

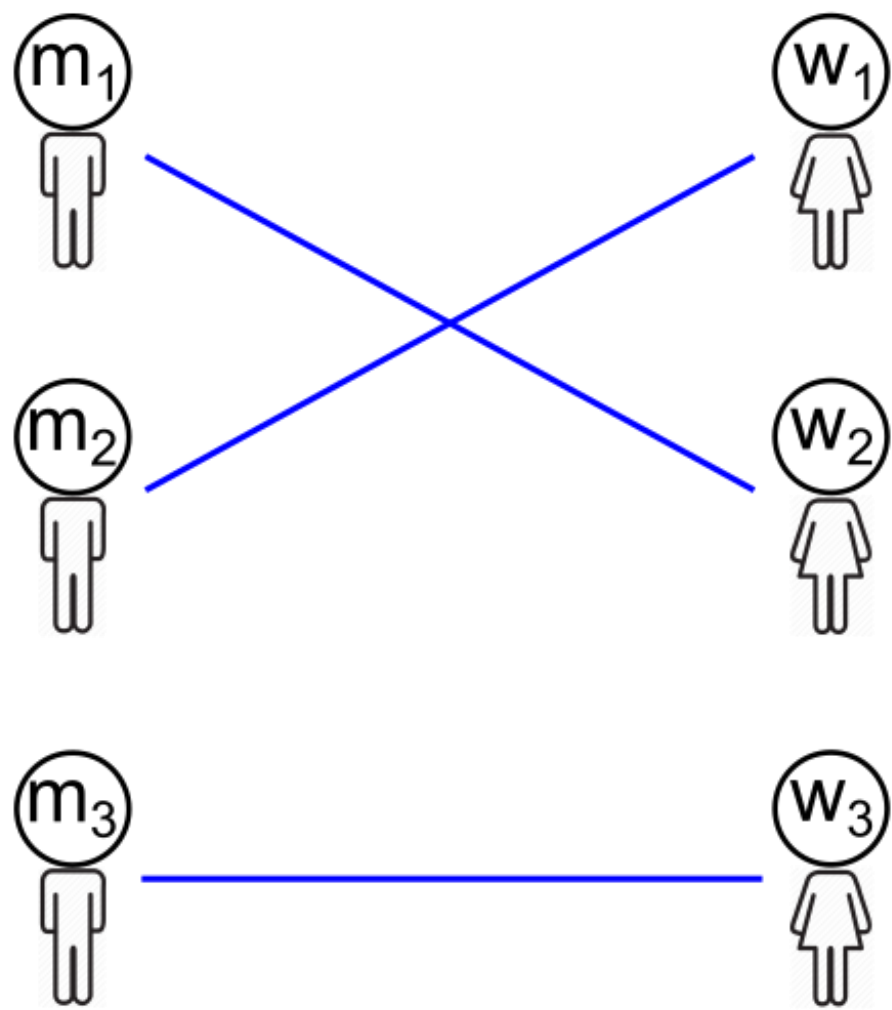
## Lecture 19

# Fairness in the Stable Matching Problem (Contd.)

# Goal for Today

Understanding the structure of the set of stable matchings through linear programming.

(This will guide us towards fair stable matchings.)



$$P = \begin{matrix} & w_1 & w_2 & w_3 \\ m_1 & \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \\ m_2 & \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \\ m_3 & \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \end{matrix}$$

# Fractional Stable Matching

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Any non-negative  $n \times n$  matrix  $X$  satisfying the following:

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# Fractional Stable Matching

Any non-negative  $n \times n$  matrix  $X$  satisfying the following:

$$X_{i,j} \geq 0 \text{ for all } i \in [n] \text{ and } j \in [n]$$

$$\sum_j X_{i,j} = 1 \text{ for all } i \in [n] \quad \textit{Every man is fully matched.}$$

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$$\sum_i X_{i,j} = 1 \text{ for all } j \in [n] \quad \textit{Every woman is fully matched.}$$

$$X_{i,j} + \sum_{k:w_k \succ_{m_i} w_j} X_{i,k} + \sum_{k:m_k \succ_{w_j} m_i} X_{k,j} \geq 1 \text{ for all } i \in [n] \text{ and } j \in [n] \quad \textit{Stability}$$

# Fractional Stable Matching

Any non-negative  $n \times n$  matrix  $X$  satisfying the following:

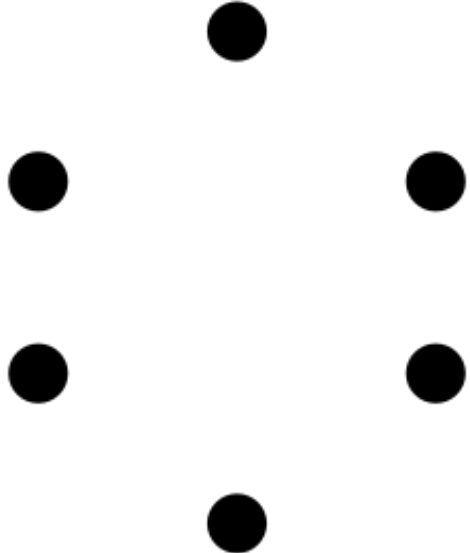
$$X_{i,j} \geq 0 \text{ for all } i \in [n] \text{ and } j \in [n]$$

$$\sum_j X_{i,j} = 1 \text{ for all } i \in [n] \quad \textit{Every man is fully matched.}$$

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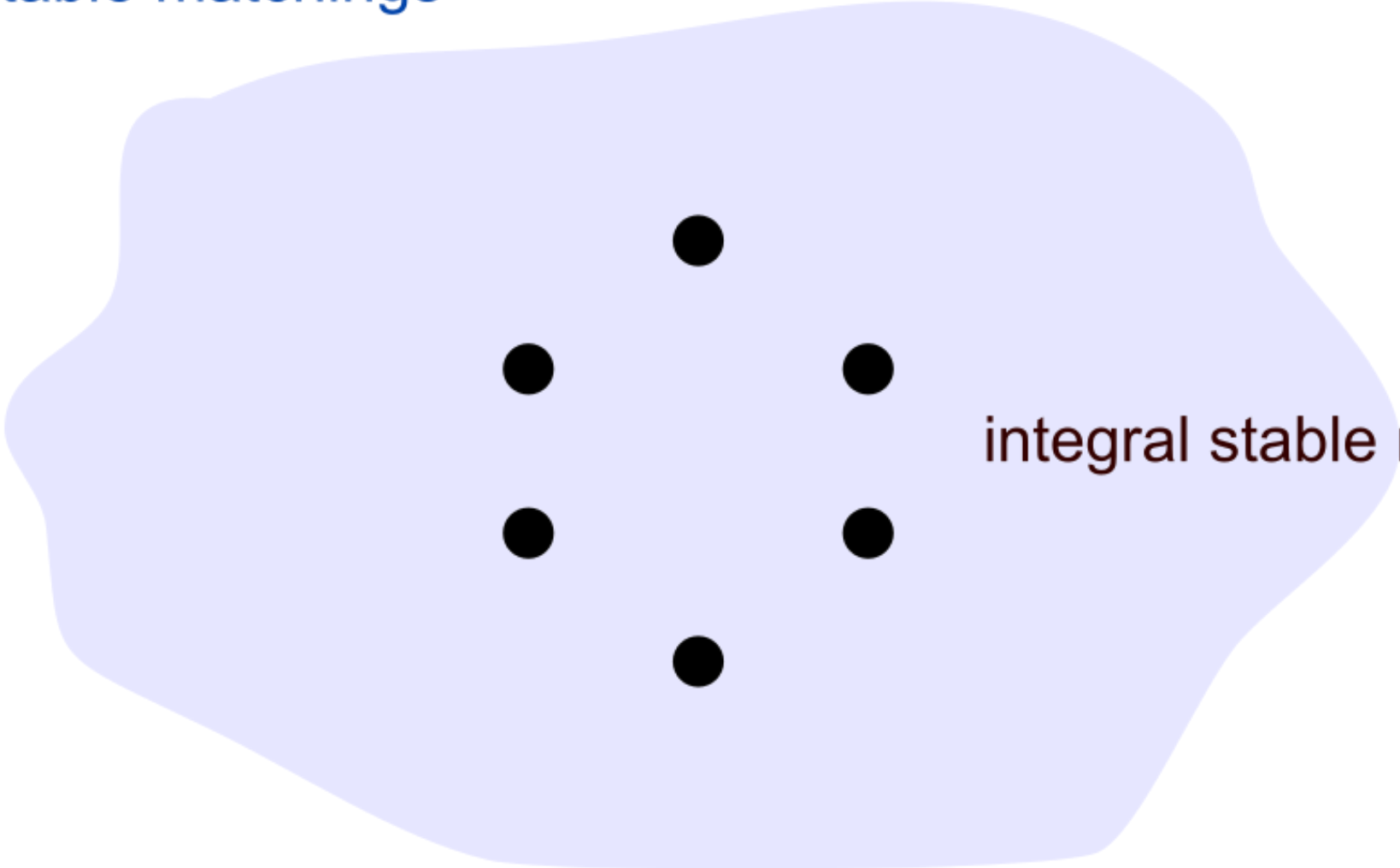
$$X_{i,j} + \sum_{k:w_k \succ_{m_i} w_j} X_{i,k} + \sum_{k:m_k \succ_{w_j} m_i} X_{k,j} \geq 1 \text{ for all } i \in [n] \text{ and } j \in [n] \quad \textit{Stability}$$

Any integral stable matching is also a fractional stable matching.



integral stable matchings

fractional stable matchings



integral stable matchings

# Fractional Stable Matching

Any non-negative  $n \times n$  matrix  $X$  satisfying the following:

Any convex combination of integral stable matchings is also a fractional stable matching.

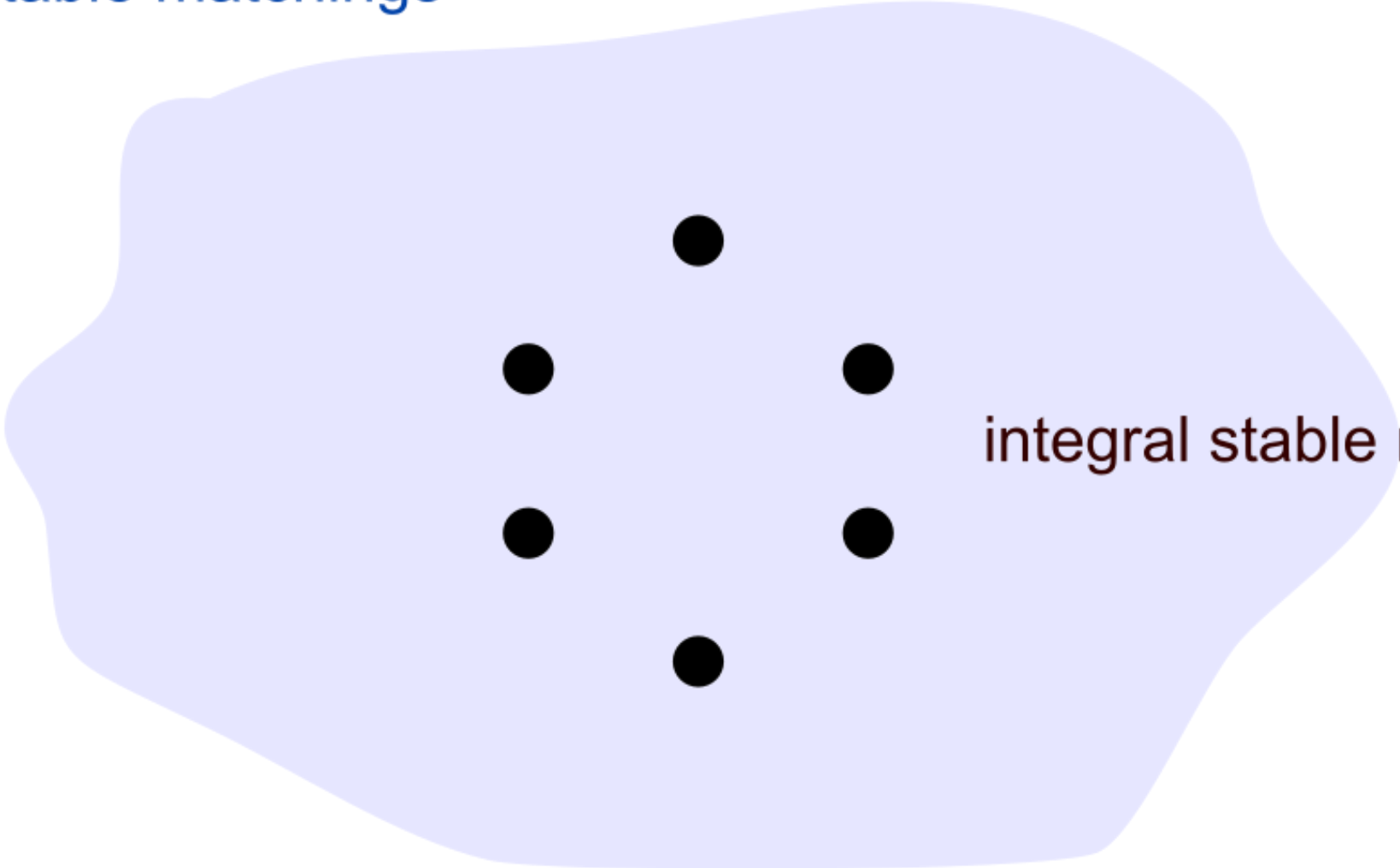
$$X = \sum_k \lambda_k P^k$$

integral stable matching

such that  $\lambda_k \geq 0$  for all  $k$  and  $\sum_k \lambda_k = 1$

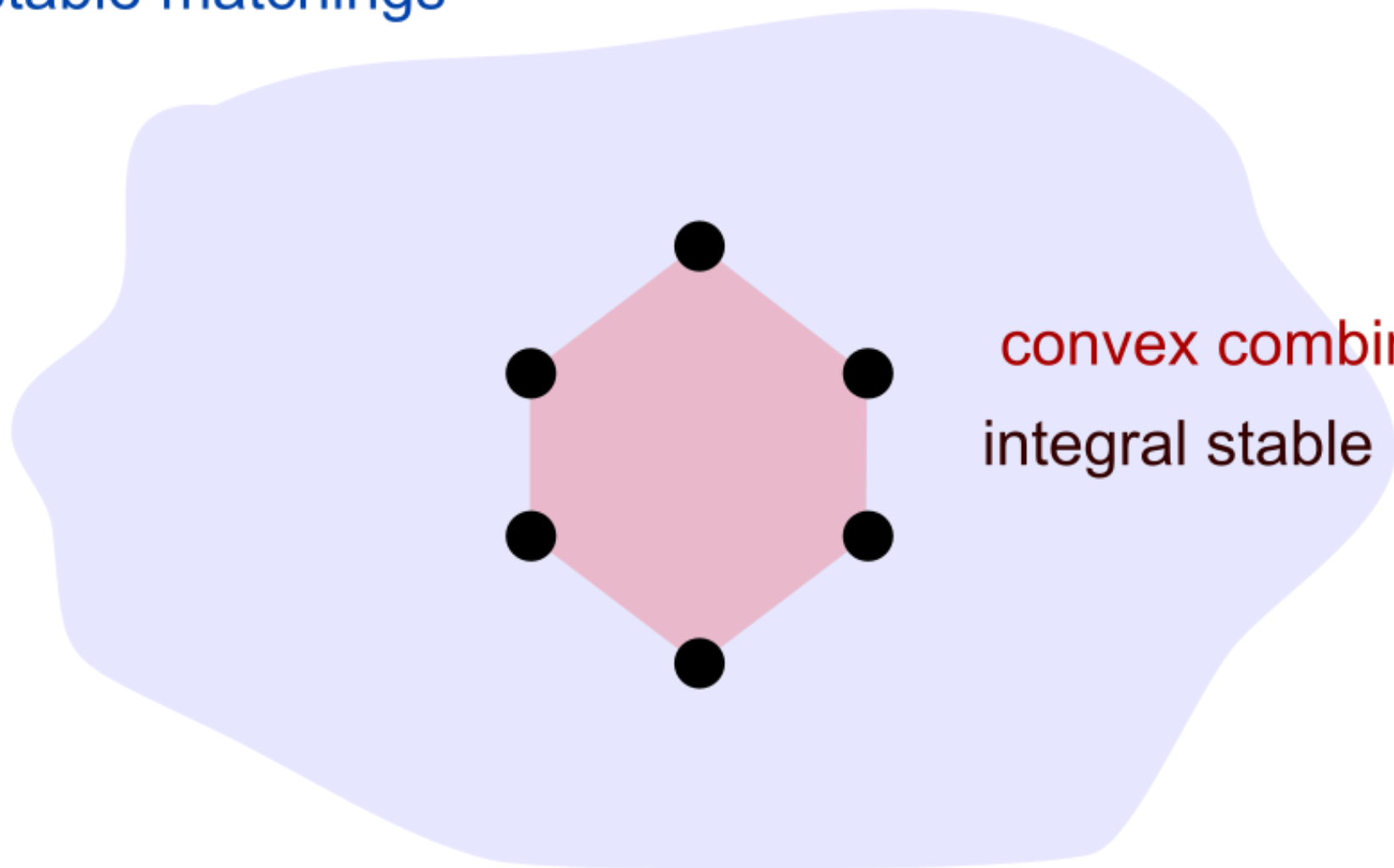
Any integral stable matching is also a fractional stable matching.

fractional stable matchings



integral stable matchings

fractional stable matchings

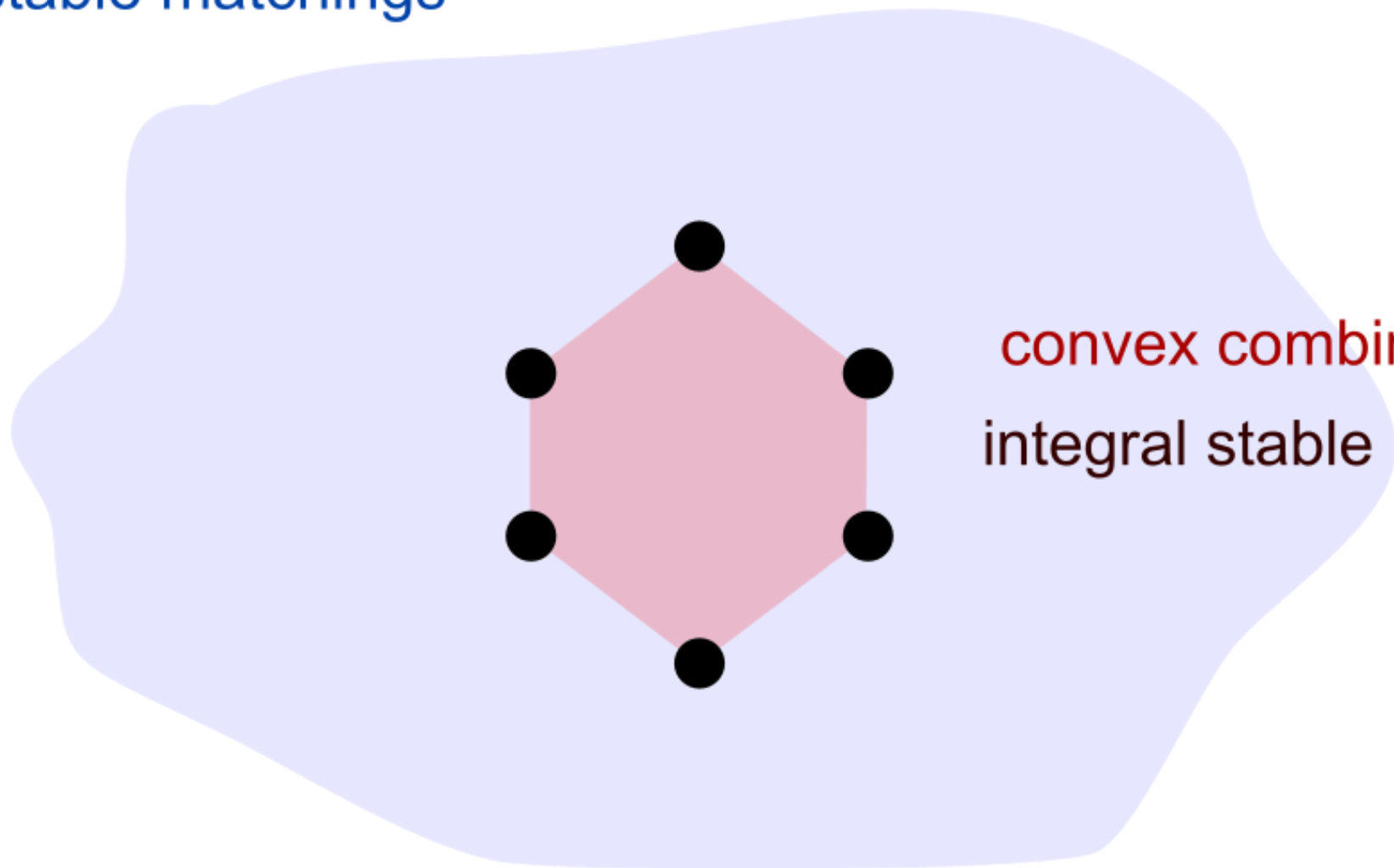


convex combinations of  
integral stable matchings

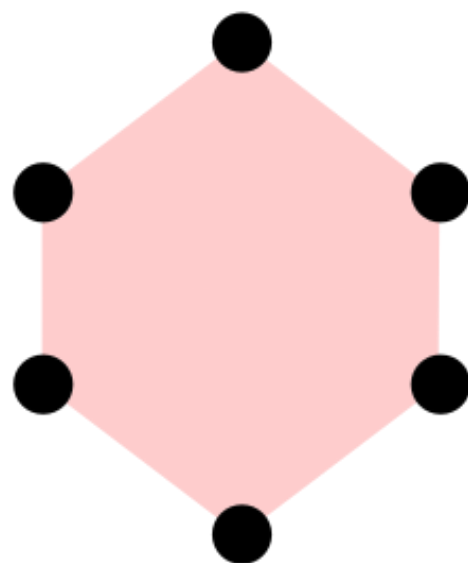
[Vande Vate, *Oper. Res. Let.* 1989]

Any fractional stable matching can be expressed as a convex combination of integral stable matchings.

fractional stable matchings



convex combinations of  
integral stable matchings



convex combinations of  
integral stable matchings

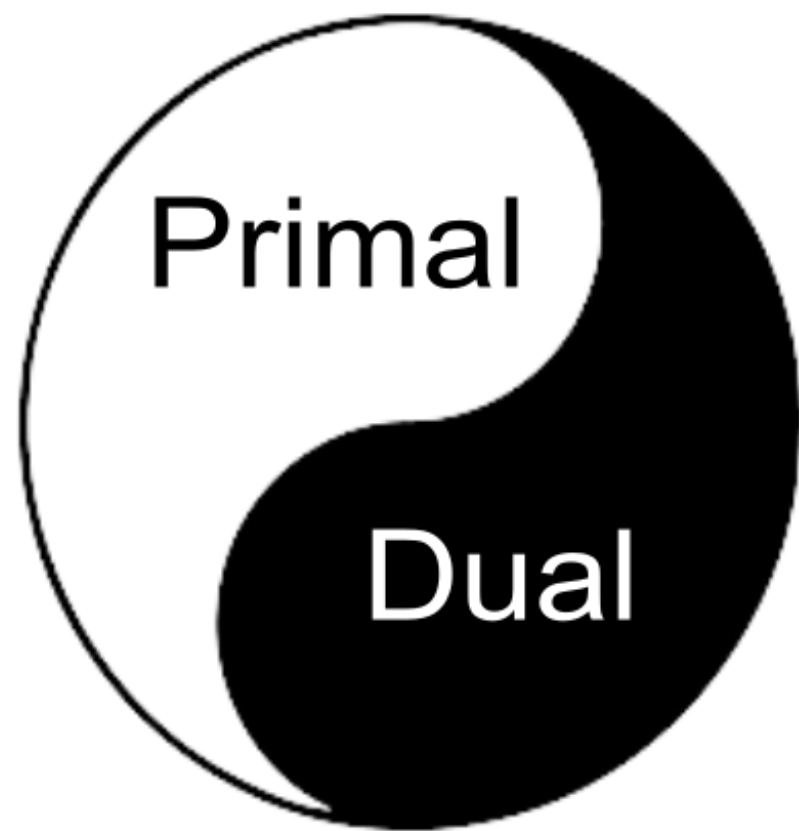
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fractional stable matchings

Any fractional stable matching can be expressed as a convex combination of integral stable matchings.

Coming up...

An elegant geometric proof that uses LP duality and its application in fair stable matchings.



# Primal

$$\max \sum_{i,j} X_{i,j}$$

$$\sum_j X_{i,j} = 1 \quad \forall i$$

$$\sum_i X_{i,j} = 1 \quad \forall j$$

$$X_{i,j} + \sum_{k:w_k \succ_{m_i} w_j} X_{i,k} + \sum_{k:m_k \succ_{w_j} m_i} X_{k,j} \geq 1 \quad \forall i, j$$

$$X_{i,j} \geq 0 \quad \forall i, j$$

# Primal

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# Primal

$$\max \sum_{i,j} X_{i,j}$$

$$\sum_j X_{i,j} = 1 \quad \forall i$$

$$\sum_i X_{i,j} = 1 \quad \forall j$$

$$-X_{i,j} - \sum_{k:w_k \succ_{m_i} w_j} X_{i,k} - \sum_{k:m_k \succ_{w_j} m_i} X_{k,j} \leq -1 \quad \forall i, j$$

$$X_{i,j} \geq 0 \quad \forall i, j$$

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$$\sum_j X_{i,j} = 1 \quad \forall i \quad \alpha_i$$

$$\sum_i X_{i,j} = 1 \quad \forall j \quad \beta_j$$

$\gamma_{i,j}$

$$-X_{i,j} - \sum_{k:w_k \succ_{m_i} w_j} X_{i,k} - \sum_{k:m_k \succ_{w_j} m_i} X_{k,j} \leq -1 \quad \forall i, j$$

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$$X_{i,j} \geq 0 \quad \forall i, j$$

Don't worry about us.

# Primal

$$\max \sum_{i,j} X_{i,j}$$

$$\sum_j X_{i,j} = 1 \quad \forall i \quad \alpha_i$$

Combine the constraints in order to construct an upper bound on the objective.

$$-X_{i,j} - \sum_{k:w_k \succ m_i} X_{i,k} - \sum_{k:m_k \succ w_j} X_{k,j} \leq -1 \quad \forall i, j \quad \gamma_{i,j}$$

$$X_{i,j} \geq 0 \quad \forall i, j$$

Don't worry about us.

