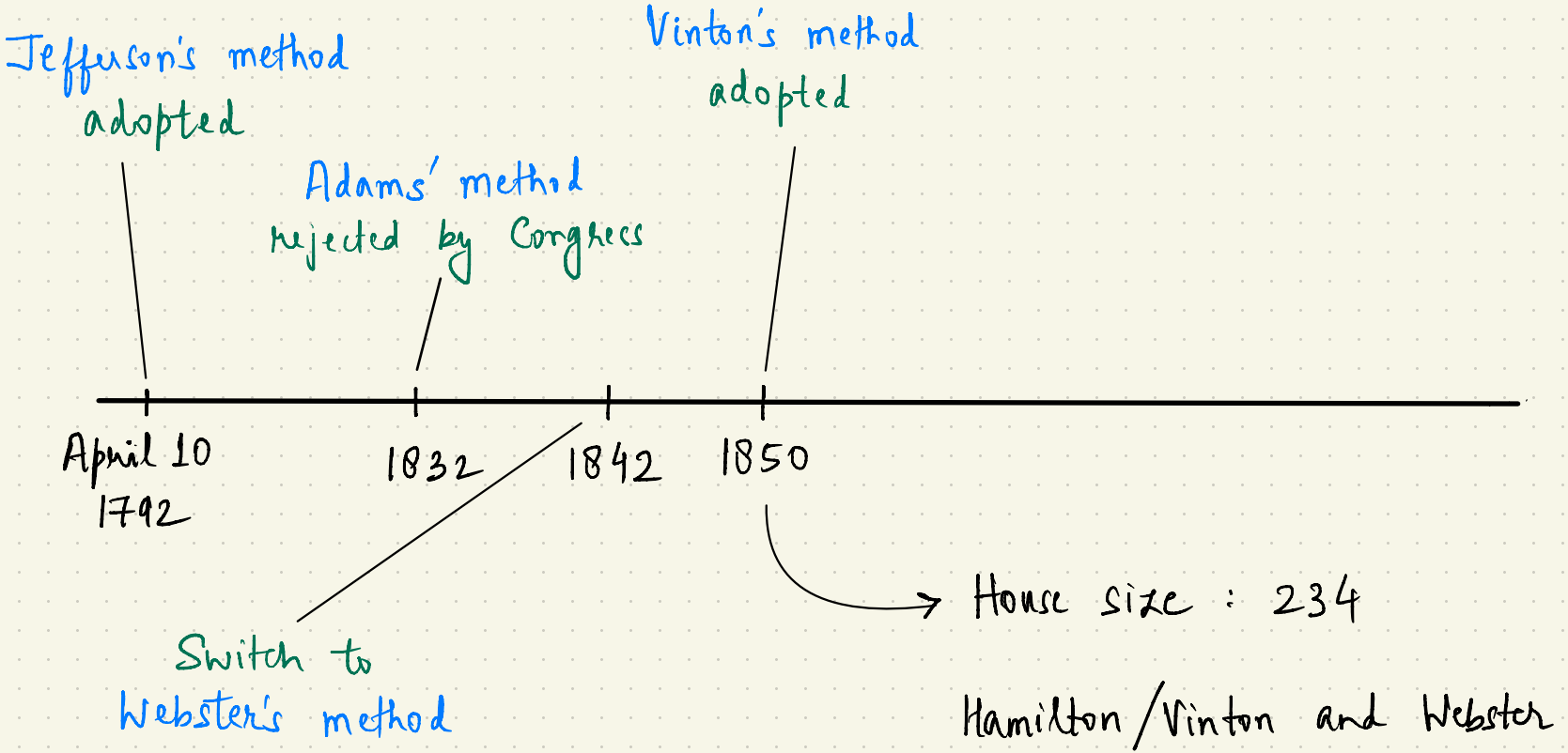

Jefferson's method adopted
Adams' method rejected by Congress
April 10 1792
1832
1842
1850
Switch to Webster's method

