COL8184: Algorithms for Fair Representation

Lecture 15

The Stable Matching Problem

Mar 30, 2026 | Rohit Vaish

Stable Matching Problem

Stable Matching Problem

Stable Matching Problem

$w_1 > w_2 > w_3$

$m_3 > m_2 > m_1$

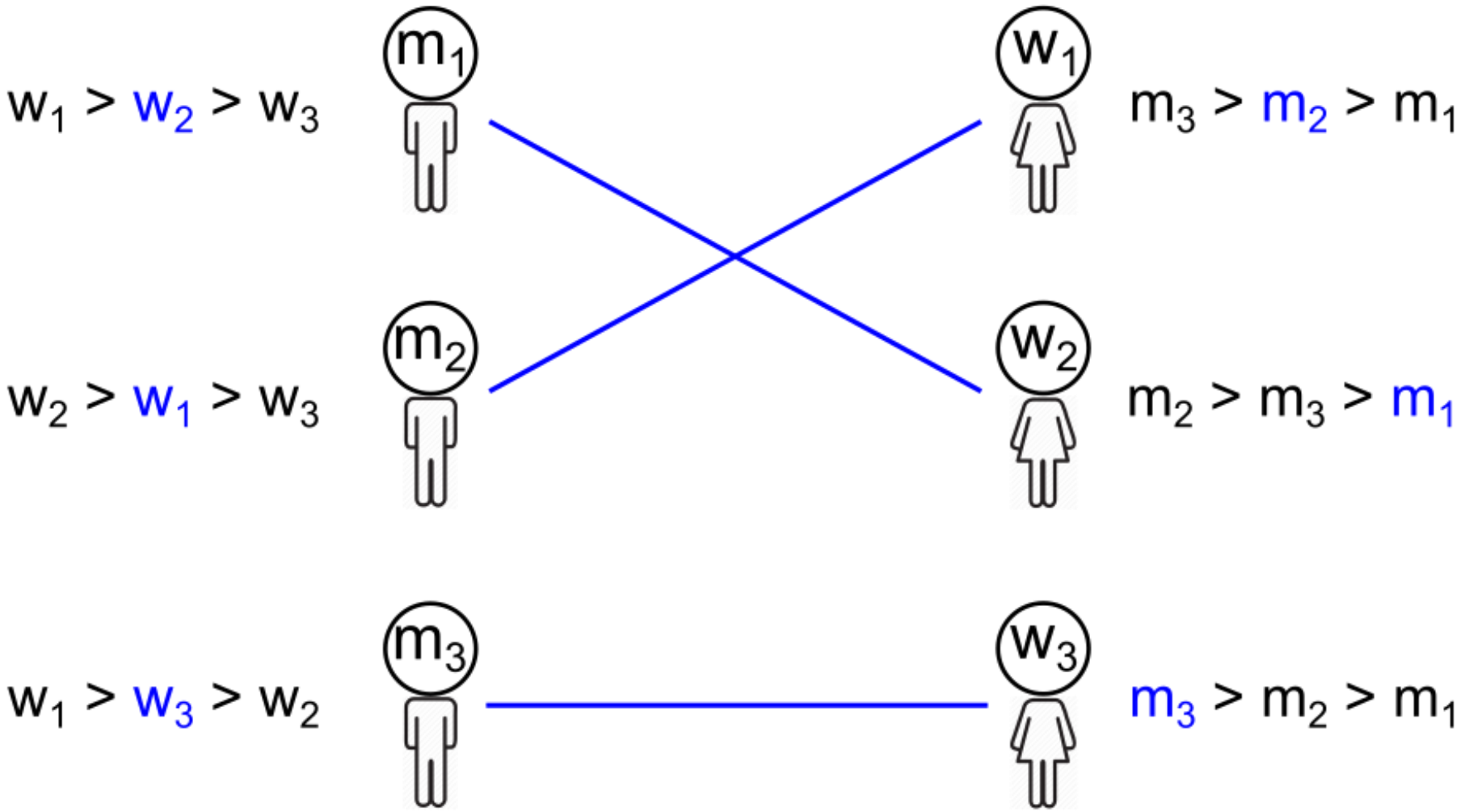
$w_2 > w_1 > w_3$

$m_2 > m_3 > m_1$

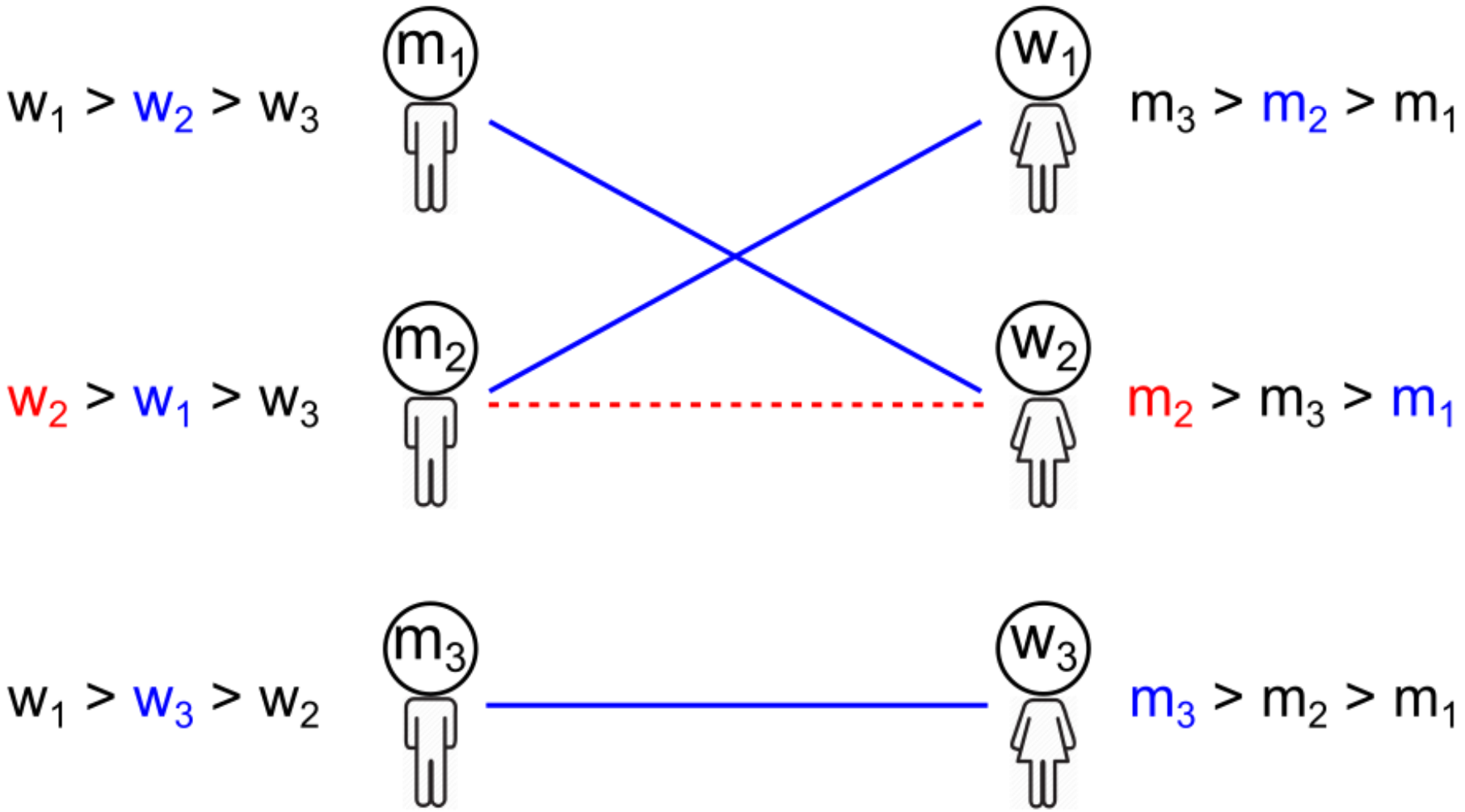
$w_1 > w_3 > w_2$

$m_3 > m_2 > m_1$

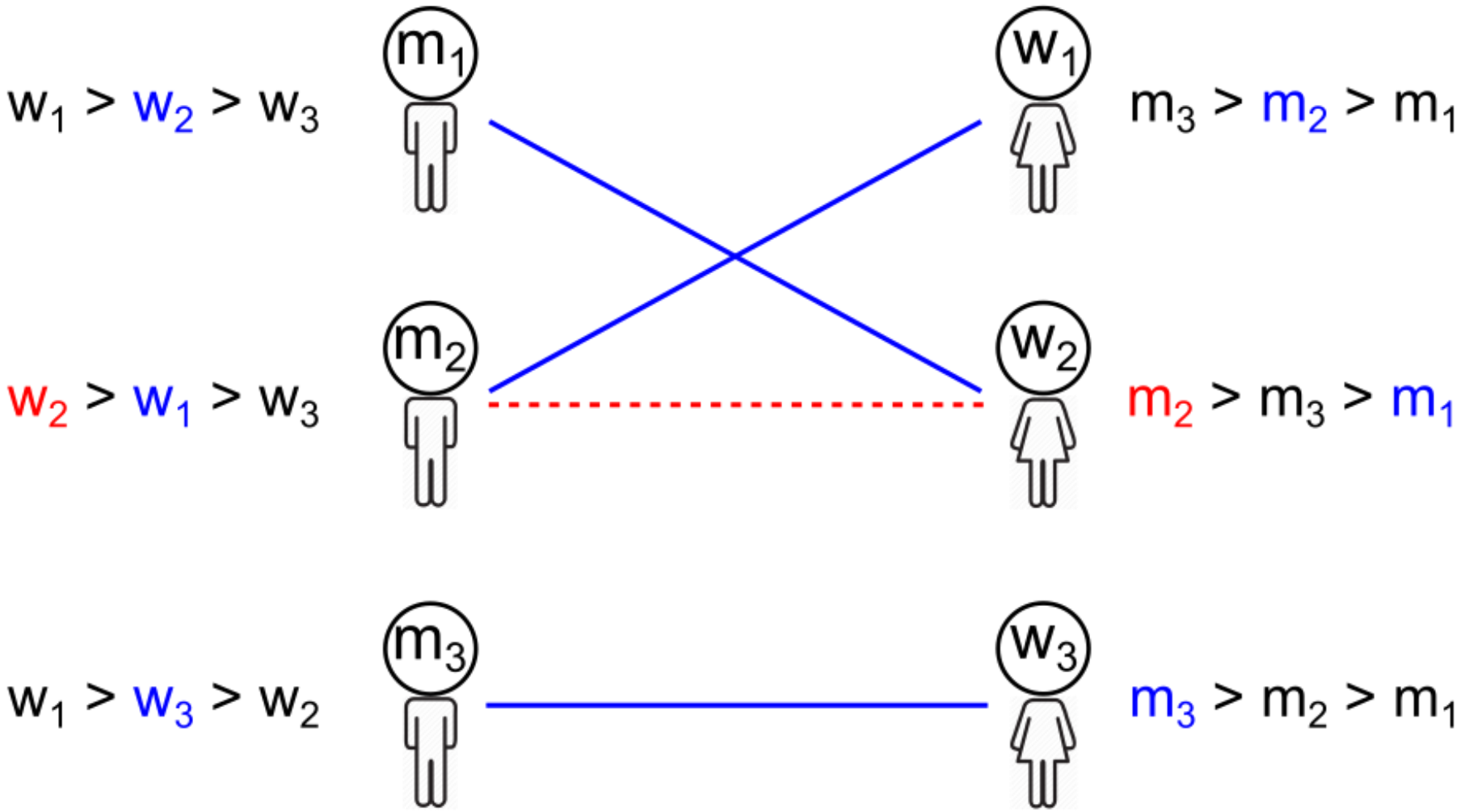
Stable Matching Problem



Stable Matching Problem

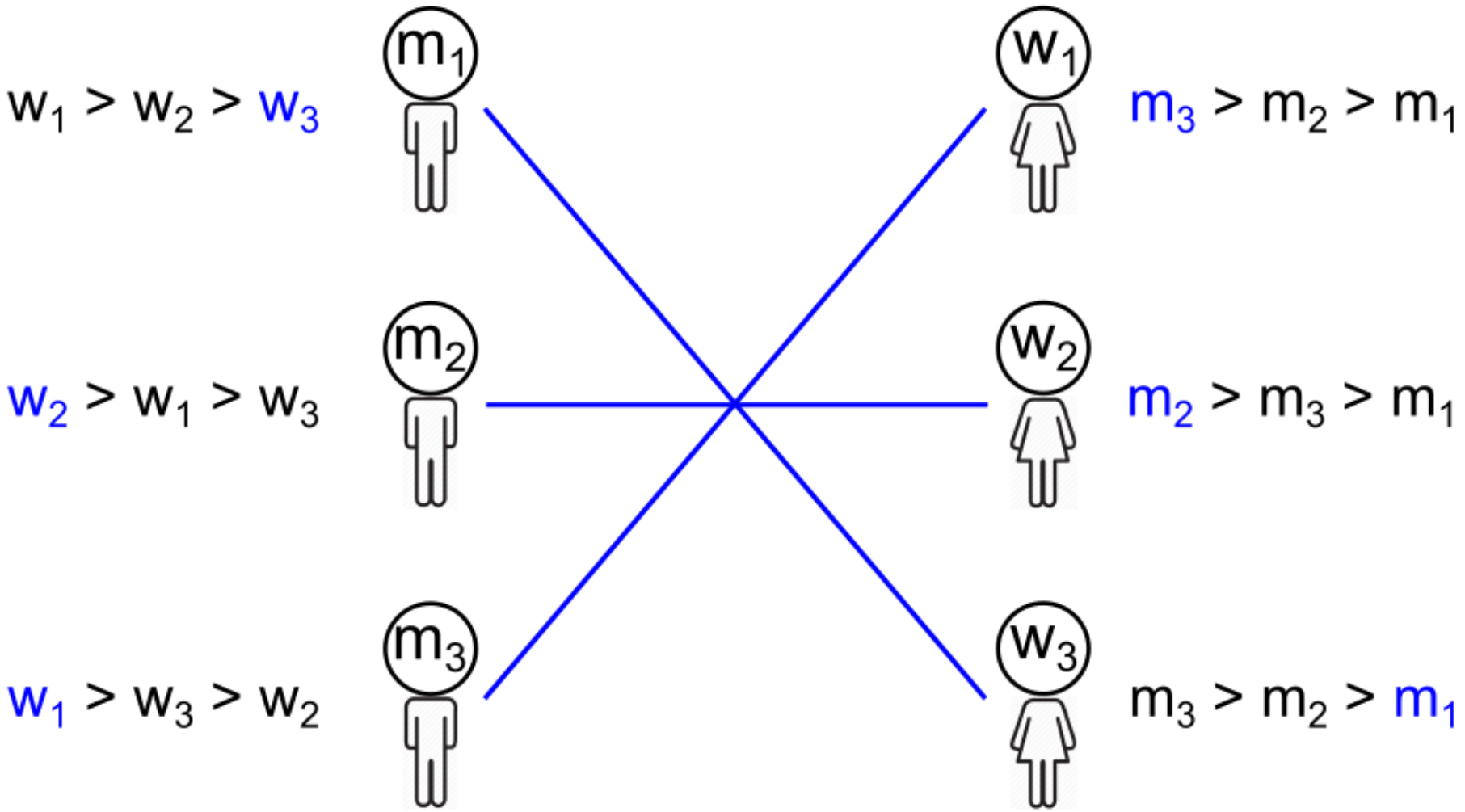


Stable Matching Problem



A matching is stable if there is no blocking pair.

Stable Matching Problem



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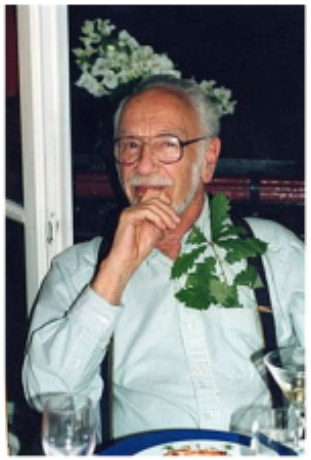


COLLEGE ADMISSIONS AND THE STABILITY OF MARRIAGE

D. GALE* AND L. S. SHAPLEY, Brown University and the RAND Corporation

Source: *The American Mathematical Monthly*, Jan., 1962, Vol. 69, No. 1 (Jan., 1962), pp. 9-15

Given any preference profile, a stable matching for that profile always exists and can be computed in polynomial time.

Deferred-Acceptance Algorithm

Deferred-Acceptance Algorithm

$w_1 > w_2 > w_3 > w_4$

$m_3 > m_2 > m_1 > m_4$

$w_1 > w_4 > w_2 > w_3$

$m_4 > m_2 > m_3 > m_1$

$w_1 > w_2 > w_3 > w_4$

$m_3 > m_2 > m_1 > m_4$

$w_1 > w_4 > w_2 > w_3$

$m_1 > m_2 > m_3 > m_4$

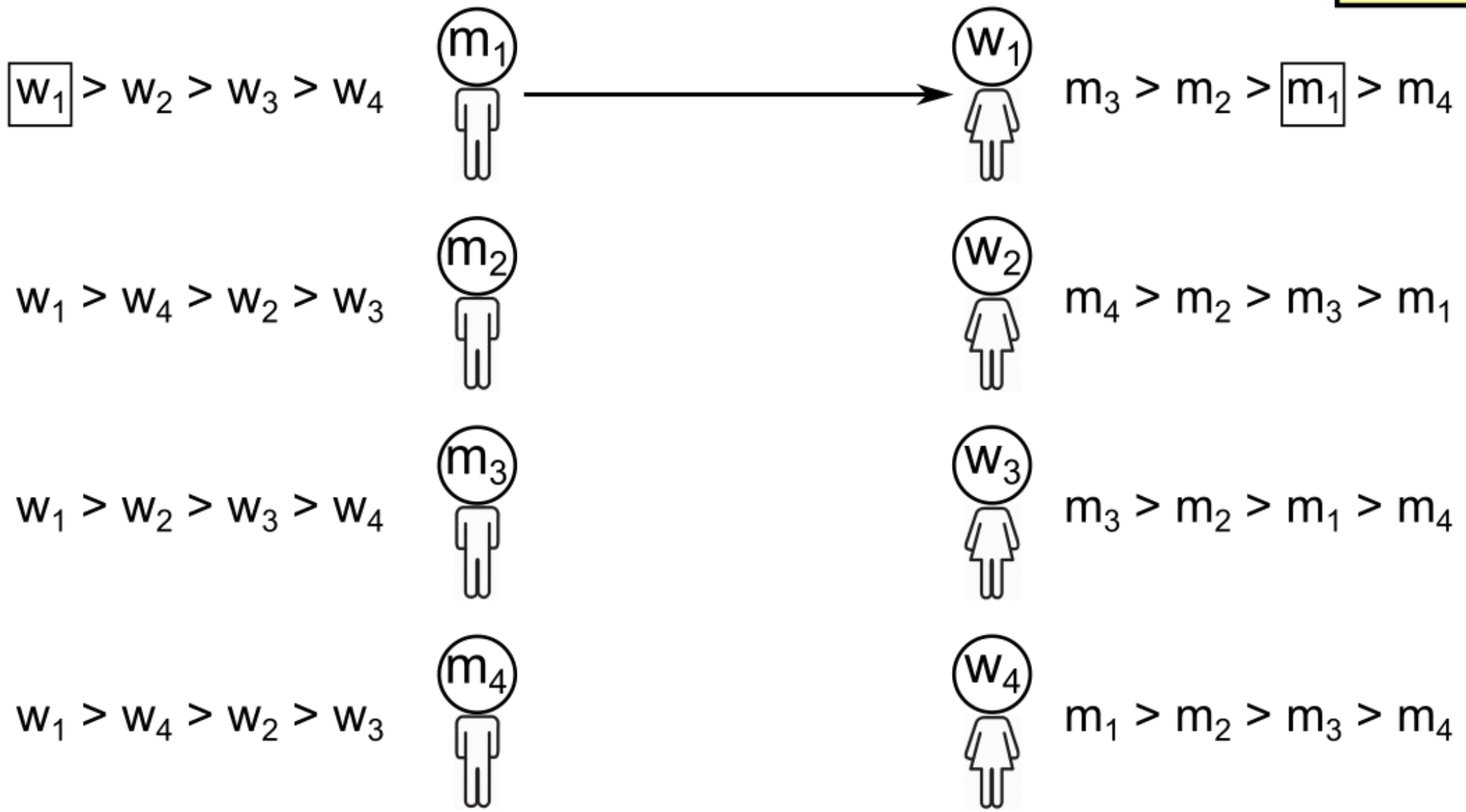
Deferred-Acceptance Algorithm

Round 1

$w_1 > w_2 > w_3 > w_4$			$m_3 > m_2 > m_1 > m_4$
$w_1 > w_4 > w_2 > w_3$			$m_4 > m_2 > m_3 > m_1$
$w_1 > w_2 > w_3 > w_4$			$m_3 > m_2 > m_1 > m_4$
$w_1 > w_4 > w_2 > w_3$			$m_1 > m_2 > m_3 > m_4$