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$$X_{i,j} \geq 0 \quad \forall i, j$$

# Primal

$$\max \sum_{i,j} X_{i,j}$$

$$\sum_{i,j} \alpha_i X_{i,j} = \sum_i \alpha_i$$

$$\sum_i X_{i,j} = 1 \quad \forall j$$

$$-X_{i,j} - \sum_{k:w_k \succ_{m_i} w_j} X_{i,k} - \sum_{k:m_k \succ_{w_j} m_i} X_{k,j} \leq -1 \quad \forall i, j$$

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$$\sum_i X_{i,j} = 1 \quad \forall j \quad \beta_j$$

$$-X_{i,j} - \sum_{k:w_k \succ_{m_i} w_j} X_{i,k} - \sum_{k:m_k \succ_{w_j} m_i} X_{k,j} \leq -1 \quad \forall i, j$$

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$$-\sum_{i,j} \left( \gamma_{i,j} X_{i,j} + \sum_{k:w_k > m_i w_j} \gamma_{i,j} X_{i,k} + \sum_{k:m_k > w_j m_i} \gamma_{i,j} X_{k,j} \right) \leq -\sum_{i,j} \gamma_{i,j}$$

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$$X_{i,j} \geq 0 \quad \forall i, j$$

as long as  
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# Primal

Let's combine these.

$$\max \sum_{i,j} X_{i,j}$$

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$$X_{i,j} \geq 0 \quad \forall i, j$$

as long as  
 $\gamma_{i,j} \geq 0 \quad \forall i, j$

---

$$\sum_{i,j} X_{i,j} \left( \alpha_i + \beta_j - \gamma_{i,j} - \sum_{k:w_j \succ_{m_i} w_k} \gamma_{i,k} - \sum_{k:m_i \succ_{w_j} m_k} \gamma_{k,j} \right)$$

$$\max \sum_{i,j} X_{i,j}$$

$$\sum_{i,j} \alpha_i X_{i,j} = \sum_i \alpha_i$$

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$$-\sum_{i,j} \left( \gamma_{i,j} X_{i,j} + \sum_{k:w_k \succ_{m_i} w_j} \gamma_{i,j} X_{i,k} + \sum_{k:m_k \succ_{w_j} m_i} \gamma_{i,j} X_{k,j} \right) \leq -\sum_{i,j} \gamma_{i,j}$$

$$X_{i,j} \geq 0 \quad \forall i, j$$

as long as  
 $\gamma_{i,j} \geq 0 \quad \forall i, j$

$$\sum_{i,j} X_{i,j} \left( \alpha_i + \beta_j - \gamma_{i,j} - \sum_{k:w_j \succ_{m_i} w_k} \gamma_{i,k} - \sum_{k:m_i \succ_{w_j} m_k} \gamma_{k,j} \right) \leq \sum_i \alpha_i + \sum_j \beta_j - \sum_{i,j} \gamma_{i,j}$$

---

$$\sum_{i,j} X_{i,j} \left( \alpha_i + \beta_j - \gamma_{i,j} - \sum_{k: w_j > m_i w_k} \gamma_{i,k} - \sum_{k: m_i > w_j m_k} \gamma_{k,j} \right) \leq \sum_i \alpha_i + \sum_j \beta_j - \sum_{i,j} \gamma_{i,j}$$

subject to this  
being at least 1

minimize this

---

$$\sum_{i,j} X_{i,j} \left( \alpha_i + \beta_j - \gamma_{i,j} - \sum_{k:w_j > m_i w_k} \gamma_{i,k} - \sum_{k:m_i > w_j m_k} \gamma_{k,j} \right) \leq \sum_i \alpha_i + \sum_j \beta_j - \sum_{i,j} \gamma_{i,j}$$

as long as  
 $\gamma_{i,j} \geq 0 \forall i, j$

subject to this  
being at least 1

minimize this

---

$$\sum_{i,j} X_{i,j} \left( \alpha_i + \beta_j - \gamma_{i,j} - \sum_{k:w_j > m_i w_k} \gamma_{i,k} - \sum_{k:m_i > w_j m_k} \gamma_{k,j} \right) \leq \sum_i \alpha_i + \sum_j \beta_j - \sum_{i,j} \gamma_{i,j}$$

# Dual

$$\min \sum_i \alpha_i + \sum_j \beta_j - \sum_{i,j} \gamma_{i,j}$$

$$\alpha_i + \beta_j - \gamma_{i,j} - \sum_{k:w_j \succ_{m_i} w_k} \gamma_{i,k} - \sum_{k:m_i \succ_{w_j} m_k} \gamma_{k,j} \geq 1 \quad \forall i, j$$

$$\gamma_{i,j} \geq 0 \quad \forall i, j$$

# Recall the Primal

$$\max \sum_{i,j} X_{i,j}$$

$$\sum_j X_{i,j} = 1 \quad \forall i$$

$$\sum_i X_{i,j} = 1 \quad \forall j$$

$$X_{i,j} + \sum_{k:w_k \succ_{m_i} w_j} X_{i,k} + \sum_{k:m_k \succ_{w_j} m_i} X_{k,j} \geq 1 \quad \forall i, j$$

$$X_{i,j} \geq 0 \quad \forall i, j$$

# Recall the Primal

$$\max \sum_{i,j} X_{i,j}$$

$$\sum_j X_{i,j} = 1 \quad \forall i$$

By Gale and Shapley's result,  
primal is always feasible!

$$X_{i,j} + \sum_{k: w_k \succ_{m_i} w_j} X_{i,k} + \sum_{k: m_k \succ_{w_j} m_i} X_{k,j} \geq 1 \quad \forall i, j$$

Let  $X_{i,j}$  be a feasible primal solution.

# Dual

$$\min \sum_i \alpha_i + \sum_j \beta_j - \sum_{i,j} \gamma_{i,j}$$

$$\alpha_i + \beta_j - \gamma_{i,j} - \sum_{k:w_j \succ_{m_i} w_k} \gamma_{i,k} - \sum_{k:m_i \succ_{w_j} m_k} \gamma_{k,j} \geq 1 \quad \forall i, j$$

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# Dual

$$\min \sum_i \alpha_i + \sum_j \beta_j - \sum_{i,j} \gamma_{i,j}$$

$$\alpha_i + \beta_j - \gamma_{i,j} - \sum_{k:w_j \succ_{m_i} w_k} \gamma_{i,k} - \sum_{k:m_i \succ_{w_j} m_k} \gamma_{k,j} \geq 1 \quad \forall i, j$$

$$\gamma_{i,j} \geq 0 \quad \forall i, j$$

---

Substitute  $\alpha_i = \sum_j X_{i,j}$ ,  $\beta_j = \sum_i X_{i,j}$ ,  $\gamma_{i,j} = X_{i,j}$  for all  $i, j$

# Dual

$$\min \sum_i \alpha_i + \sum_j \beta_j - \sum_{i,j} \gamma_{i,j}$$

$$\alpha_i + \beta_j - \gamma_{i,j} - \sum_{k:w_j \succ_{m_i} w_k} \gamma_{i,k} - \sum_{k:m_i \succ_{w_j} m_k} \gamma_{k,j} \geq 1 \quad \forall i, j$$

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# Dual

$$\sum_{i,j} X_{i,j}$$

$$\alpha_i + \beta_j - \gamma_{i,j} - \sum_{k:w_j \succ_{m_i} w_k} \gamma_{i,k} - \sum_{k:m_i \succ_{w_j} m_k} \gamma_{k,j} \geq 1 \quad \forall i, j$$

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$$\sum_{i,j} X_{i,j}$$

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# Dual

$$\sum_{i,j} X_{i,j}$$

$$\sum_j X_{i,j} + \sum_i X_{i,j} - X_{i,j} - \sum_{k:w_j \succ_{m_i} w_k} X_{i,k} - \sum_{k:m_i \succ_{w_j} m_k} X_{k,j} \geq 1 \quad \forall i, j$$

$$\gamma_{i,j} \geq 0 \quad \forall i, j$$

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Substitute  $\alpha_i = \sum_j X_{i,j}$ ,  $\beta_j = \sum_i X_{i,j}$ ,  $\gamma_{i,j} = X_{i,j}$  for all  $i, j$

# Dual

$$\sum_{i,j} X_{i,j}$$

$$\sum_j X_{i,j} + \sum_i X_{i,j} - X_{i,j} - \sum_{k:w_j \succ_{m_i} w_k} X_{i,k} - \sum_{k:m_i \succ_{w_j} m_k} X_{k,j} \geq 1 \quad \forall i, j$$

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Substitute  $\alpha_i = \sum_j X_{i,j}$ ,  $\beta_j = \sum_i X_{i,j}$ ,  $\gamma_{i,j} = X_{i,j}$  for all  $i, j$

# Dual

$$\sum_{i,j} X_{i,j}$$

$$\sum_{k:w_k \succ_{m_i} w_j} X_{i,k} + \sum_i X_{i,j} - \sum_{k:m_i \succ_{w_j} m_k} X_{k,j} \geq 1 \quad \forall i, j$$

$$\gamma_{i,j} \geq 0 \quad \forall i, j$$

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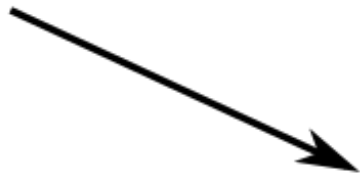
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# Dual

Stability constraint  
from primal



$$\sum_{i,j} X_{i,j}$$

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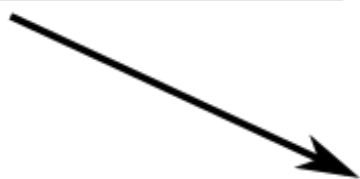
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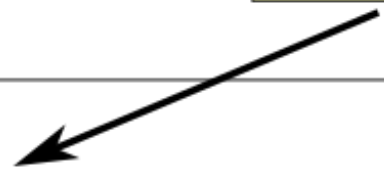


$$\sum_{i,j} X_{i,j}$$

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Dual feasible!



Substitute  $\alpha_i = \sum_j X_{i,j}$ ,  $\beta_j = \sum_i X_{i,j}$ ,  $\gamma_{i,j} = X_{i,j}$  for all  $i, j$

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Stability constraint  
from primal

Objective is equal  
to that in the primal

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Dual feasible!

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# Dual

Stability constraint  
from primal

Objective is equal  
to that in the primal

$$\sum_{i,j} X_{i,j}$$

By strong duality:

$X_{i,j}$  must be *primal optimal*, and  $X_{k,j} + X_{i,j} \geq 1 \quad \forall i,j$

$\alpha_i = \sum_j X_{i,j}$ ,  $\beta_j = \sum_i X_{i,j}$ ,  $\gamma_{i,j} = X_{i,j}$  must be *dual optimal*.

Dual feasible!

Substitute  $\alpha_i = \sum_j X_{i,j}$ ,  $\beta_j = \sum_i X_{i,j}$ ,  $\gamma_{i,j} = X_{i,j}$  for all  $i,j$

# Dual

Stability constraint  
from primal

Objective is equal  
to that in the primal

By complementary slackness:

For any primal feasible  $X$ ,

$$X_{i,j} > 0 \Rightarrow X_{i,j} + \sum_{k: w_k > m_i w_j} X_{i,k} + \sum_{k: m_k > w_j m_i} X_{k,j} = 1.$$

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Substitute  $\alpha_i = \sum_j X_{i,j}$ ,  $\beta_j = \sum_i X_{i,j}$ ,  $\gamma_{i,j} = X_{i,j}$  for all  $i, j$

[Vande Vate, *Oper. Res. Let.* 1989]

Any fractional stable matching can be expressed as a convex combination of integral stable matchings.

[Vande Vate, *Oper. Res. Let.* 1989]

Any fractional stable matching can be expressed as a convex combination of integral stable matchings.

Proof by picture (and LP duality)

[Teo and Sethuraman, *MOR* 1998].

[Vande Vate, *Oper. Res. Let.* 1989]

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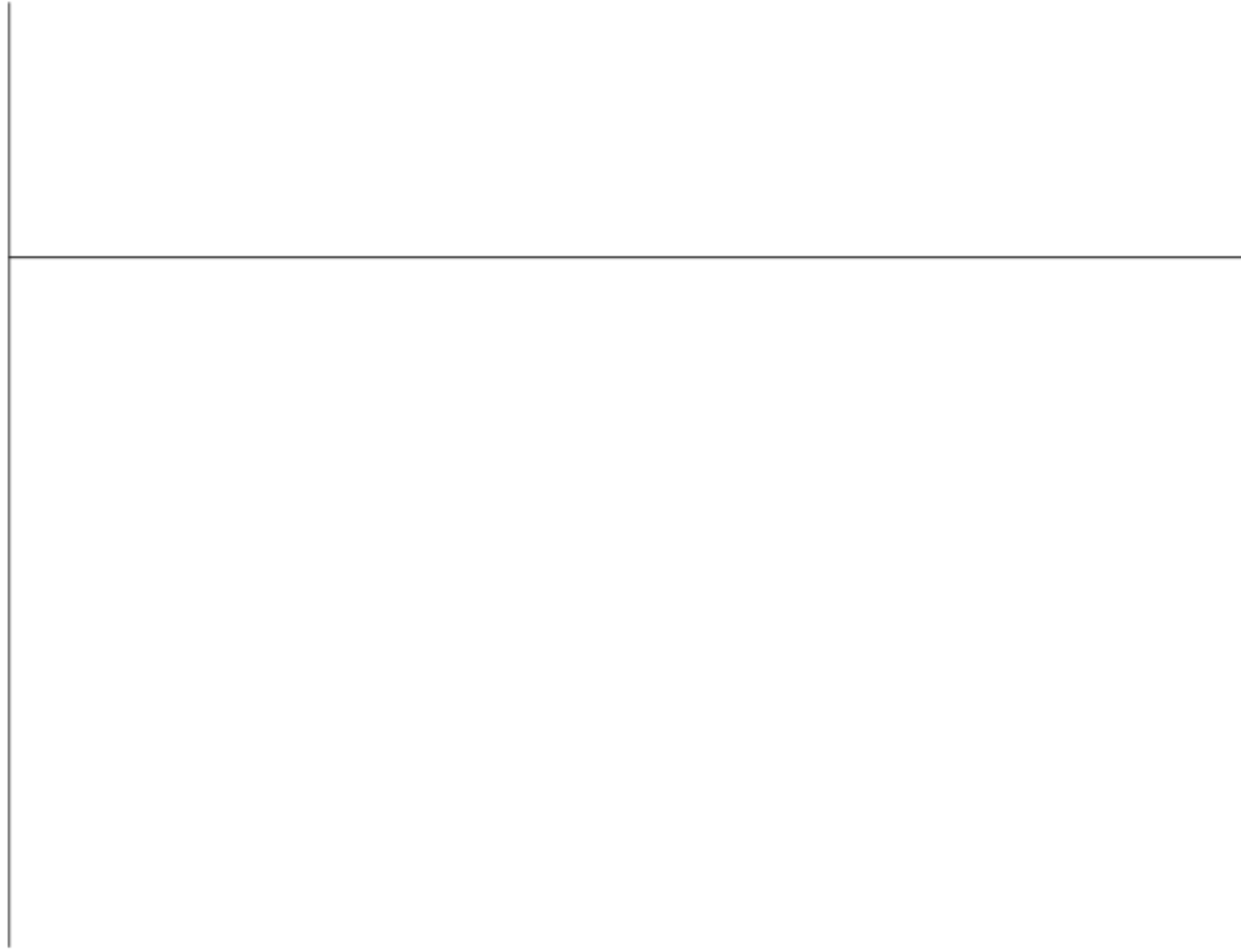
Recall complementary slackness:

For any primal feasible  $X$ ,

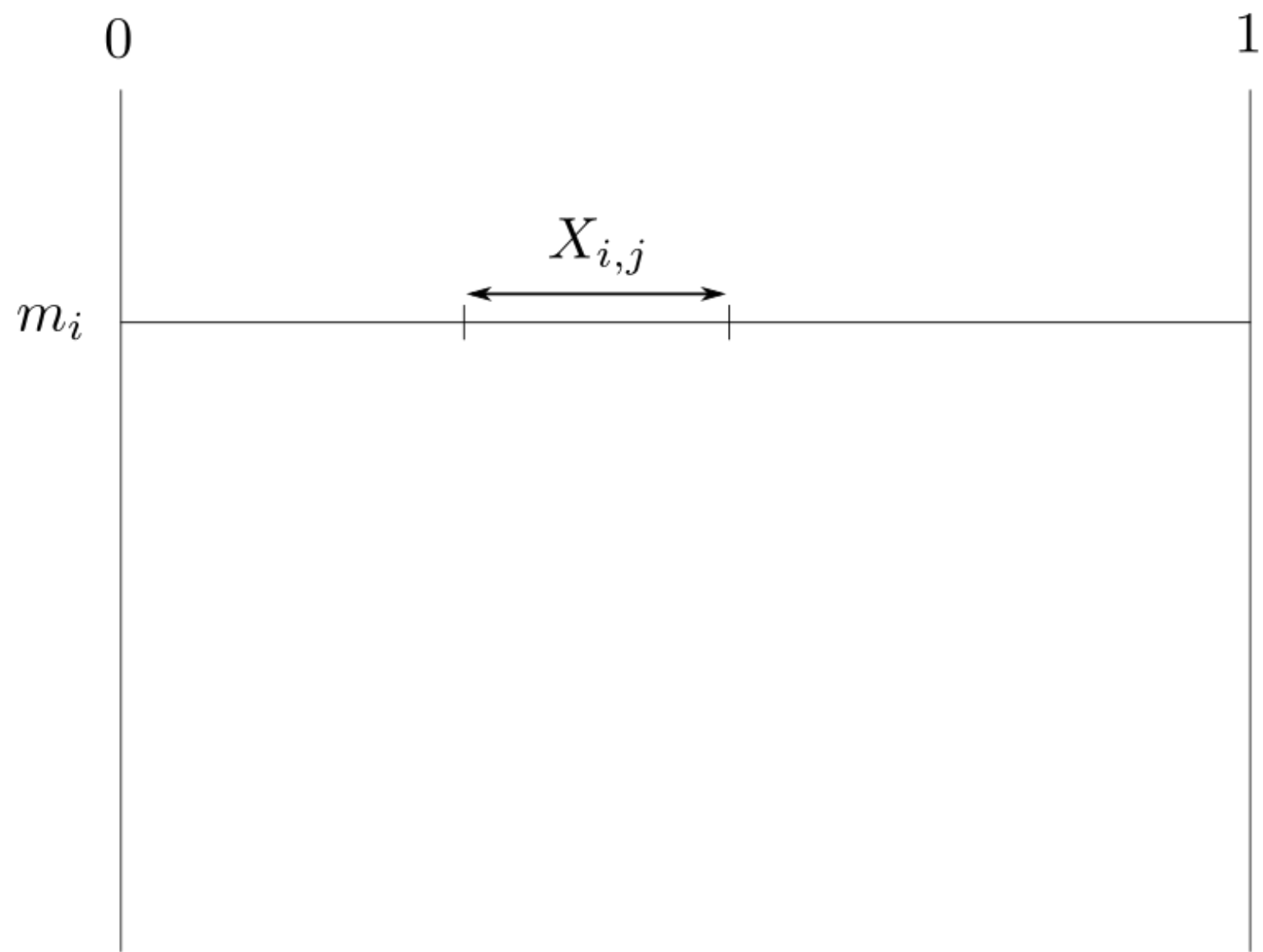
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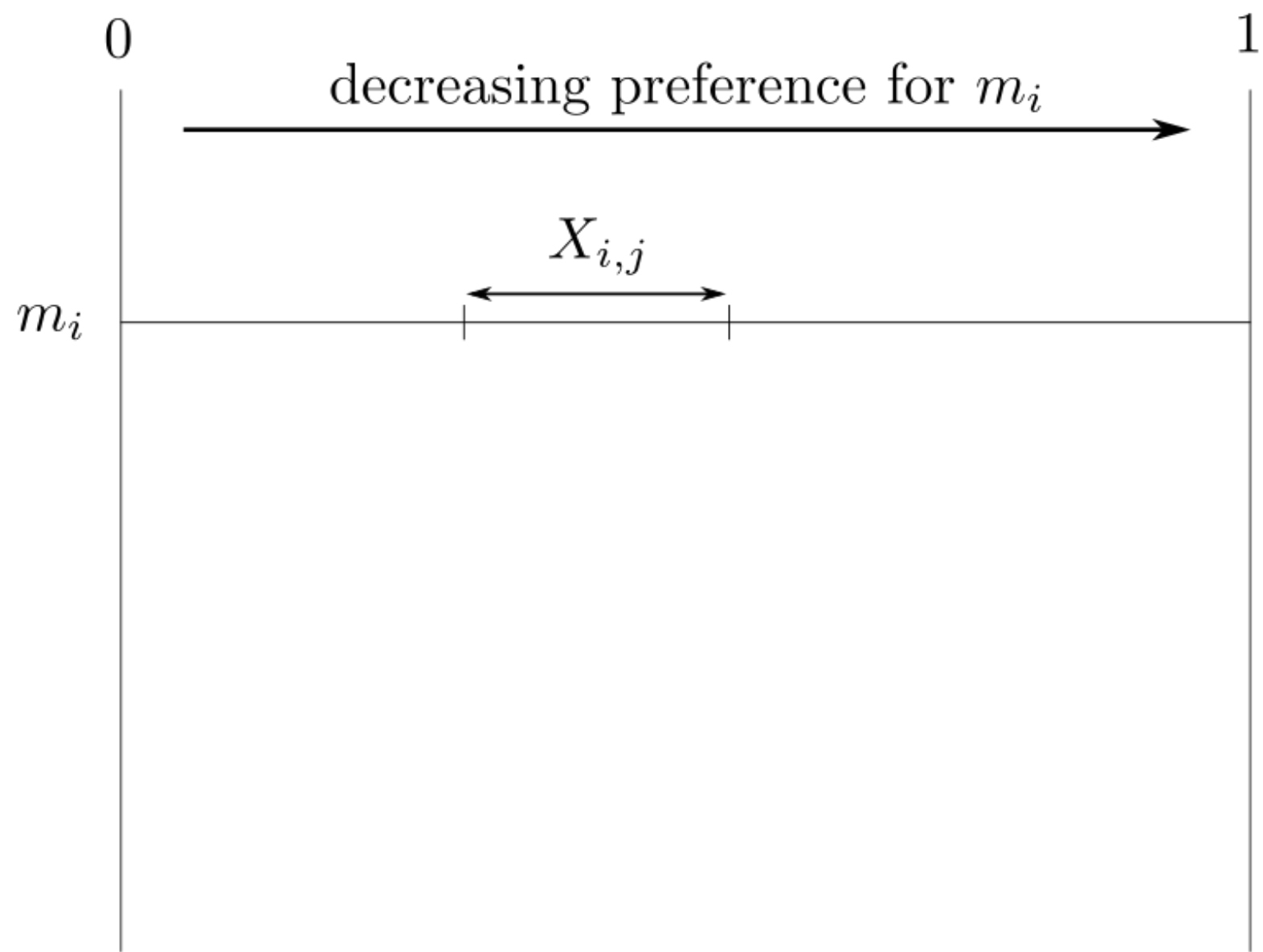
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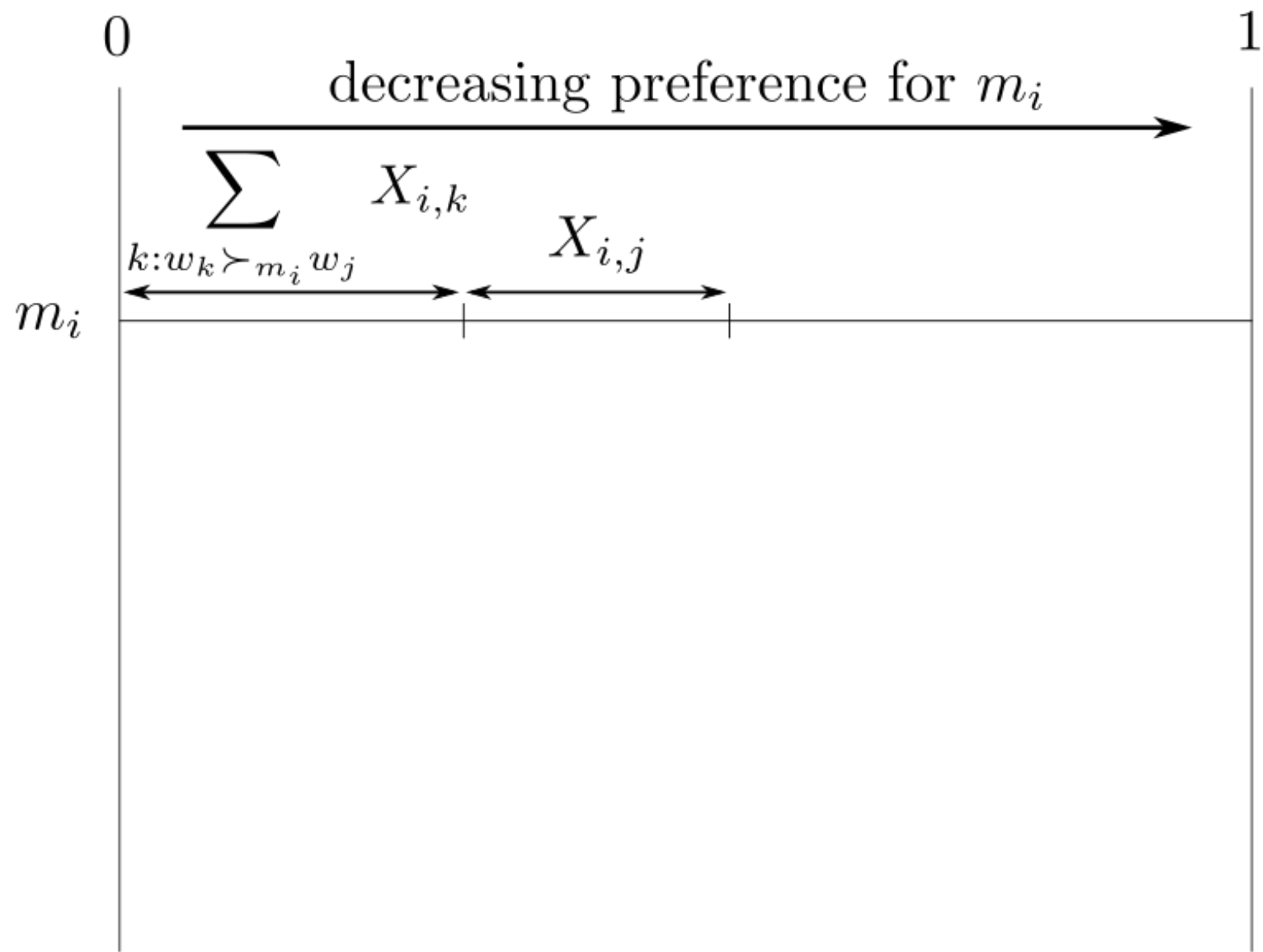
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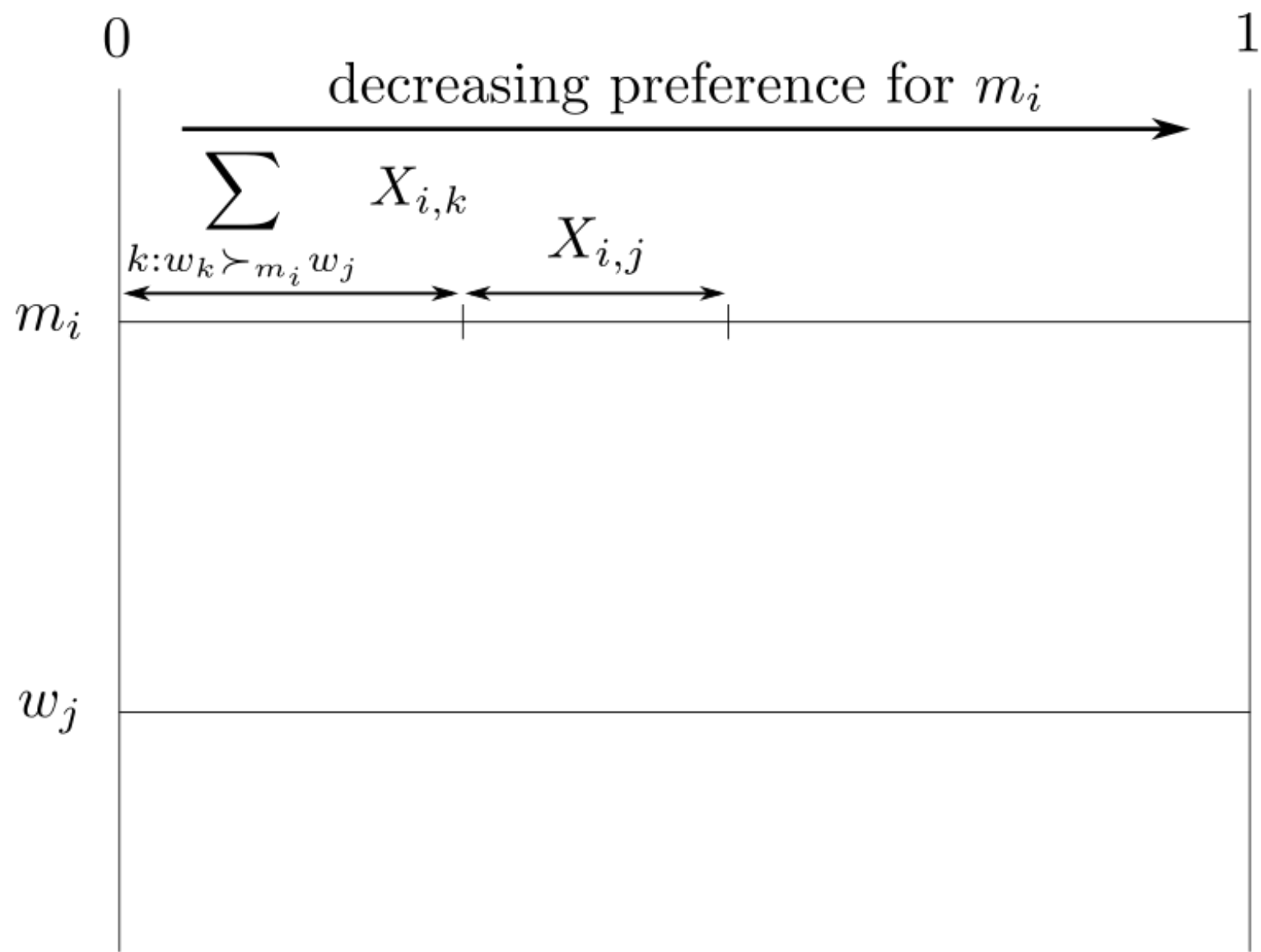