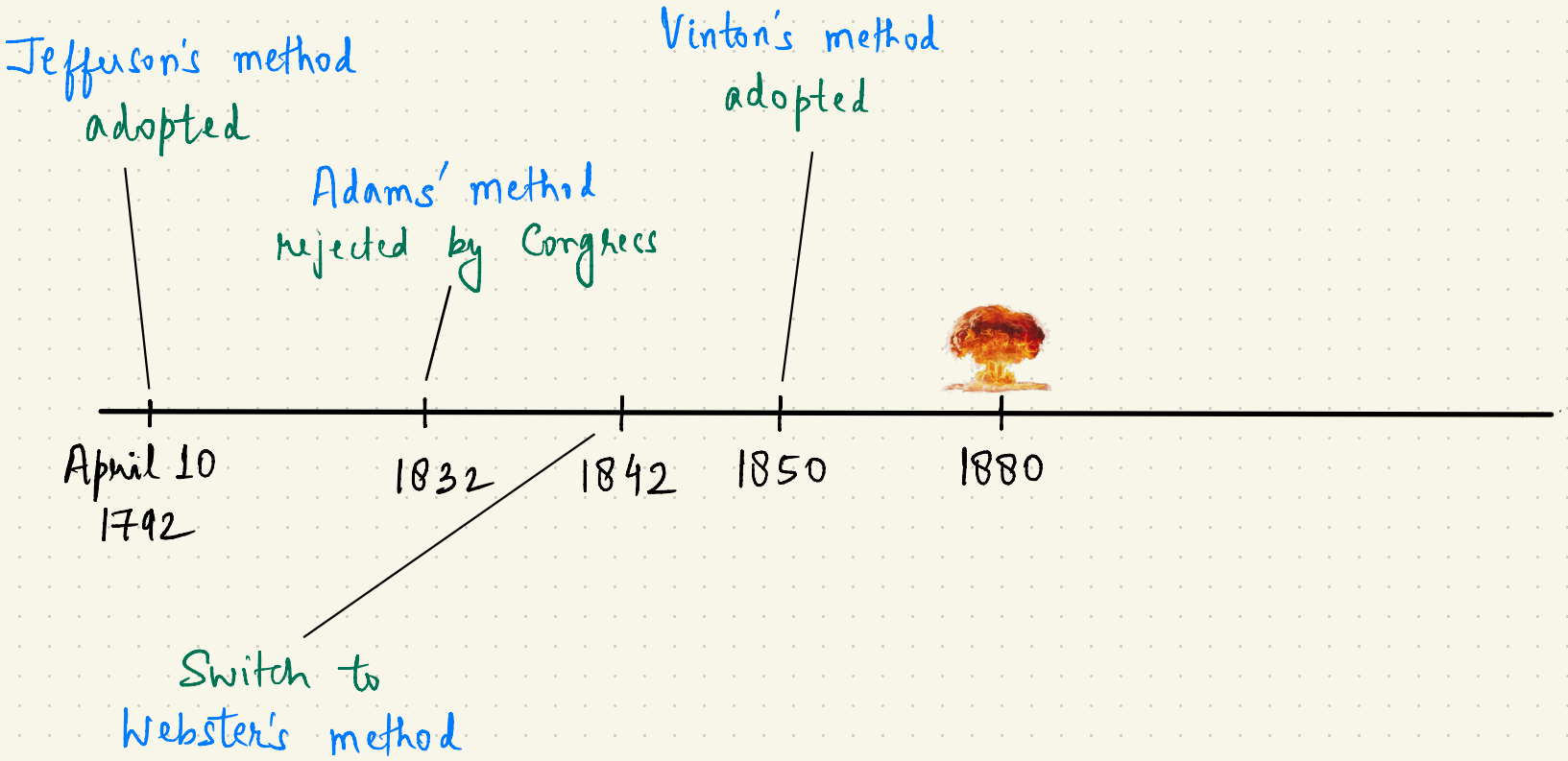
SAMUEL VINTON

* Member of House of Representatives
* Argued in favor of adopting a permanent apportionment method
* Proposed a method identical to Hamilton's method.

1792 – 1862


Jefferson's method adopted
Vinton's method adopted
Adams' method rejected by Congress
April 10 1792
1832
1842
1850
Switch to Webster's method


Jefferson's method adopted
Vinton's method adopted
Adams' method rejected by Congress
April 10 1792
1832
1842
1850
Switch to Webster's method
House size : 234
Hamilton/Vinton and Webster outcomes coincided


Jefferson's method adopted
Vinton's method adopted
Adams' method rejected by Congress
April 10 1792
1832
1842
1850
1880
Switch to Webster's method

* Chief Clerk of Census Office

* Using 1880 Census results,
  Calculated Hamilton / Vinton outcomes
  for all house sizes between
  275 and 350.

C.W. SEATON

299 seats are to be allocated. Total population is ~~49,713,370~~ 49,371,340 and the appropriate divisor is 165,120.

	Alabama	*Texas*	*Illinois*
Population	1,262,505	1,591,749	3,077,871
"Raw" allocation	7.646	9.640	18.640
Seats in first round	7	9	18
Fractional part	0.646	0.640	0.640
Additional seats	1	0	0
Total seats	8	9	18

Now 300 seats are to be allocated. The appropriate divisor is 164,580.

	Alabama	*Texas*	*Illinois*
Population	1,262,505	1,591,749	3,077,871
"Raw" allocation	7.671	9.672	18.701
Seats in first round	7	9	18
Fractional part	0.671	0.672	0.701
Additional seats	0	1	1
Total seats	7	10	19

Alabama loses one seat; Texas and Illinois each gain a seat.

House size grows but Alabama loses a seat!