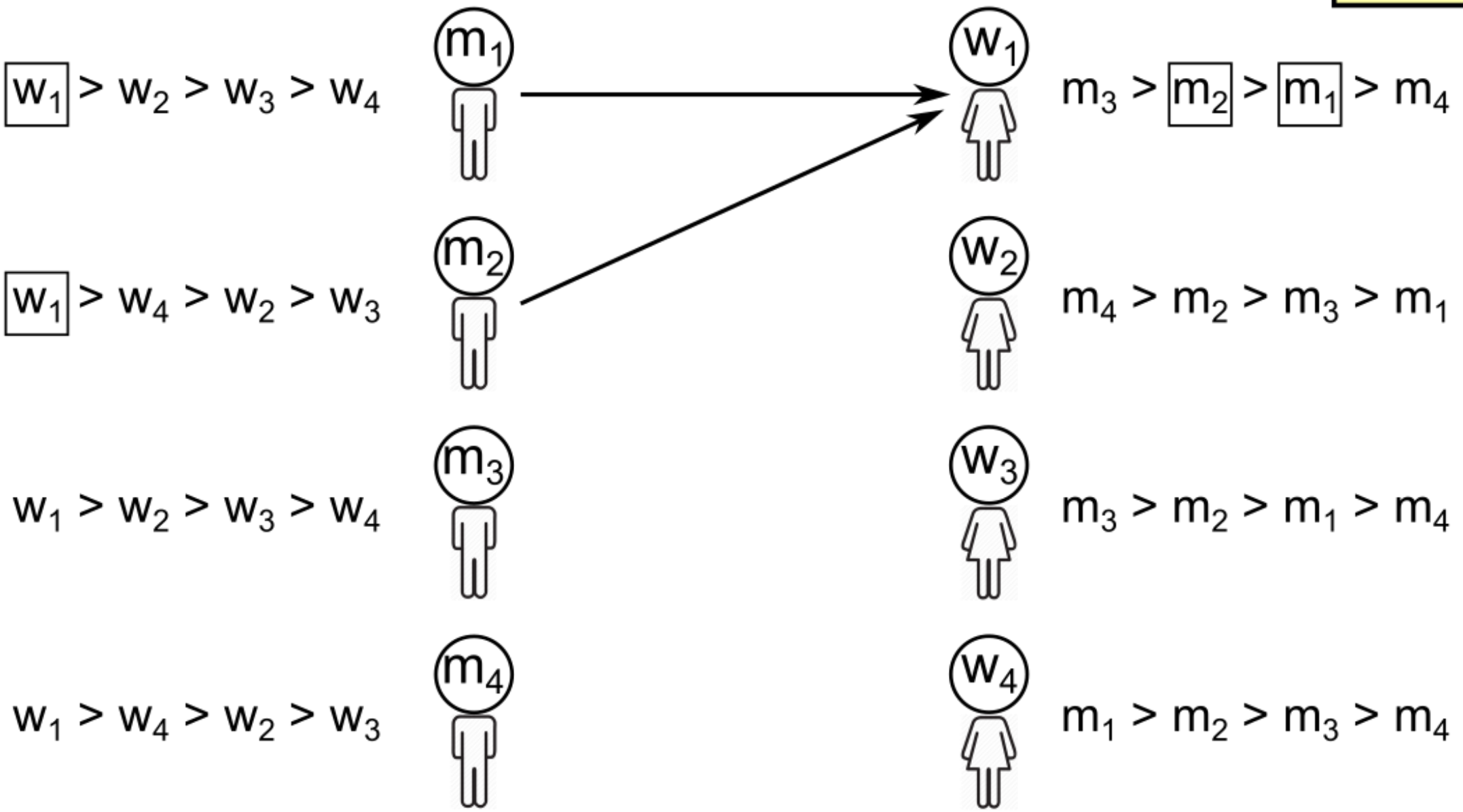
Deferred-Acceptance Algorithm

Round 1



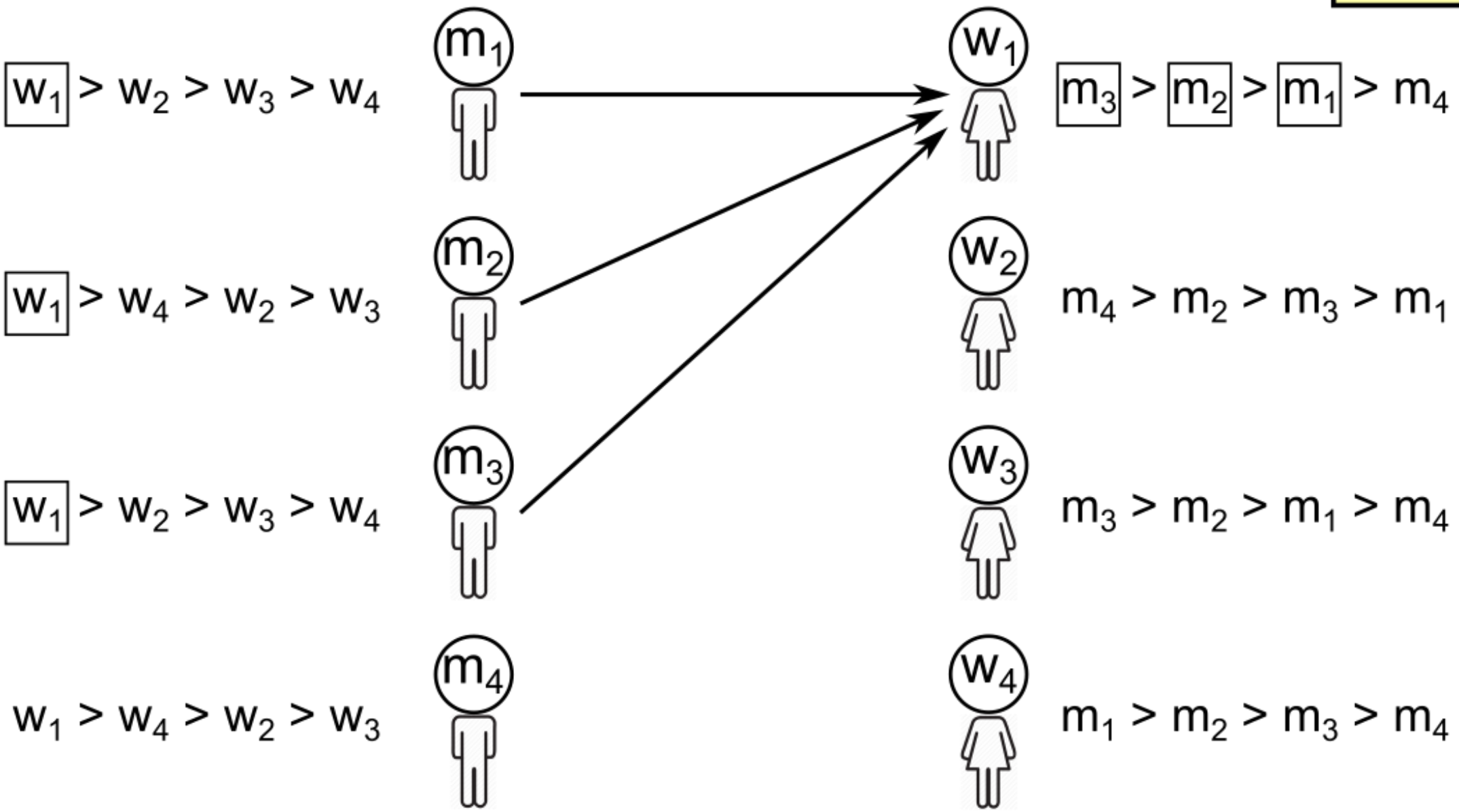
Deferred-Acceptance Algorithm

Round 1



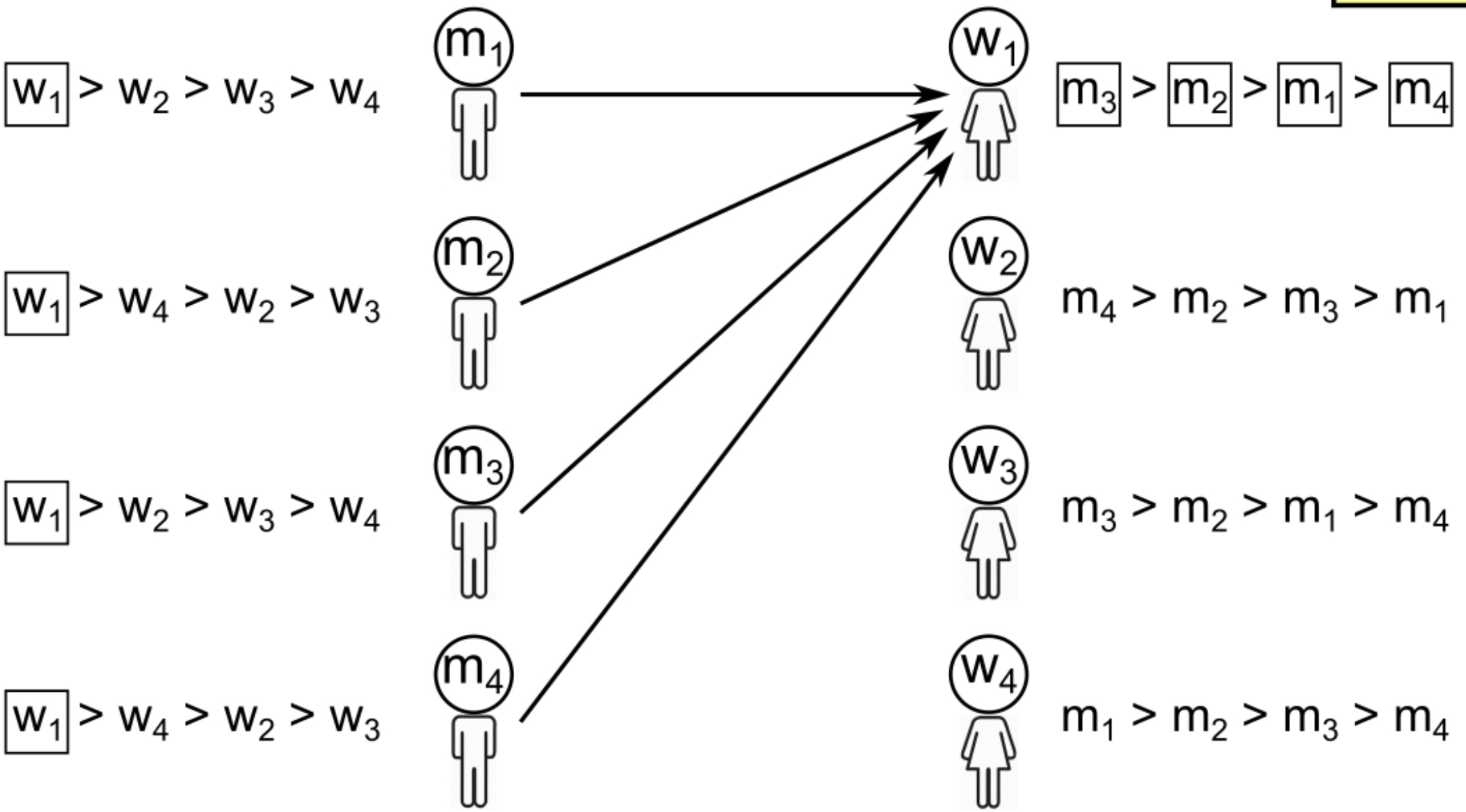
Deferred-Acceptance Algorithm

Round 1



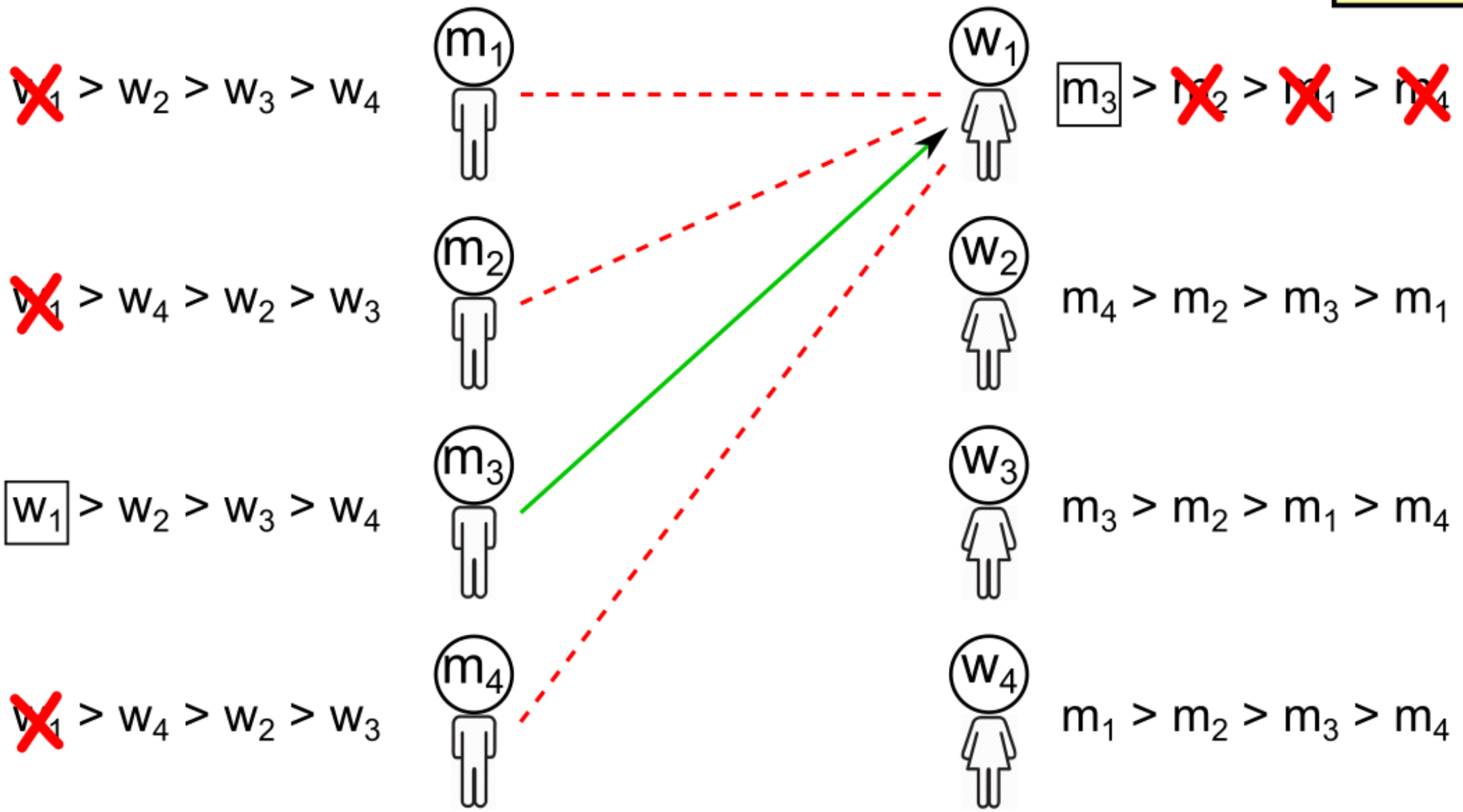
Deferred-Acceptance Algorithm

Round 1



Deferred-Acceptance Algorithm

Round 1



Deferred-Acceptance Algorithm

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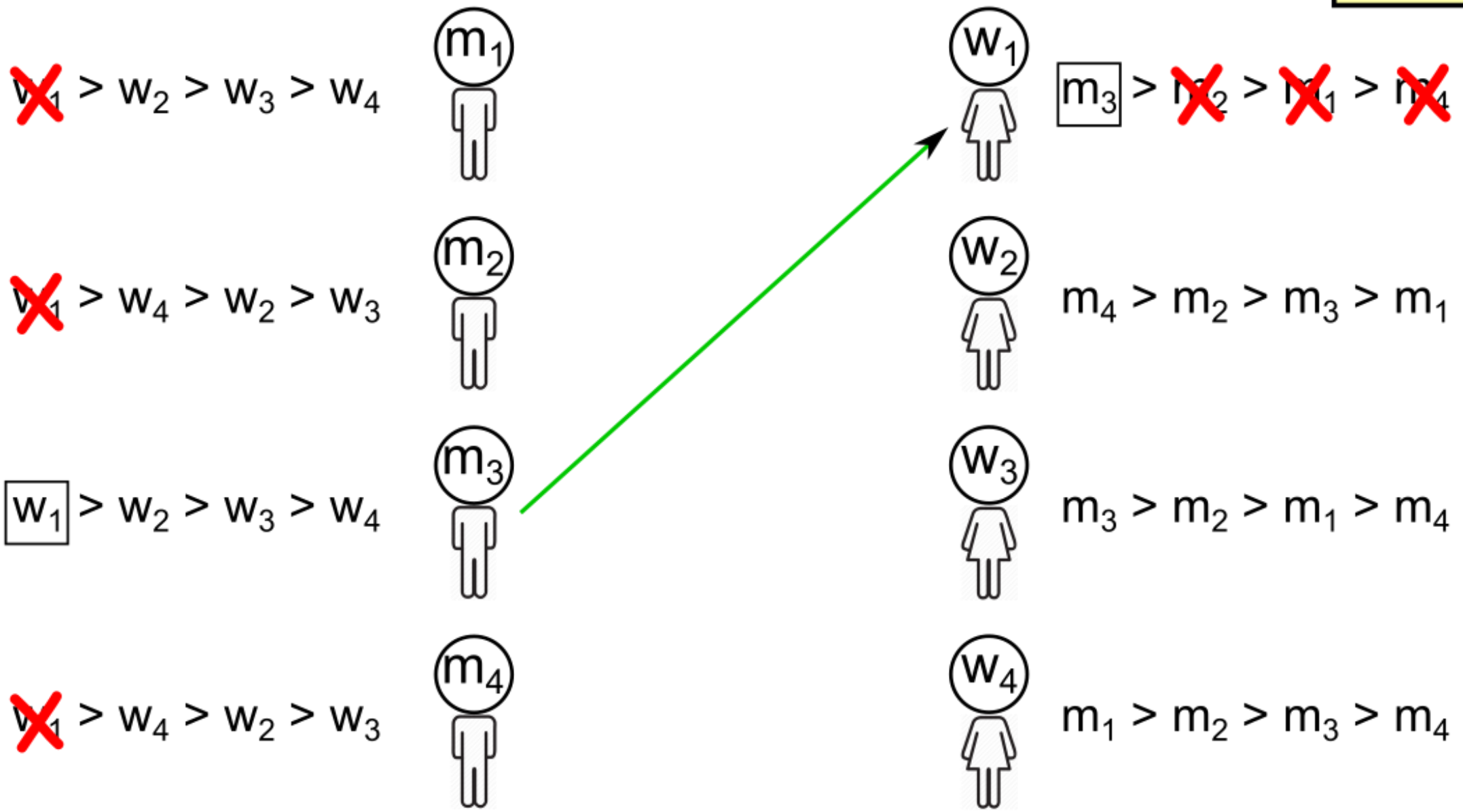
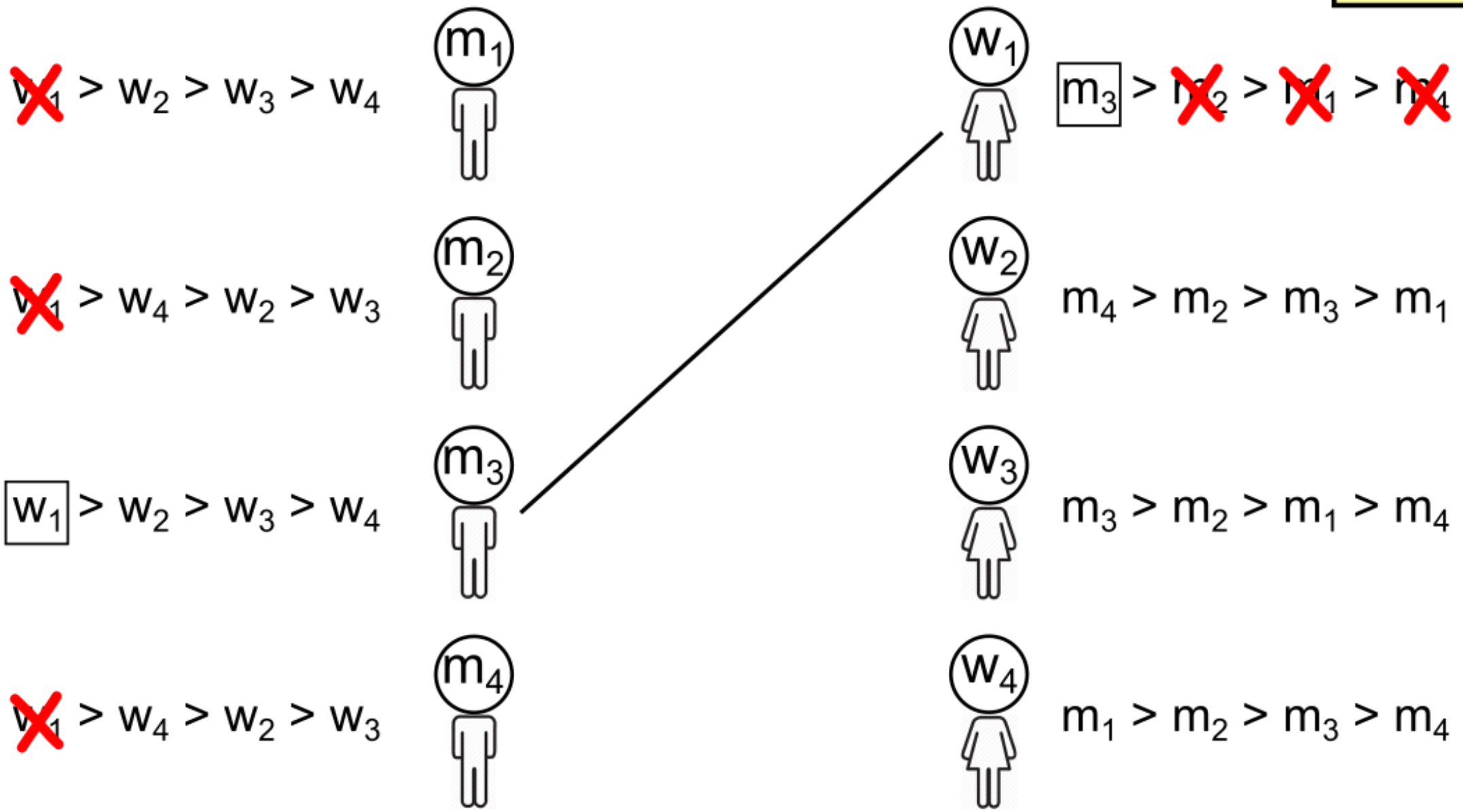


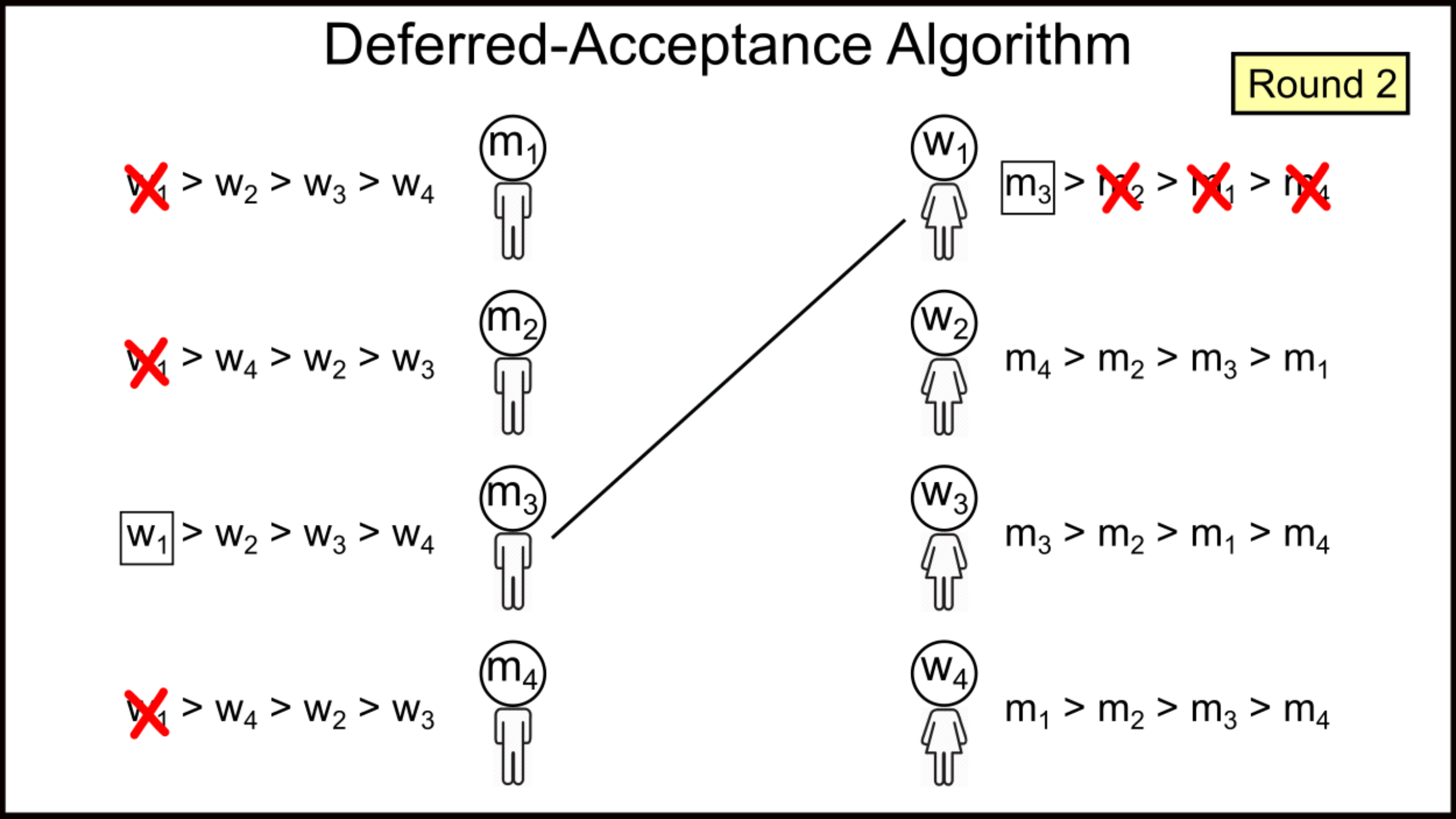
Deferred-Acceptance Algorithm

Round 1



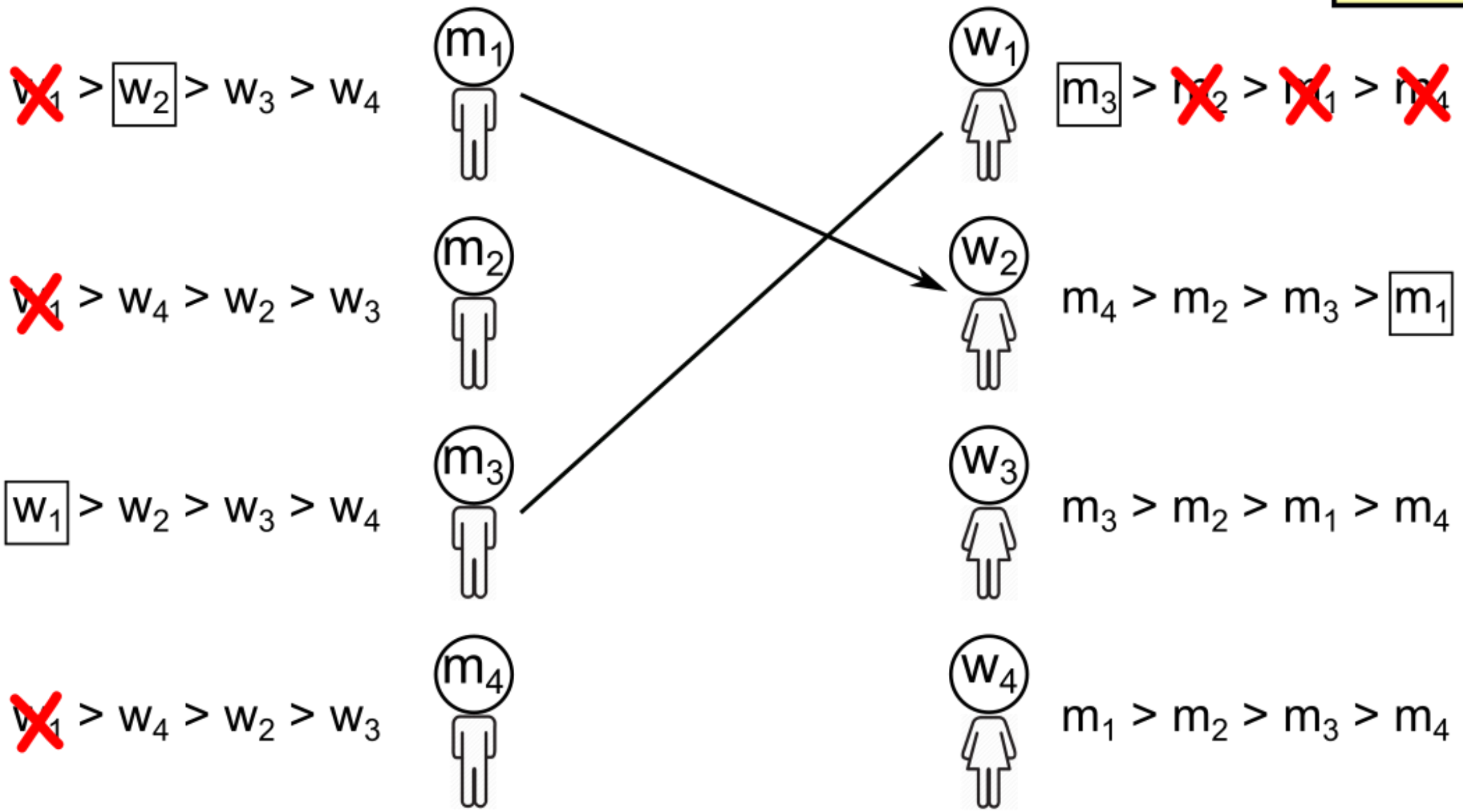
Deferred-Acceptance Algorithm

Round 2



Deferred-Acceptance Algorithm

Round 2



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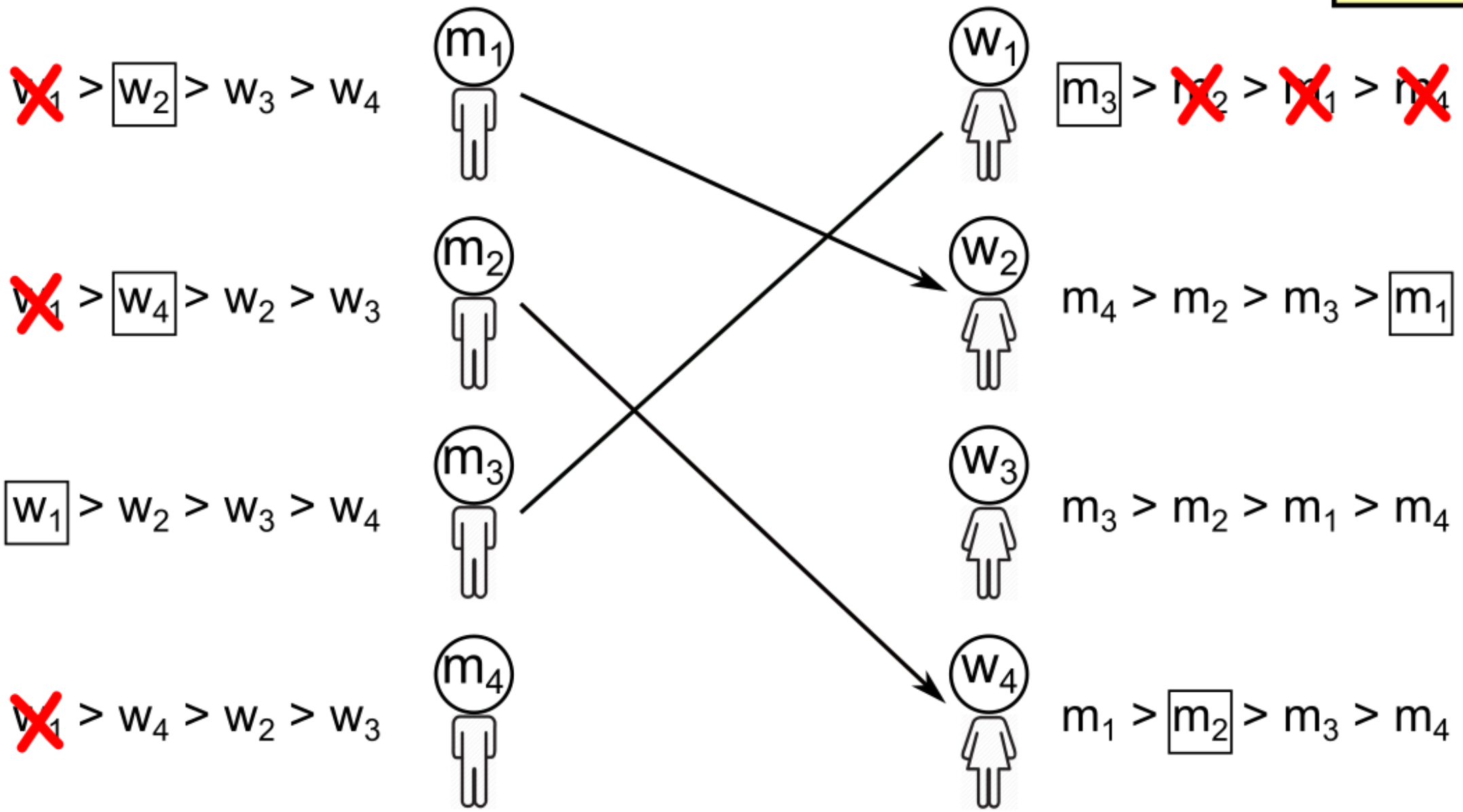