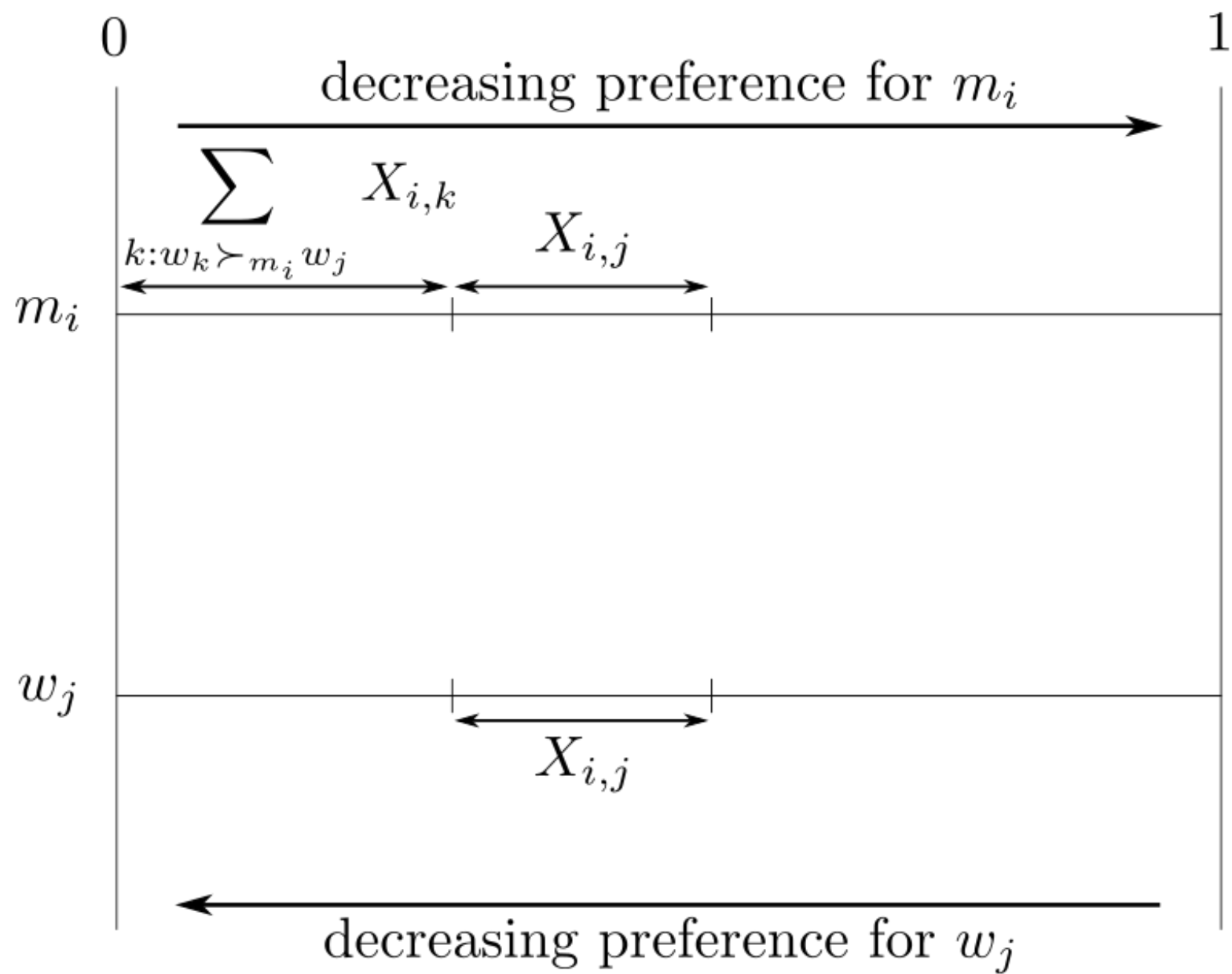


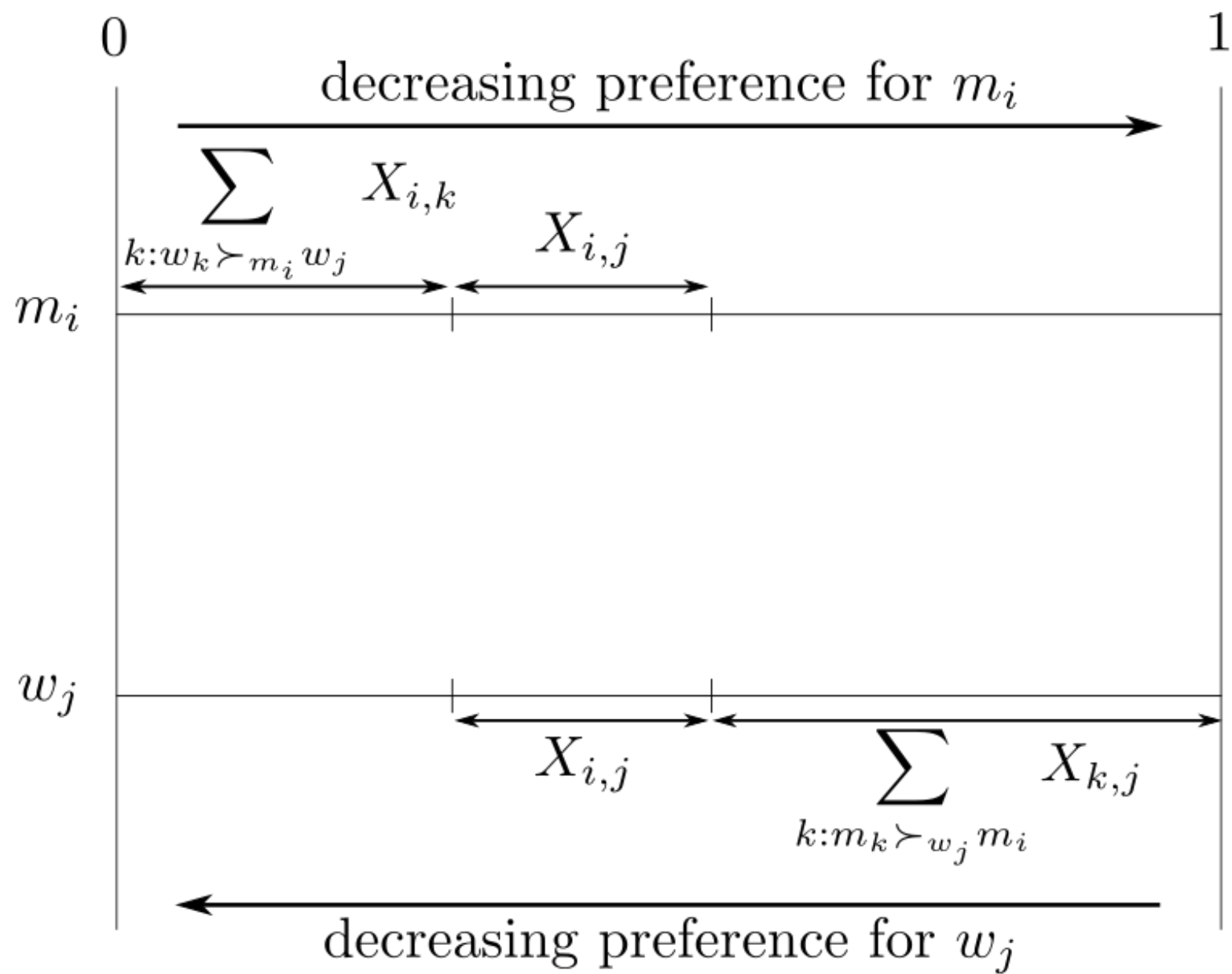


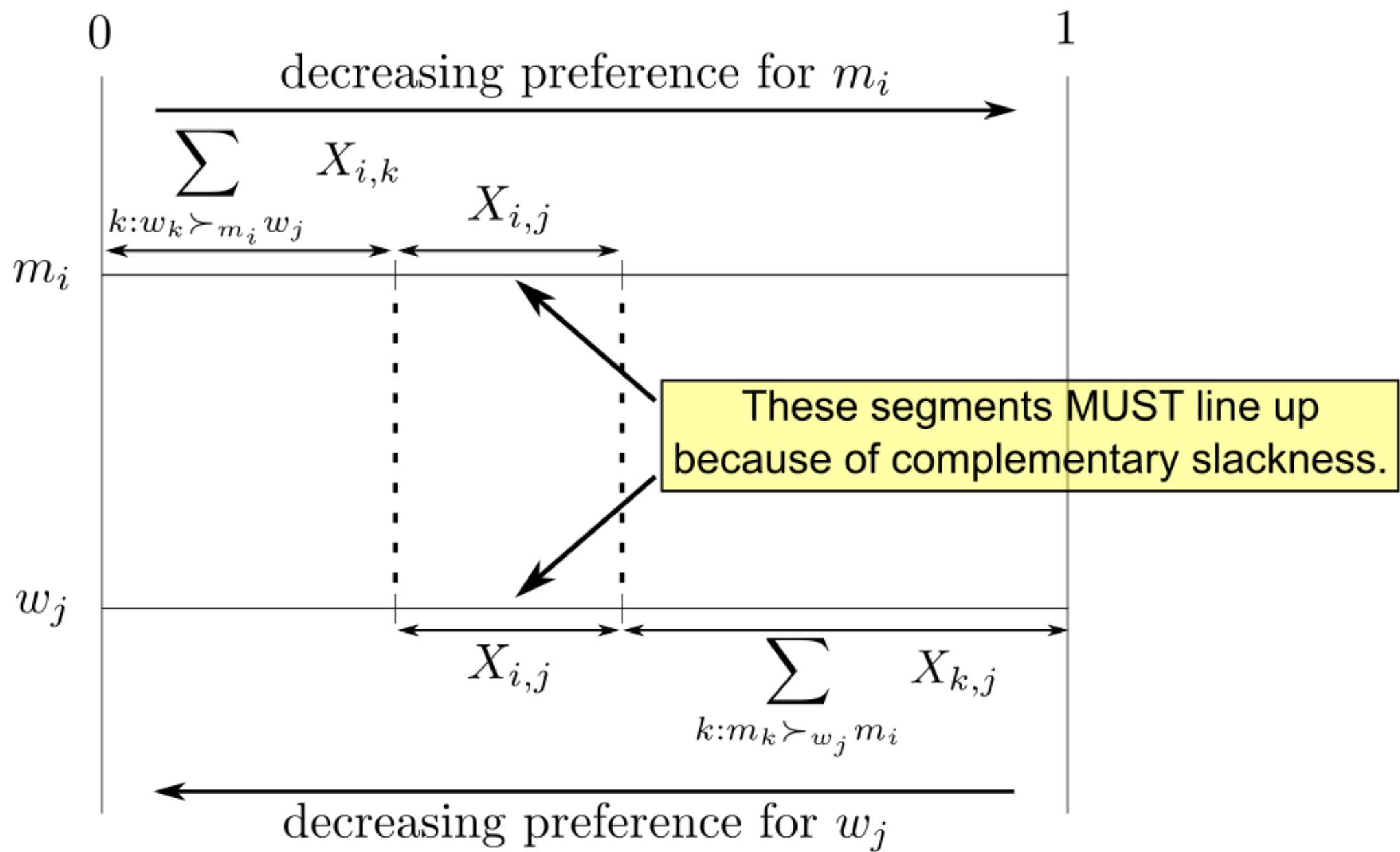


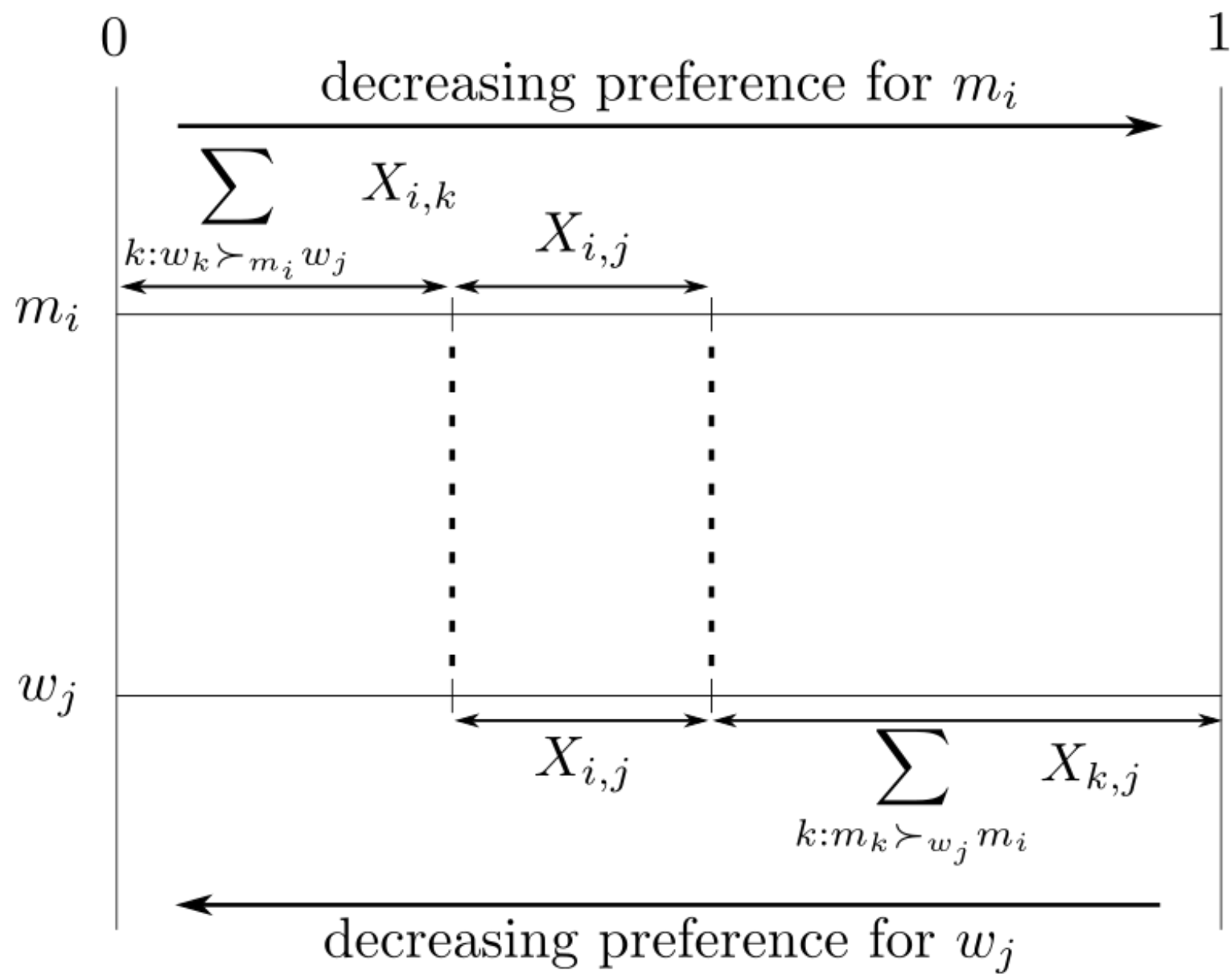


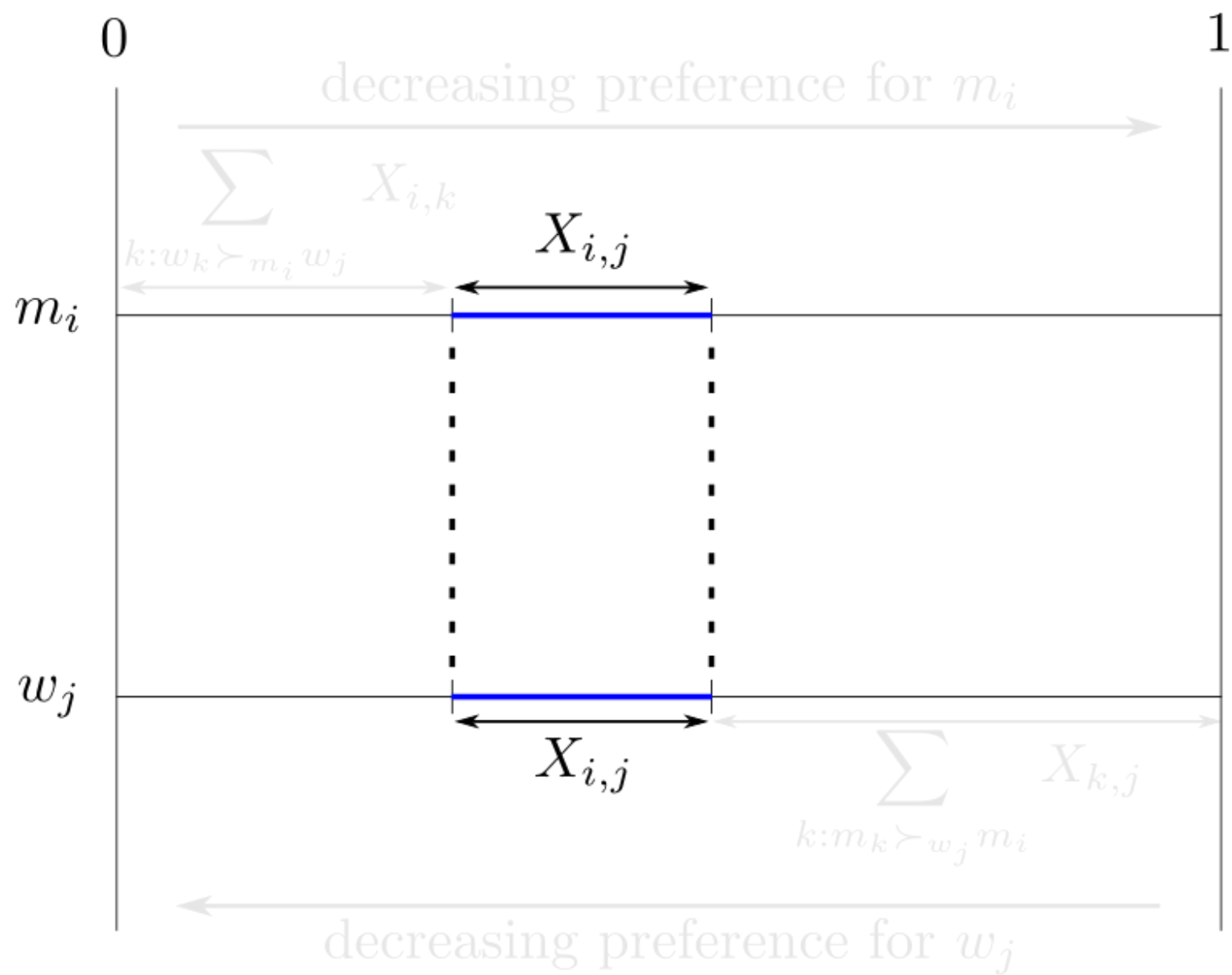


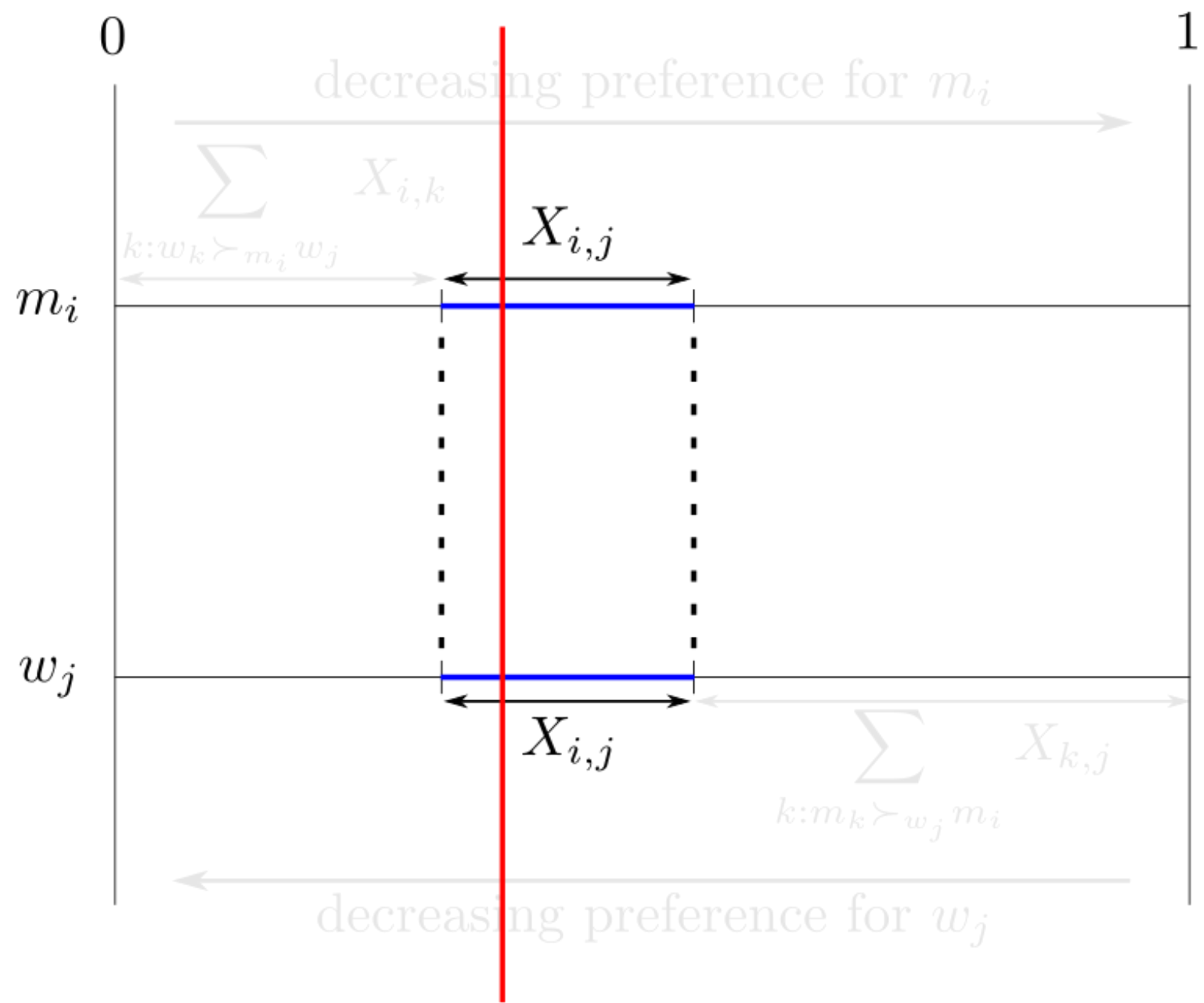


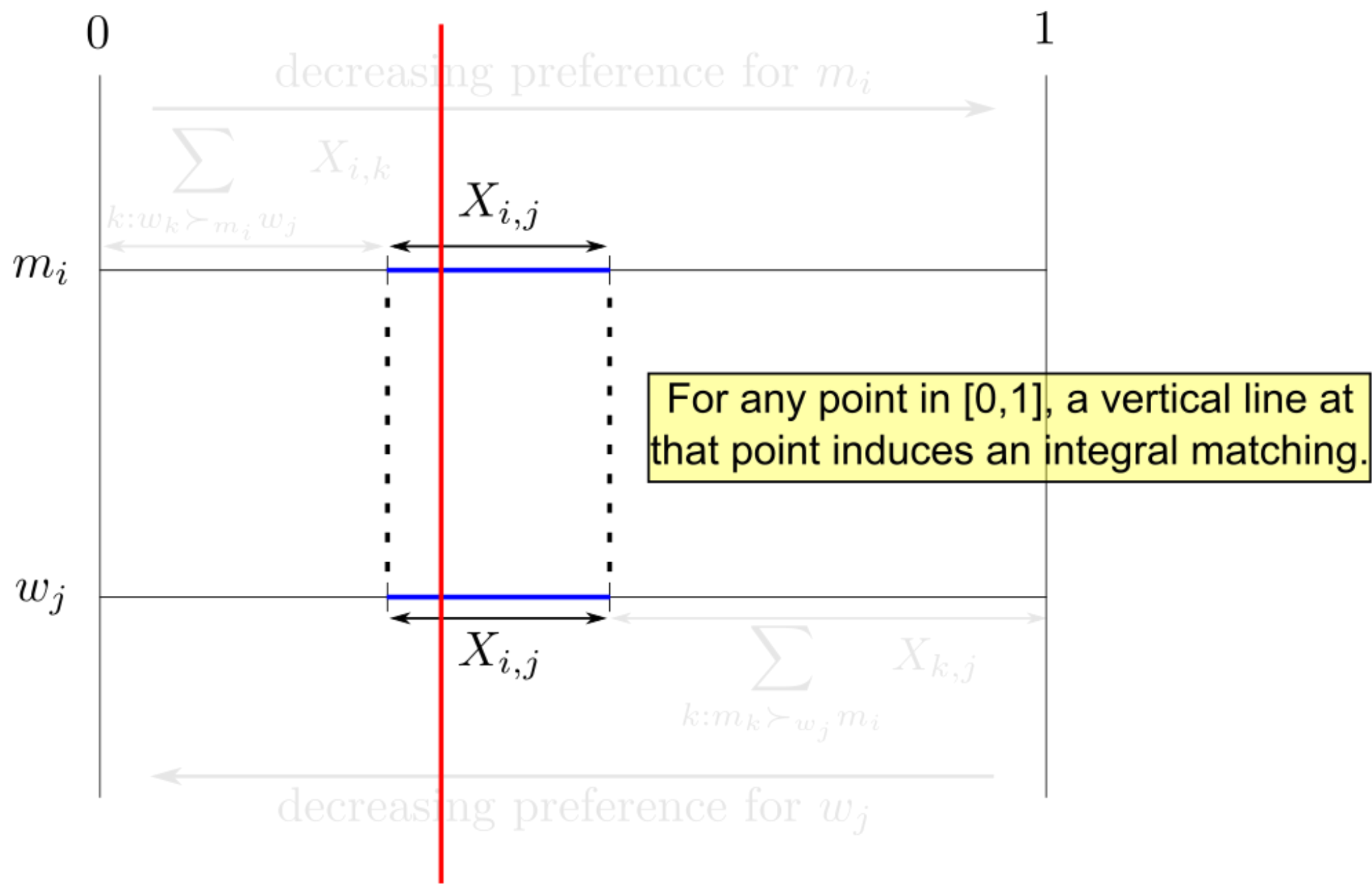


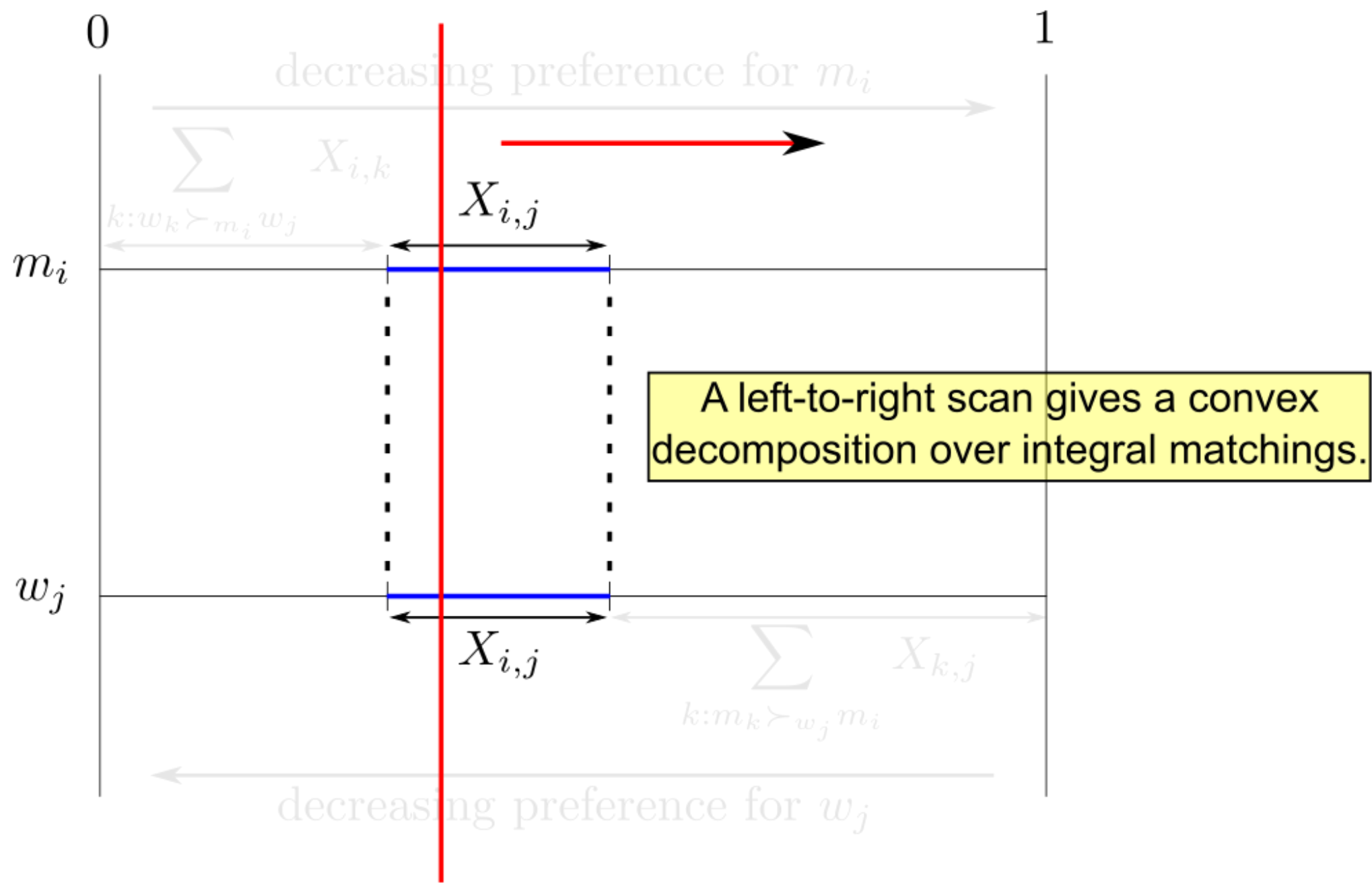


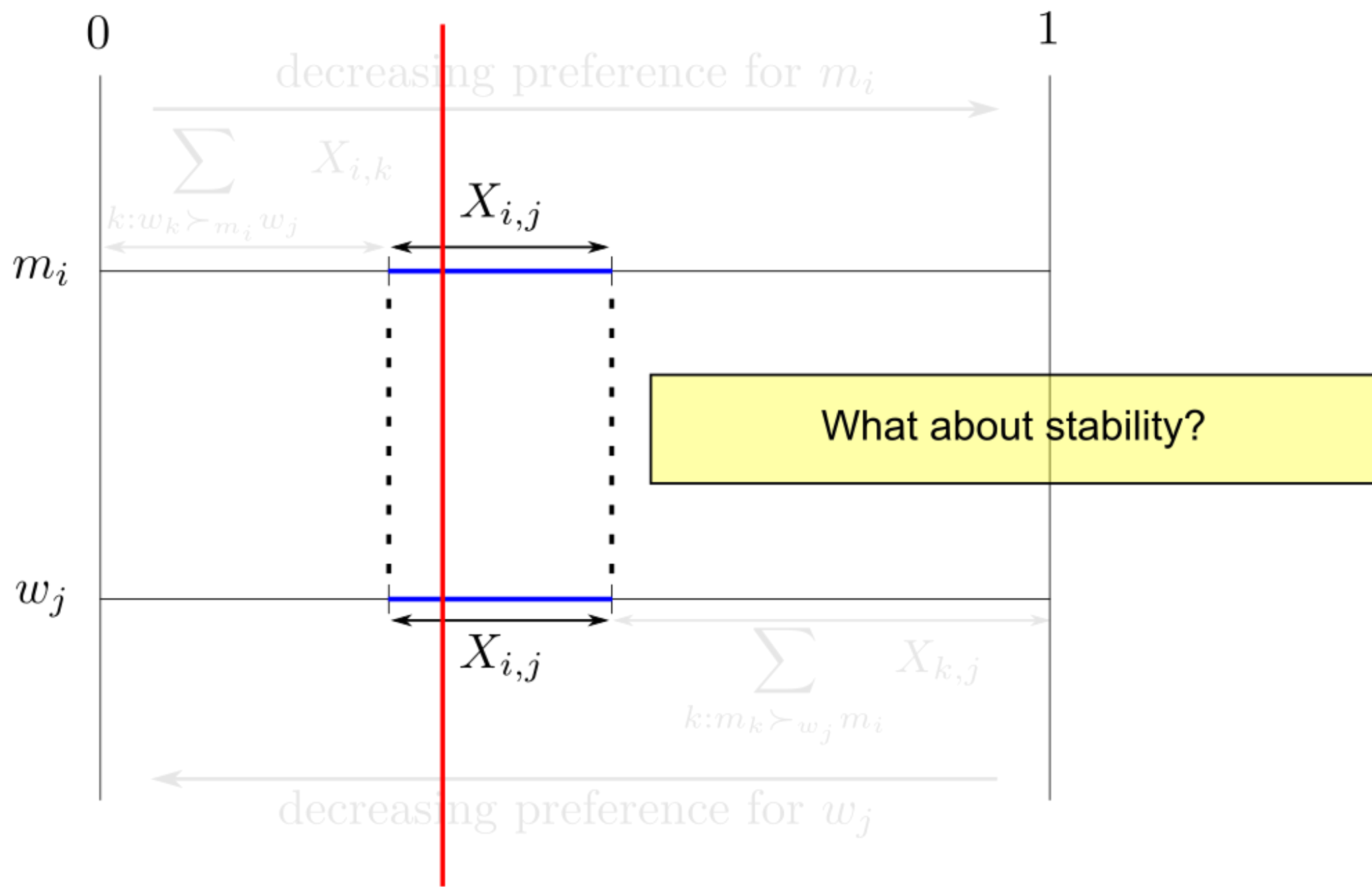


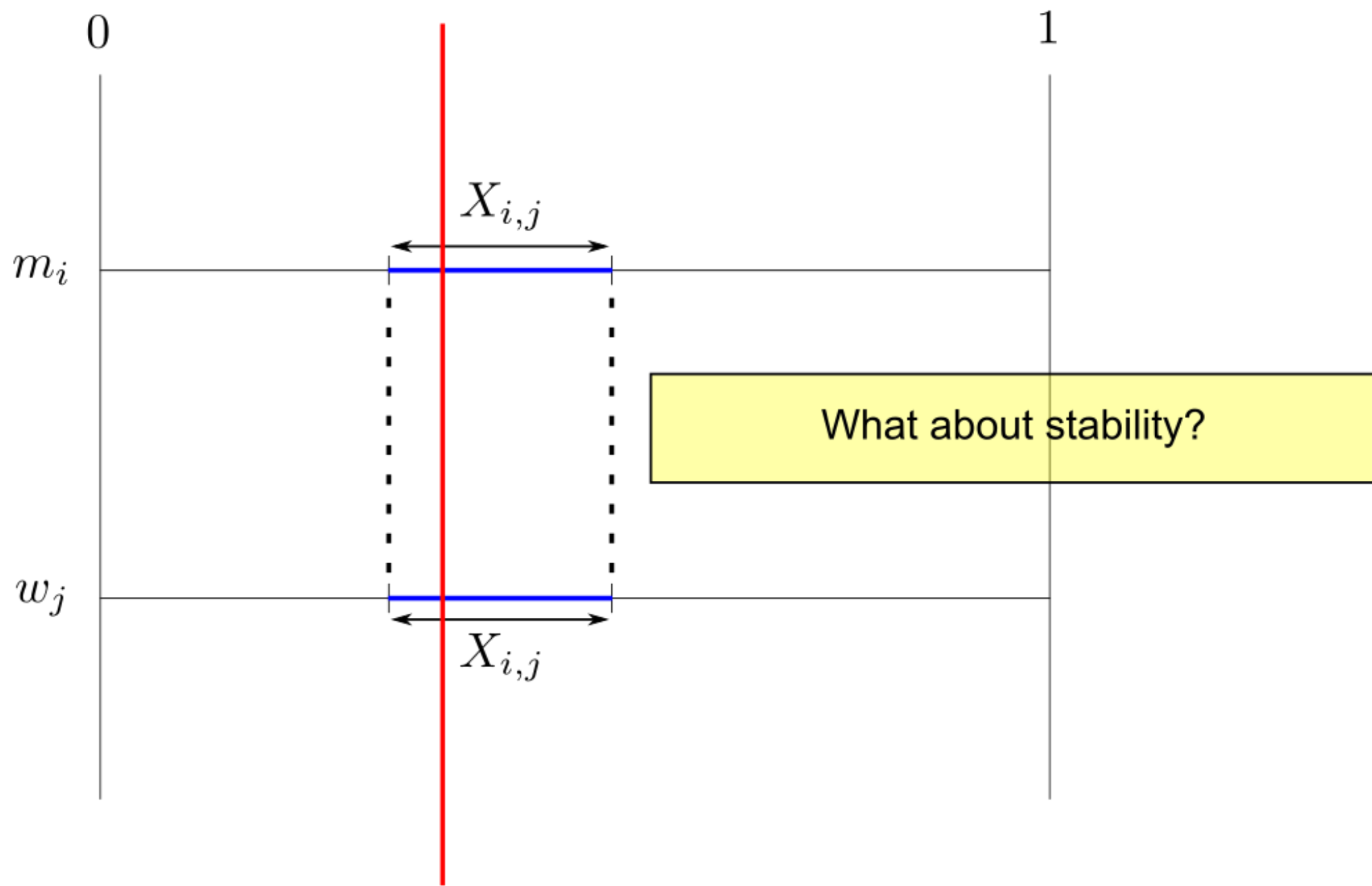


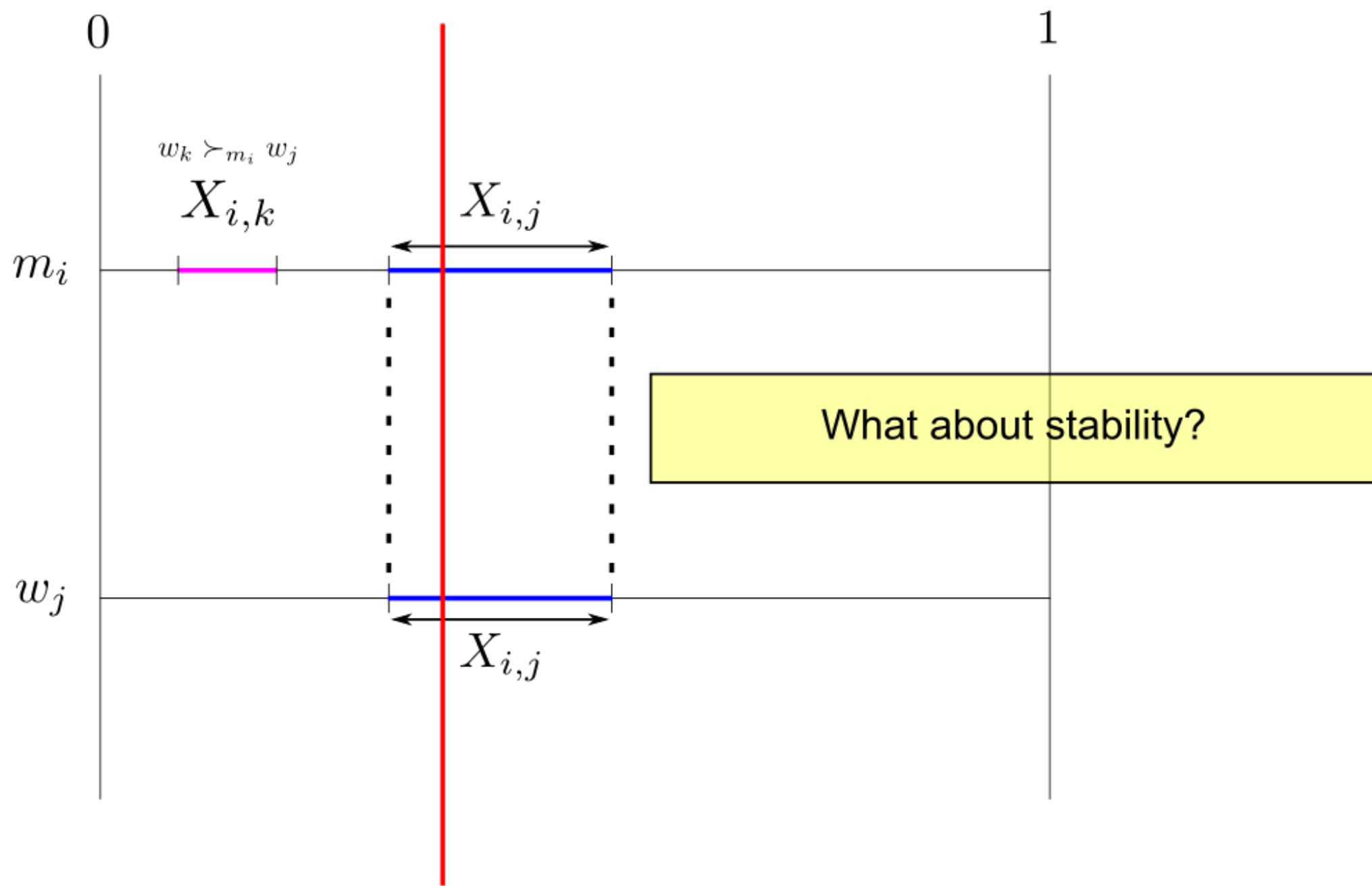


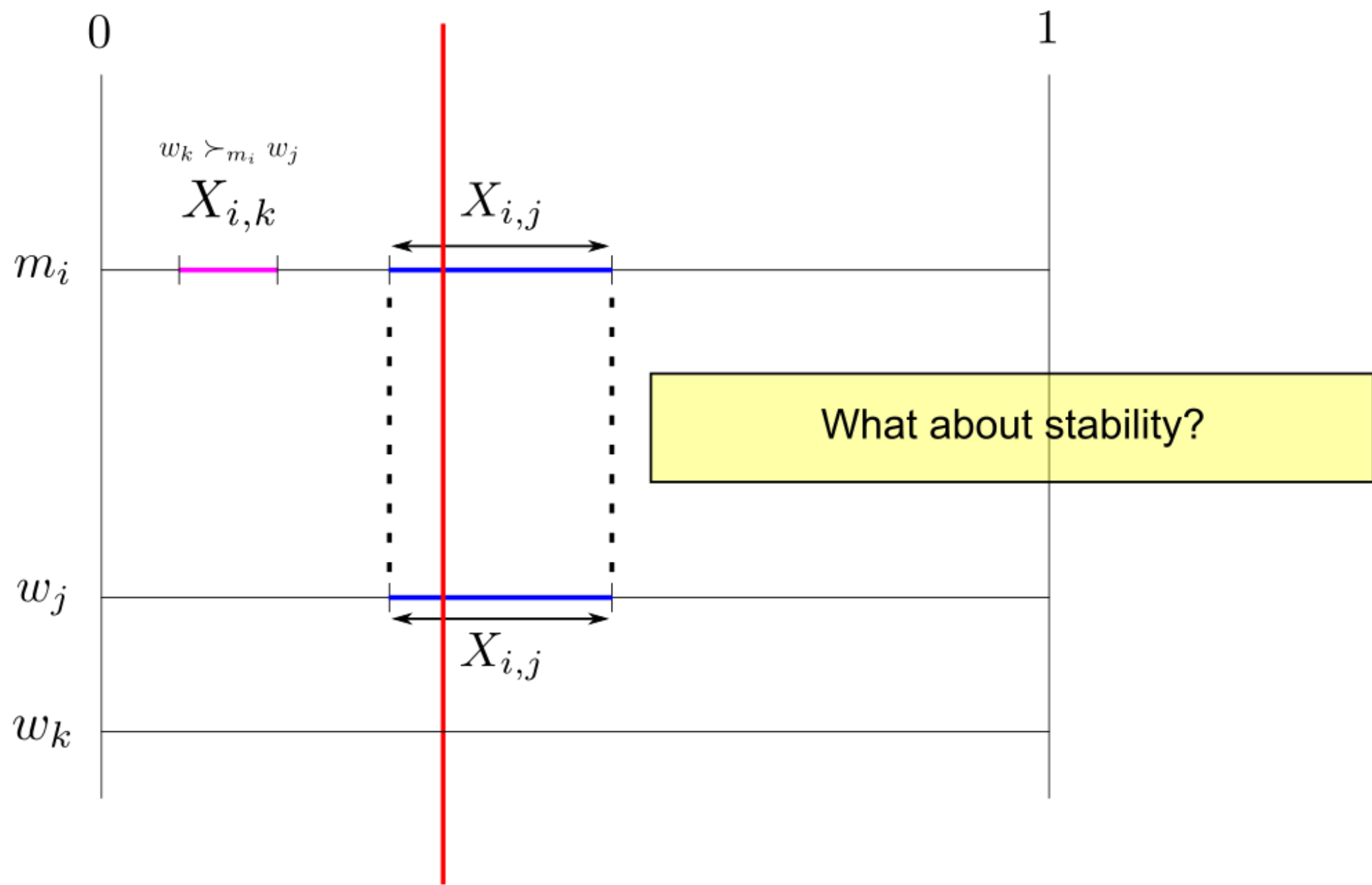


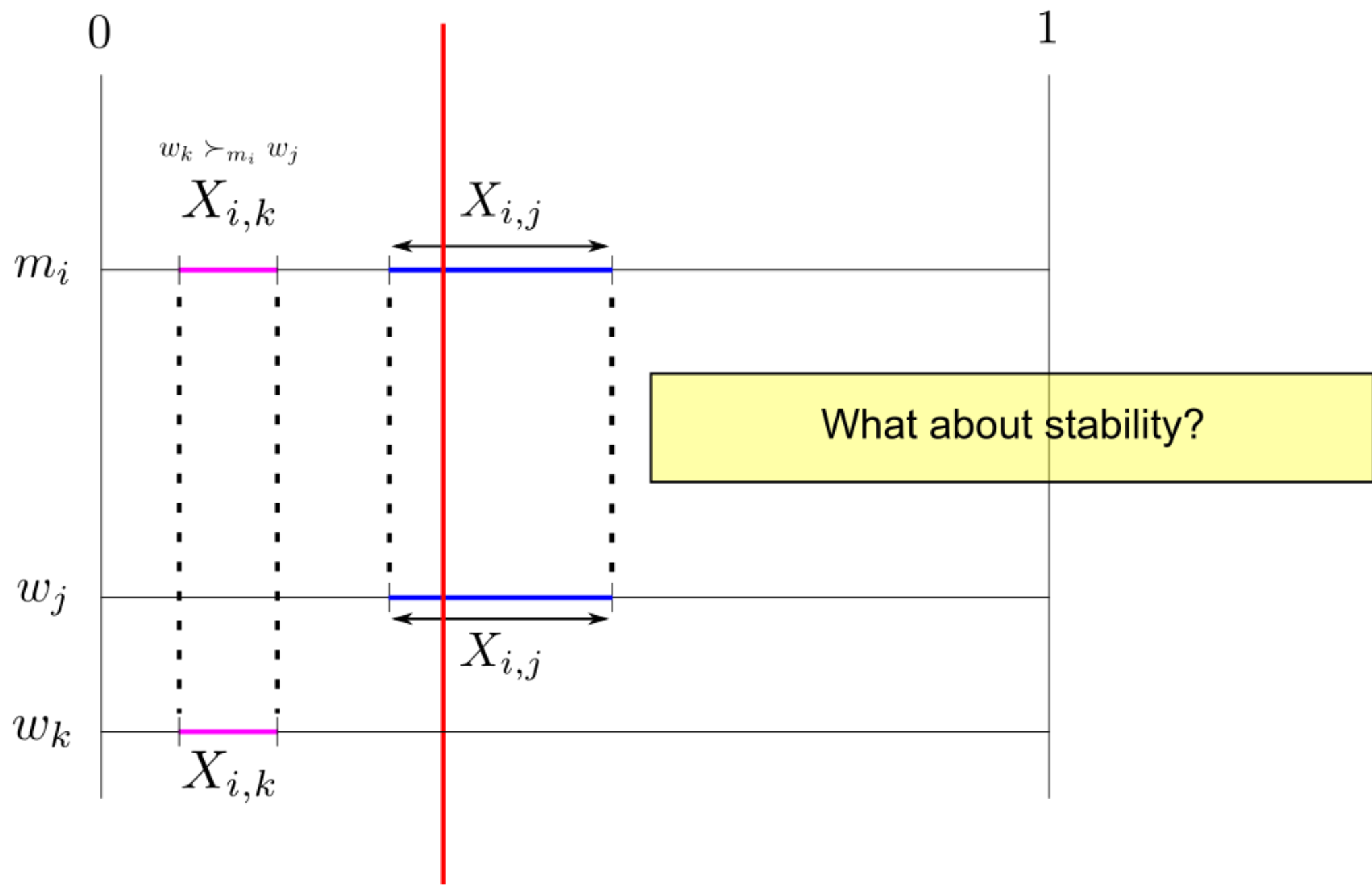


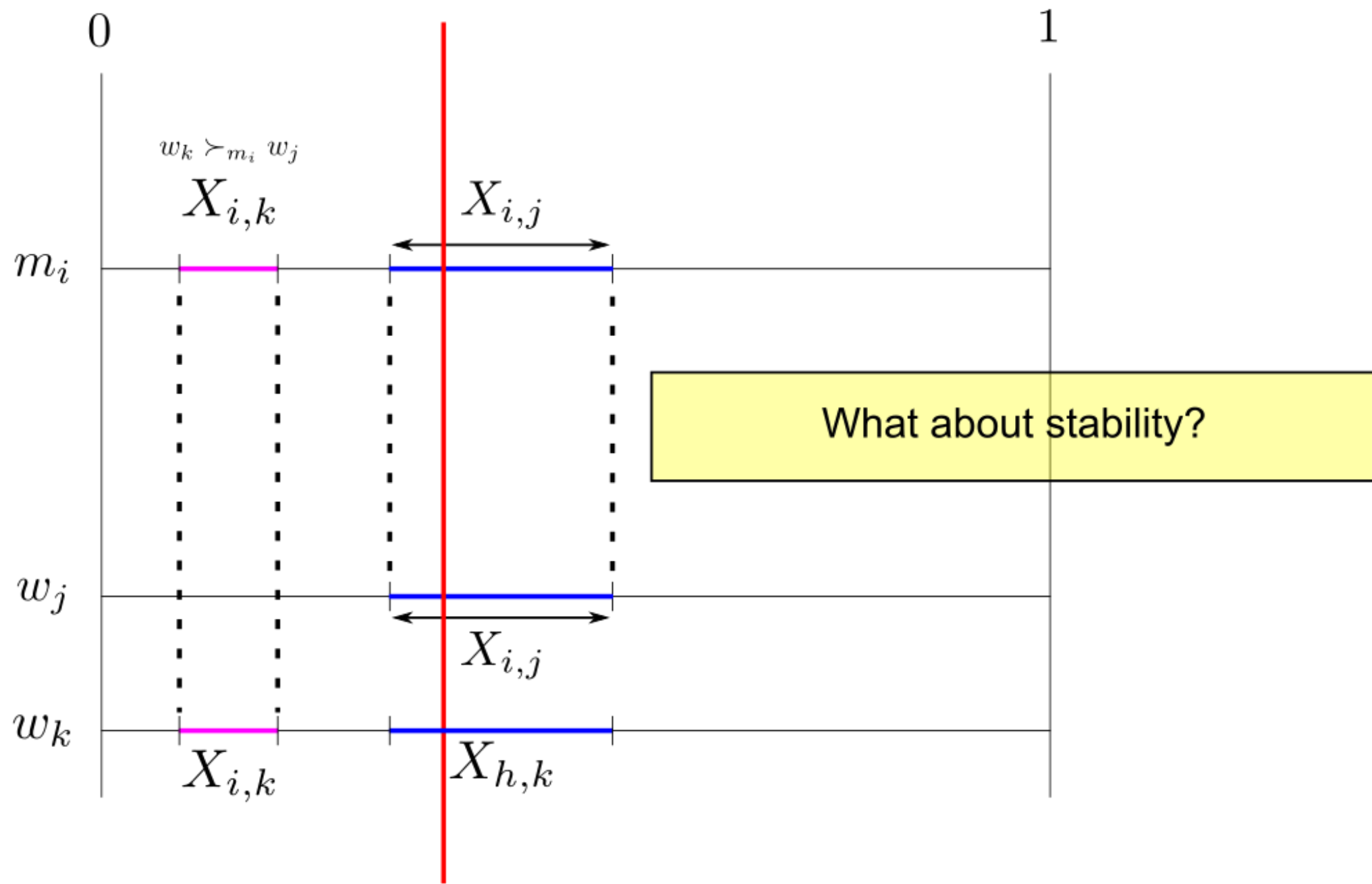


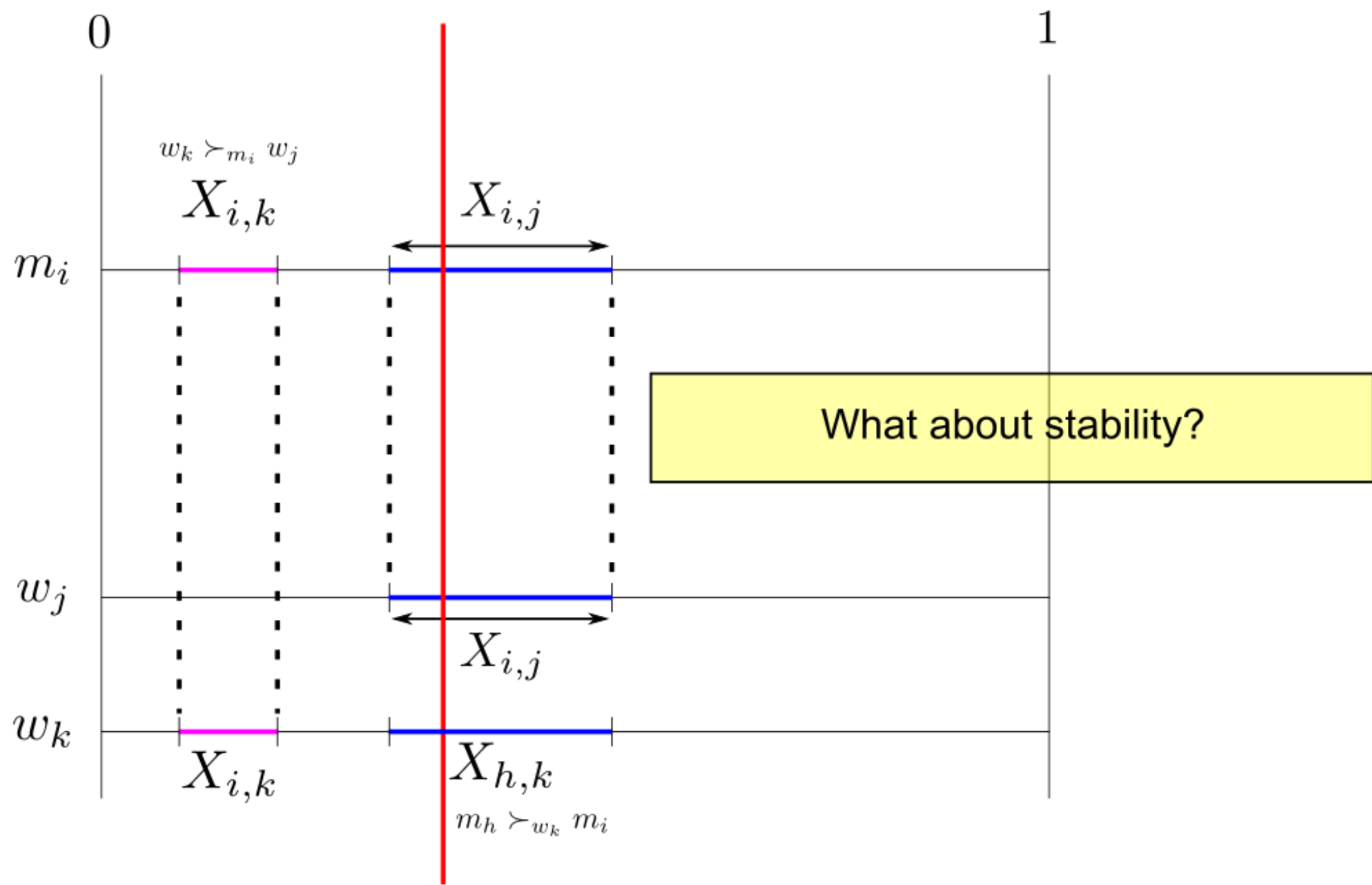


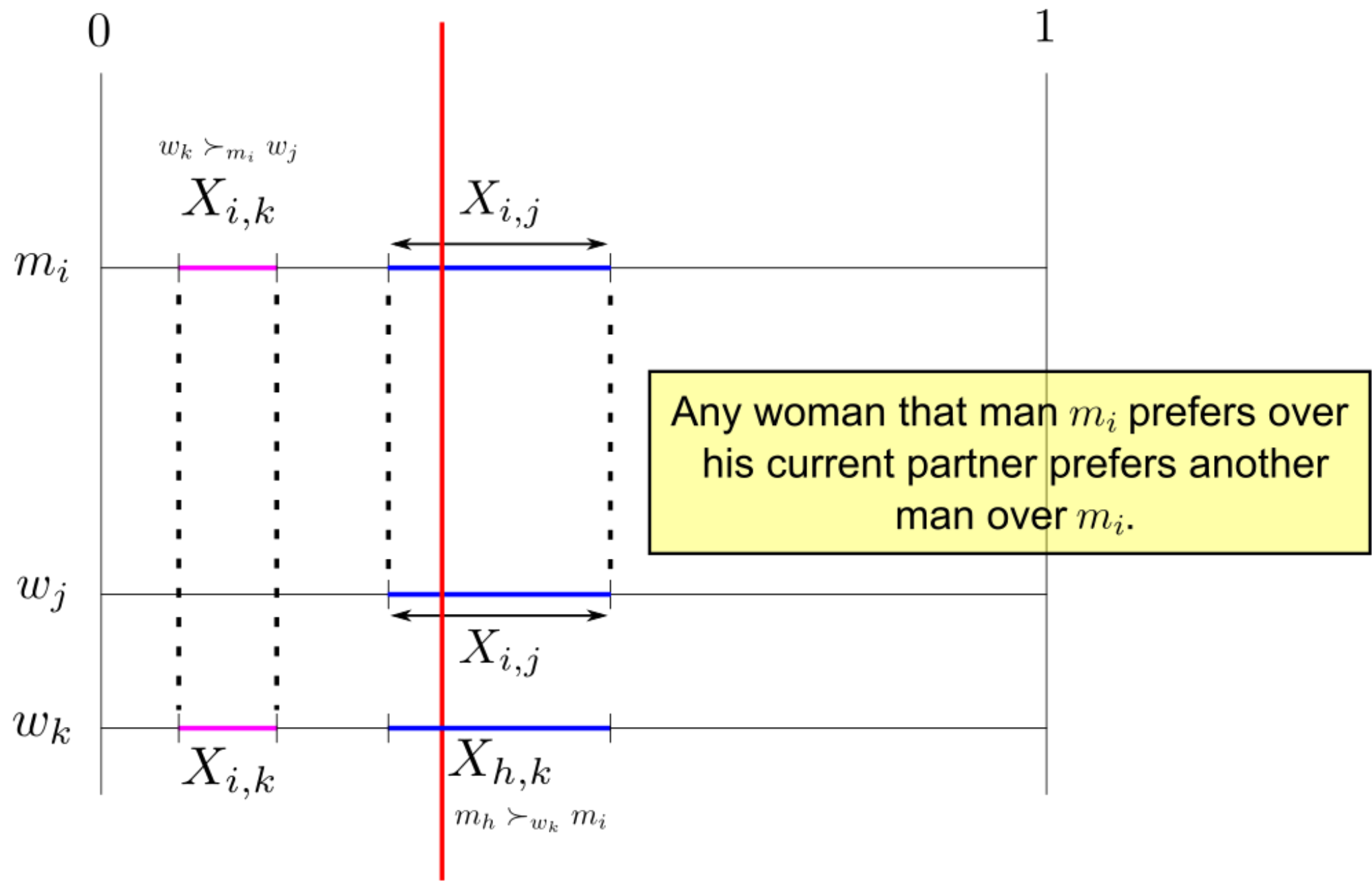


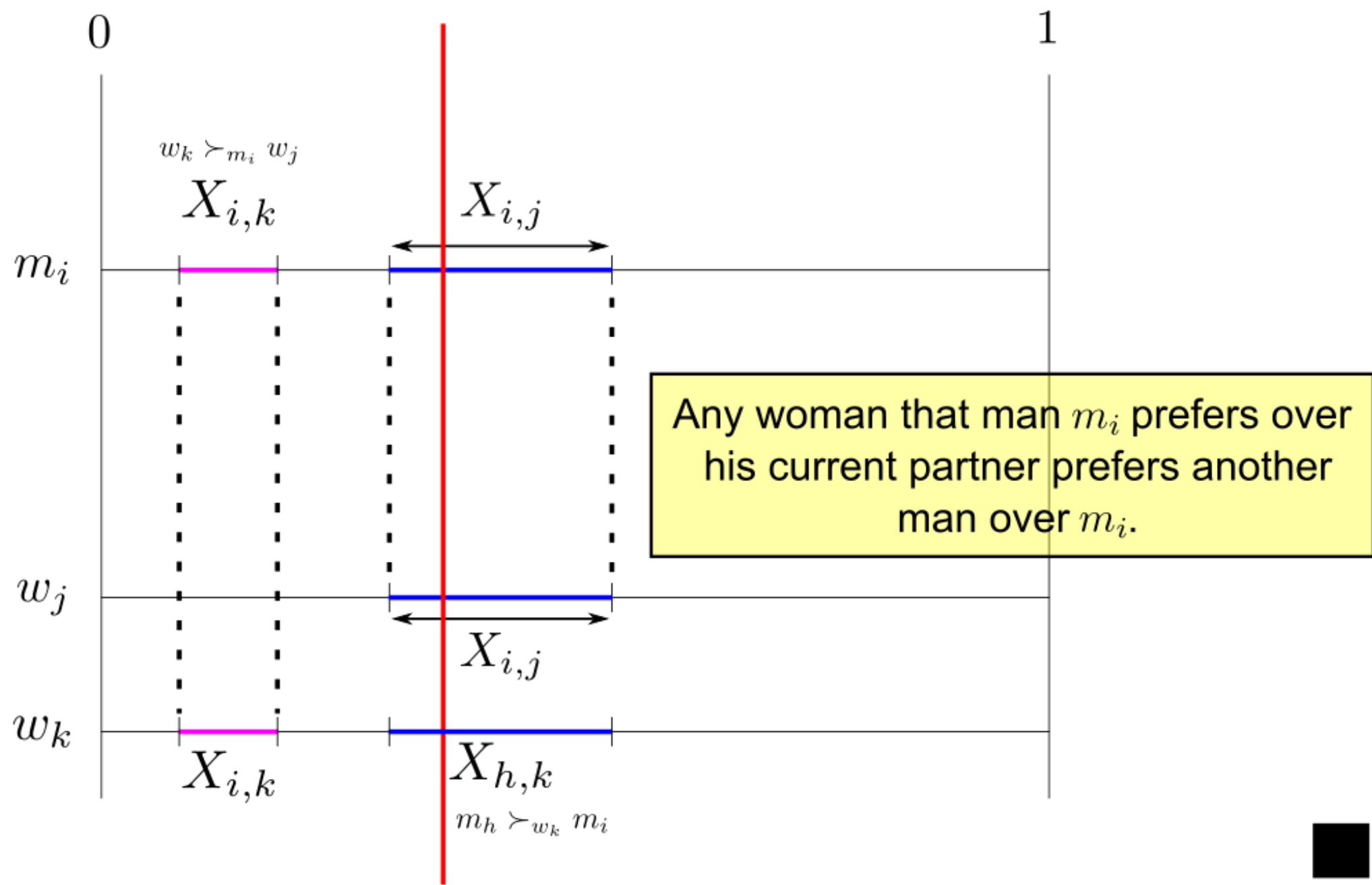








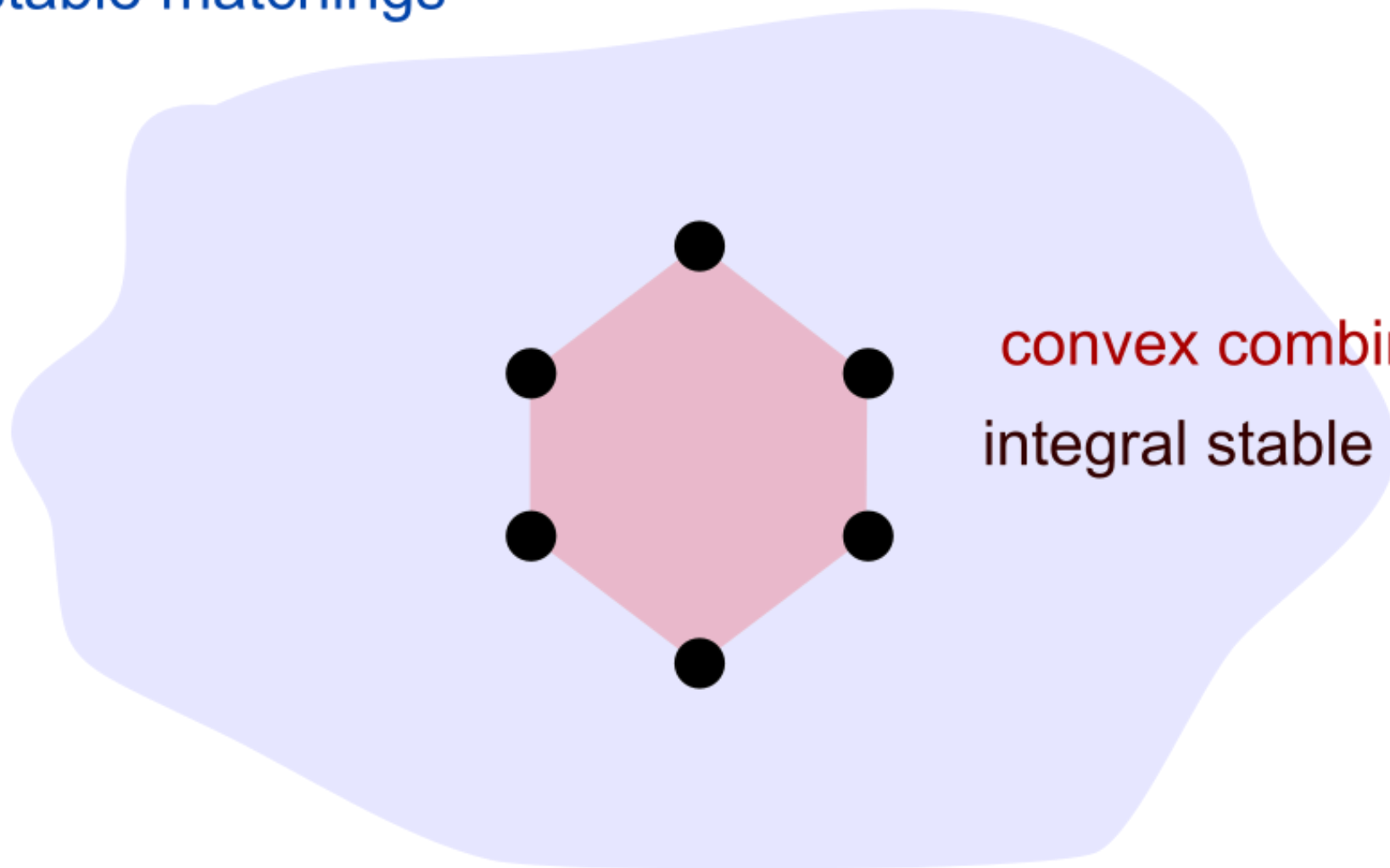




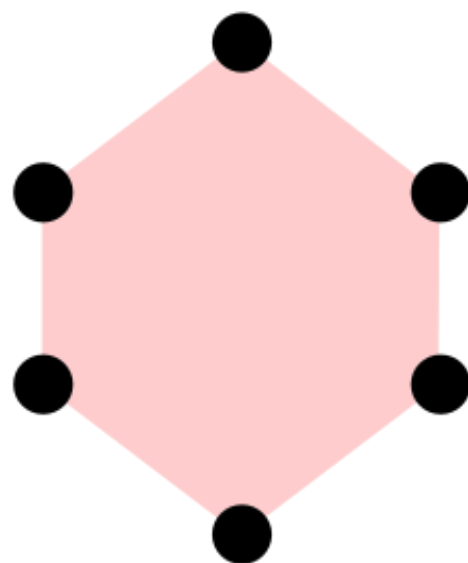
[Vande Vate, *Oper. Res. Let.* 1989]