299 seats are to be allocated. Total population is ~~49,713,370~~ 49,371,340 and the appropriate divisor is 165,120.

	Alabama	*Texas*	*Illinois*
Population	1,262,505	1,591,749	3,077,871
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Fractional part	0.646	0.640	0.640
Additional seats	1	0	0
Total seats	8	9	18

Now 300 seats are to be allocated. The appropriate divisor is 164,580.

	Alabama	*Texas*	*Illinois*
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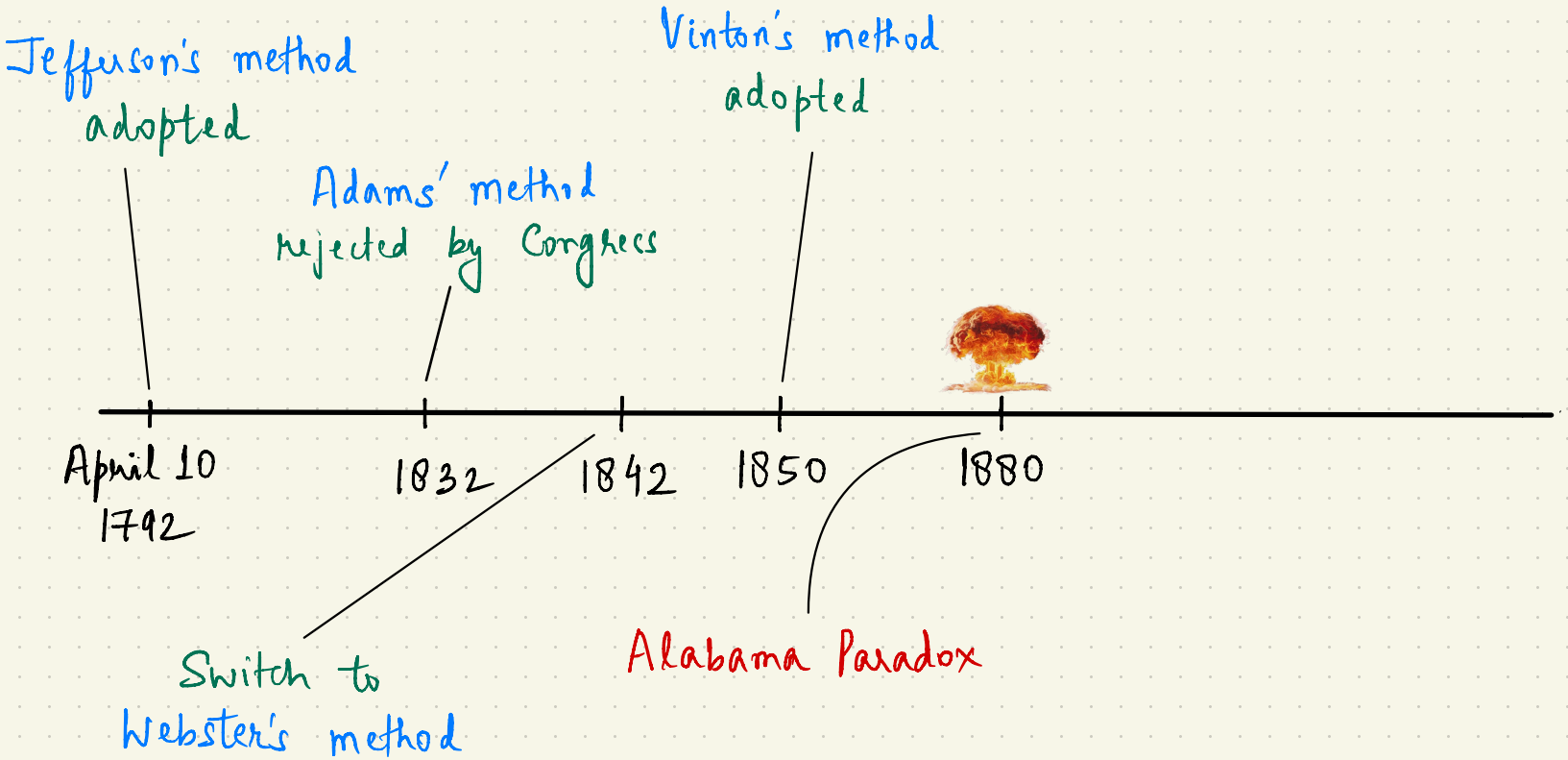
"Raw" allocation	7.671	9.672	18.701
Seats in first round	7	9	18
Fractional part	0.671	0.672	0.701
Additional seats	0	1	1
Total seats	7	10	19

Alabama loses one seat; Texas and Illinois each gain a seat.

Why? Larger states (Texas, Illinois) outpace Alabama.


Jefferson's method adopted
Vinton's method adopted
Adams' method rejected by Congress
April 10 1792
1832
1842
1850
1880
Switch to Webster's method
Alabama Paradox