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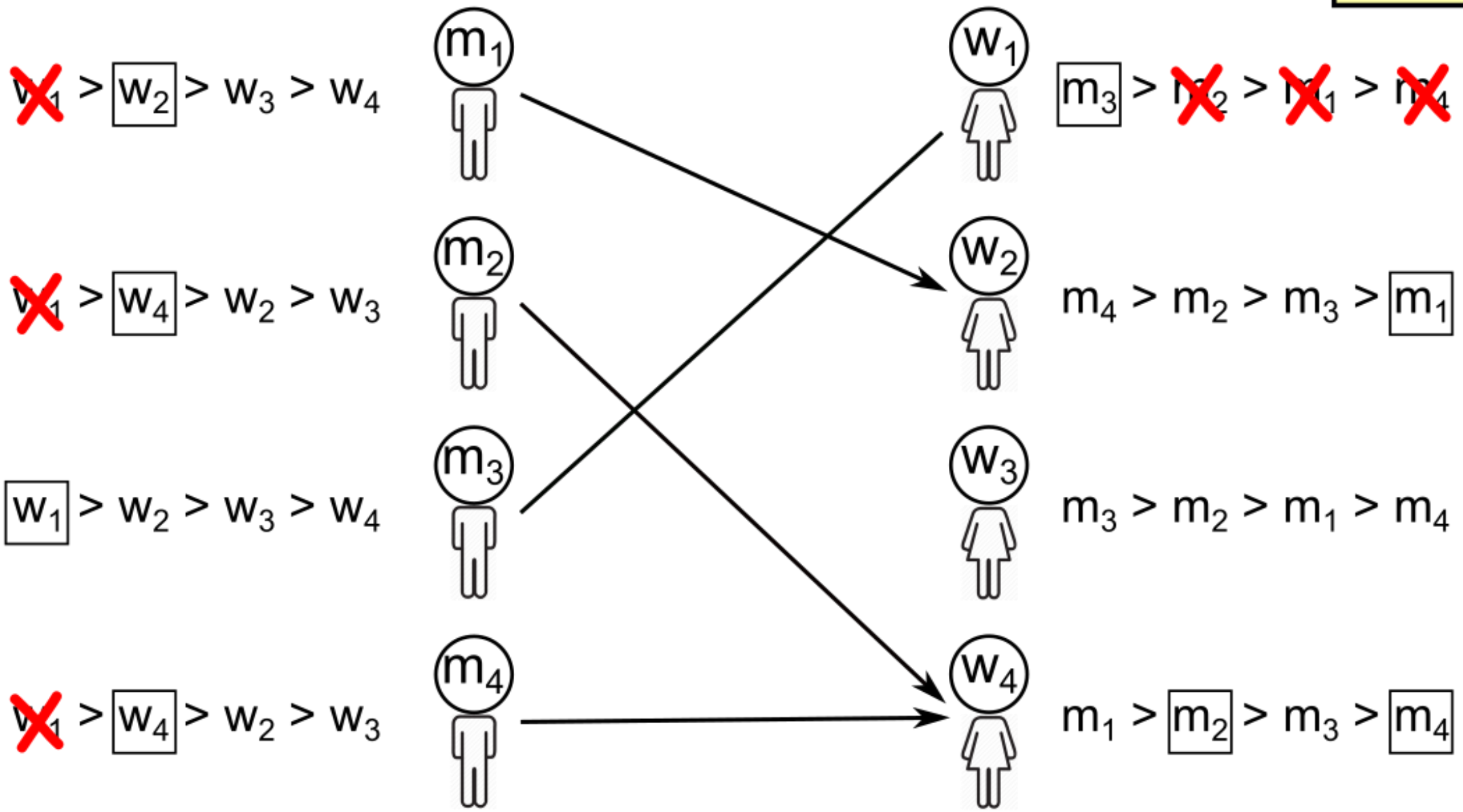

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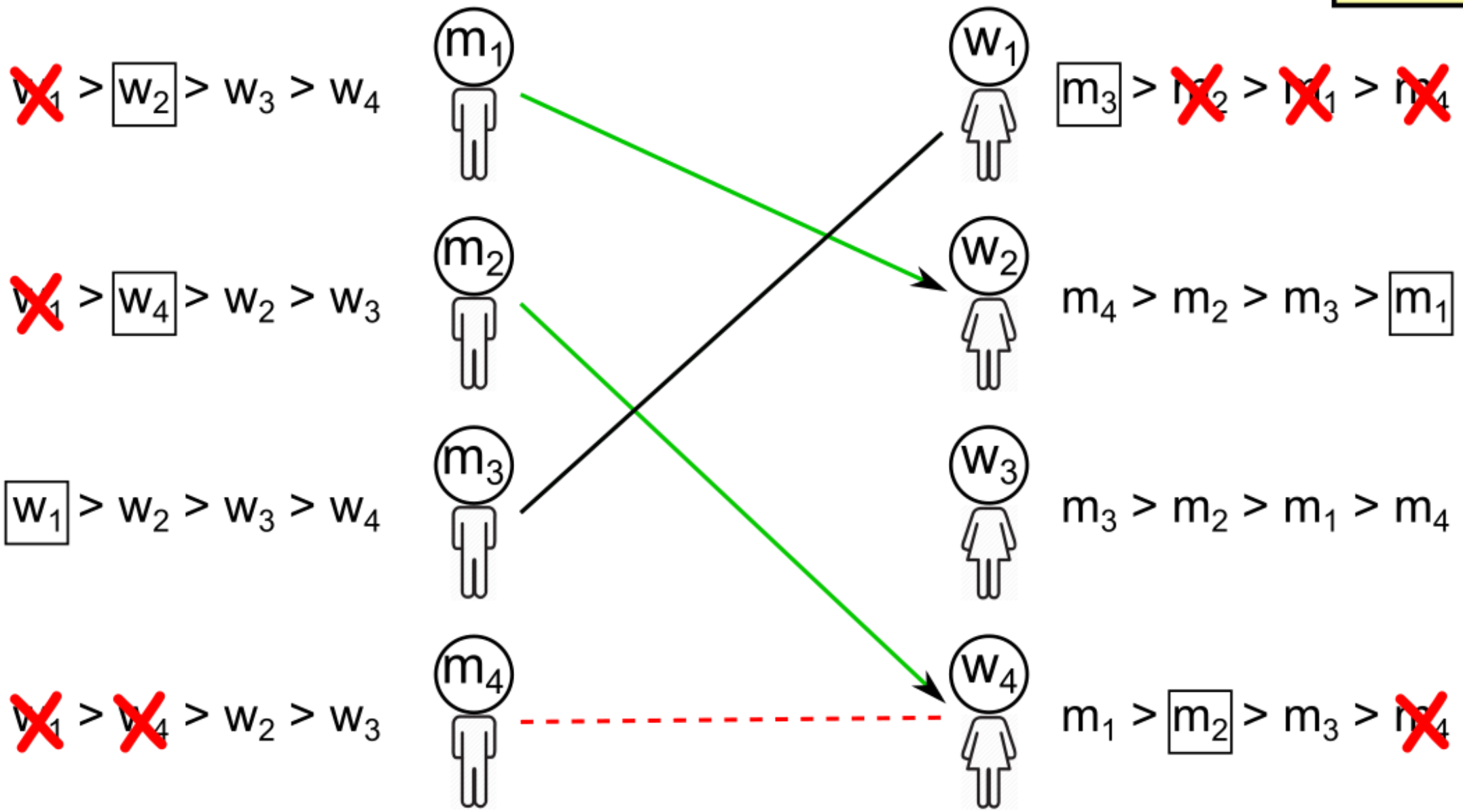

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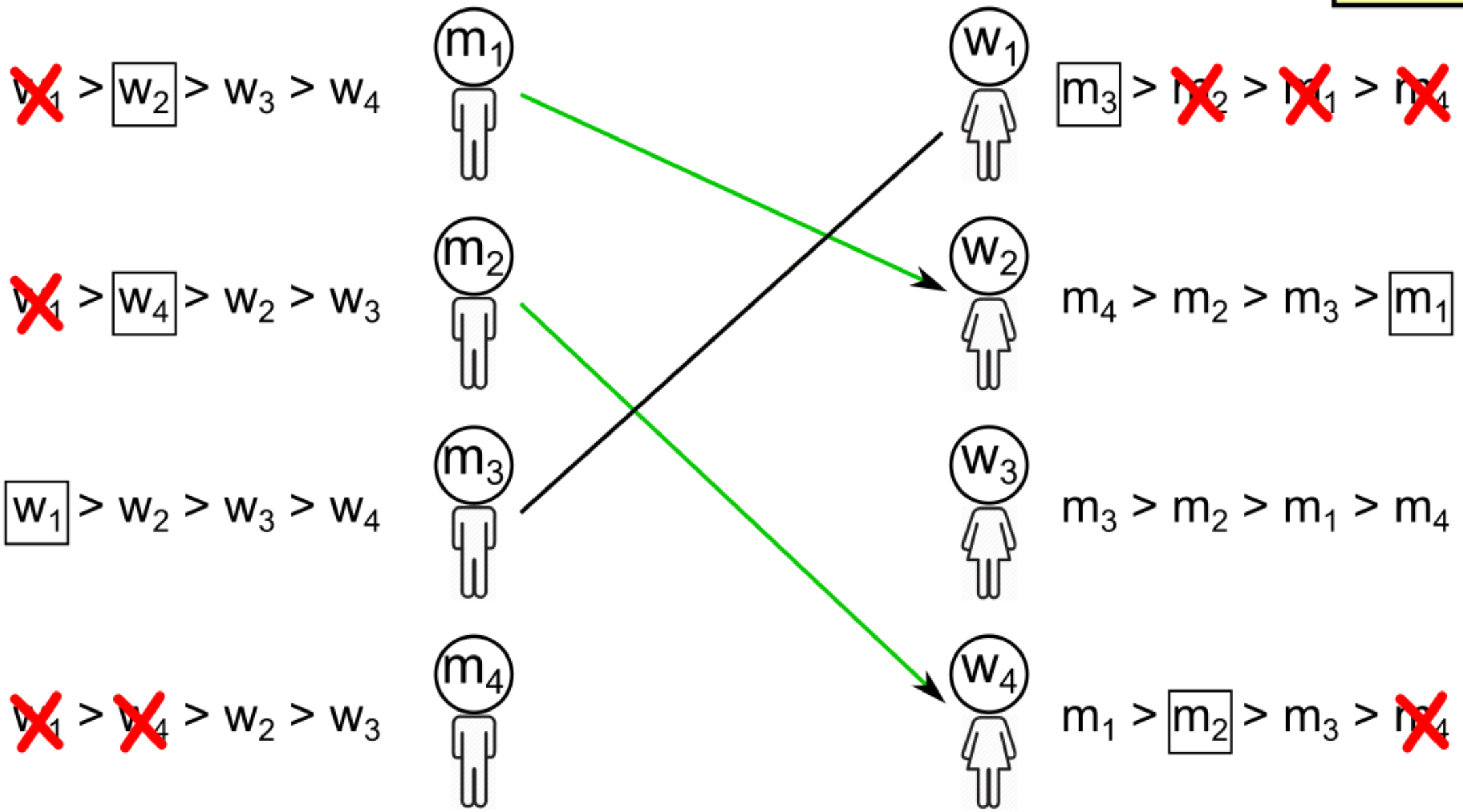

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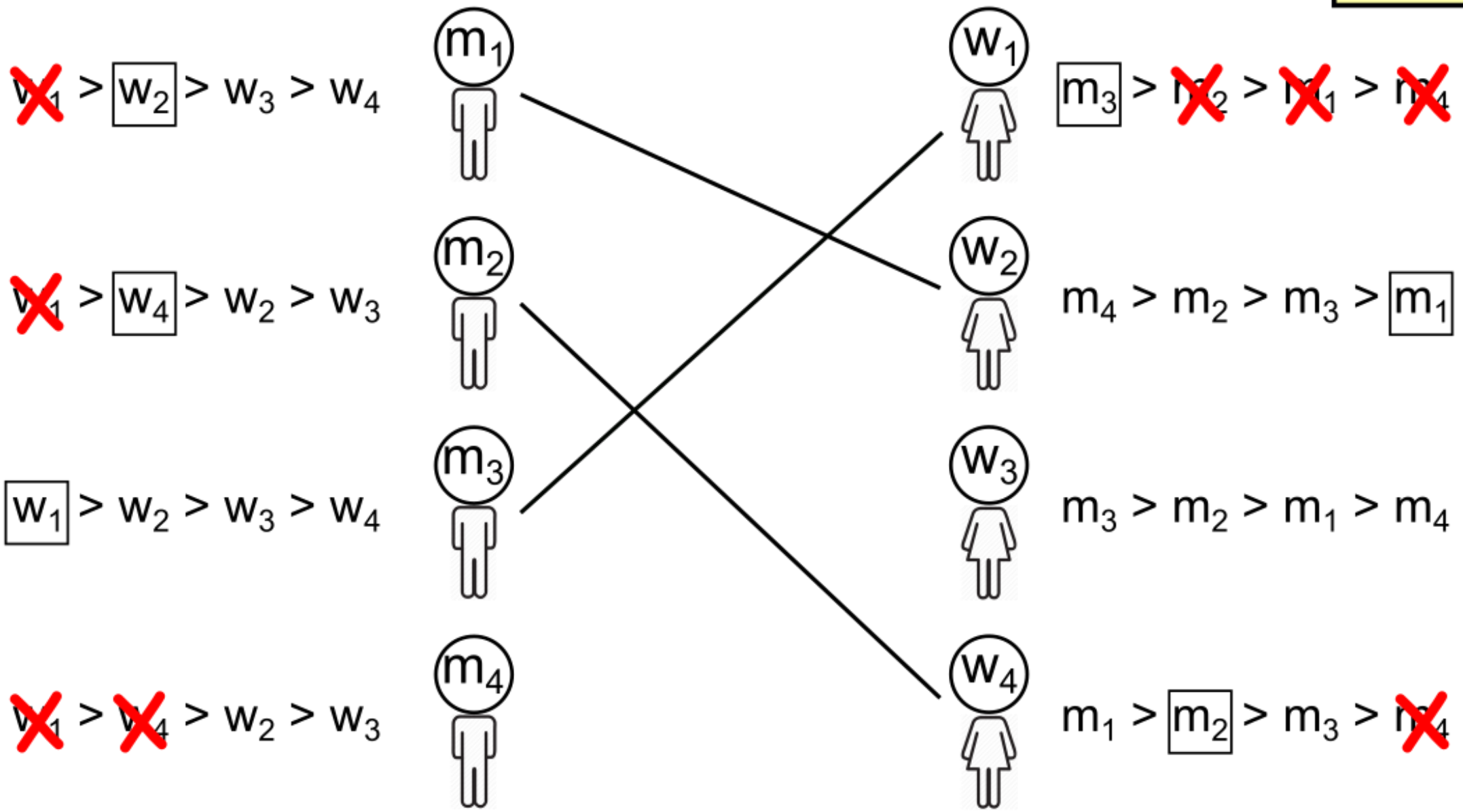

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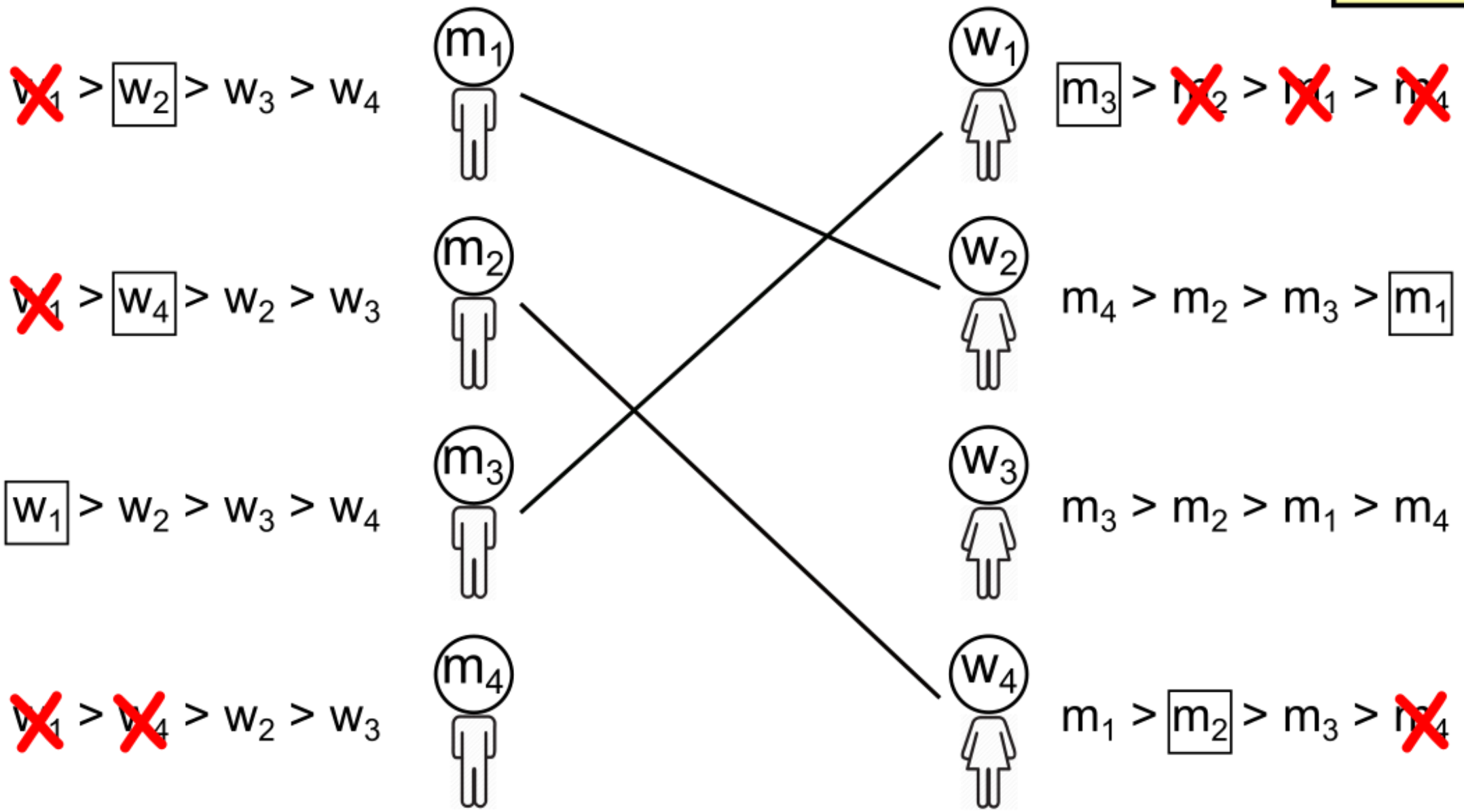

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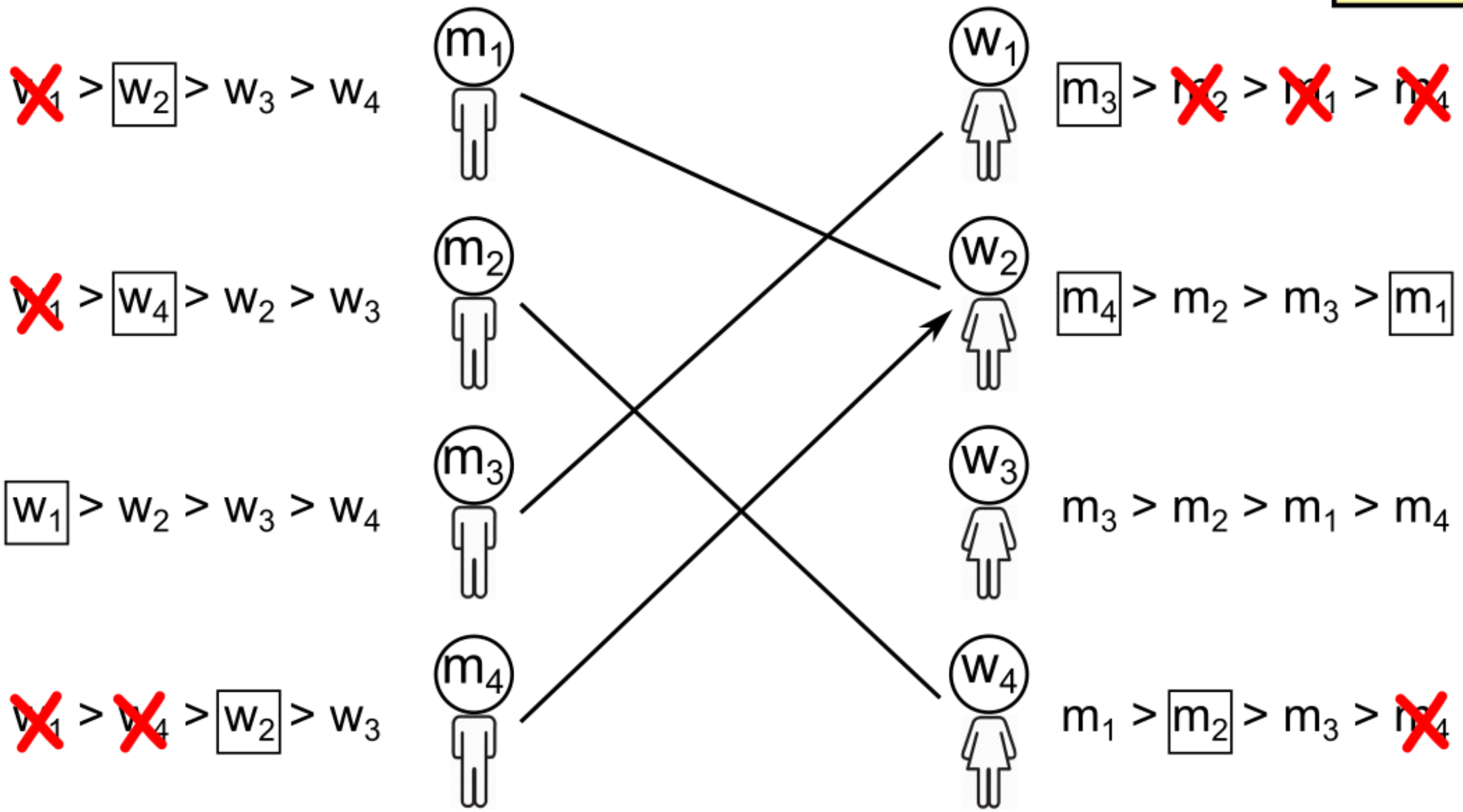

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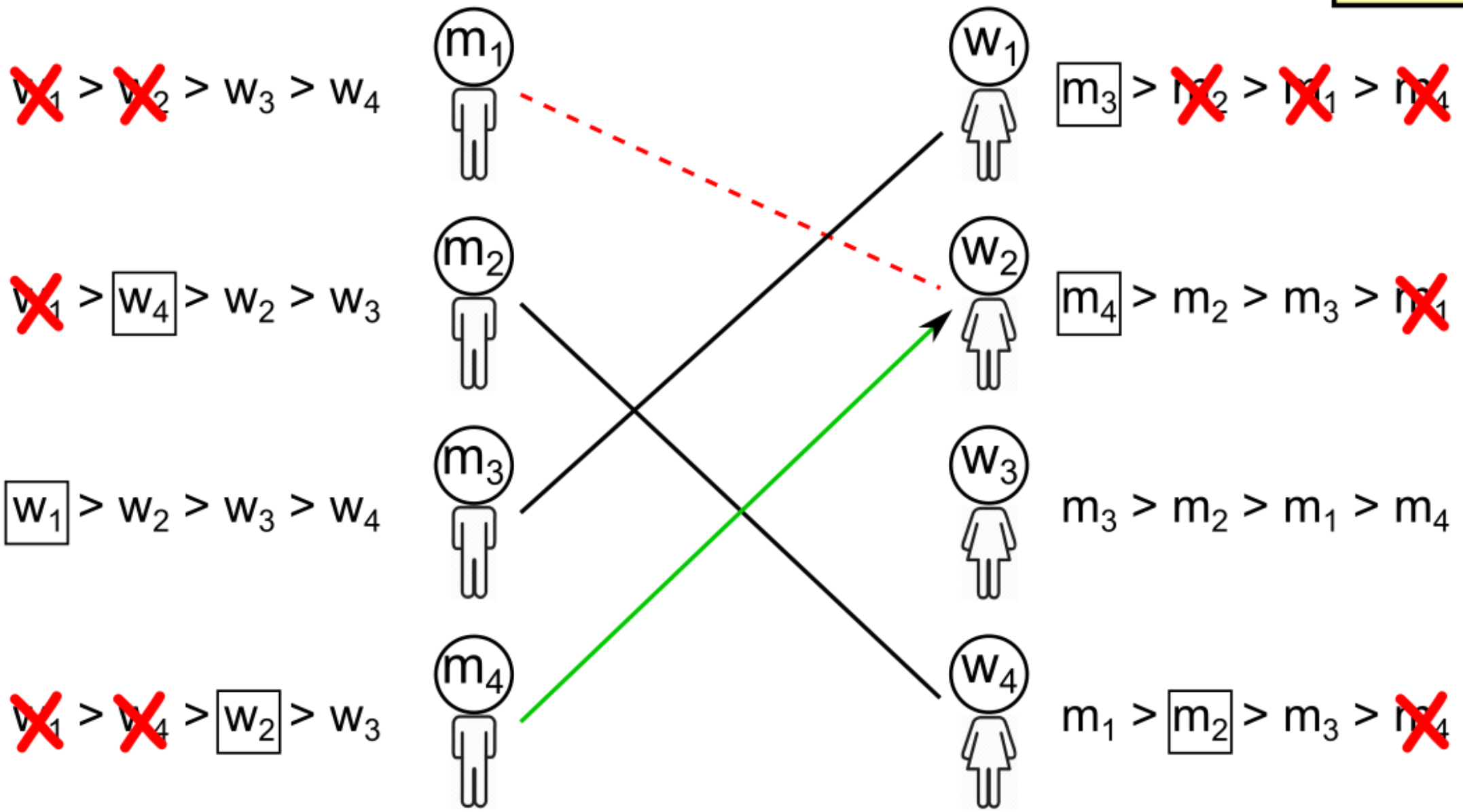

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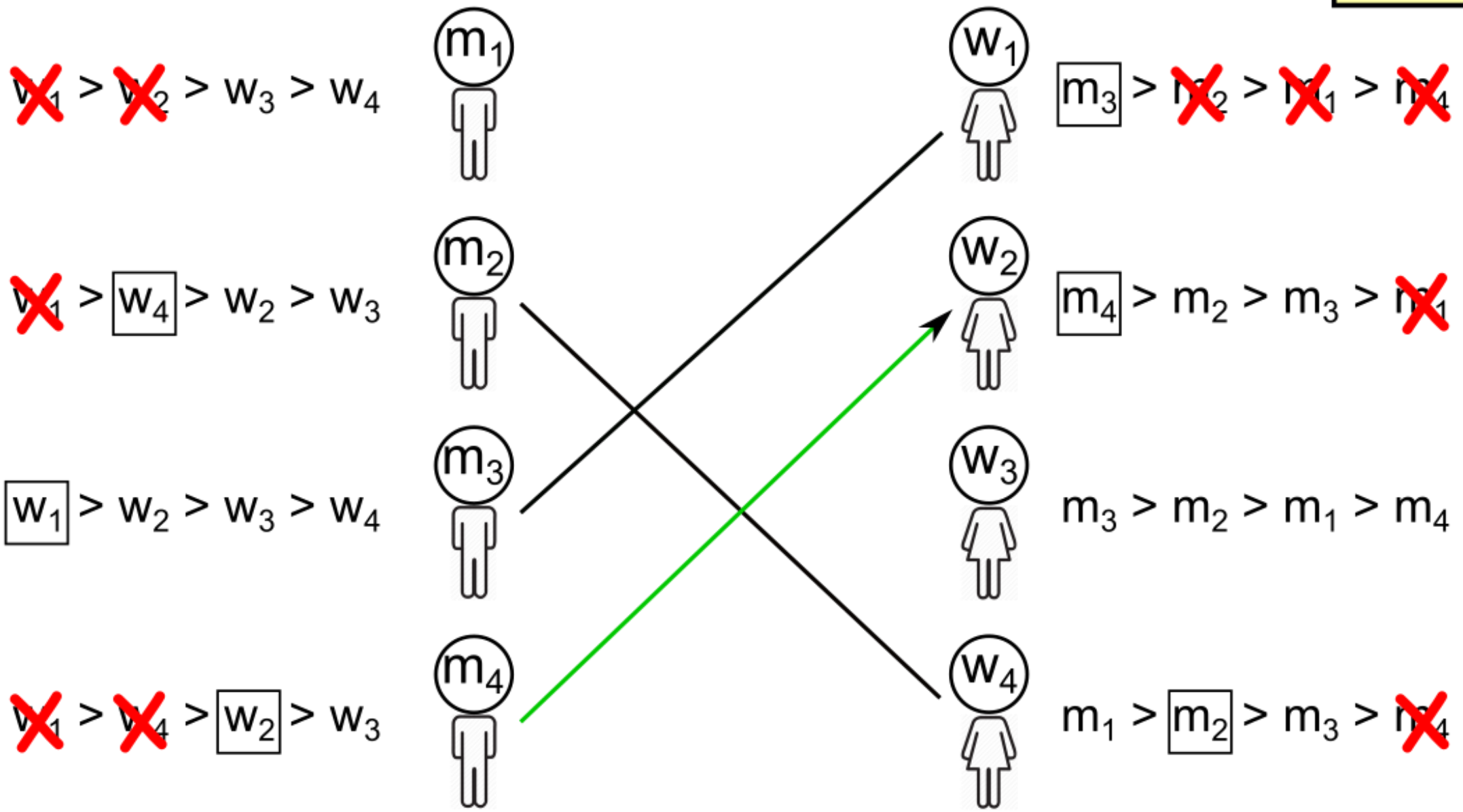