Any fractional stable matching can be expressed as a convex combination of integral stable matchings.

fractional stable matchings



convex combinations of  
integral stable matchings



convex combinations of  
integral stable matchings

=

fractional stable matchings

Let us use the decomposition technique to show the existence of a "fair" matching.

# A "Fair" Stable Matching

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Suppose there are  $L$  stable matchings  $\mu_1, \dots, \mu_L$  for a given instance.

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For each man  $m_i$ , define his *median rank* as:

$$\text{med}(m_i) = \text{median}(\text{rank of } \mu_1(m_i) \text{ in } \succ_{m_i}, \dots, \text{rank of } \mu_L(m_i) \text{ in } \succ_{m_i}).$$

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**Median mapping:** Each agent points to its median rank agent.

1,1,1,1 | 4,4,3,3

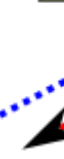
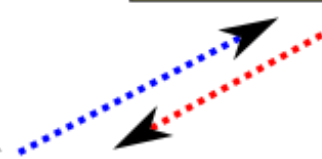
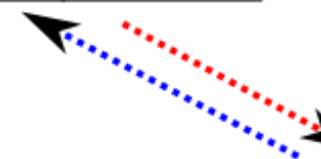
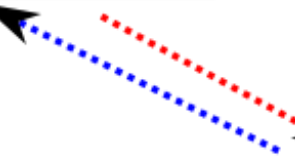
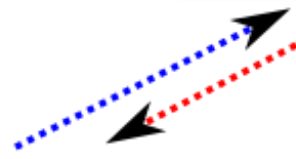
1,1,2,2 | 3,3,3,3

1,2,3,2 | 3,2,1,3

2,1,2,3 | 2,3,3,1

2,2,3,3 | 2,2,1,1

3,4,3,3 | 1,1,1,1



1,1,1,1 | 4,4,3,3

$m_1$     1    1    1    2    2    3

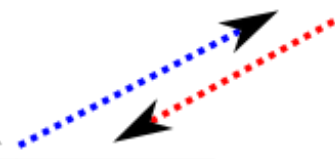
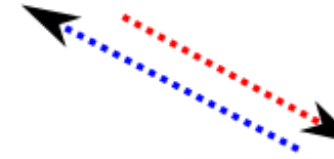
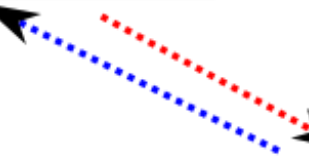
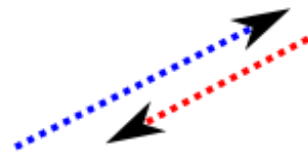
1,1,2,2 | 3,3,3,3

1,2,3,2 | 3,2,1,3

2,1,2,3 | 2,3,3,1

2,2,3,3 | 2,2,1,1

3,4,3,3 | 1,1,1,1



1,1,1,1 | 4,4,3,3

1,1,2,2 | 3,3,3,3

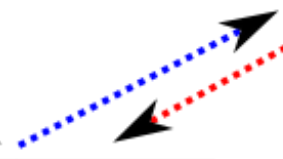
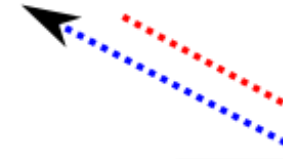
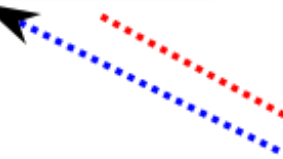
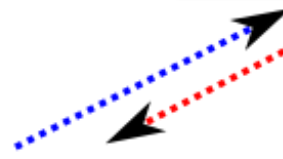
1,2,3,2 | 3,2,1,3