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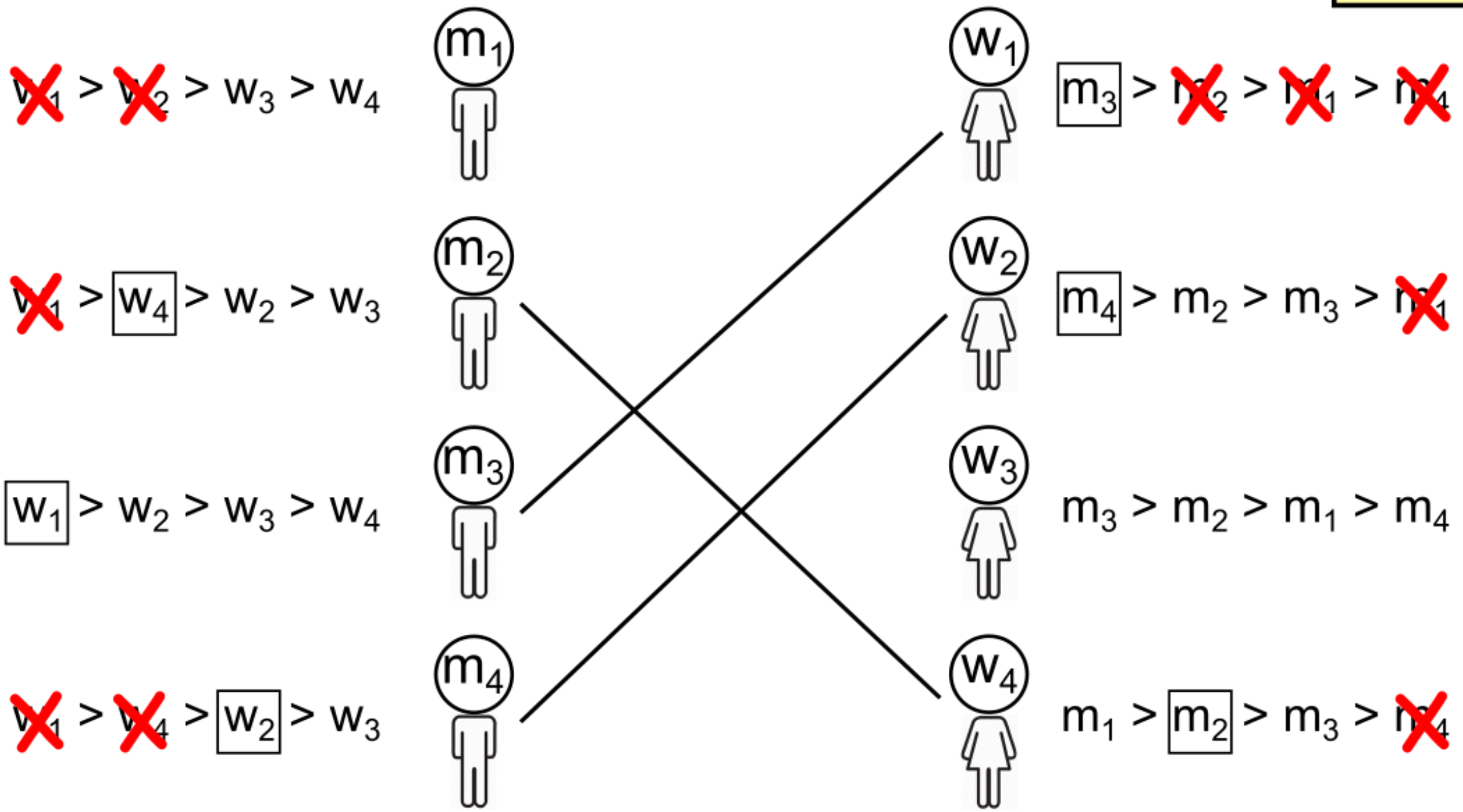
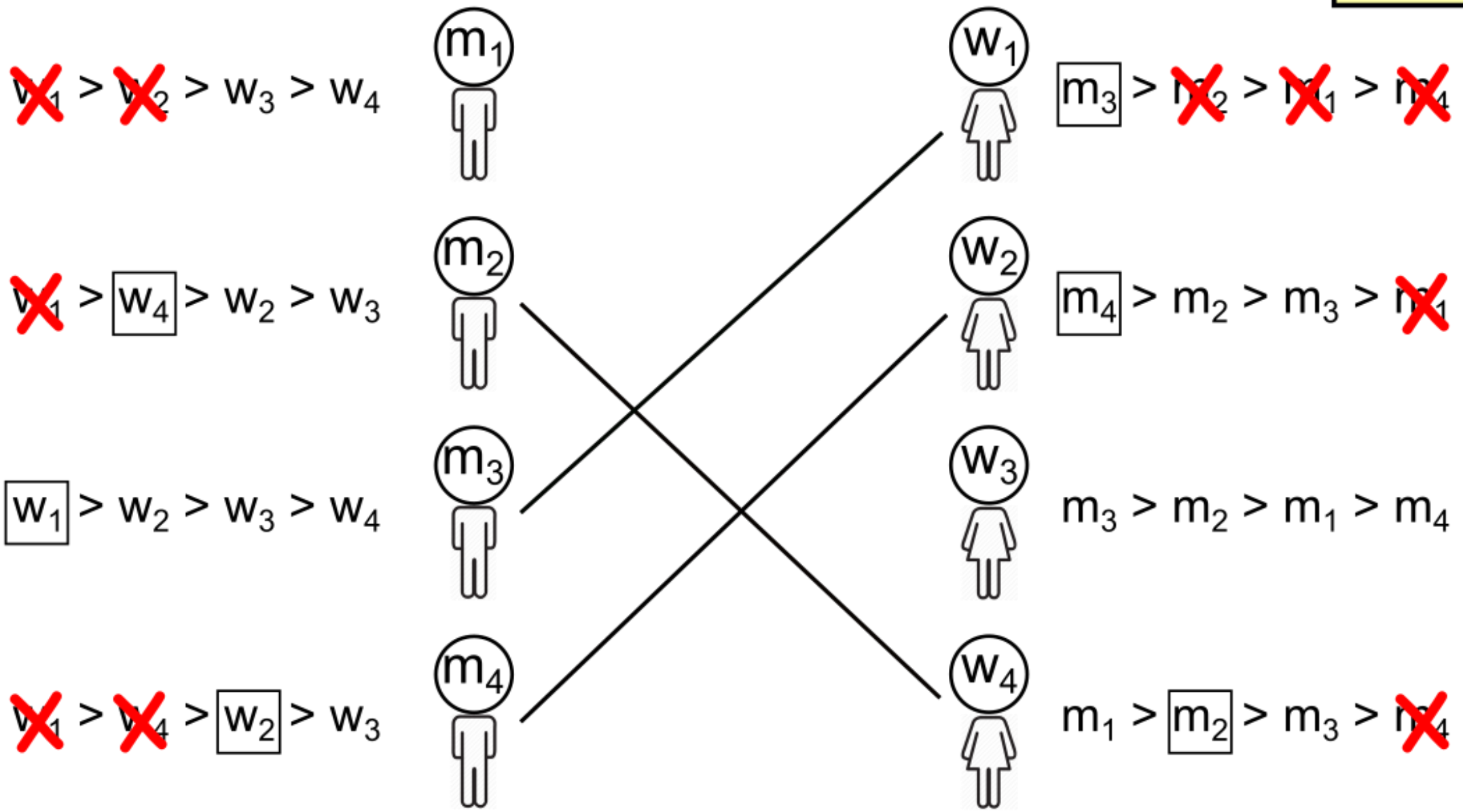


Deferred-Acceptance Algorithm

Round 4



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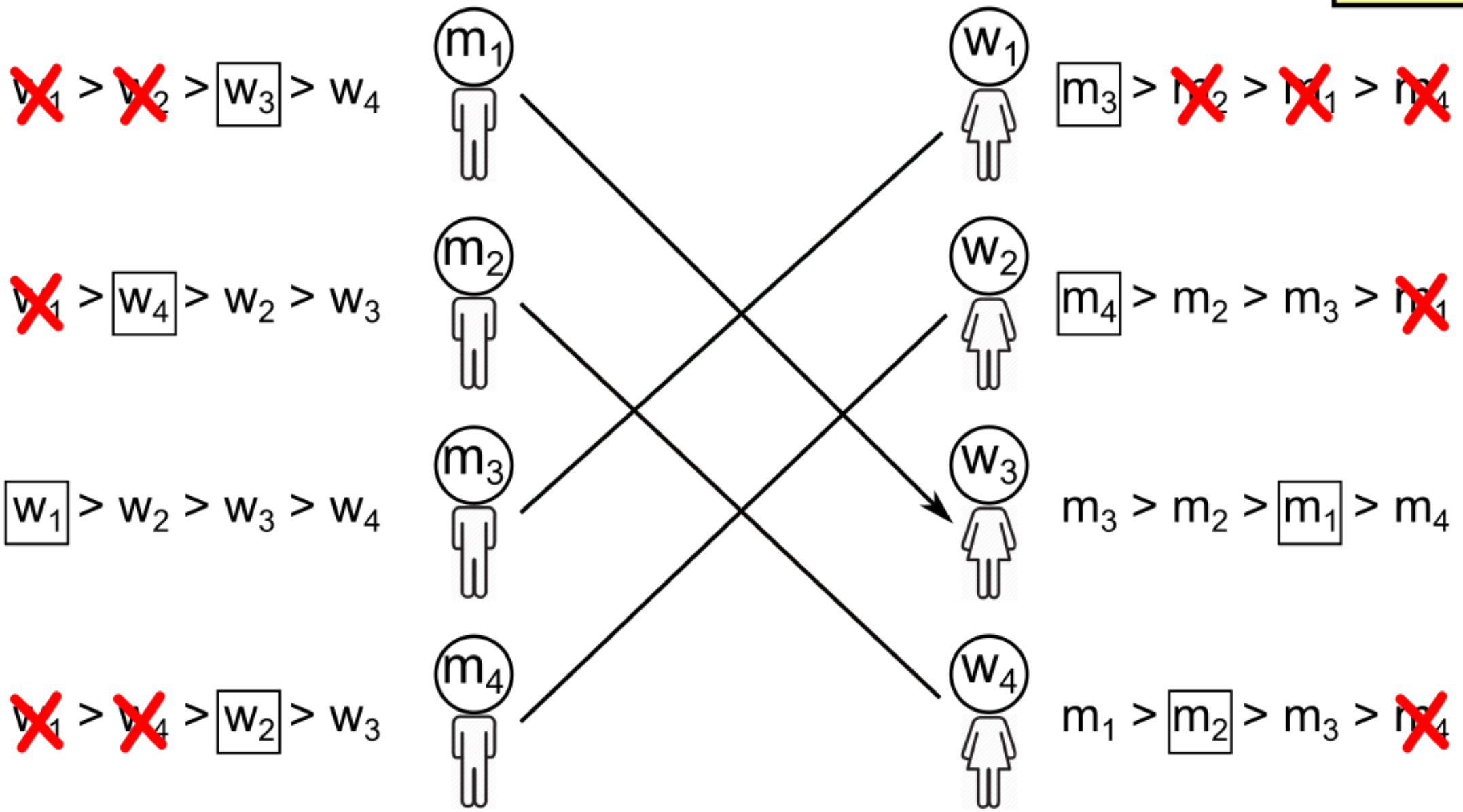

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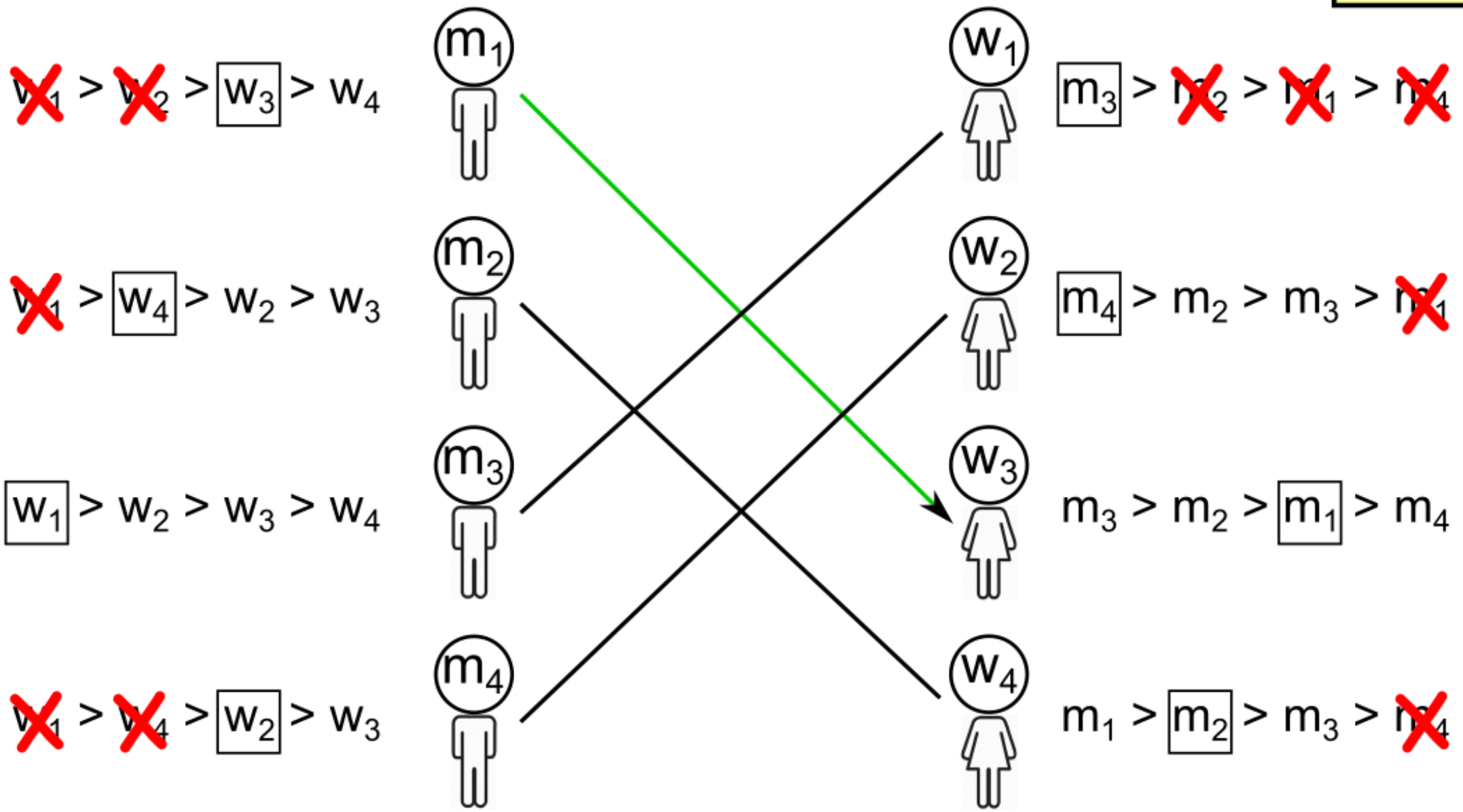

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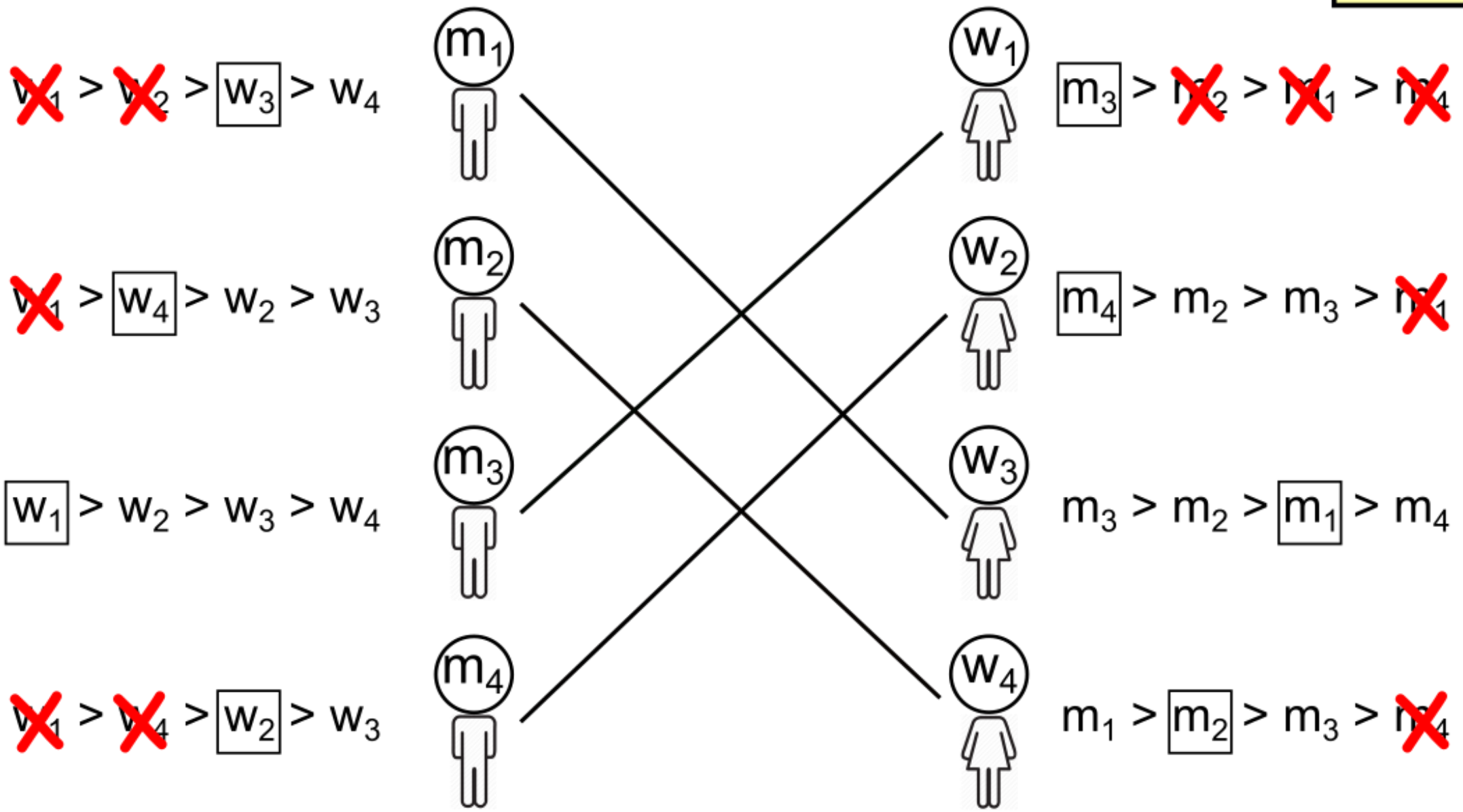

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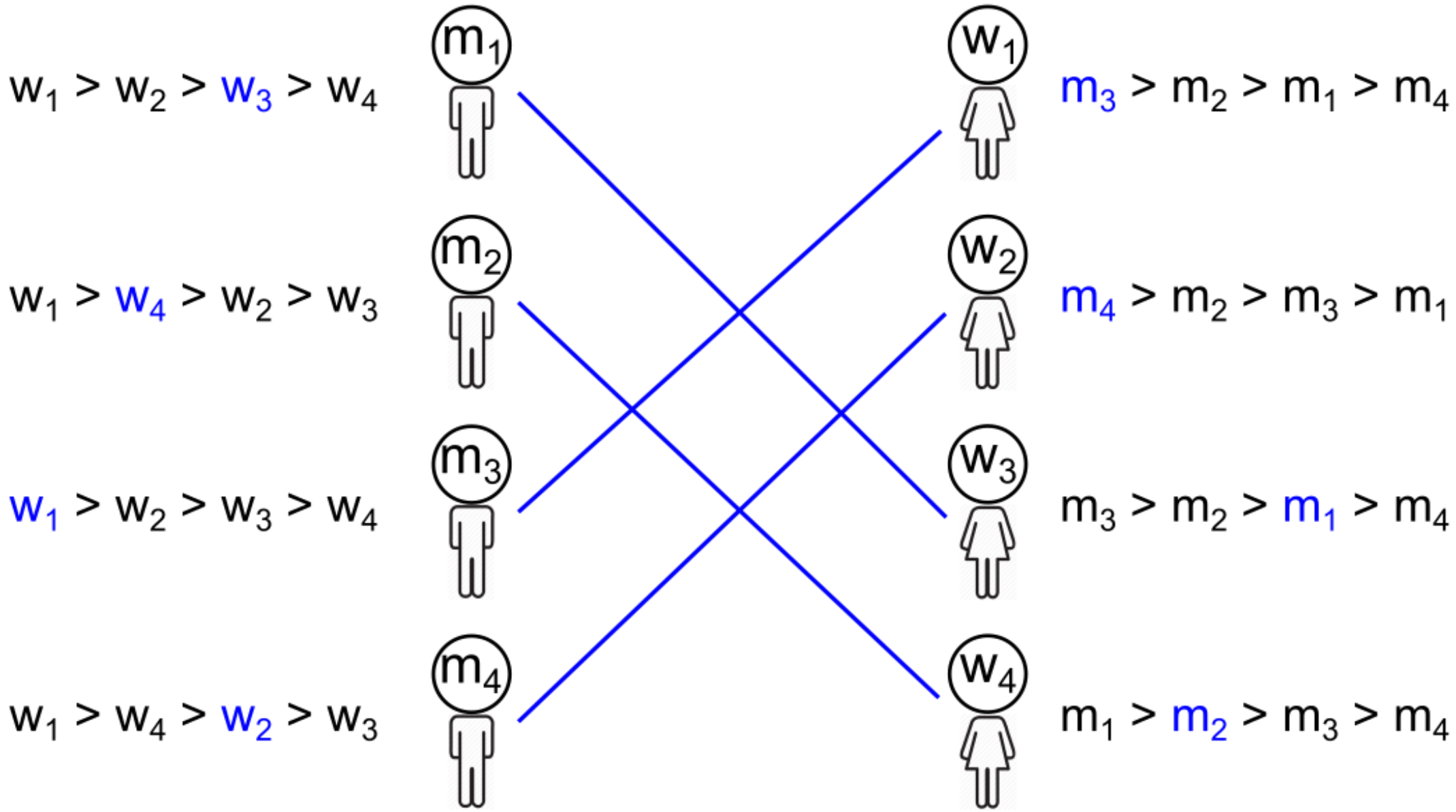

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Deferred-acceptance algorithm terminates in polynomial time.

2 Does the deferred-acceptance algorithm always find a perfect matching?

2

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Once tentatively matched, a woman never becomes unmatched.

3 Does the deferred-acceptance algorithm always find a stable matching?

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Men make proposals in decreasing order of their preference.
So, m must have proposed to (and been rejected by) w.

Then, w must have received a better-than-m proposal in some round.

Women only "trade up" during the DA algorithm.

Applications


HOSPITAL


UNHCR

Trends in Computational Social Choice

CHAPTER 18

Applications of Matching Models under Preferences

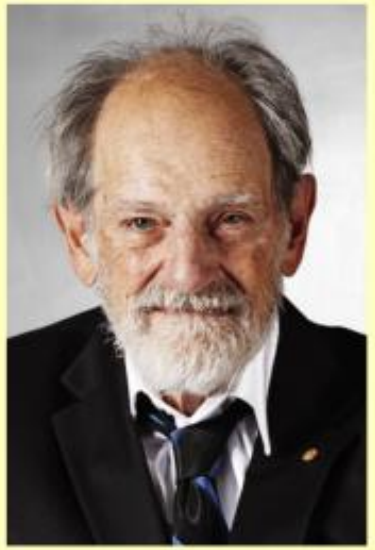
Péter Biró

18.1 Introduction

Matching problems under preferences have been studied widely in mathematics, computer science and economics, starting with the seminal paper by Gale and Shapley (1962). A comprehensive survey on this topic was published also in Chapter 14 of the Handbook of Computational Social Choice (Klaus et al., 2016), and for the interested reader we recommend consulting the following four comprehensive books on the computational (Gusfield and Irving, 1989; Manlove, 2013) and game-theoretical, market design aspects (Roth and Sotomayor, 1990; Roth, 2015) of this topic. In this chapter our goal is to give a general overview of the related applications.

Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel 2012

Alvin E. Roth



Lloyd S. Shapley

"*for the theory of stable allocations and the practice of market design*."

Structure of the Set of Stable Matchings

$w_4 > w_1 > w_2 > w_3$

w_1

$m_2 > m_1 > m_4 > m_3$

$w_3 > w_2 > w_4 > w_1$

w_2

$m_1 > m_2 > m_3 > m_4$

$w_1 > w_2 > w_3 > w_4$

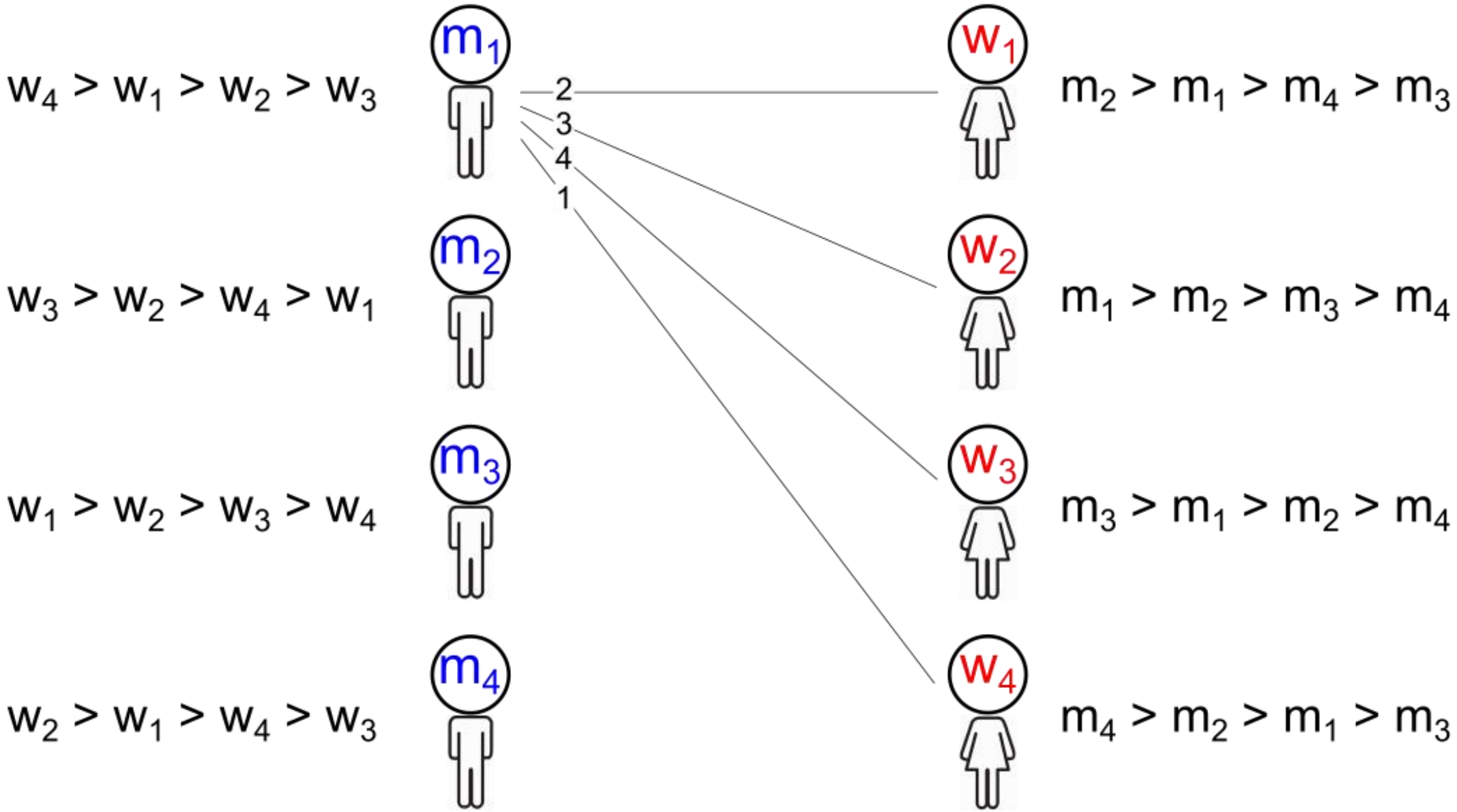
w_3

$m_3 > m_1 > m_2 > m_4$

$w_2 > w_1 > w_4 > w_3$

w_4

$m_4 > m_2 > m_1 > m_3$

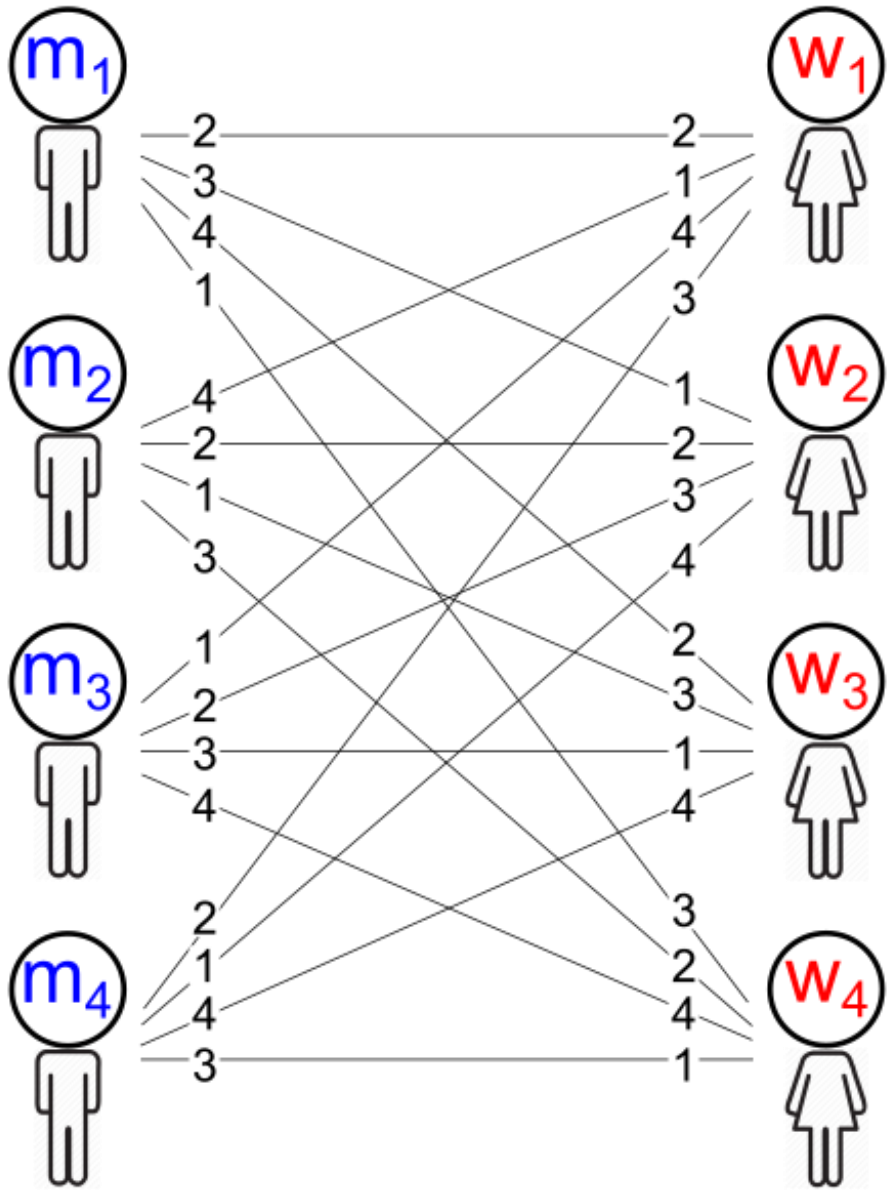

$w_4 > w_1 > w_2 > w_3$
m_1
2
3
4
1
w_1
$m_2 > m_1 > m_4 > m_3$
$w_3 > w_2 > w_4 > w_1$
m_2
w_2
$m_1 > m_2 > m_3 > m_4$
$w_1 > w_2 > w_3 > w_4$
m_3
w_3
$m_3 > m_1 > m_2 > m_4$
$w_2 > w_1 > w_4 > w_3$
m_4
w_4
$m_4 > m_2 > m_1 > m_3$

$w_4 > w_1 > w_2 > w_3$

$w_3 > w_2 > w_4 > w_1$

$w_1 > w_2 > w_3 > w_4$

$w_2 > w_1 > w_4 > w_3$

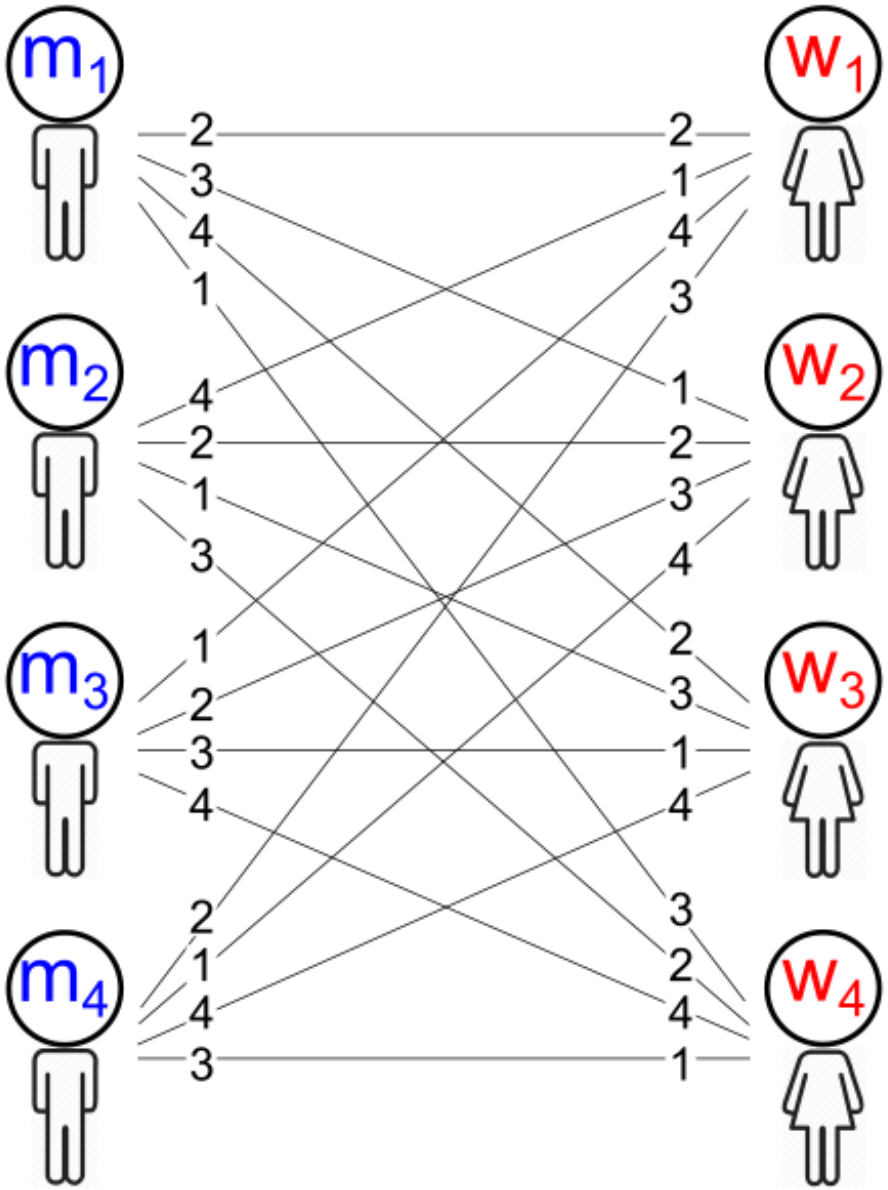


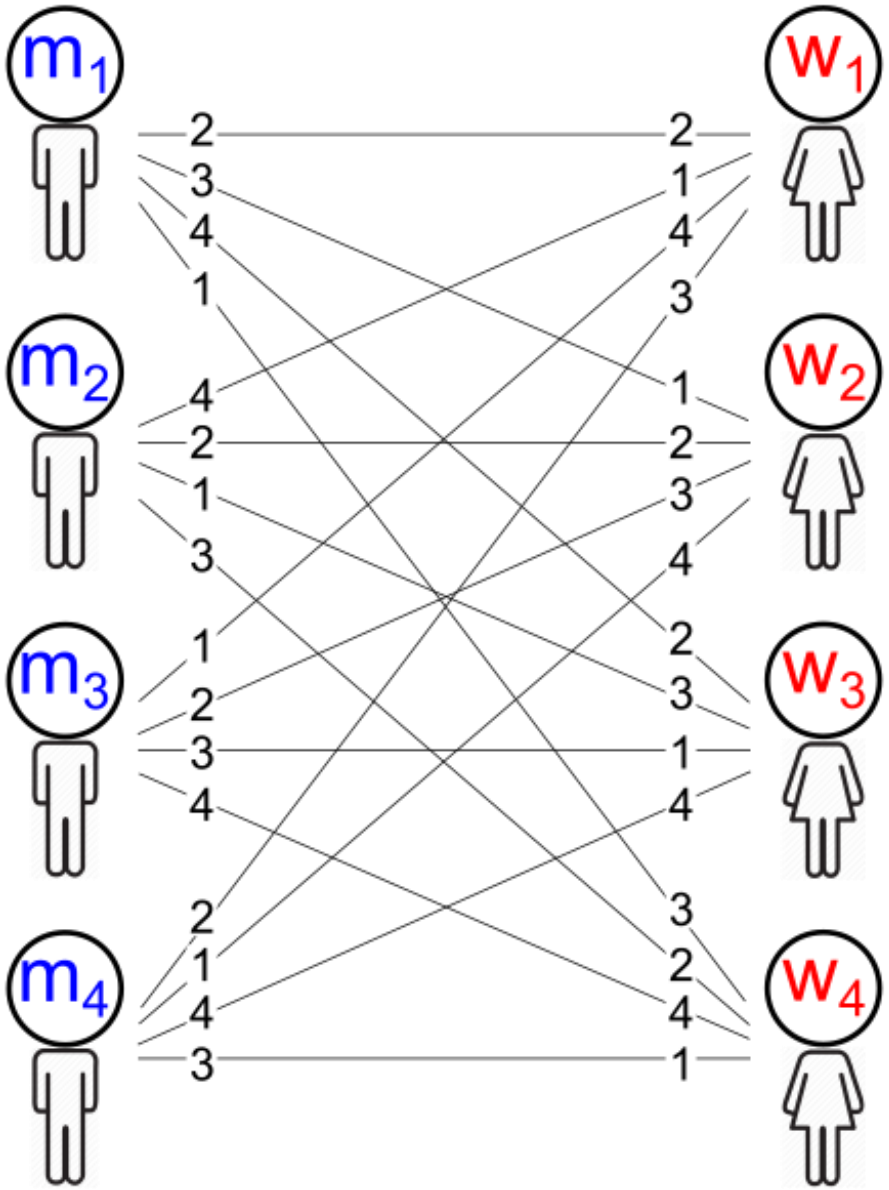
$m_2 > m_1 > m_4 > m_3$

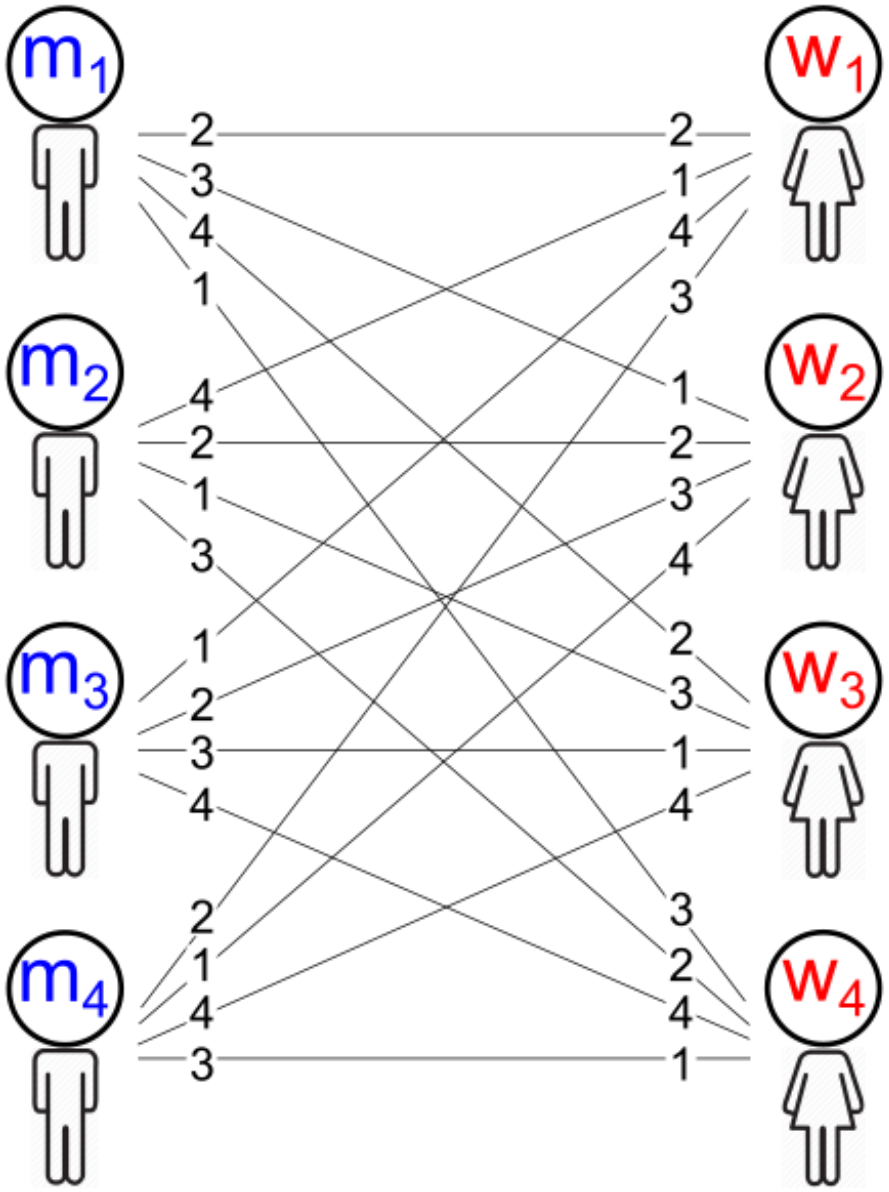
$m_1 > m_2 > m_3 > m_4$

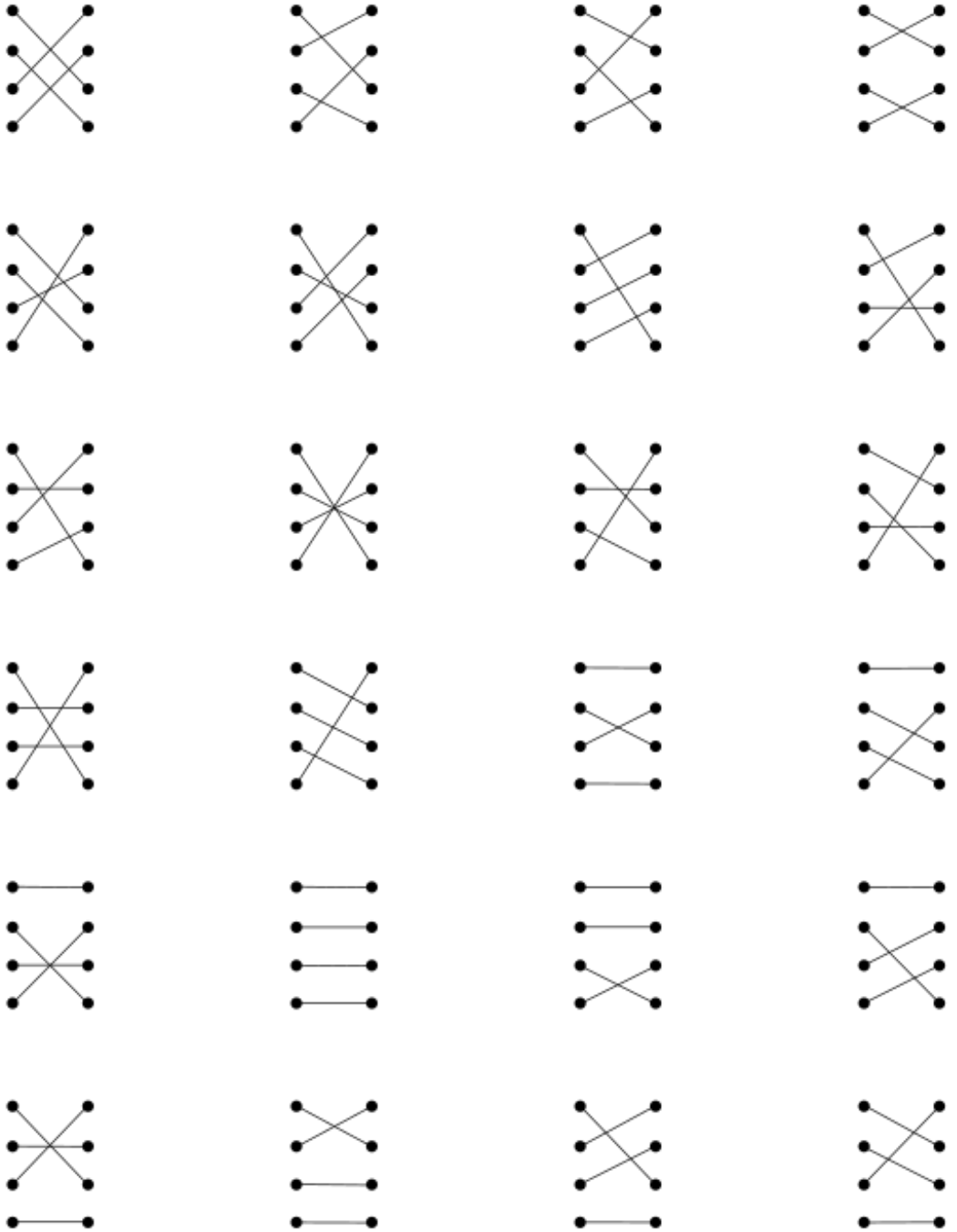
$m_3 > m_1 > m_2 > m_4$

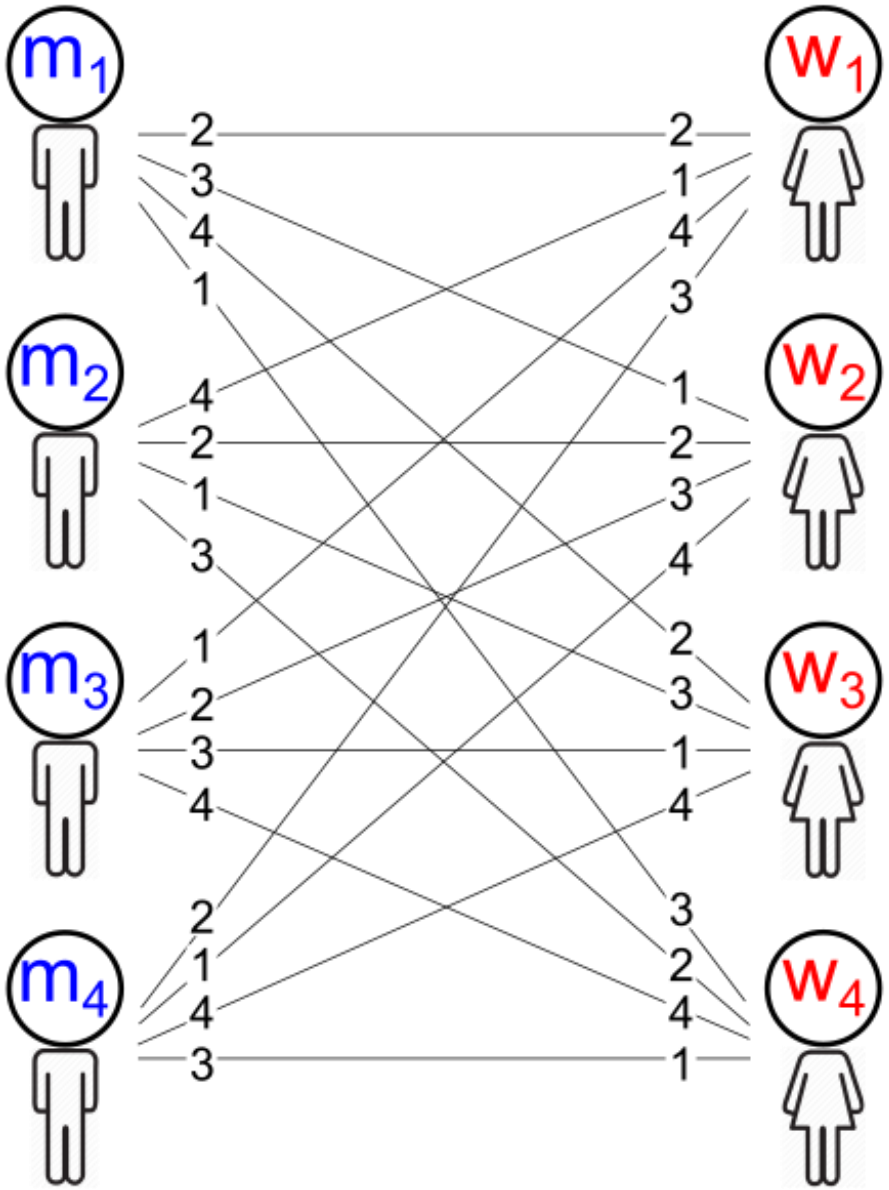
$m_4 > m_2 > m_1 > m_3$

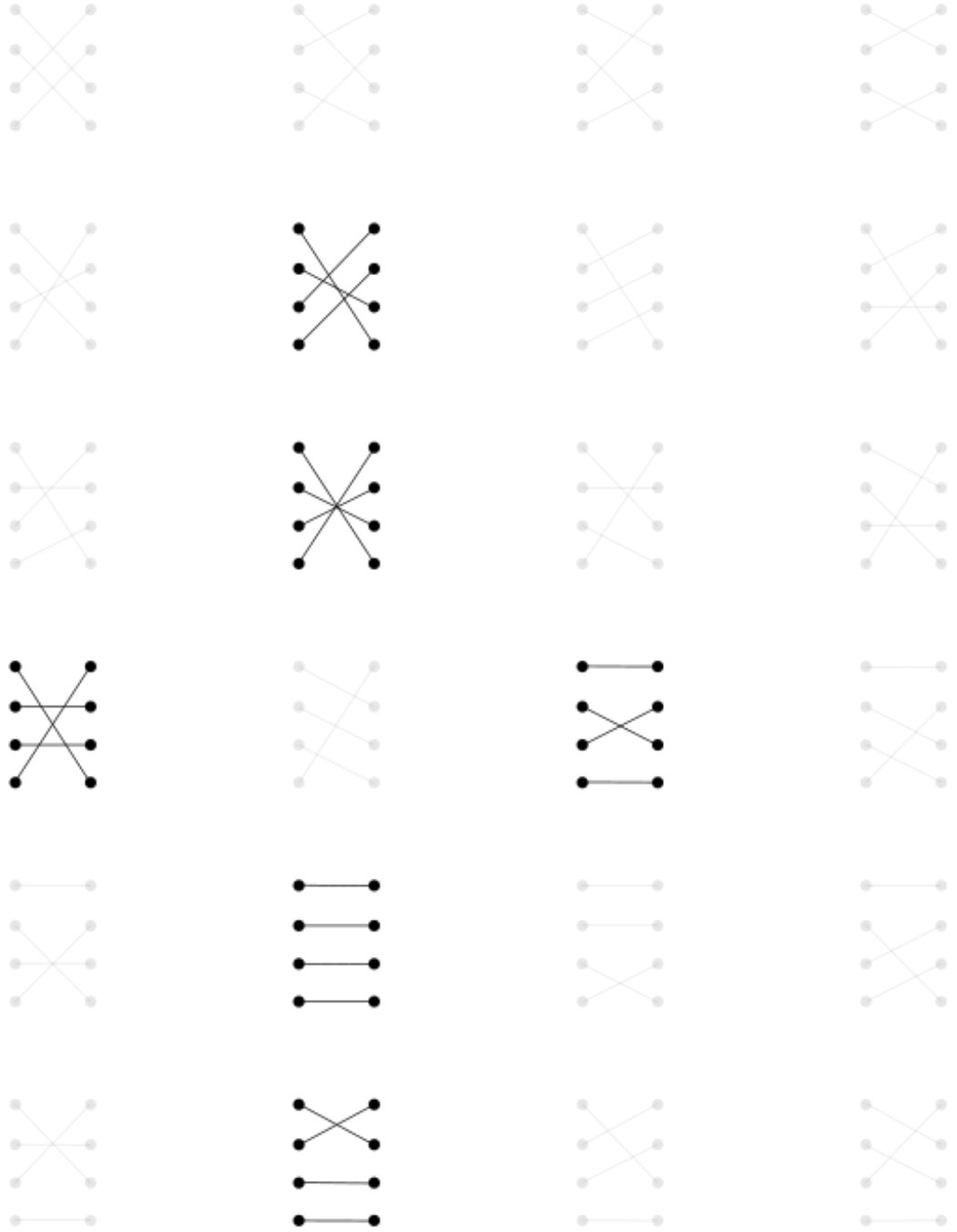

m1
m2
m3
m4
w1
w2
w3
w4
2 3 4 1
4 2 1 3
1 2 3 4
2 1 4 3
2 1 4 3
1 2 3 4
2 3 1 4
3 2 4 1