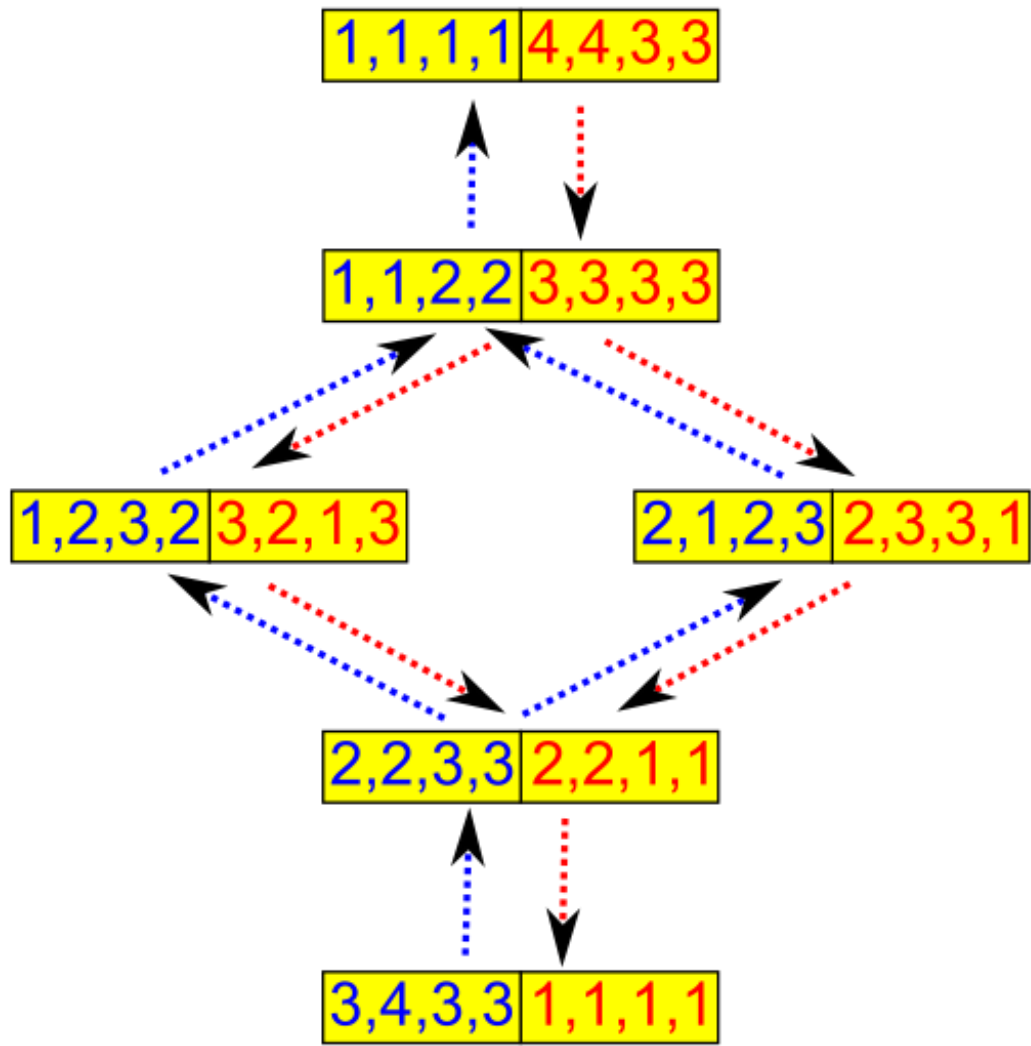
2,1,2,3 | 2,3,3,1

2,2,3,3 | 2,2,1,1

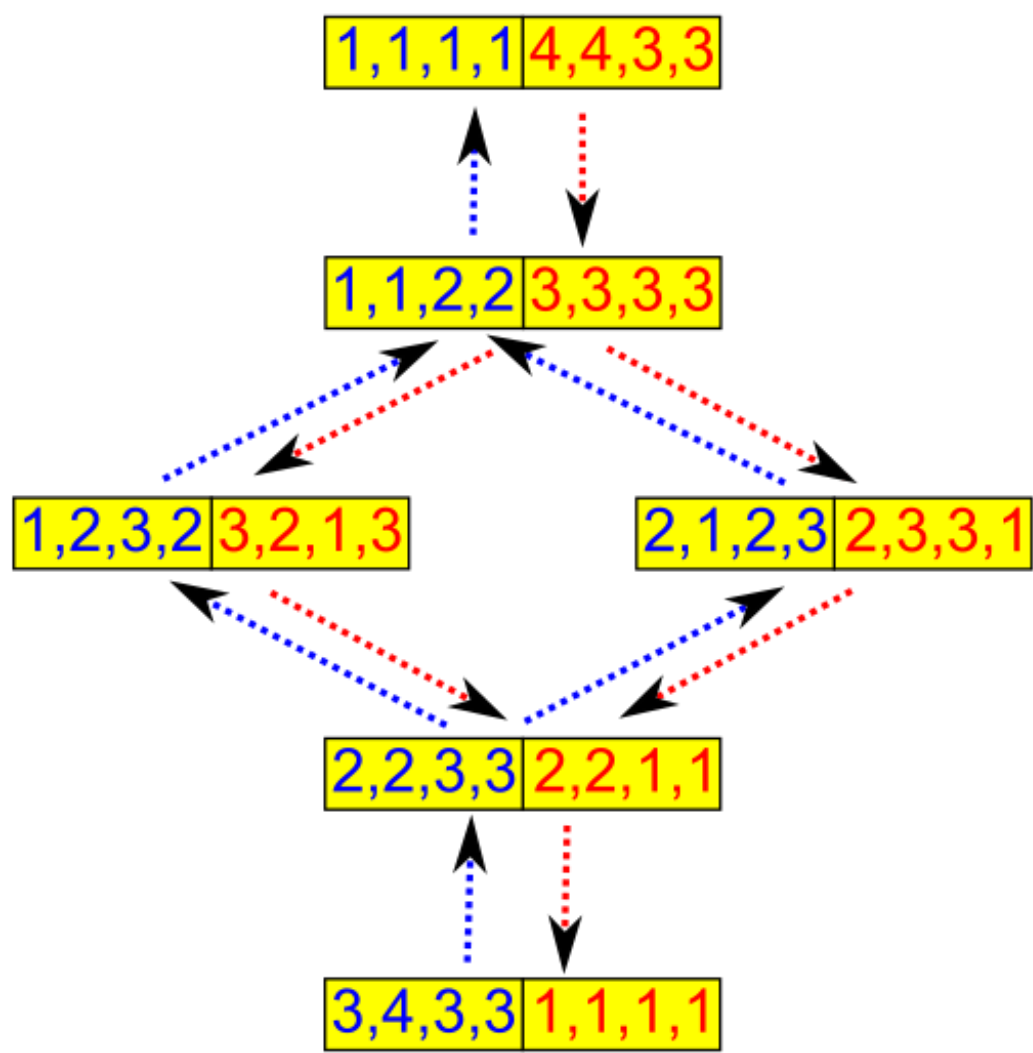
3,4,3,3 | 1,1,1,1

$m_1$	1	1	1	2	2	3
$m_2$	1	1	1	2	2	4

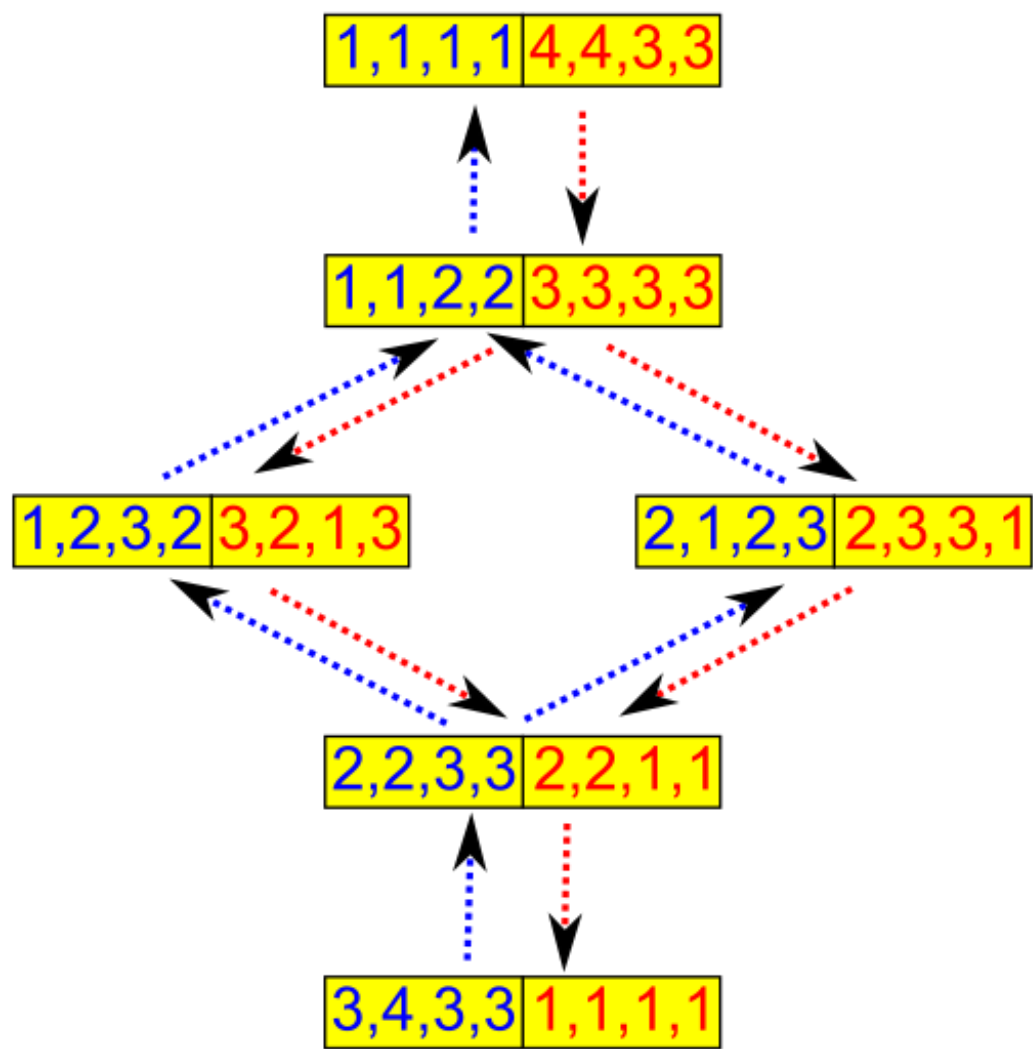




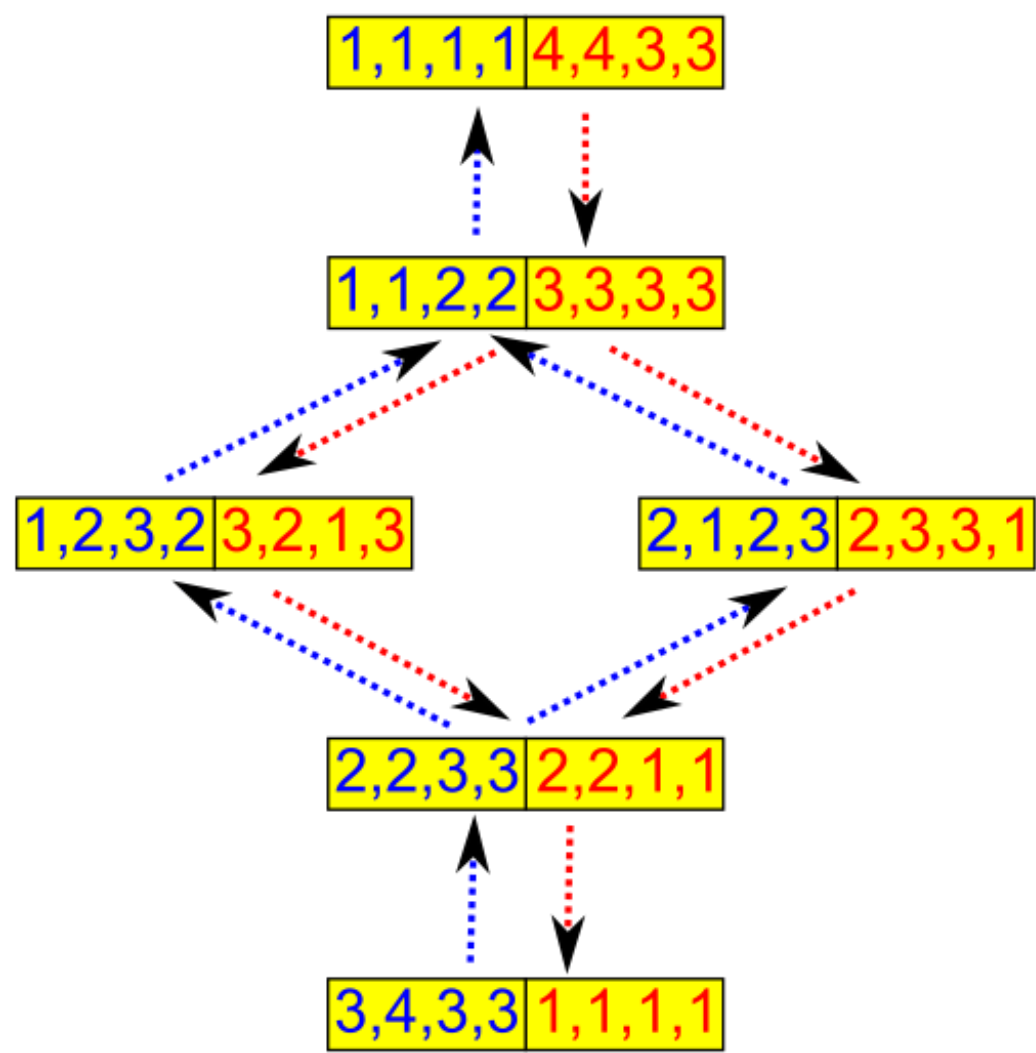
$m_1$	1	1	1	2	2	3
$m_2$	1	1	1	2	2	4
$m_3$	1	2	2	3	3	3
$m_4$	1	2	2	3	3	3



$m_1$	1	1	1	2	2	3
$m_2$	1	1	1	2	2	4
$m_3$	1	2	2	3	3	3
$m_4$	1	2	2	3	3	3
$w_1$	4	3	3	2	2	1
$w_2$	4	3	3	2	2	1
$w_3$	3	3	3	1	1	1
$w_4$	3	3	3	1	1	1



$m_1$	1	1	1	2	2	3
$m_2$	1	1	1	2	2	4
$m_3$	1	2	2	3	3	3
$m_4$	1	2	2	3	3	3
$w_1$	4	3	3	2	2	1
$w_2$	4	3	3	2	2	1
$w_3$	3	3	3	1	1	1
$w_4$	3	3	3	1	1	1



$m_1$	1	1	1	2	2	3
$m_2$	1	1	1	2	2	4
$m_3$	1	2	2	3	3	3
$m_4$	1	2	2	3	3	3
$w_1$	4	3	3	2	2	1
$w_2$	4	3	3	2	2	1
$w_3$	3	3	3	1	1	1
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# A "Fair" Stable Matching

Suppose there are  $L$  stable matchings  $\mu_1, \dots, \mu_L$  for a given instance.

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For each woman  $w_j$ , define her *median rank* as:

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[Teo and Sethuraman, *MOR* 1998]

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Proof by example.

1,1,1,1 | 4,4,3,3

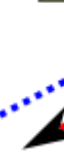
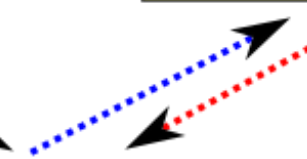
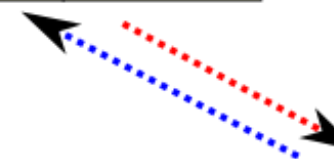
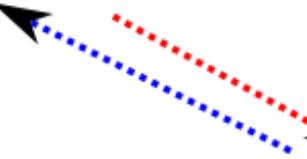
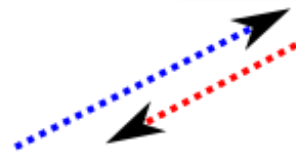
1,1,2,2 | 3,3,3,3

1,2,3,2 | 3,2,1,3

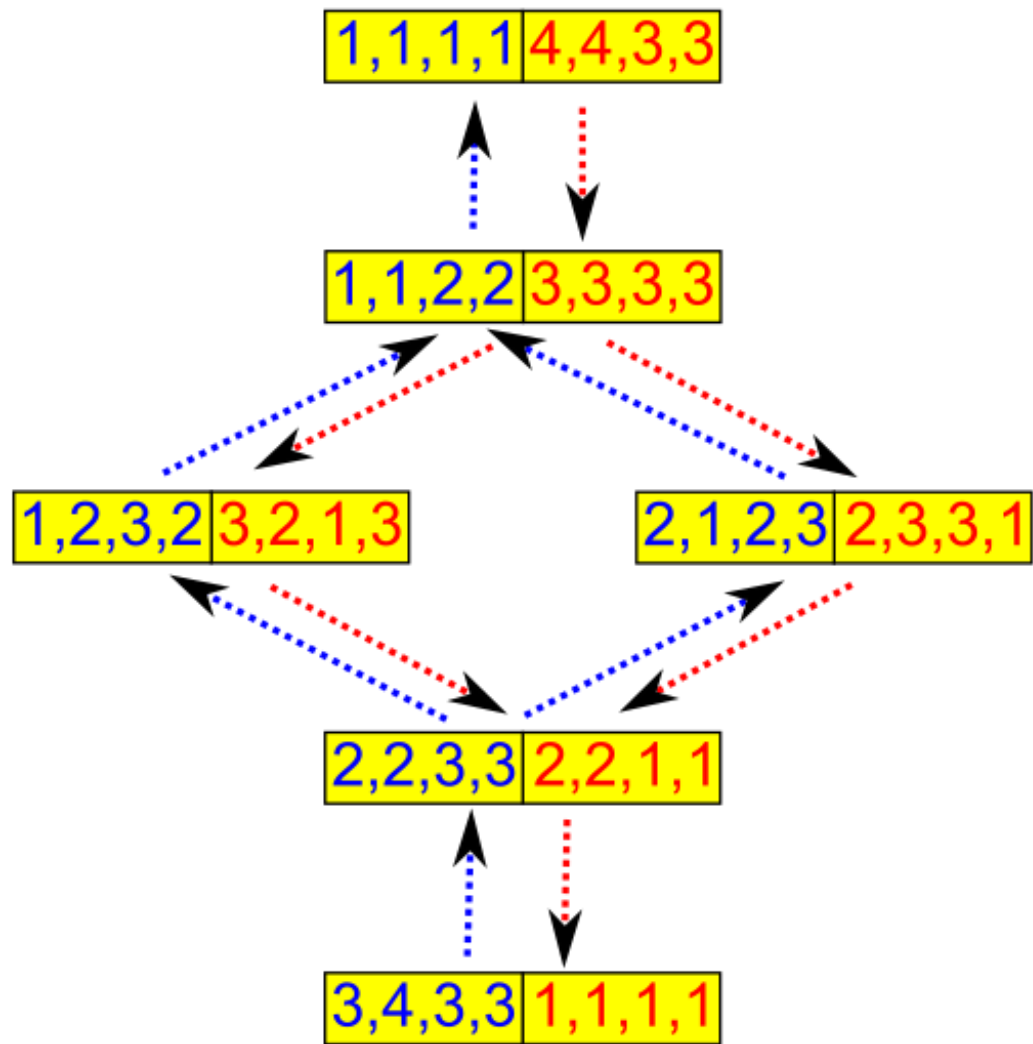
2,1,2,3 | 2,3,3,1

2,2,3,3 | 2,2,1,1

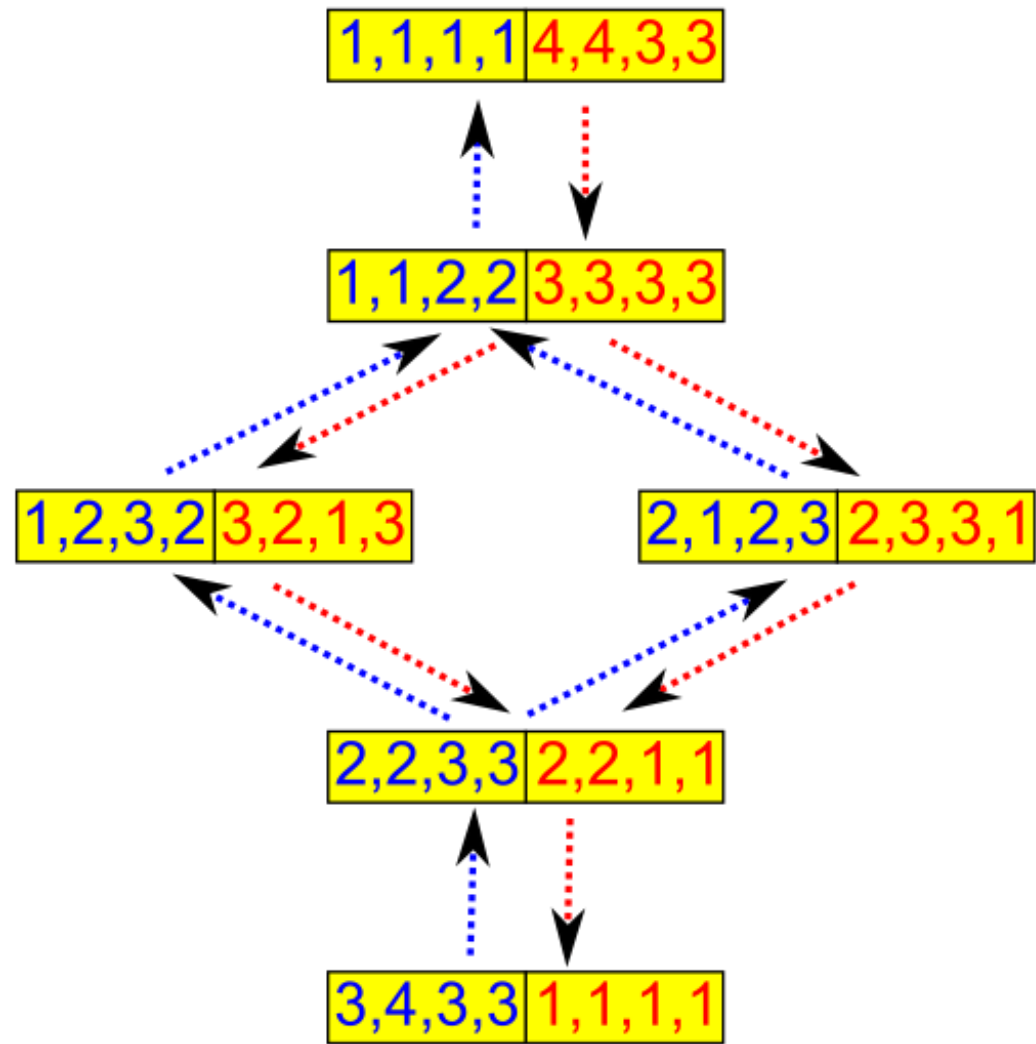
3,4,3,3 | 1,1,1,1



Consider a uniform combination of all integral stable matchings.



Consider a uniform combination of all integral stable matchings.



$m_1$

1

1

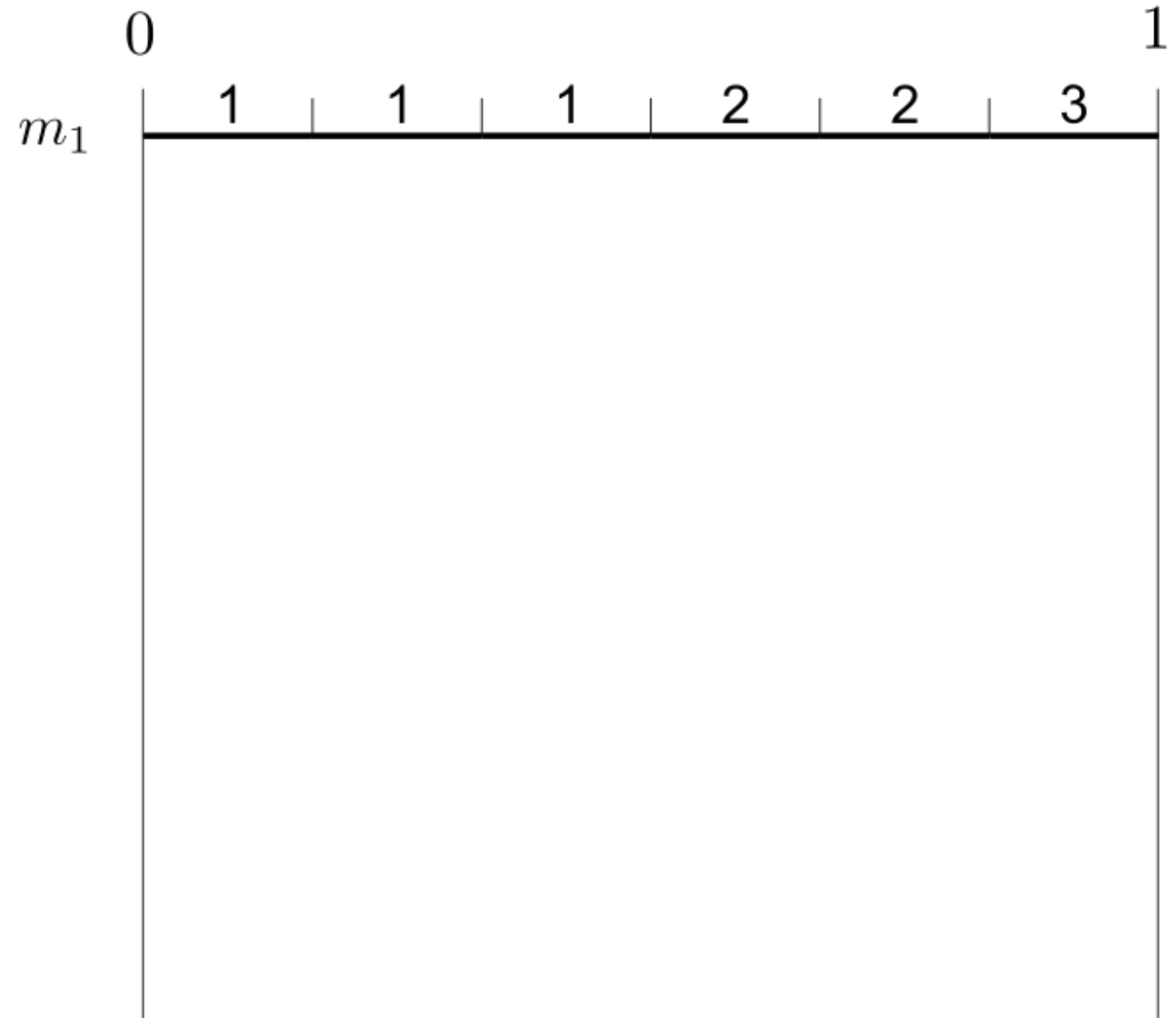
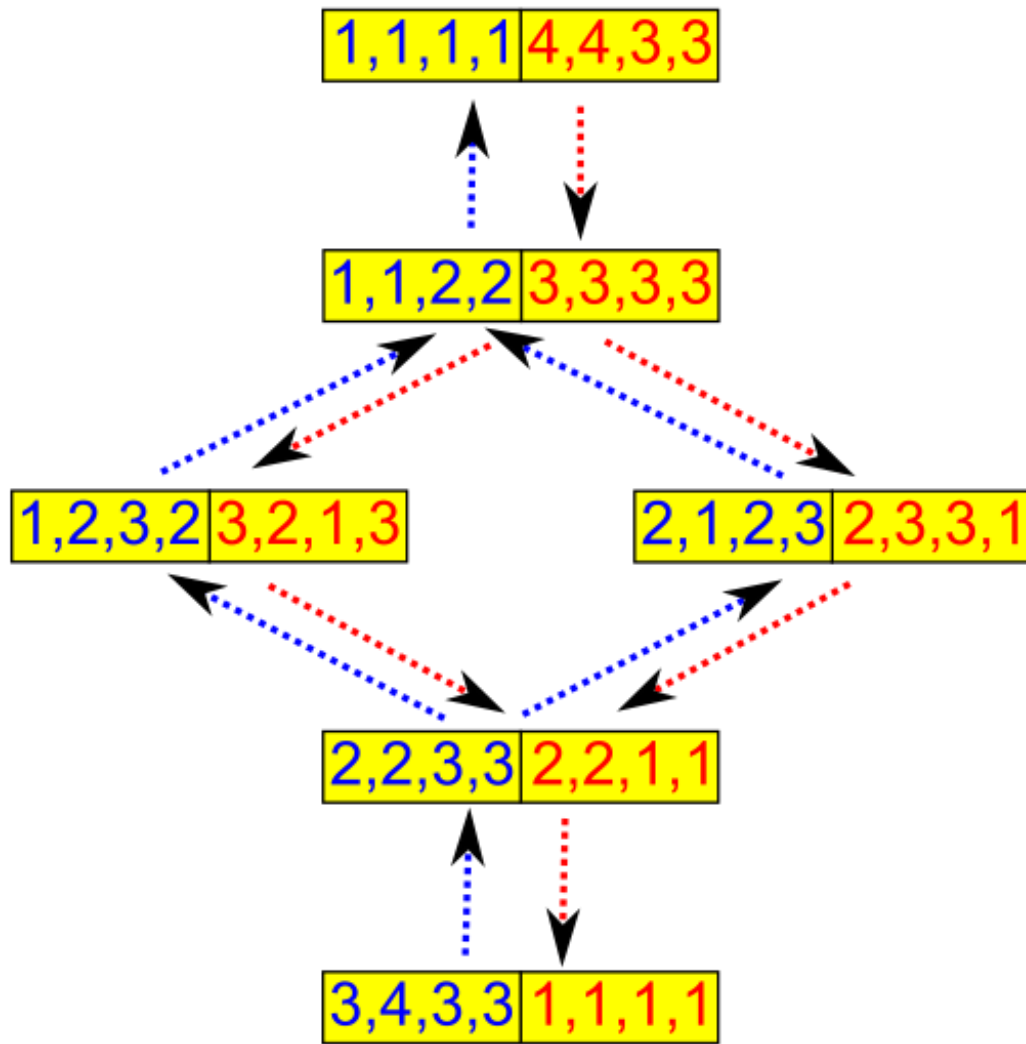
1

2

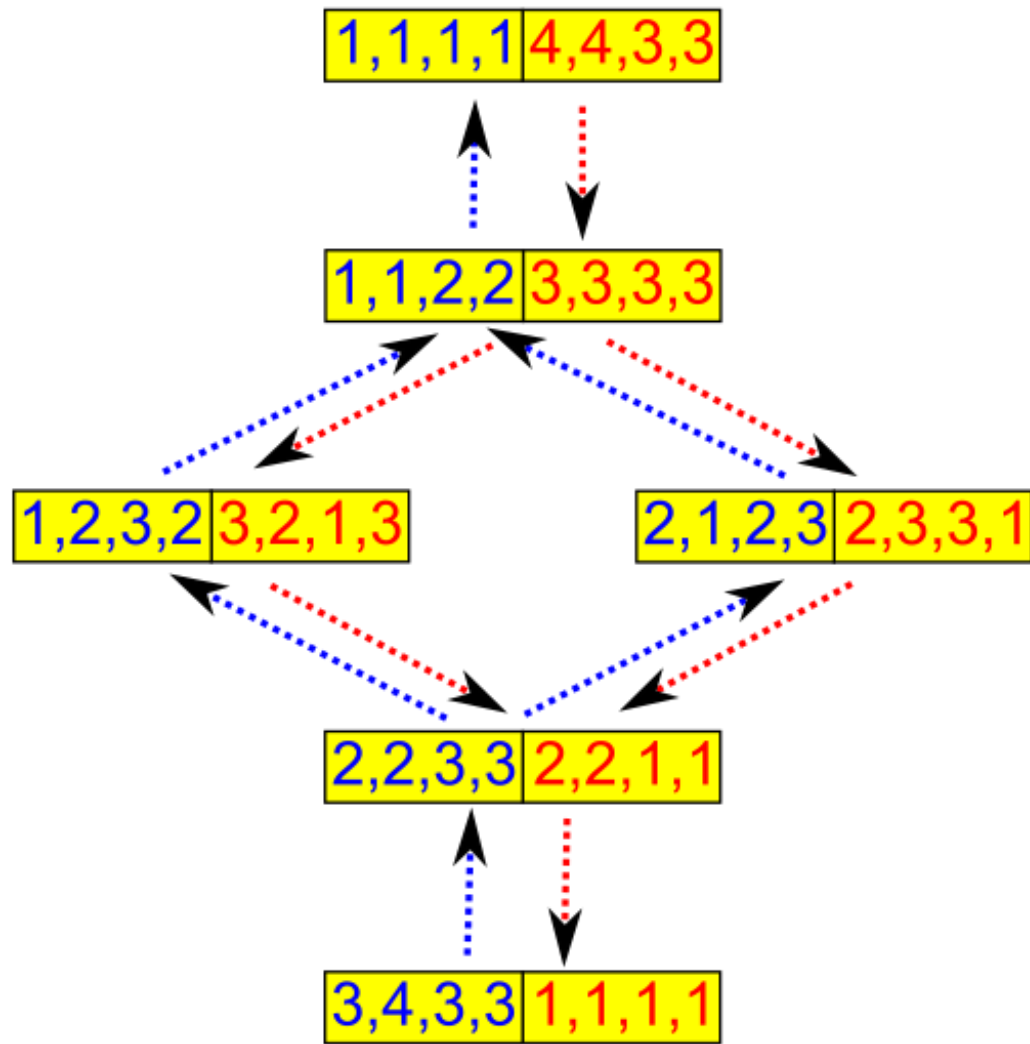
2

3

Consider a uniform combination of all integral stable matchings.