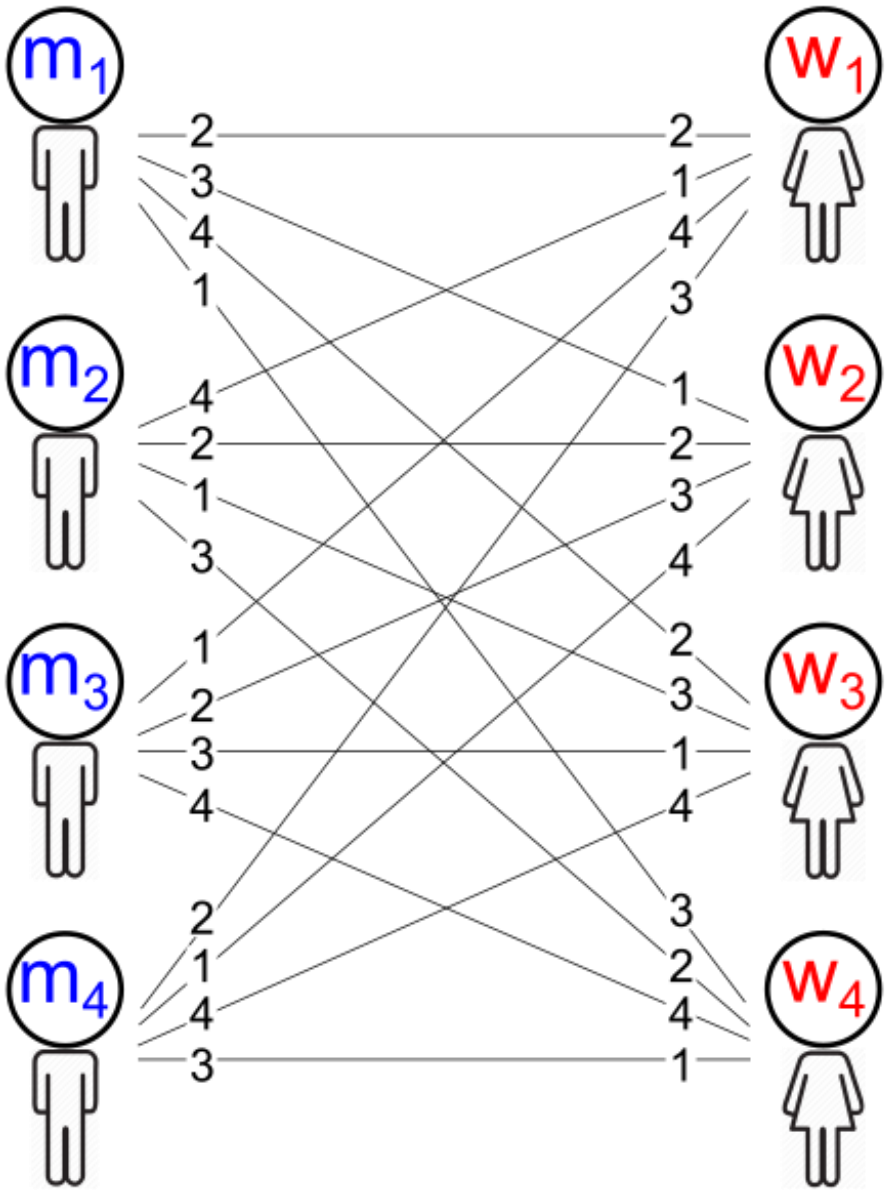

m1
m2
m3
m4
w1
w2
w3
w4
2 3 4 1
4 2 1 3
1 2 3 4
2 1 4 3
2 1 4 3
1 2 3 4
2 3 1 4
3 2 4 1


m1
m2
m3
m4
w1
w2
w3
w4
2 3 4 1
4 2 1 3
1 2 3 4
2 1 4 3
2 1 4 3
1 2 3 4
2 3 1 4
3 2 4 1




m1
m2
m3
m4
w1
w2
w3
w4
2 3 4 1
4 2 1 3
1 2 3 4
2 1 4 3
2 1 4 3
1 2 3 4
2 3 1 4
3 2 4 1




m1
m2
m3
m4
w1
w2
w3
w4
2 3 4 1
4 2 1 3
1 2 3 4
2 1 4 3
2 1 4 3
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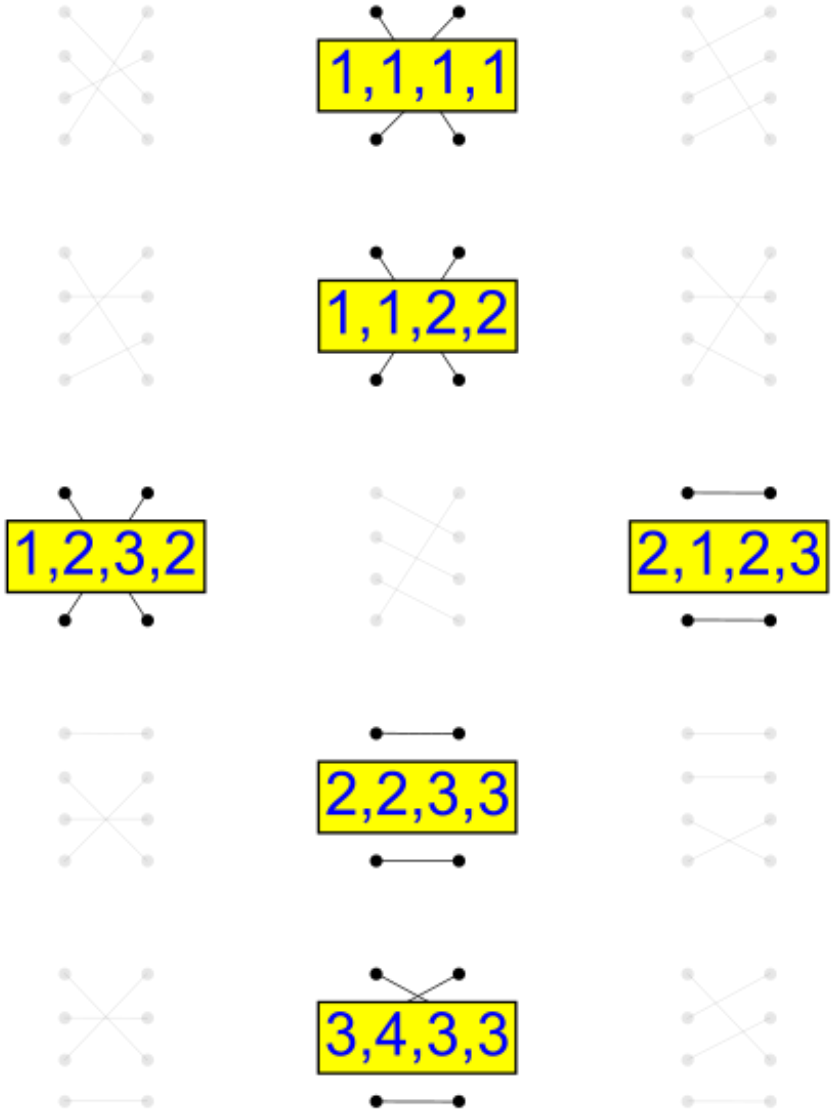
3 2 4 1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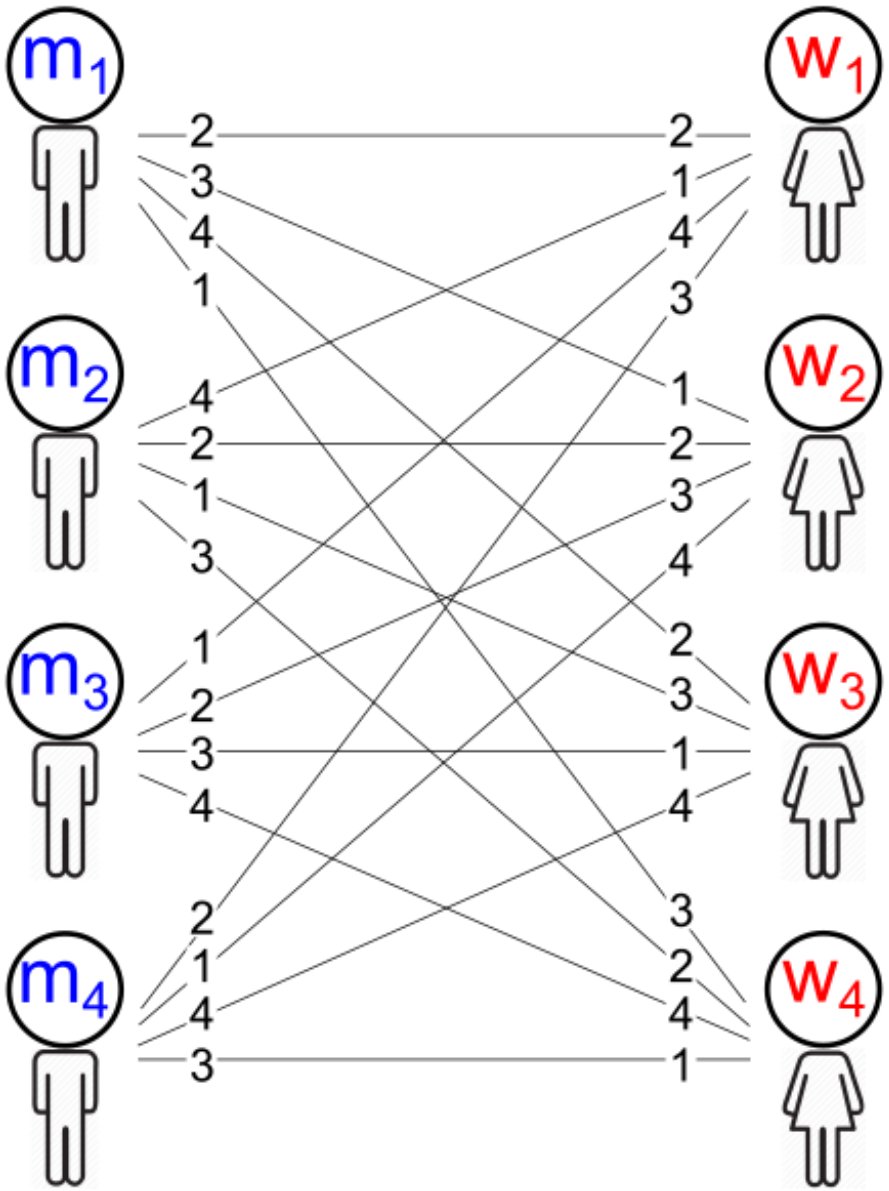

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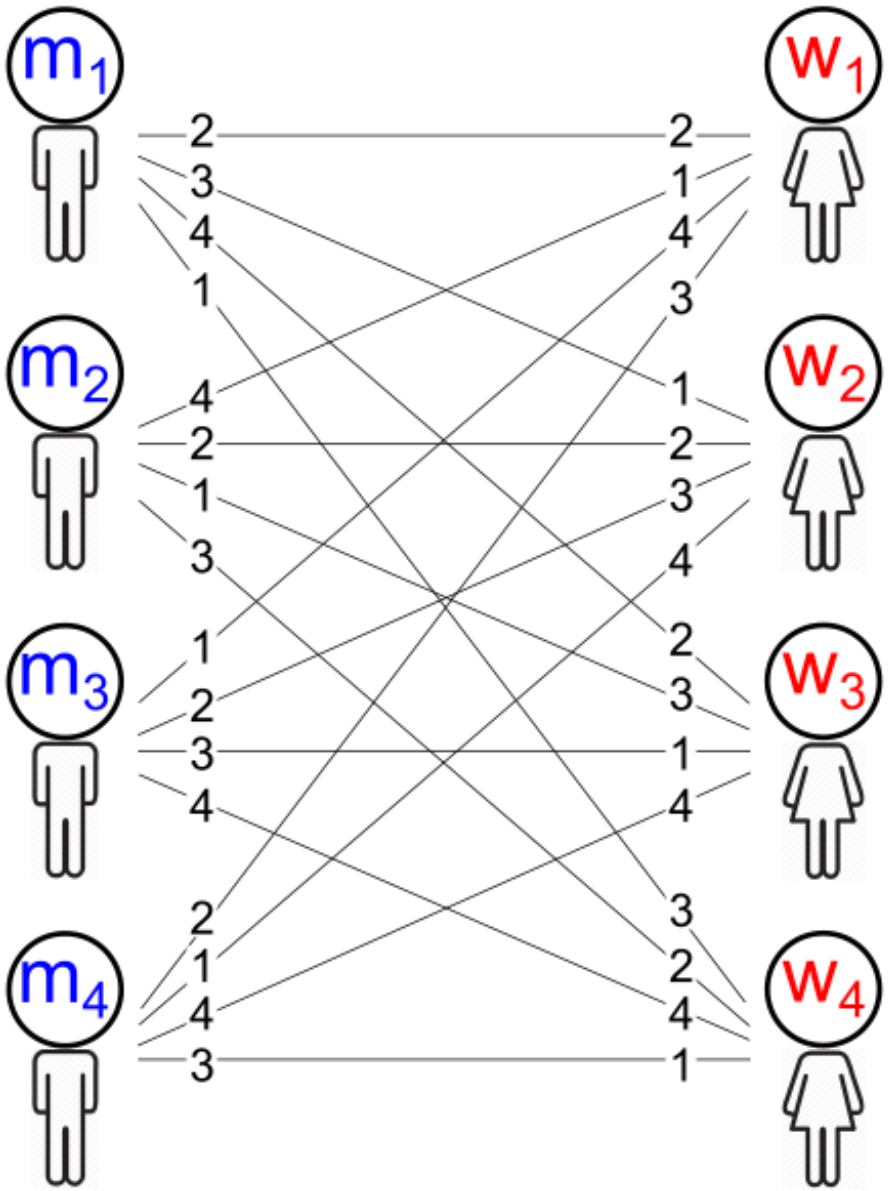


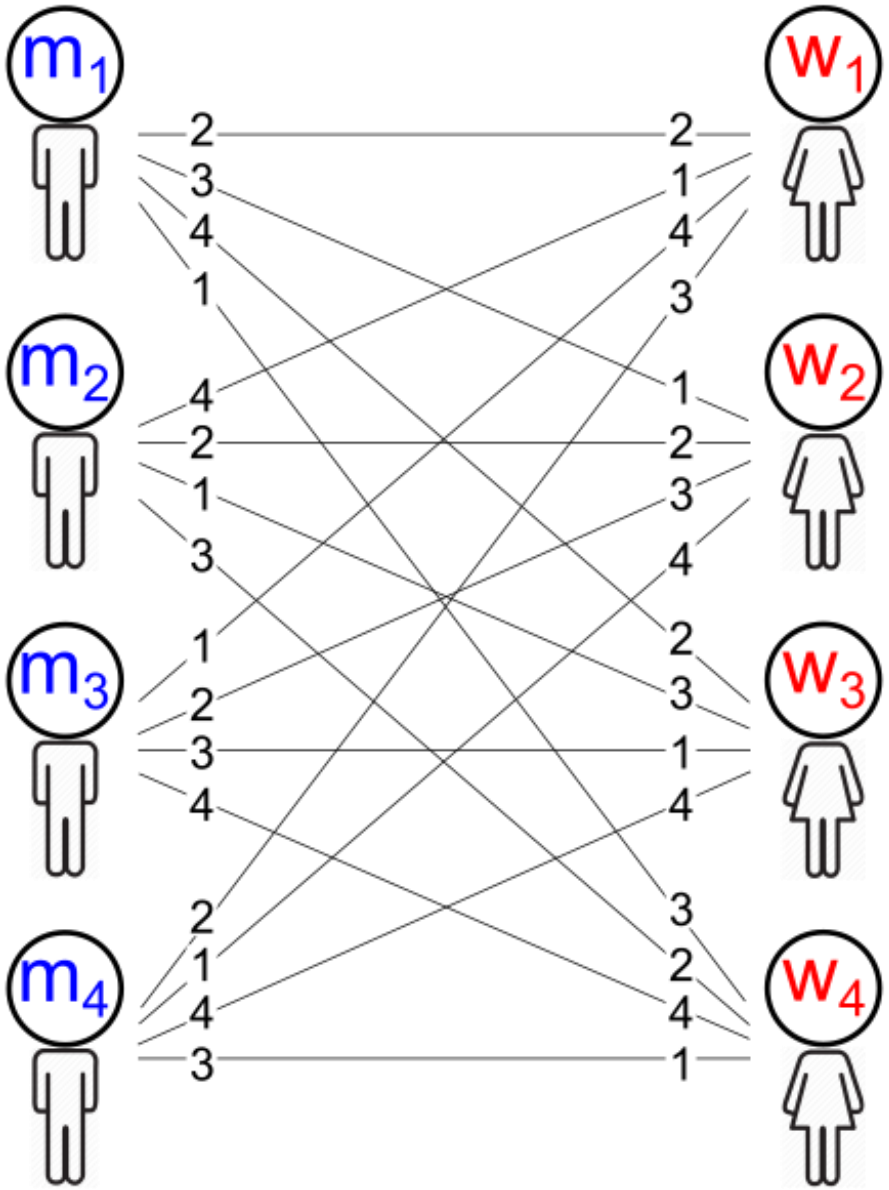
Men-optimal 1,1,1,1

1,1,2,2

1,2,3,2 2,1,2,3

2,2,3,3

Men-pessimal 3,4,3,3


m1
m2
m3
m4
w1
w2
w3
w4
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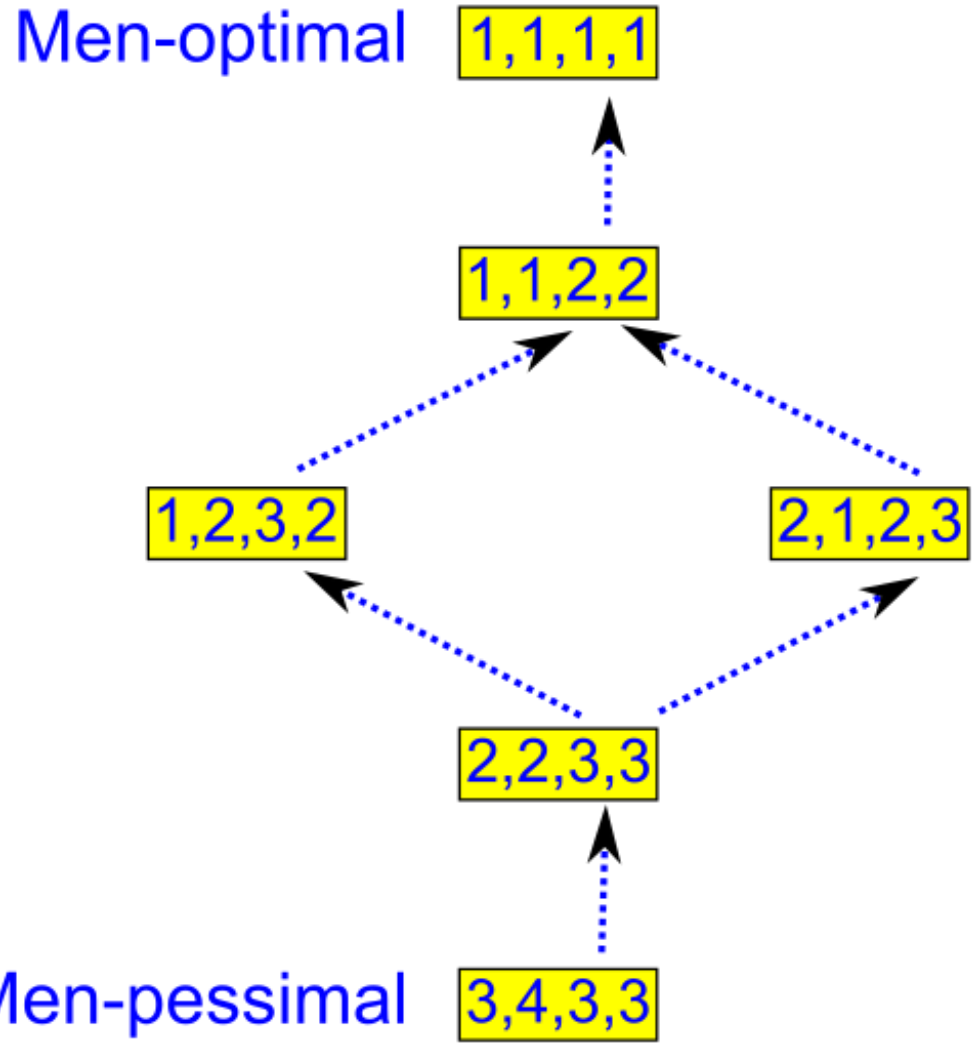
Men-optimal
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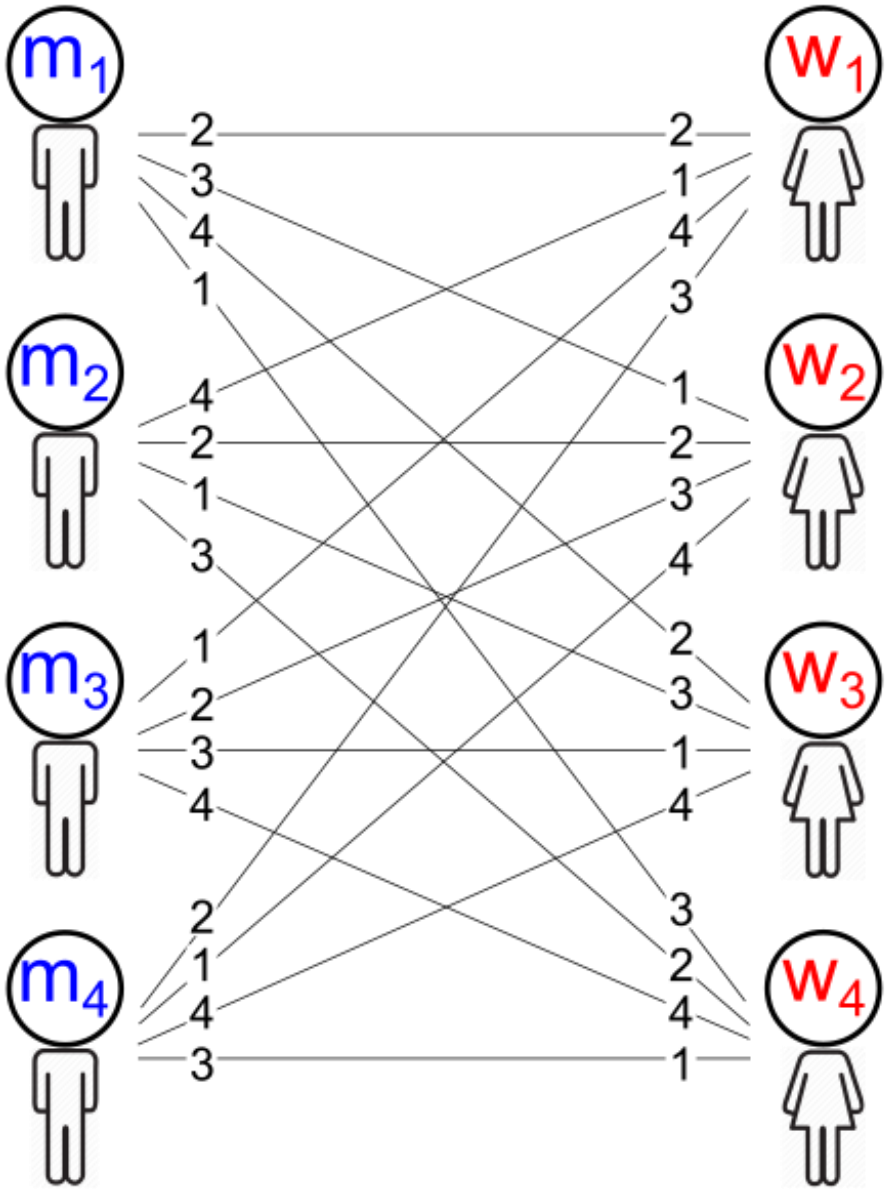
m1
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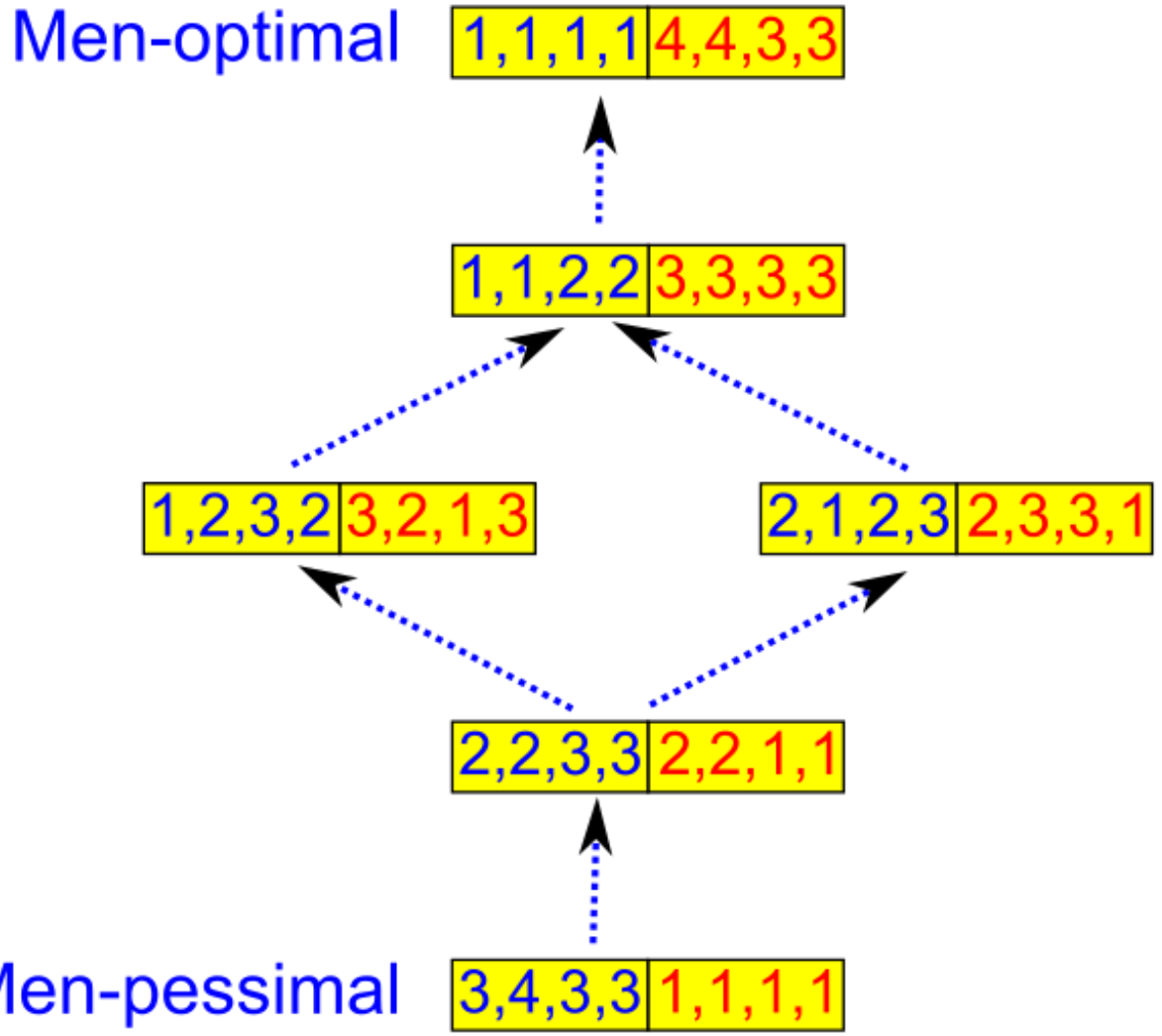
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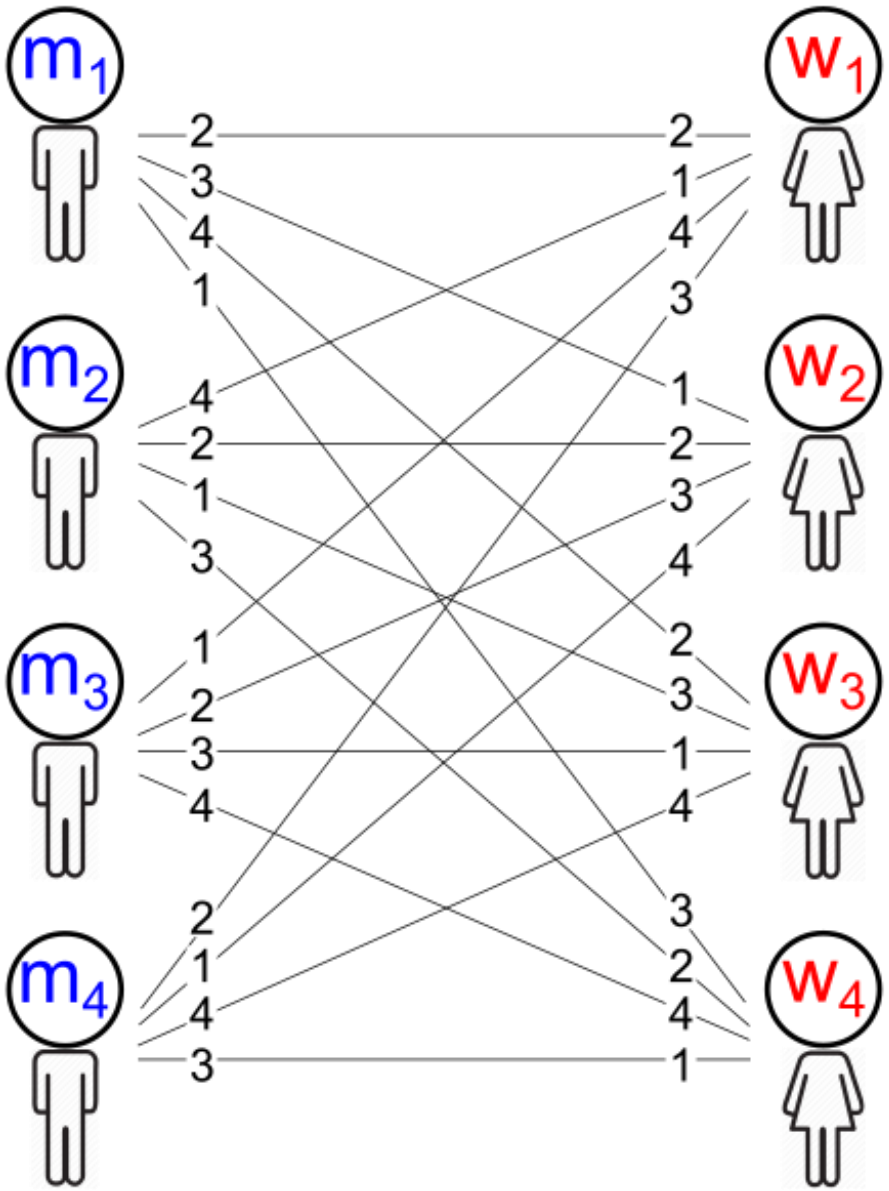
m1
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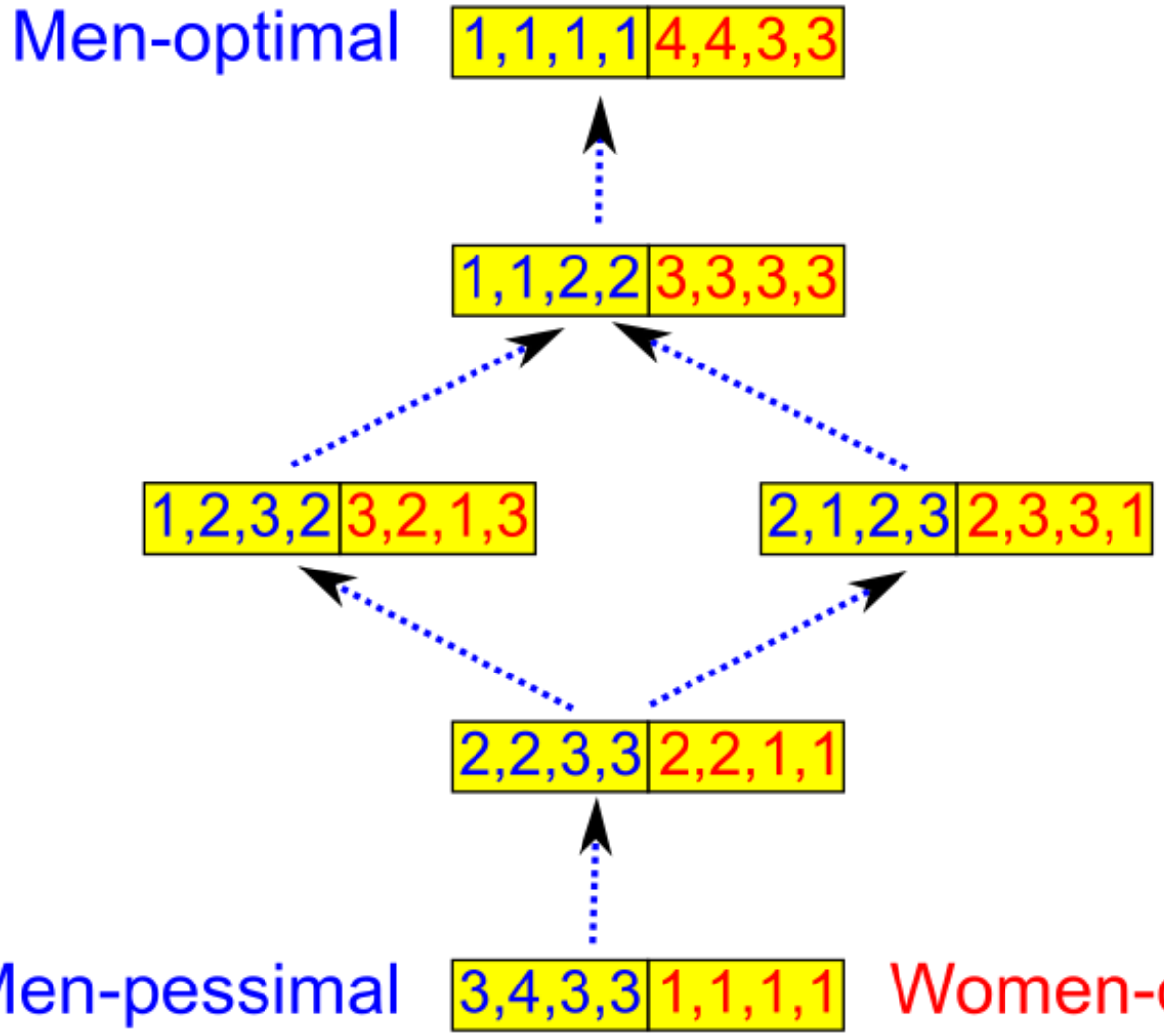
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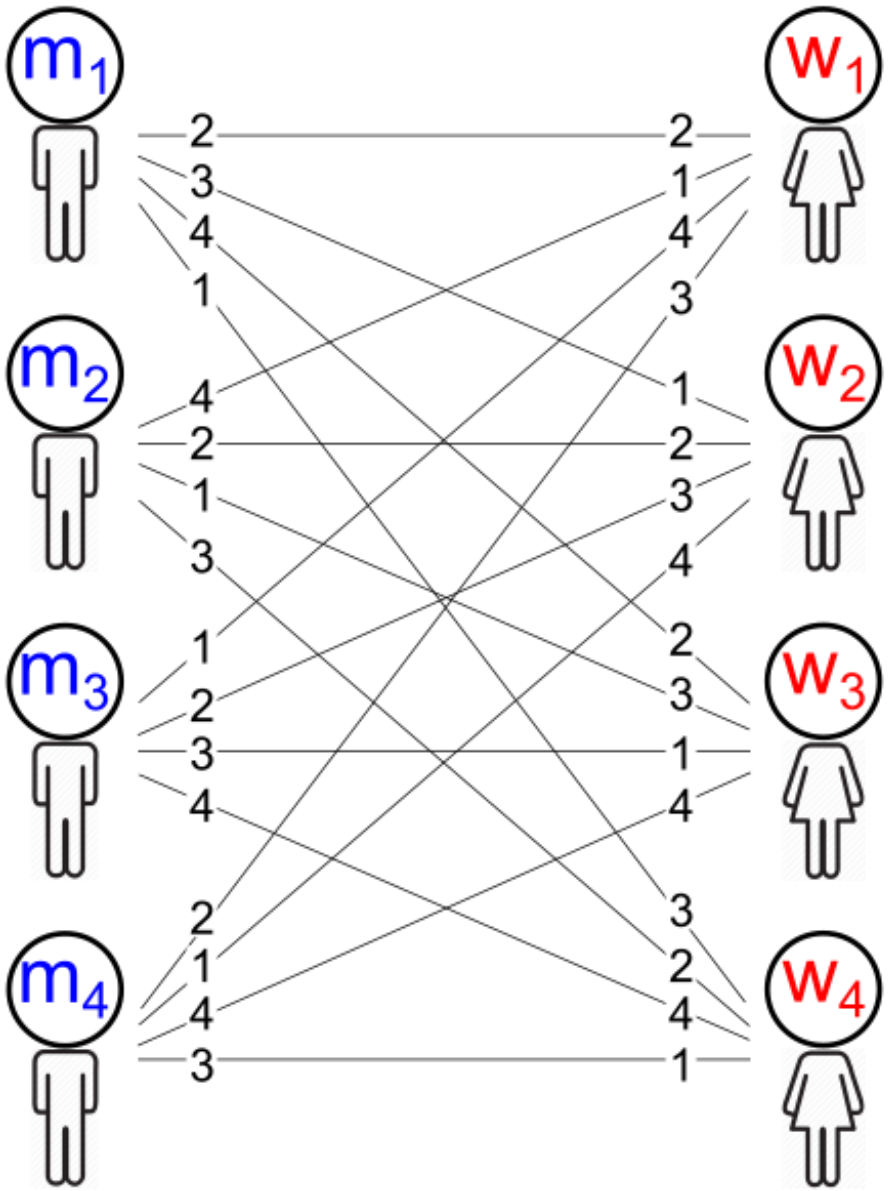
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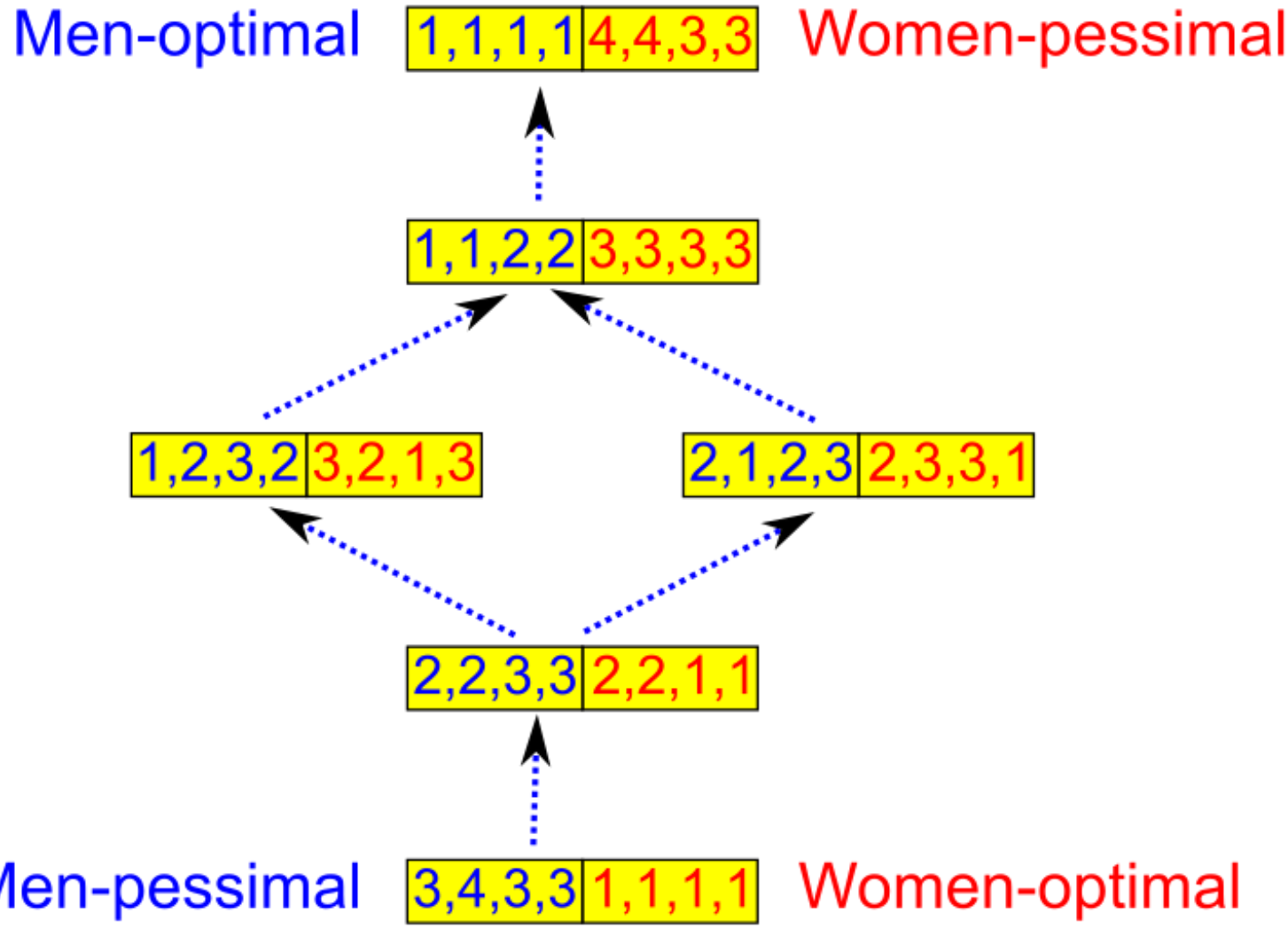
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1,2,3,2 3,2,1,3
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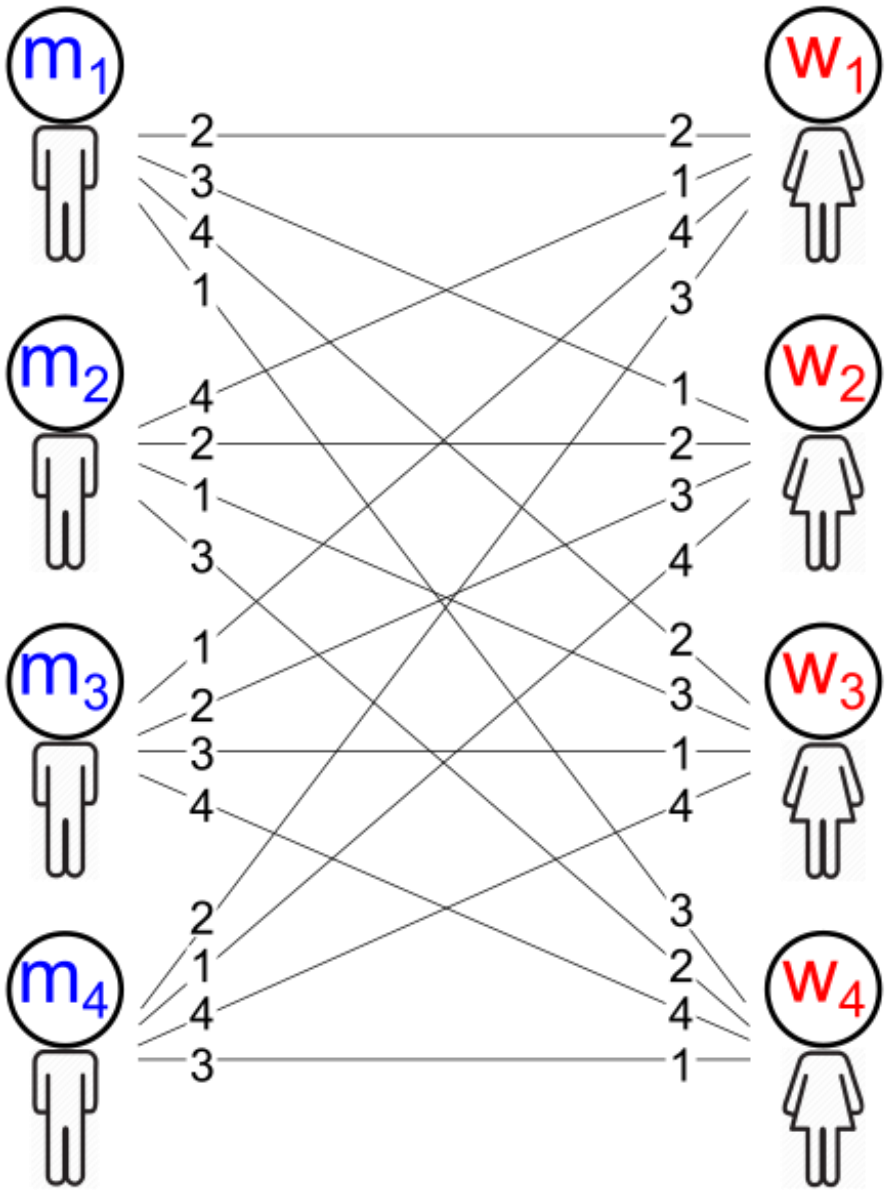
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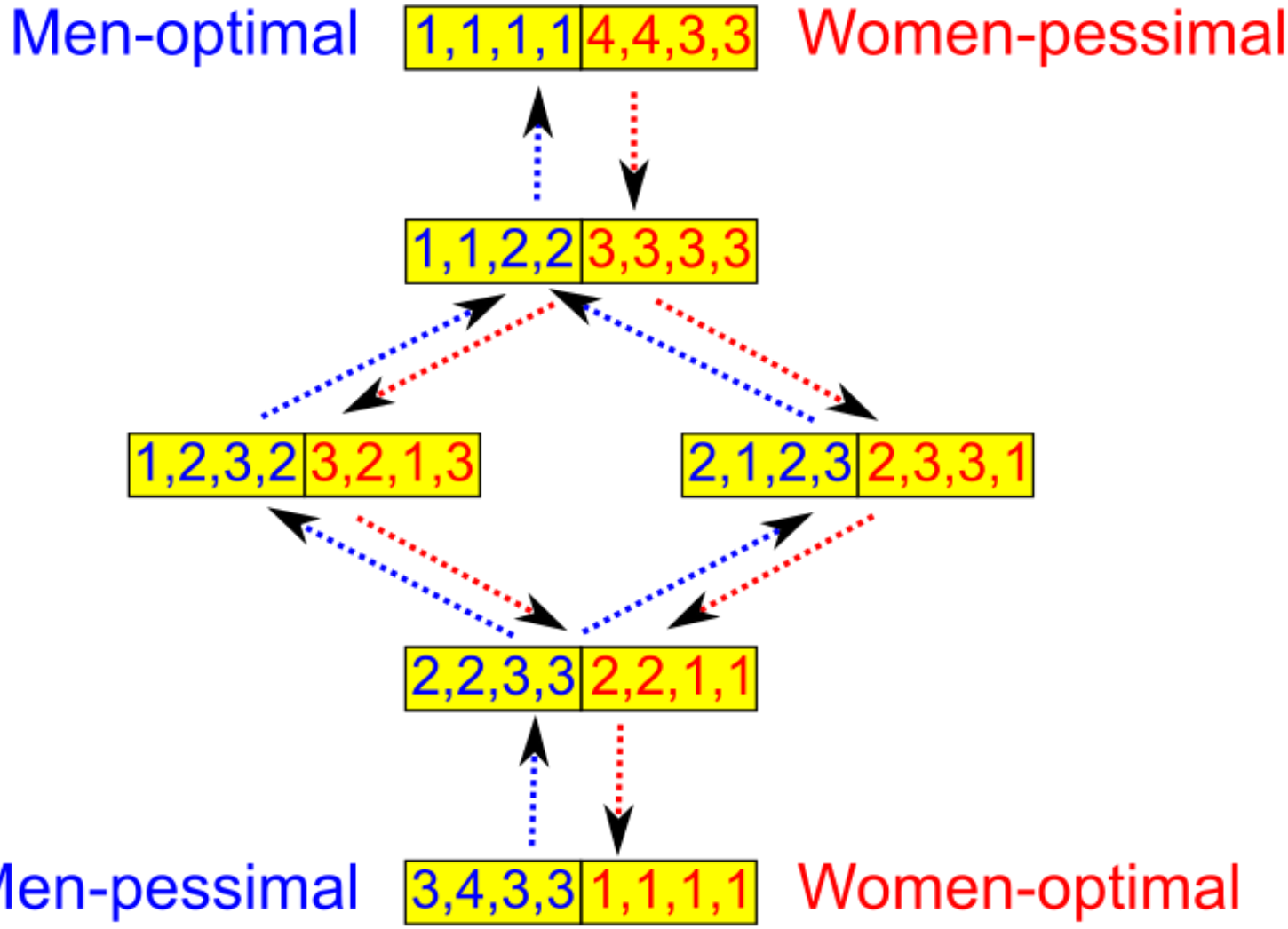
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Women-pessimal
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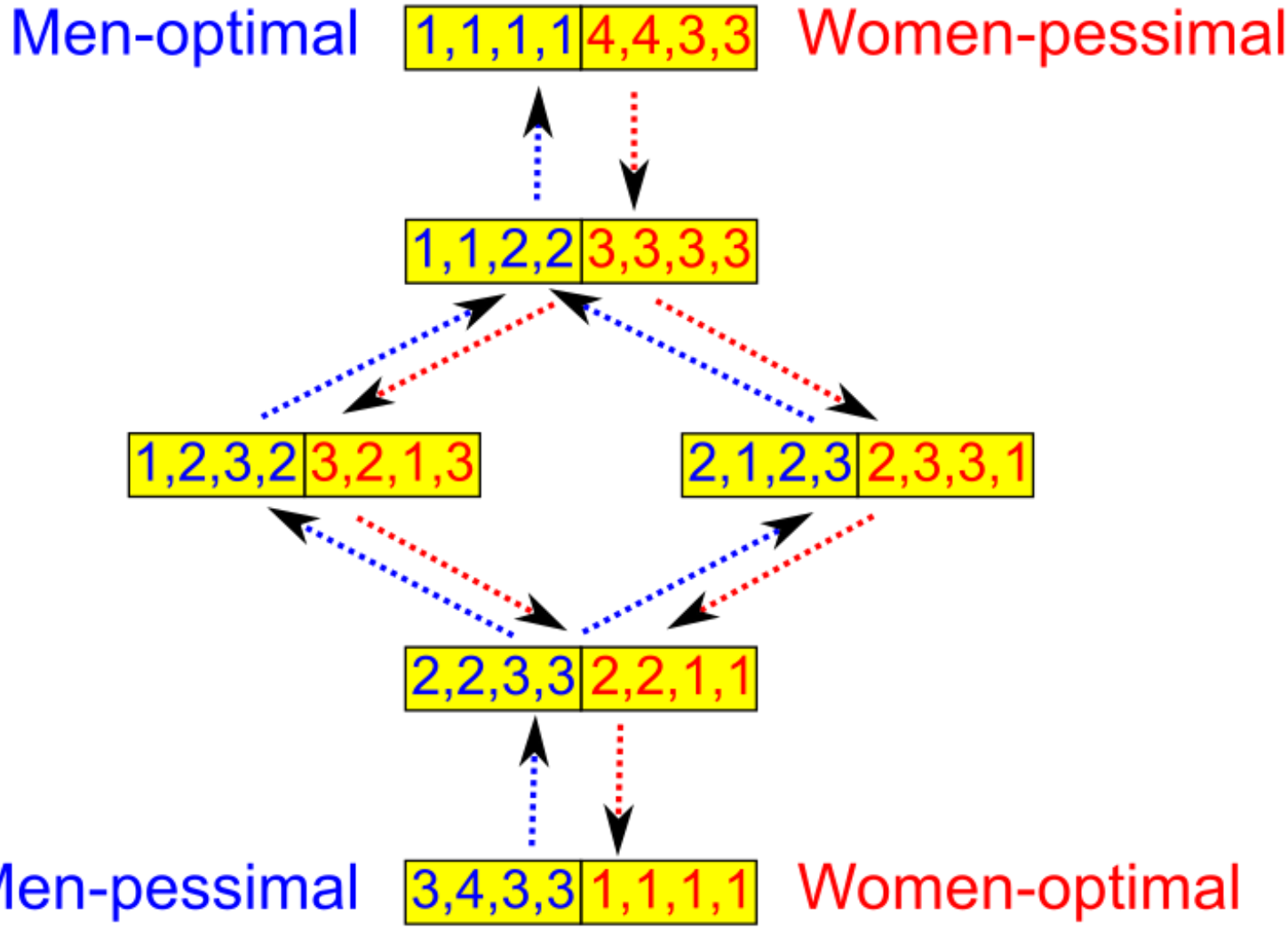
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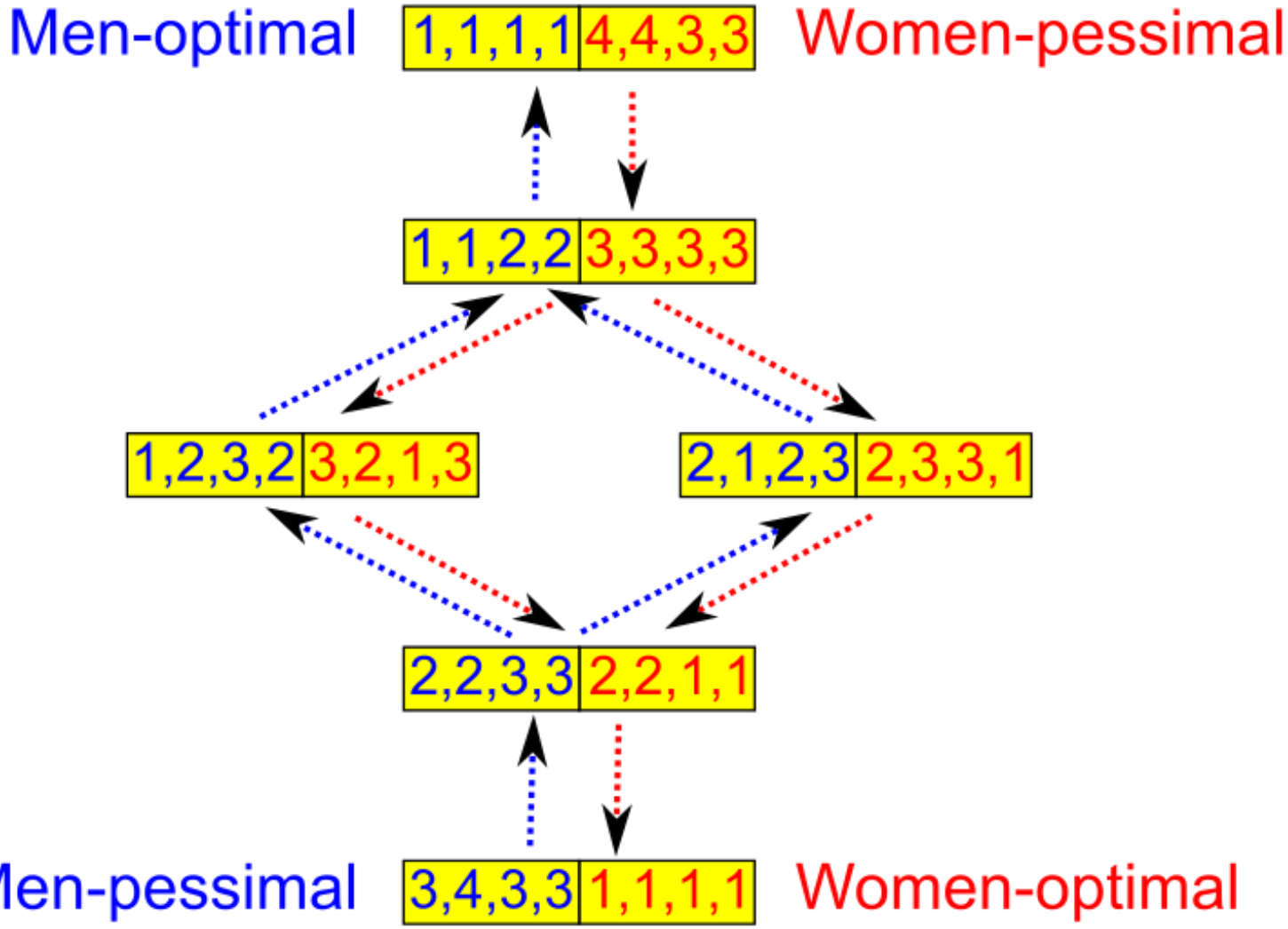
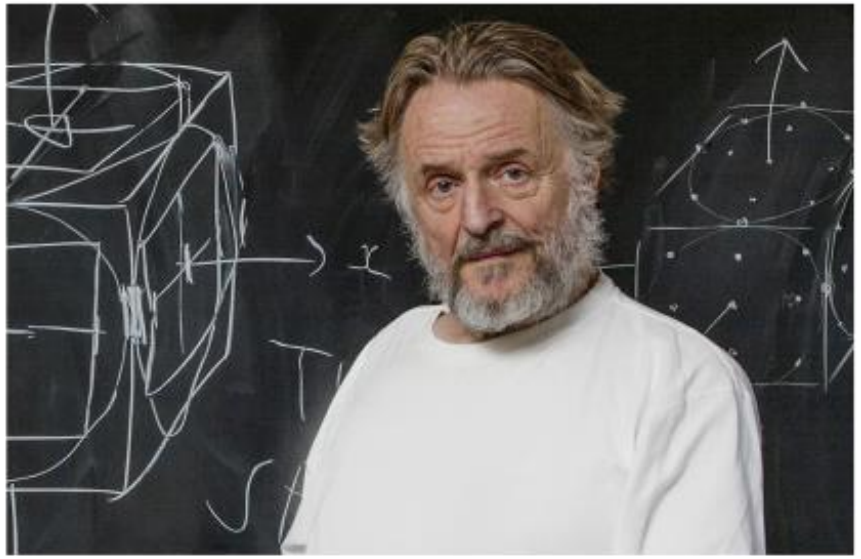
The Lattice of Stable Matchings