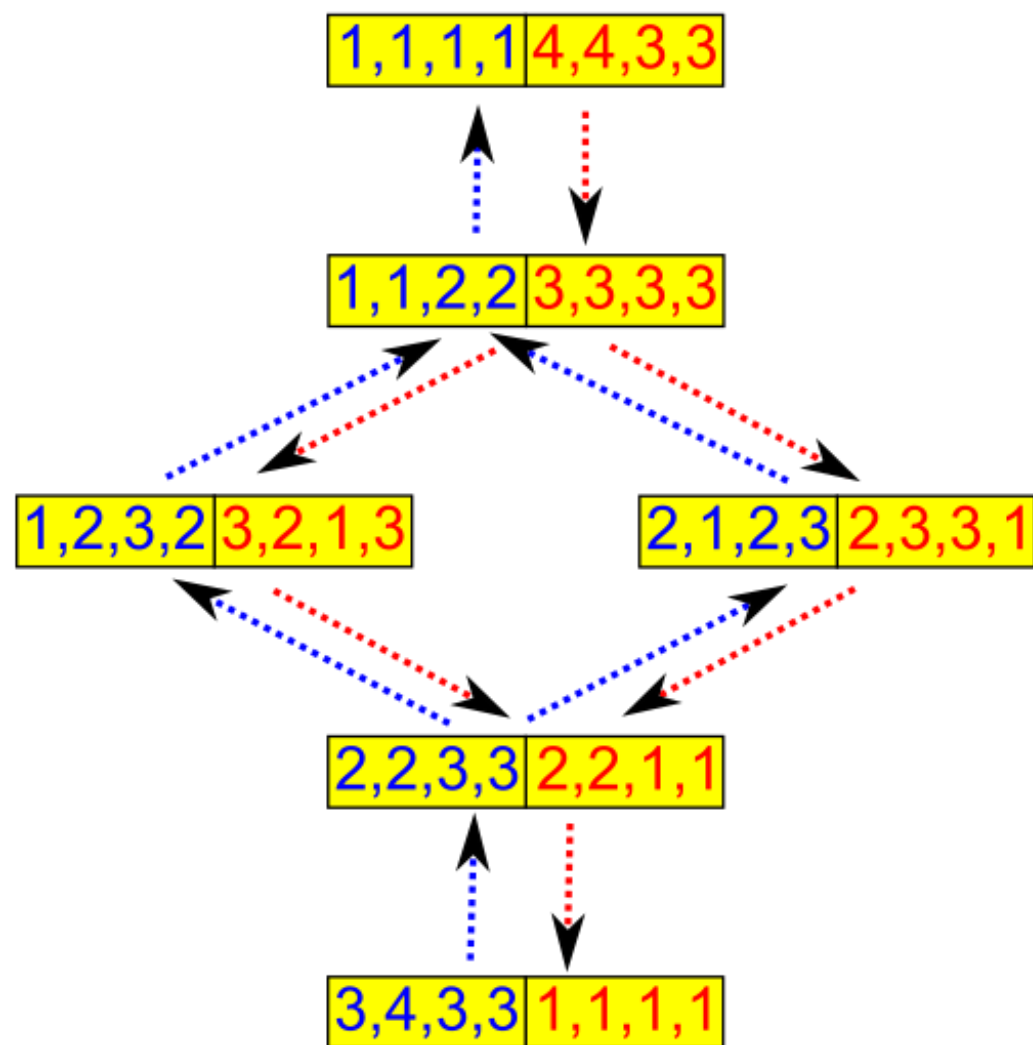


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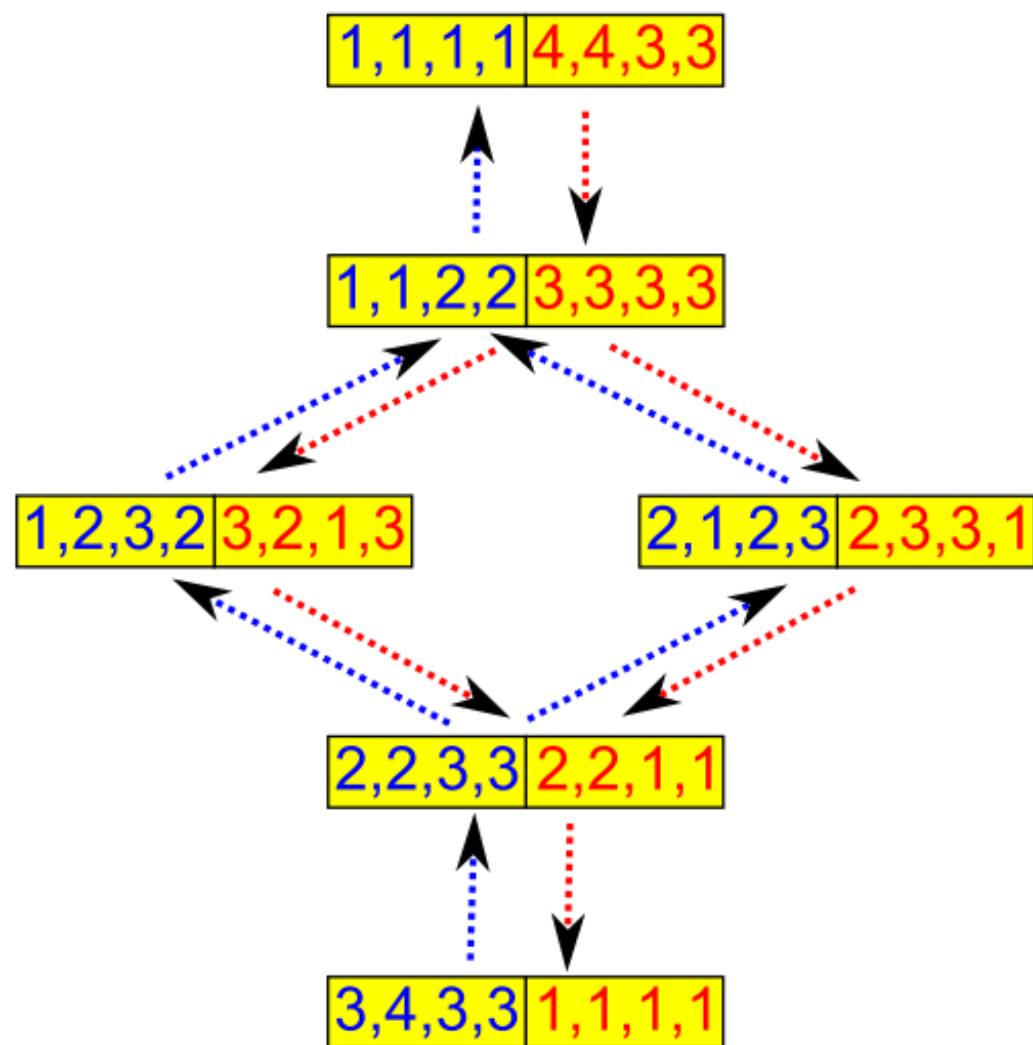
	0					1
$m_1$	1	1	1	2	2	3
$m_2$	1	1	1	2	2	4

Consider a uniform combination of all integral stable matchings.



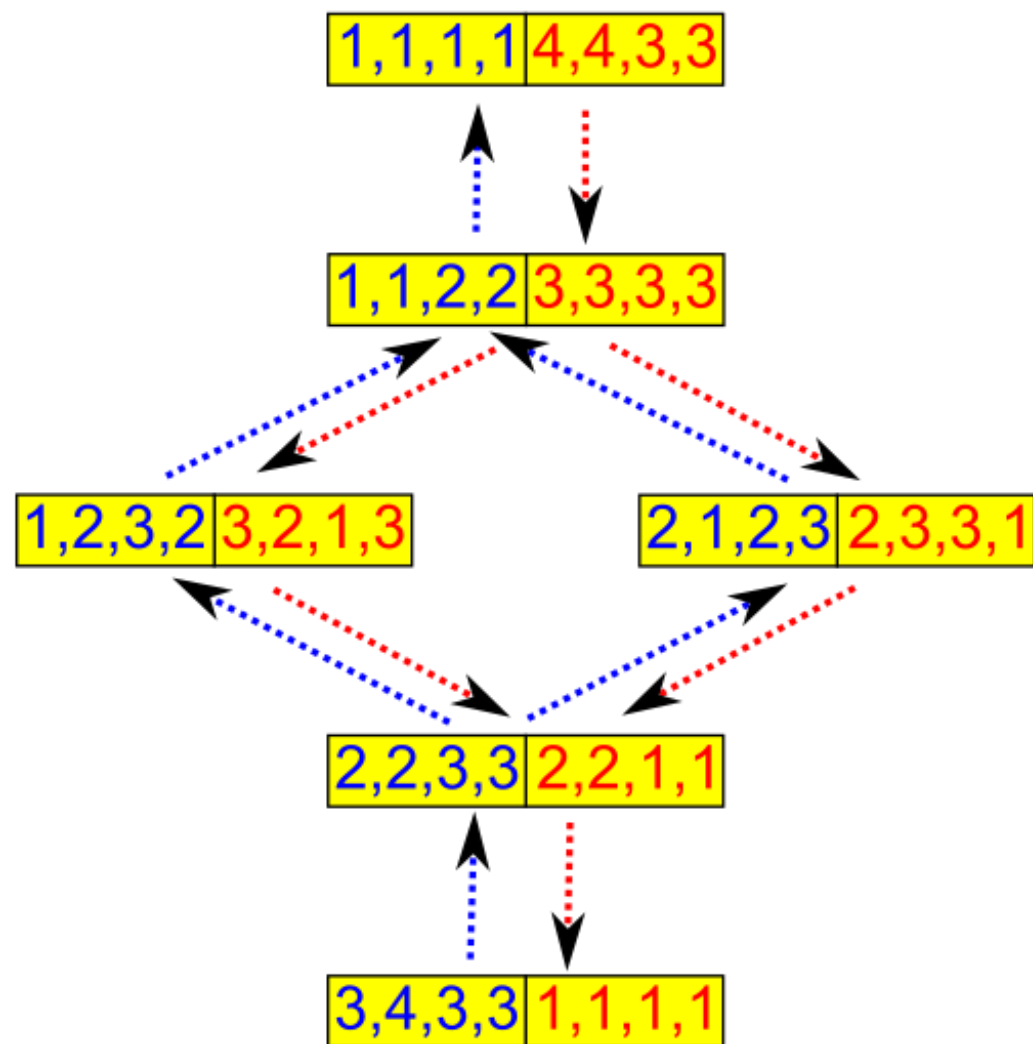
	0					1
$m_1$	1	1	1	2	2	3
$m_2$	1	1	1	2	2	4
$m_3$	1	2	2	3	3	3
$m_4$	1	2	2	3	3	3
$w_1$	4	3	3	2	2	1
$w_2$	4	3	3	2	2	1
$w_3$	3	3	3	1	1	1
$w_4$	3	3	3	1	1	1

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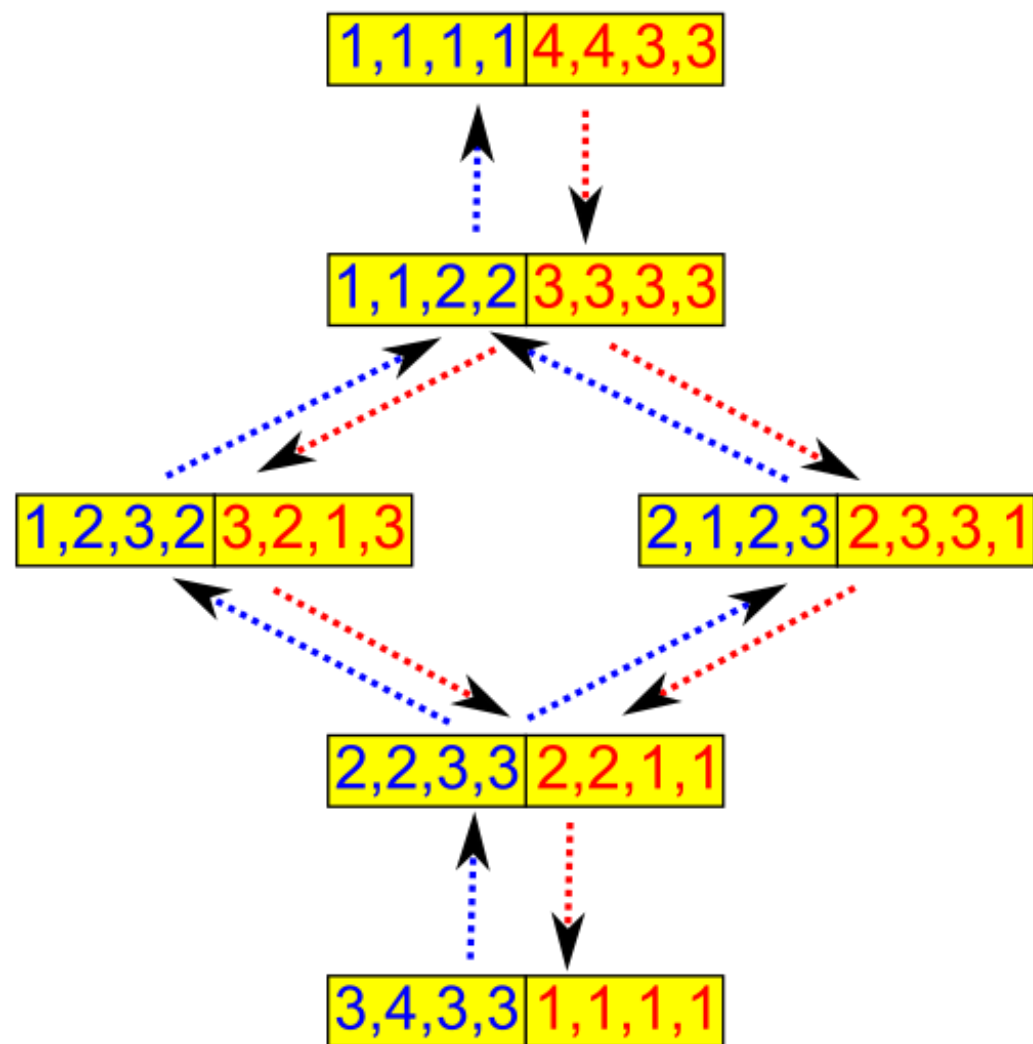
	0					1
$m_1$	1	1	1	2	2	3
$m_2$	1	1	1	2	2	4
$m_3$	1	2	2	3	3	3
$m_4$	1	2	2	3	3	3
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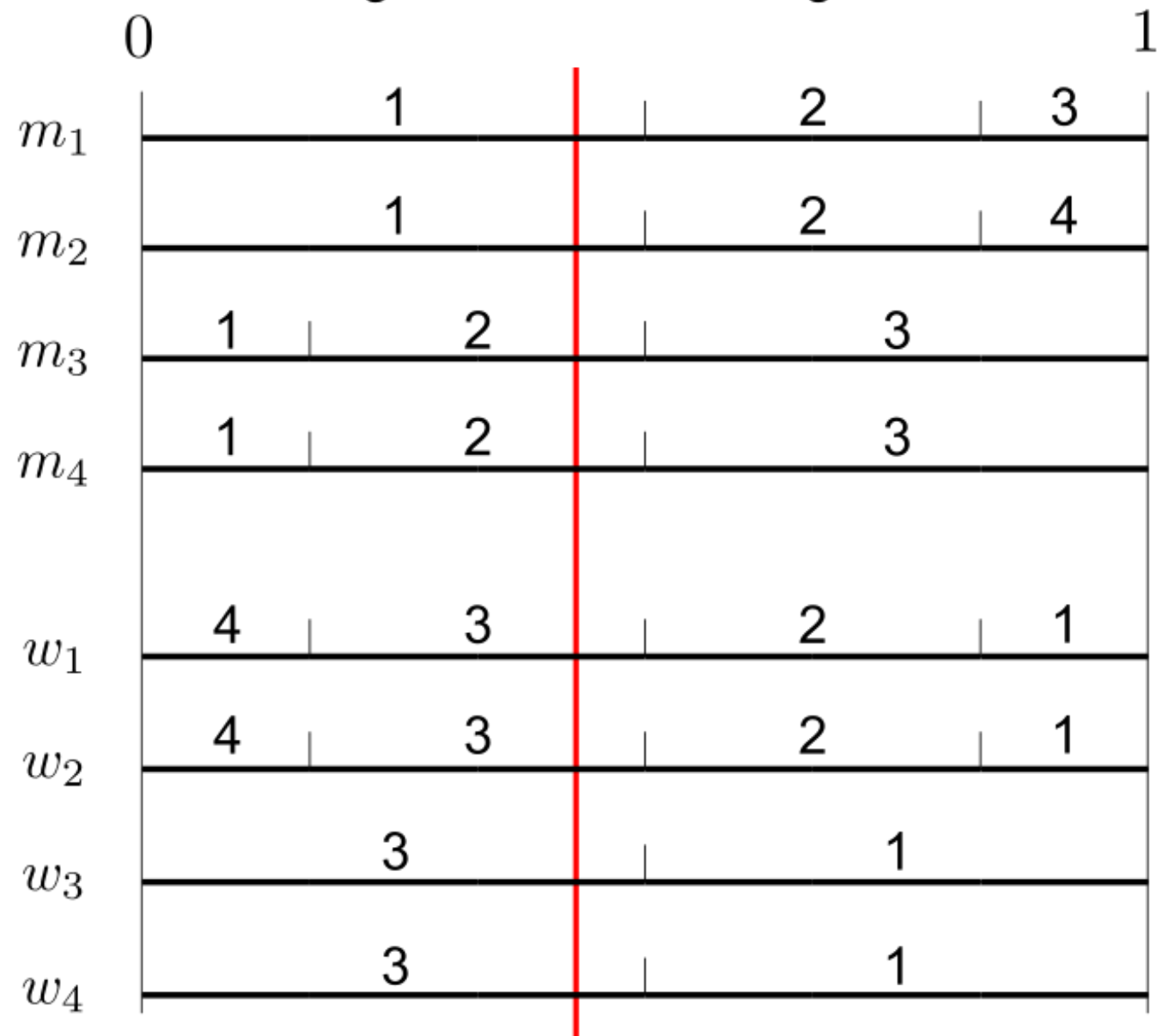
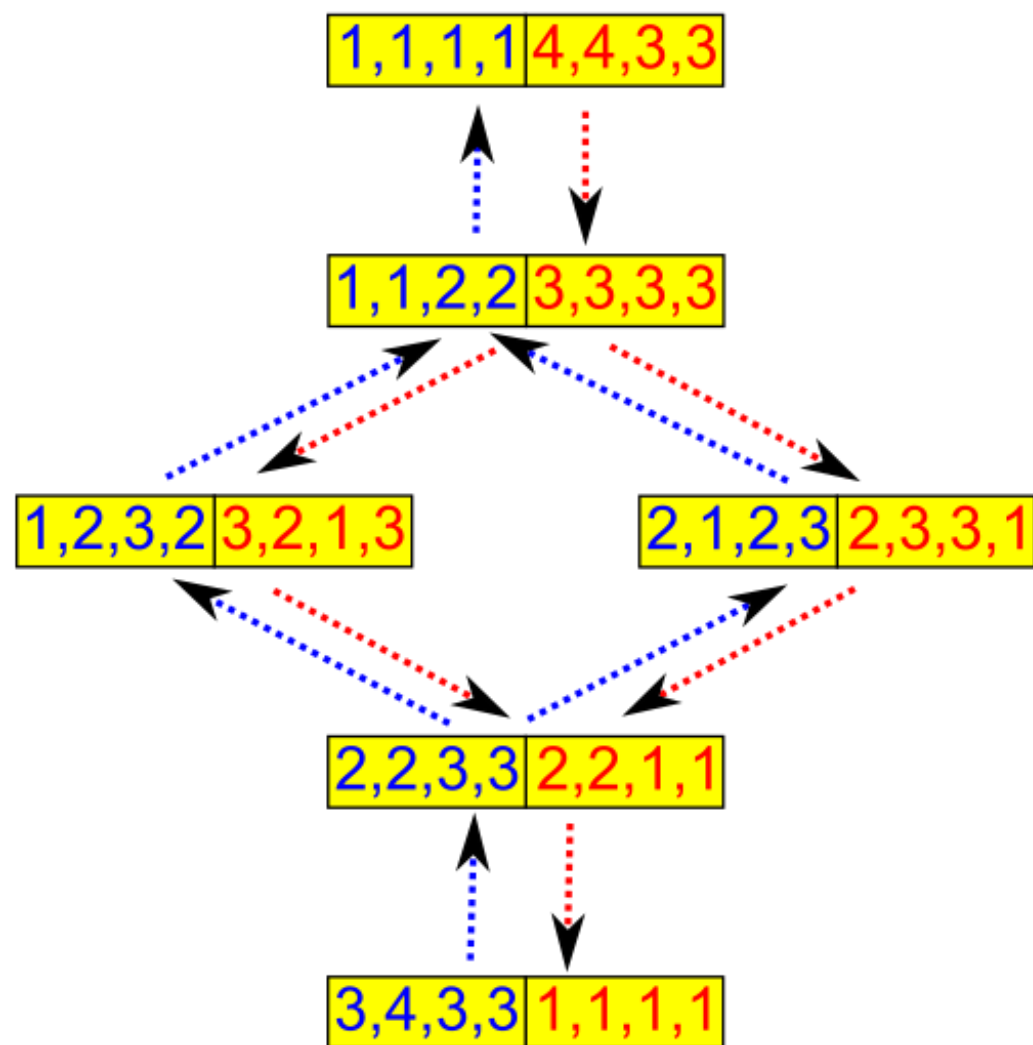
	0					1
$m_1$		1		2	2	3
$m_2$	1	1	1	2	2	4
$m_3$	1	2	2	3	3	3
$m_4$	1	2	2	3	3	3
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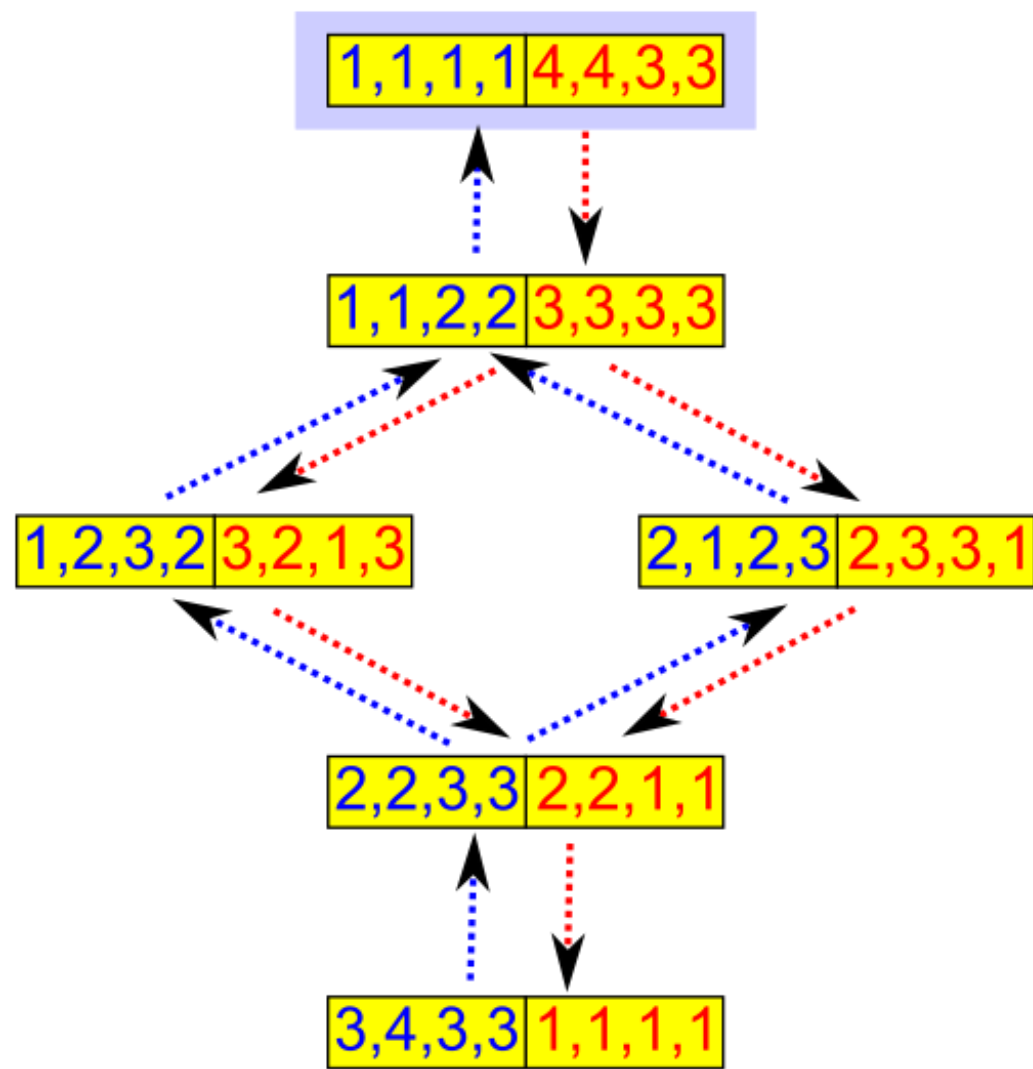
	0				1	
$m_1$		1		2	3	
$m_2$		1		2	4	
$m_3$	1		2		3	
$m_4$	1		2		3	
$w_1$	4		3		2	1
$w_2$	4		3		2	1
$w_3$		3			1	
$w_4$		3			1	

Consider a uniform combination of all integral stable matchings.



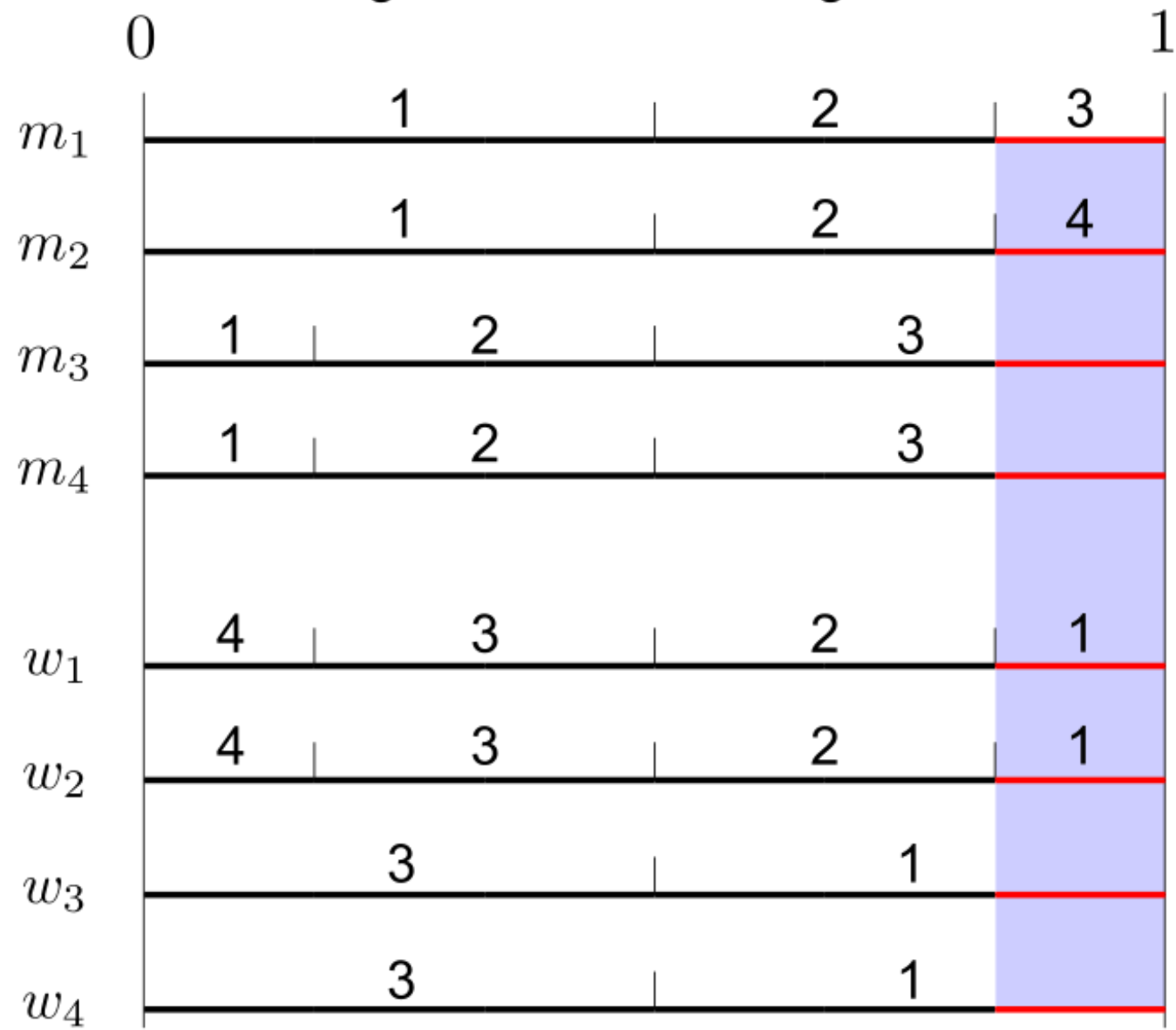
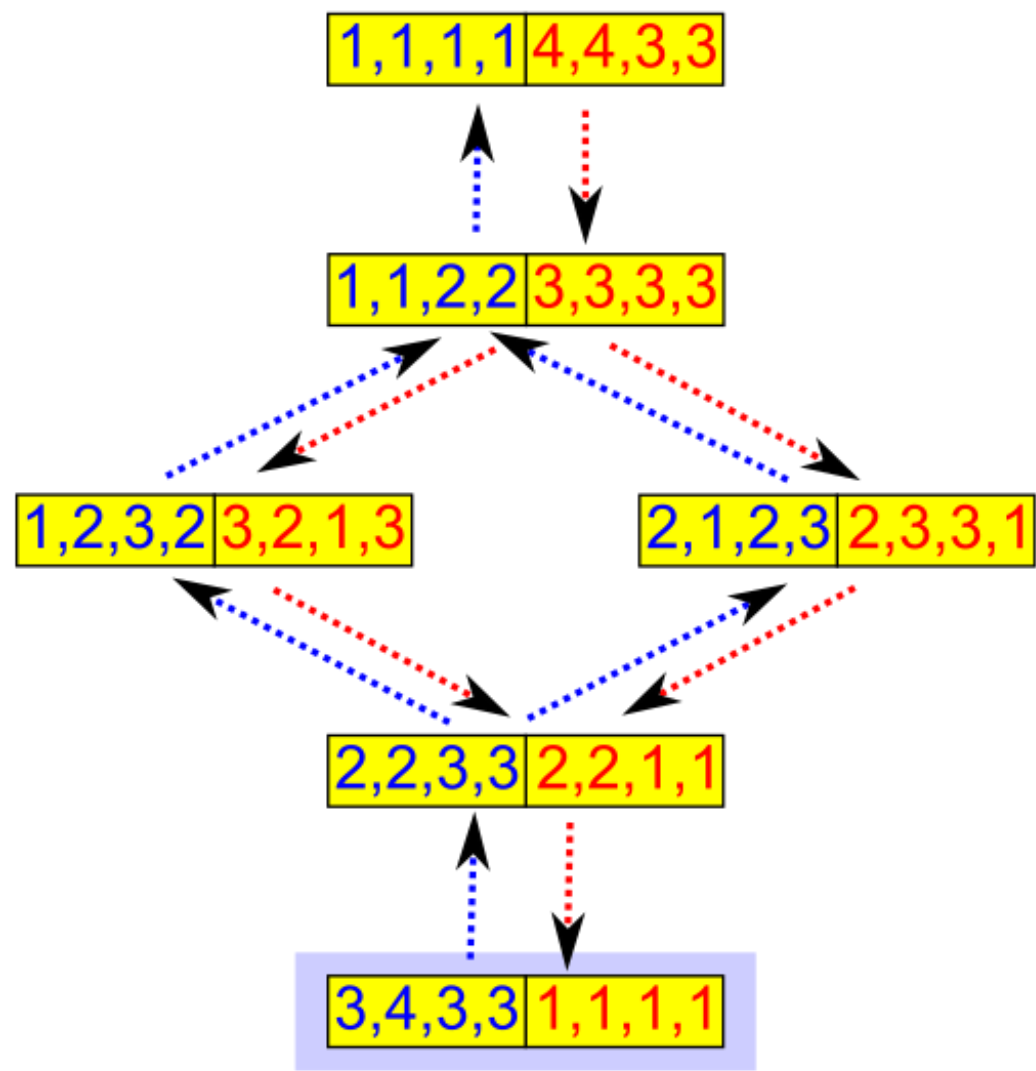
A vertical cut at any point in  $[0,1]$  induces a stable matching.

Consider a uniform combination of all integral stable matchings.

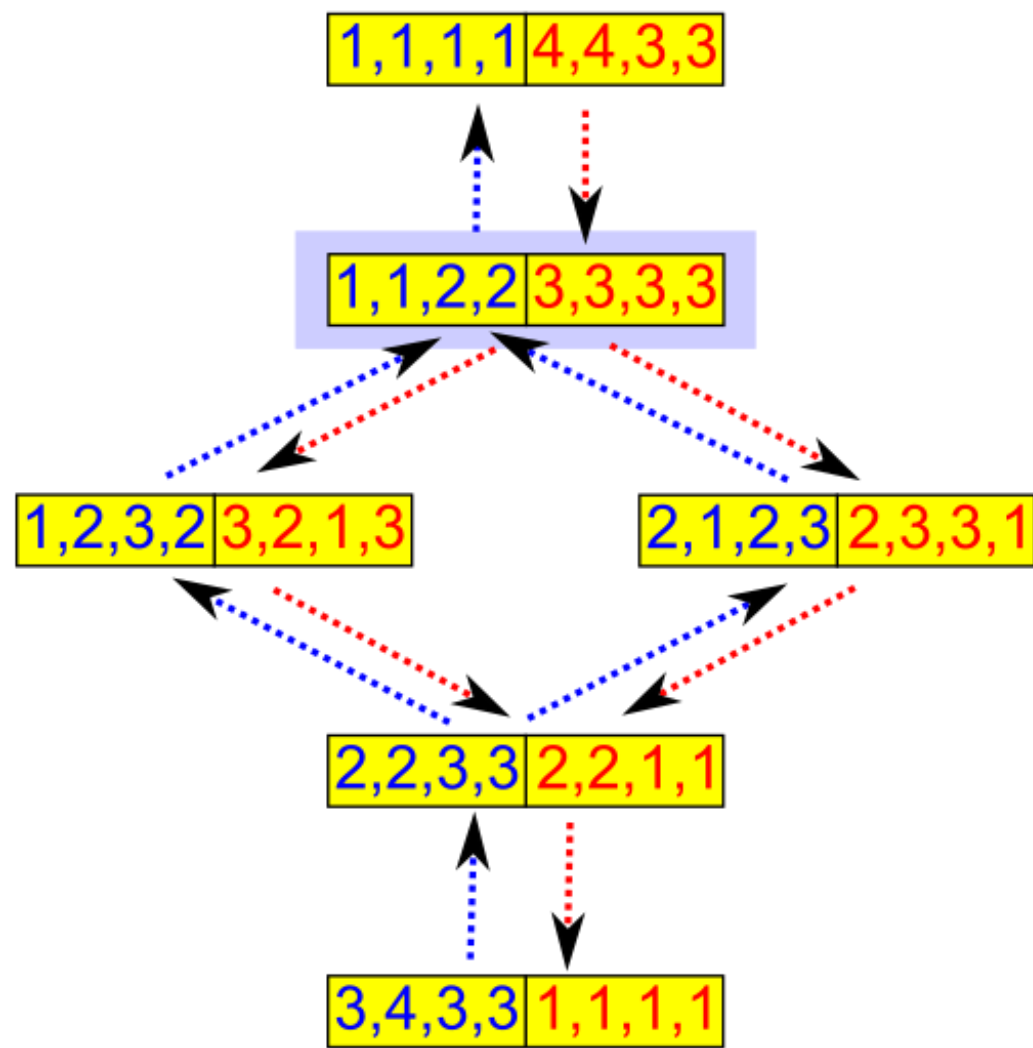


	0				1	
$m_1$		1		2	3	
$m_2$		1		2	4	
$m_3$	1		2		3	
$m_4$	1		2		3	
$w_1$	4		3		2	1
$w_2$	4		3		2	1
$w_3$		3			1	
$w_4$		3			1	

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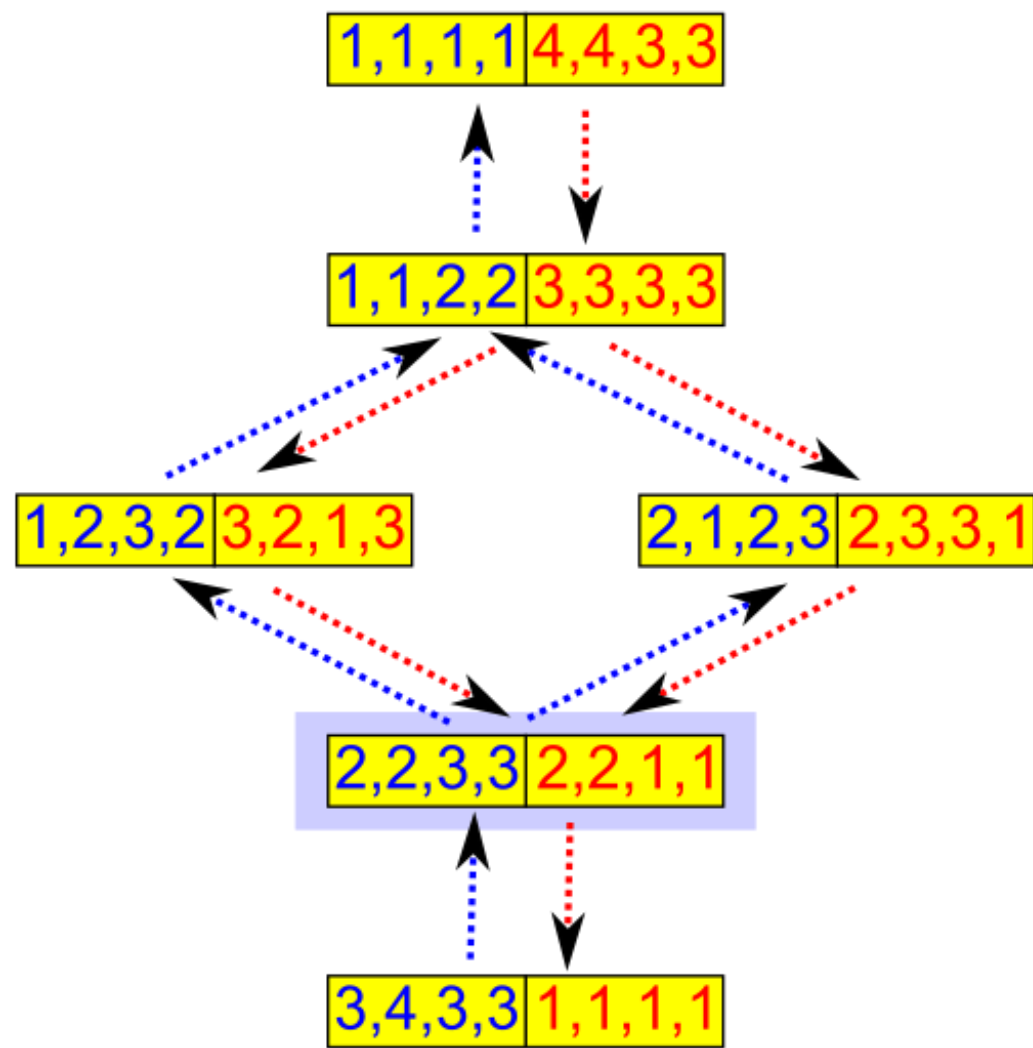


Consider a uniform combination of all integral stable matchings.



	0		1
$m_1$		1	2 3
$m_2$		1	2 4
$m_3$	1	2	3
$m_4$	1	2	3
$w_1$	4	3	2 1
$w_2$	4	3	2 1
$w_3$		3	1
$w_4$		3	1

Consider a uniform combination of all integral stable matchings.



	0		1
$m_1$	1	2	3
$m_2$	1	2	4
$m_3$	1	2	3
$m_4$	1	2	3
$w_1$	4	3	2
$w_2$	4	3	2
$w_3$	3	1	1
$w_4$	3	1	1

A vertical column of cells is shaded light blue, spanning from the top row to the bottom row, between the columns labeled 0 and 1.

# Bad News

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[Cheng, *Algorithmica* 2010]

Computing a median stable matching is **#P-hard**.

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[Cheng, *Algorithmica* 2010]

Computing a median stable matching is **#P-hard**.

as hard as...

- computing the permanent of a 0-1 matrix
- counting the number of perfect matchings in a bipartite graph
- counting the number of stable matchings for a given instance
- and many others...

# Other Notions of Fairness

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## Egalitarian

[Irving, Leather, and Gusfield, 1987]

a stable matching that minimizes the  
*average rank* of matched partners  
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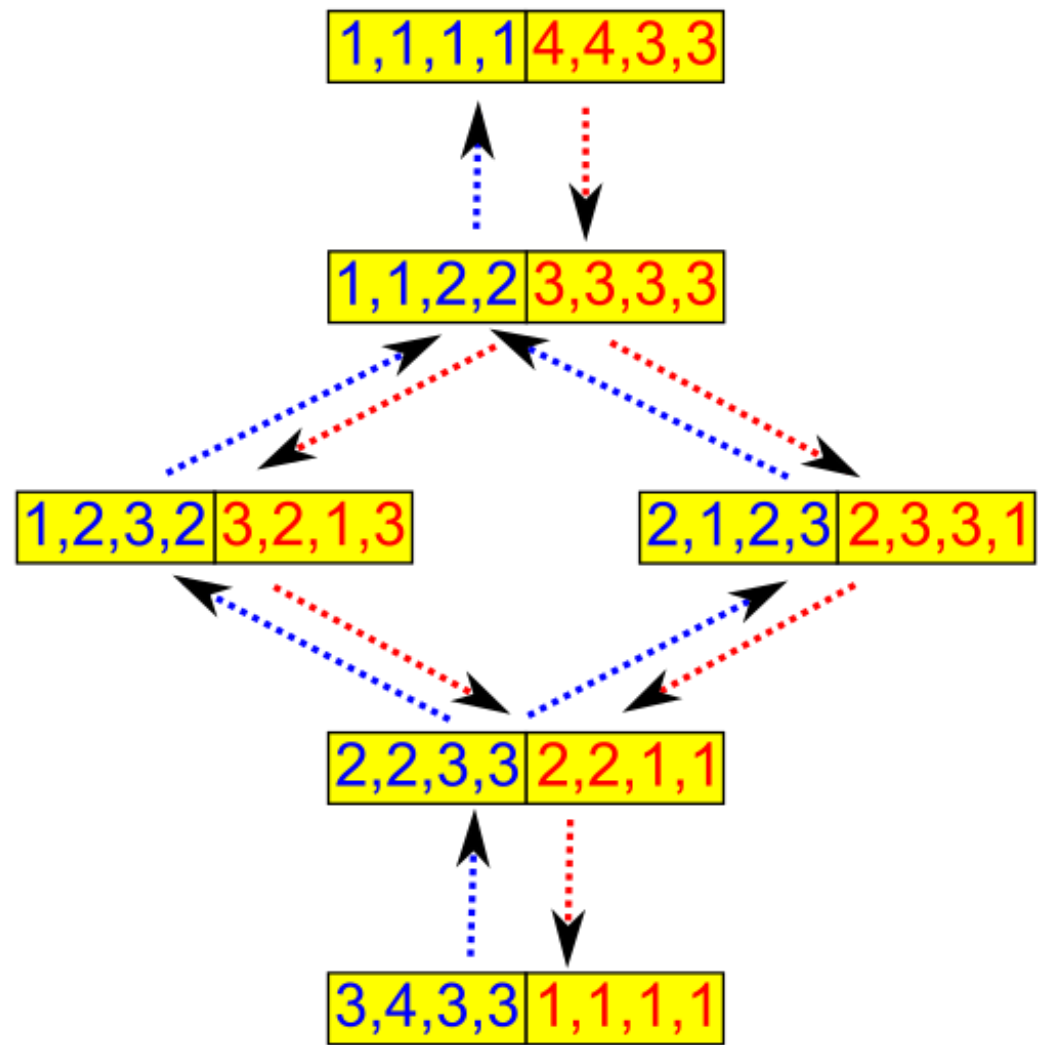
Polynomial time

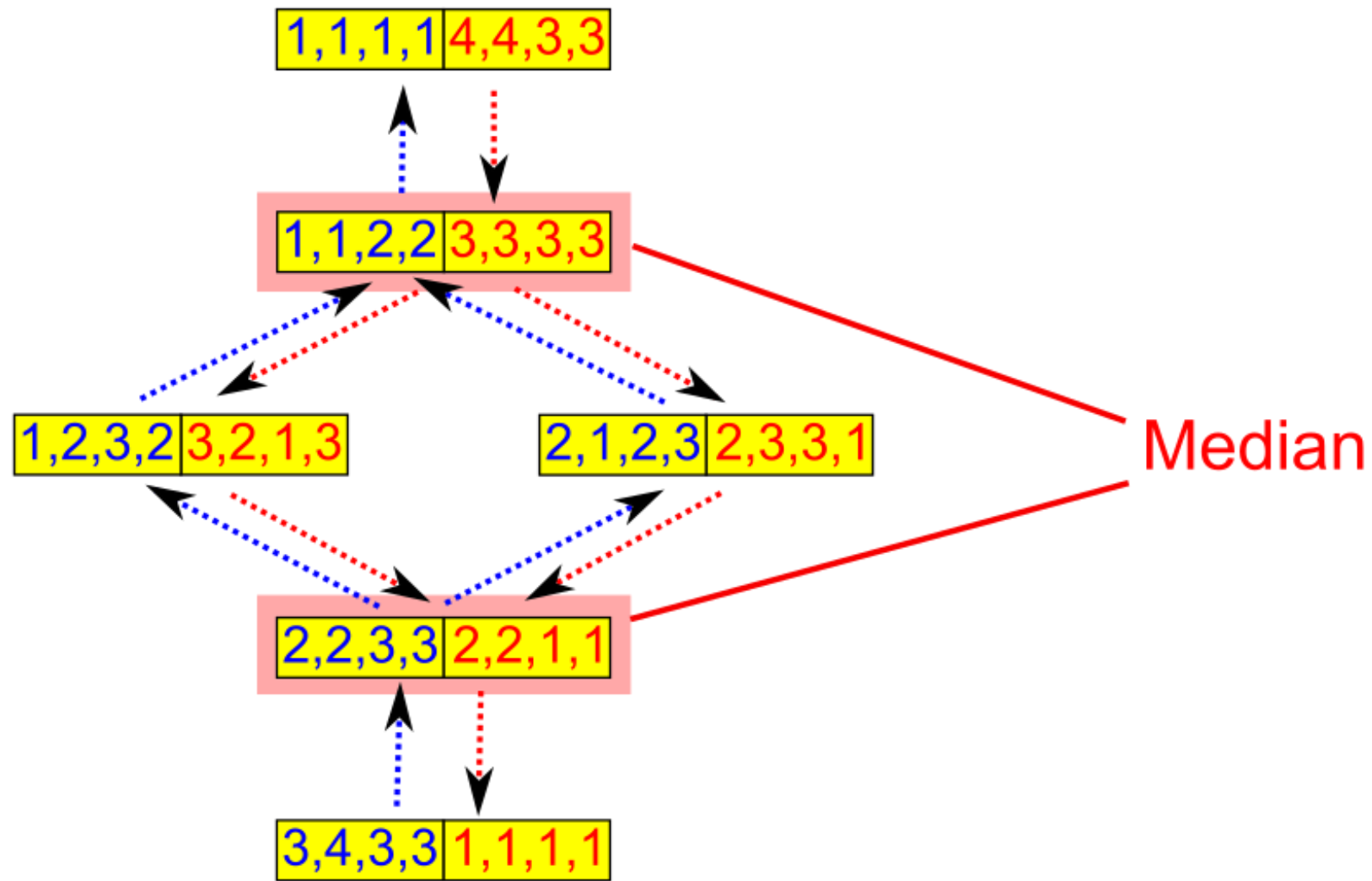
## Minimum regret

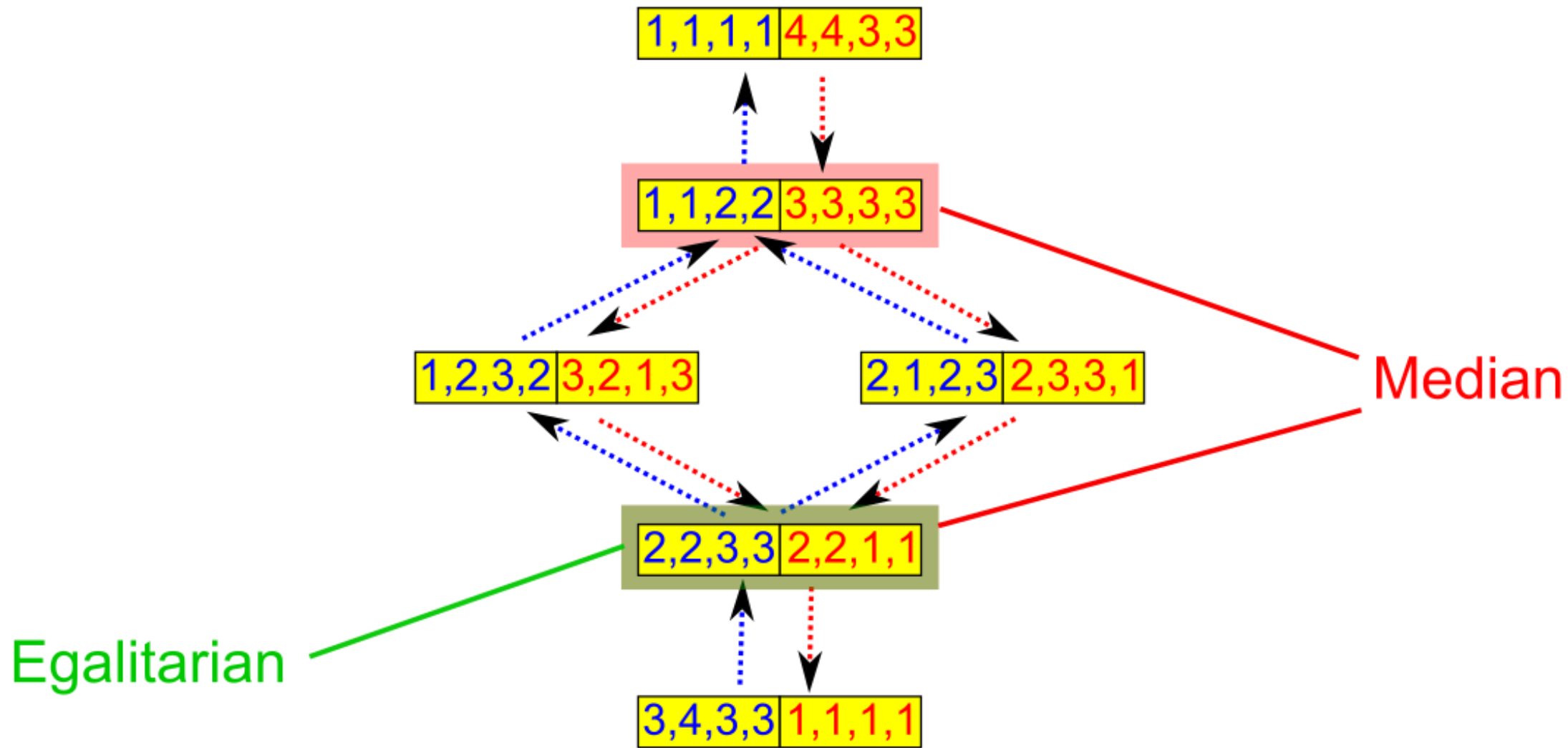
[Knuth, 1976 (attributed to Selkow)]

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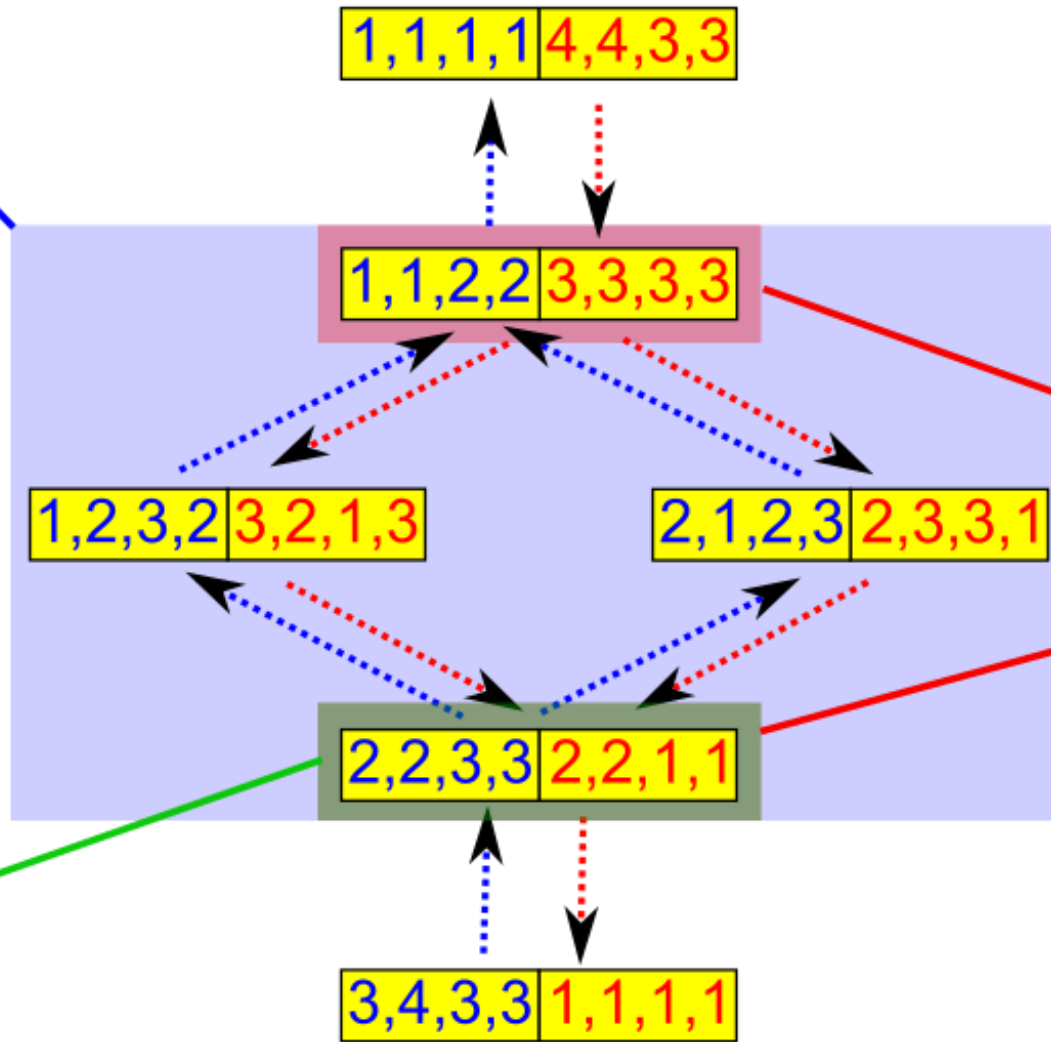
Polynomial time







Minimum Regret



Median

Egalitarian

# References

- Slides by Christine Cheng on “Fair Stable Matchings”.

<https://www.optimalmatching.com/MATCHUP2015/slides/ChristineCheng.pdf>

- Linear programming-based formulation of the stable matching problem

John Vande Vate

“Linear Programming Brings Marital Bliss”

*Operations Research Letters*, 8(3), 1989 pg 147-153

# References

- A median stable matching always exists.

Chung-Piaw Teo and Jay Sethuraman

“The Geometry of Fractional Stable Matchings and Its Applications”

*Mathematics of Operations Research*, 23(4), 1998 pg 874-891

- Computing a median stable matching is #P-hard.

Christine Cheng

“Understanding the Generalized Median Stable Matchings”

*Algorithmica*, 58, 2010 pg 34-51

# References

- An algorithm for computing an egalitarian stable matching.

Robert Irving, Paul Leather, and Dan Gusfield

“An Efficient Algorithm for the Optimal Stable Marriage Problem”

*Journal of the ACM*, 34, 1987 pg 532-543