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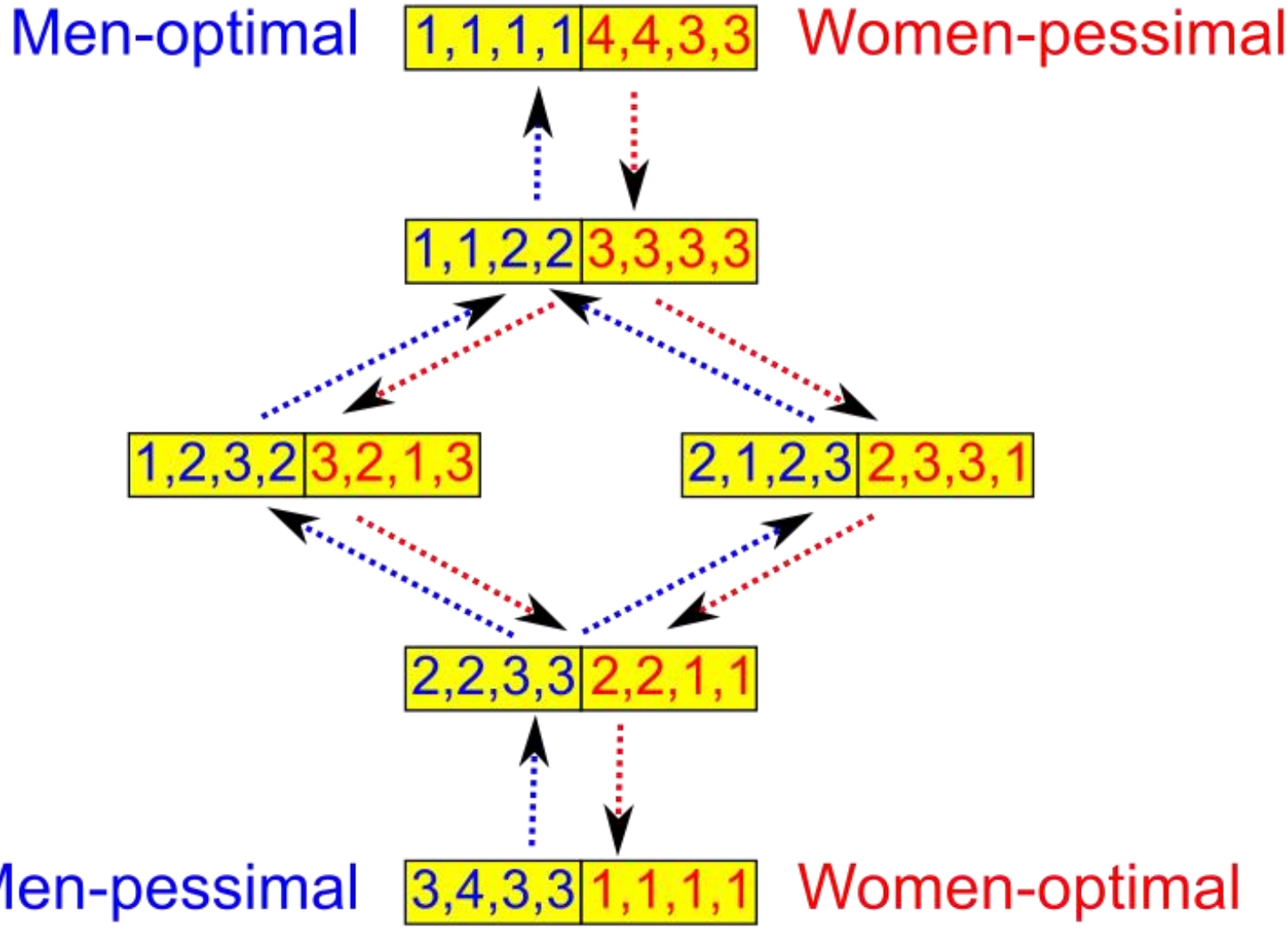

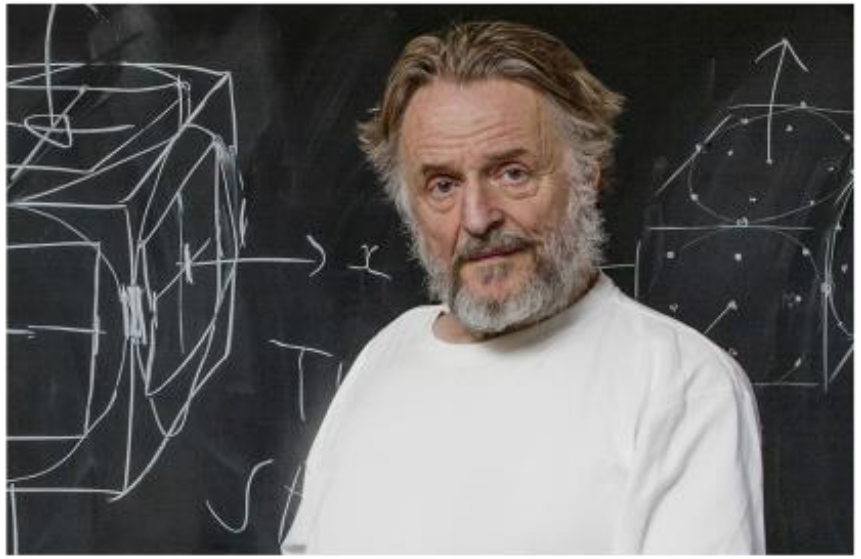

John H. Conway

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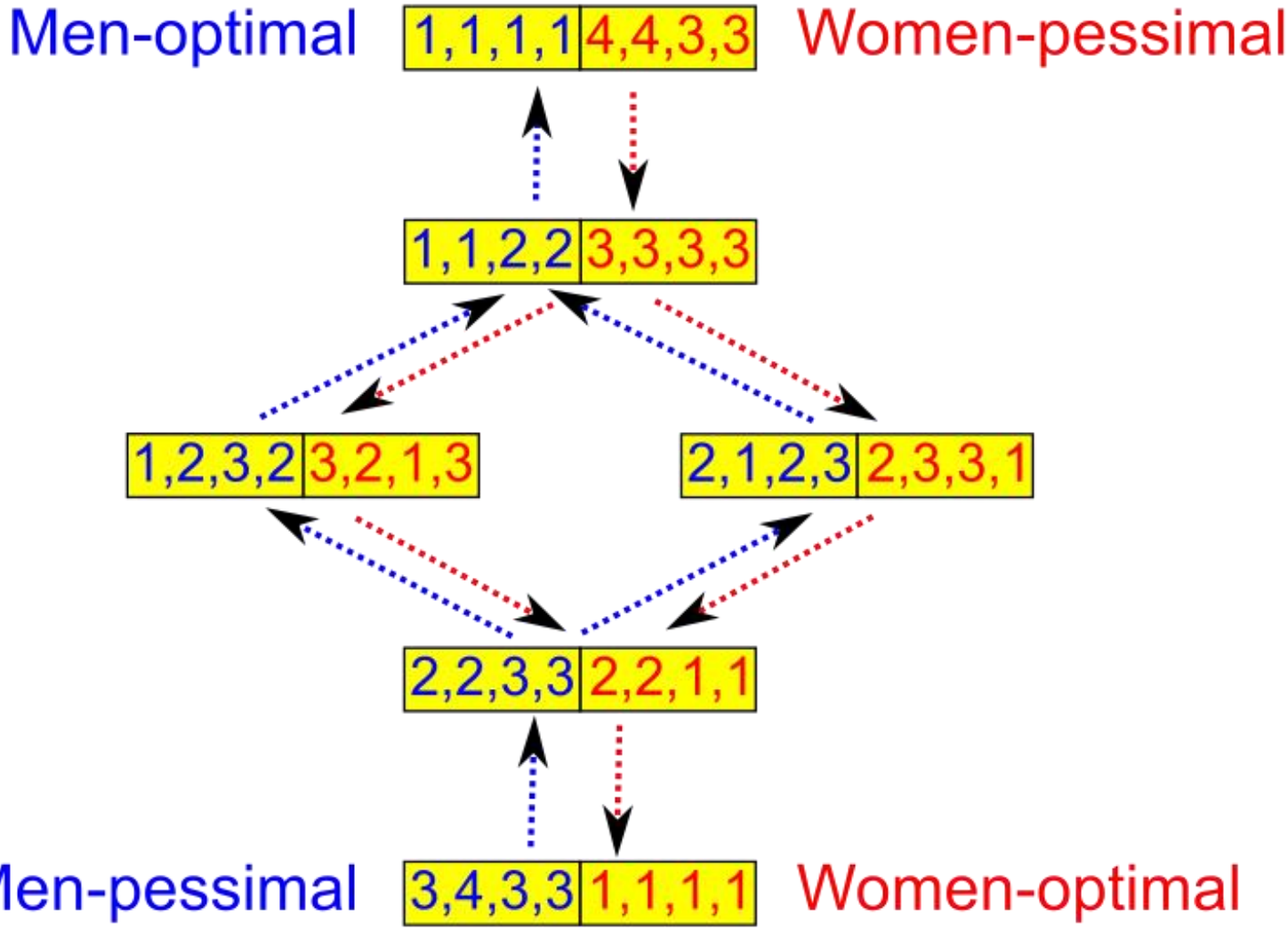

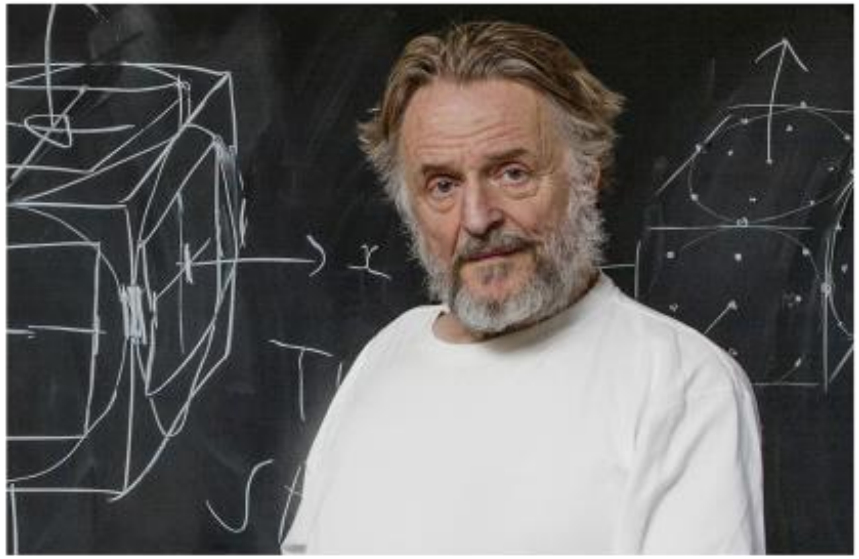

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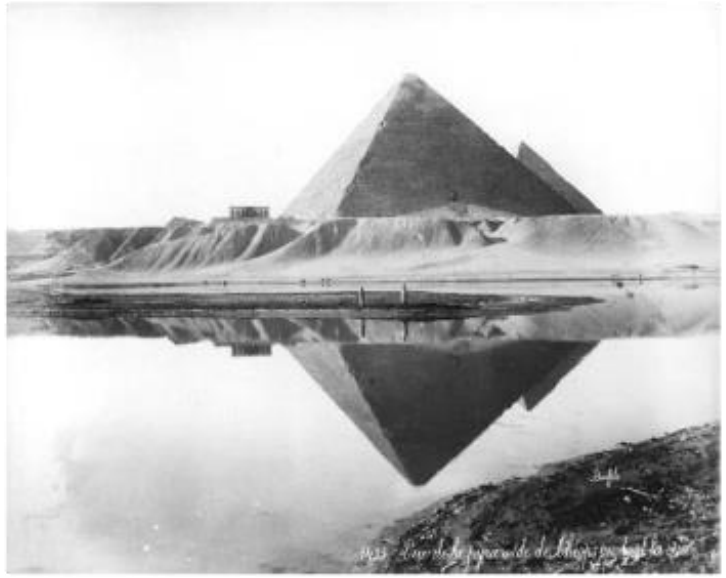

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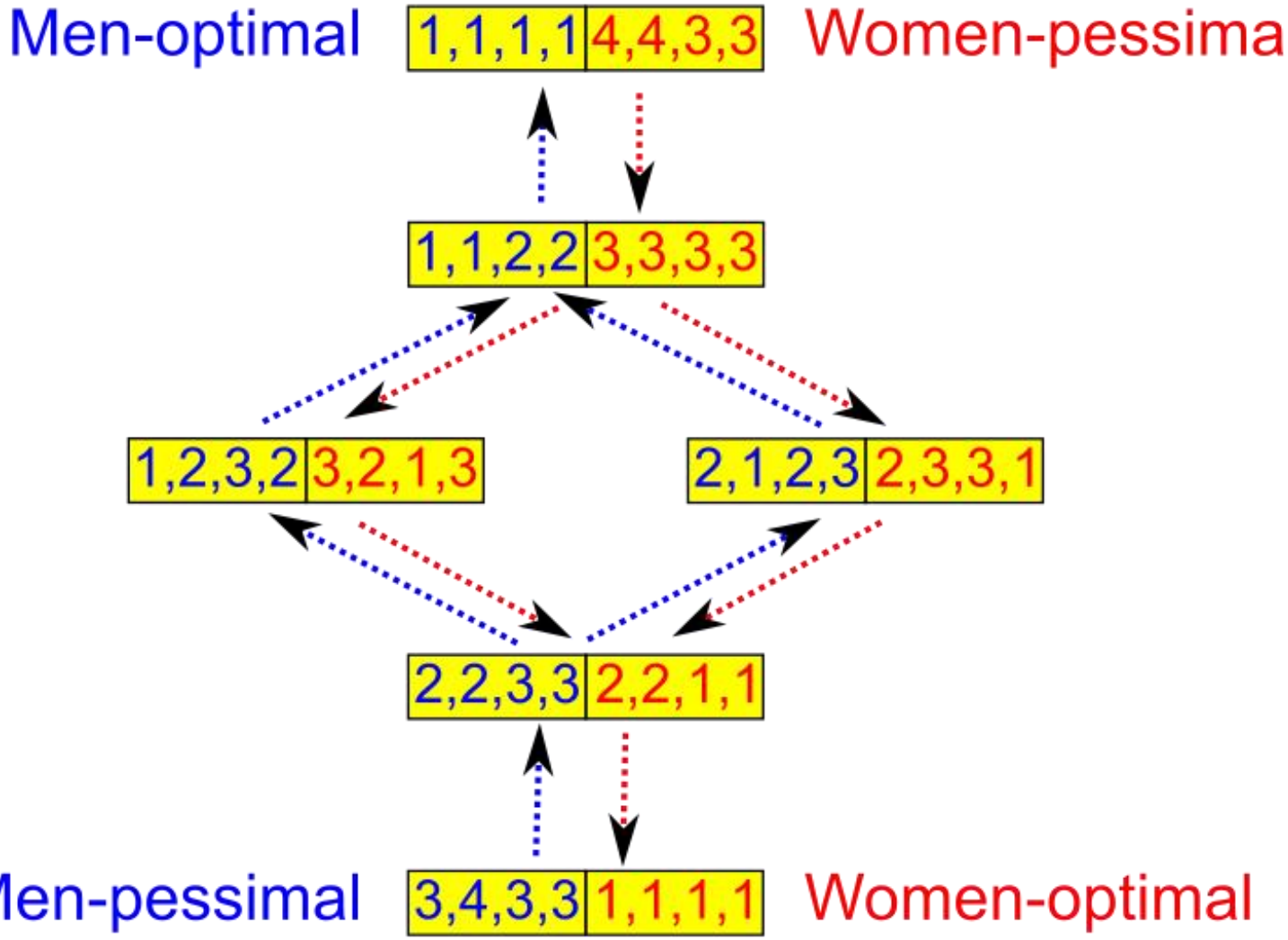

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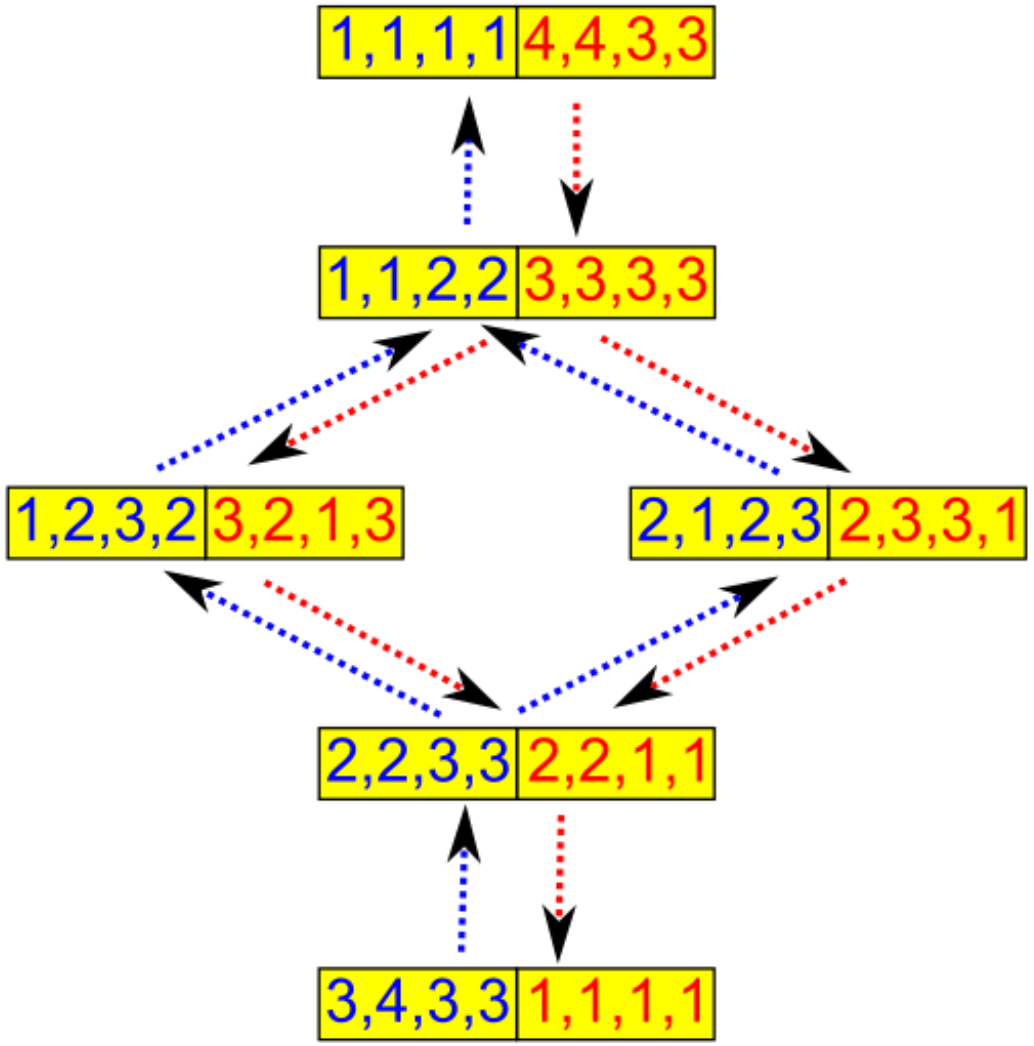
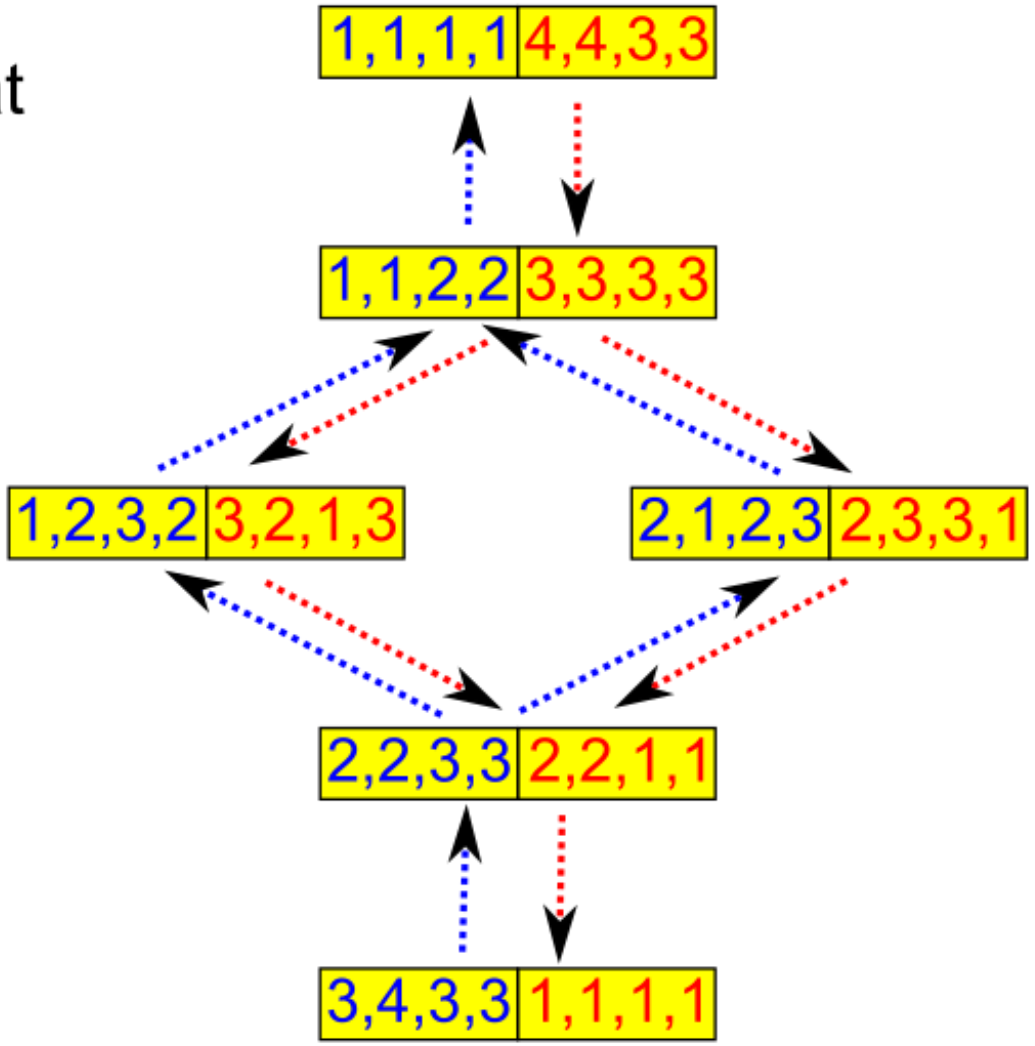


Some Observations

Consensus

There is a stable matching that all men find at least as good as *any other* stable matching, and one that they find at least as bad.
(Analogously for the women.)



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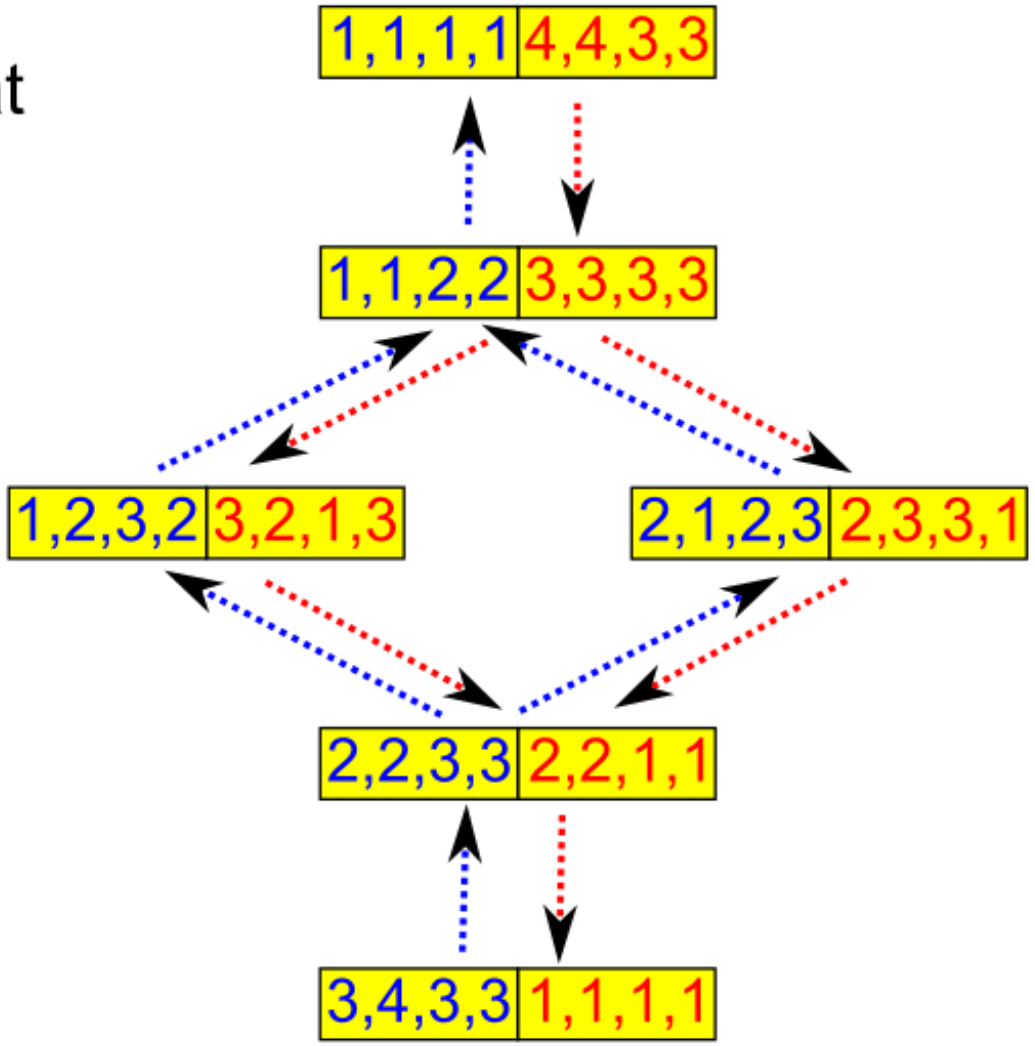
For any distinct stable matchings P and Q, if all men find P at least as good as Q, then all women find Q at least as good as P (and vice versa).



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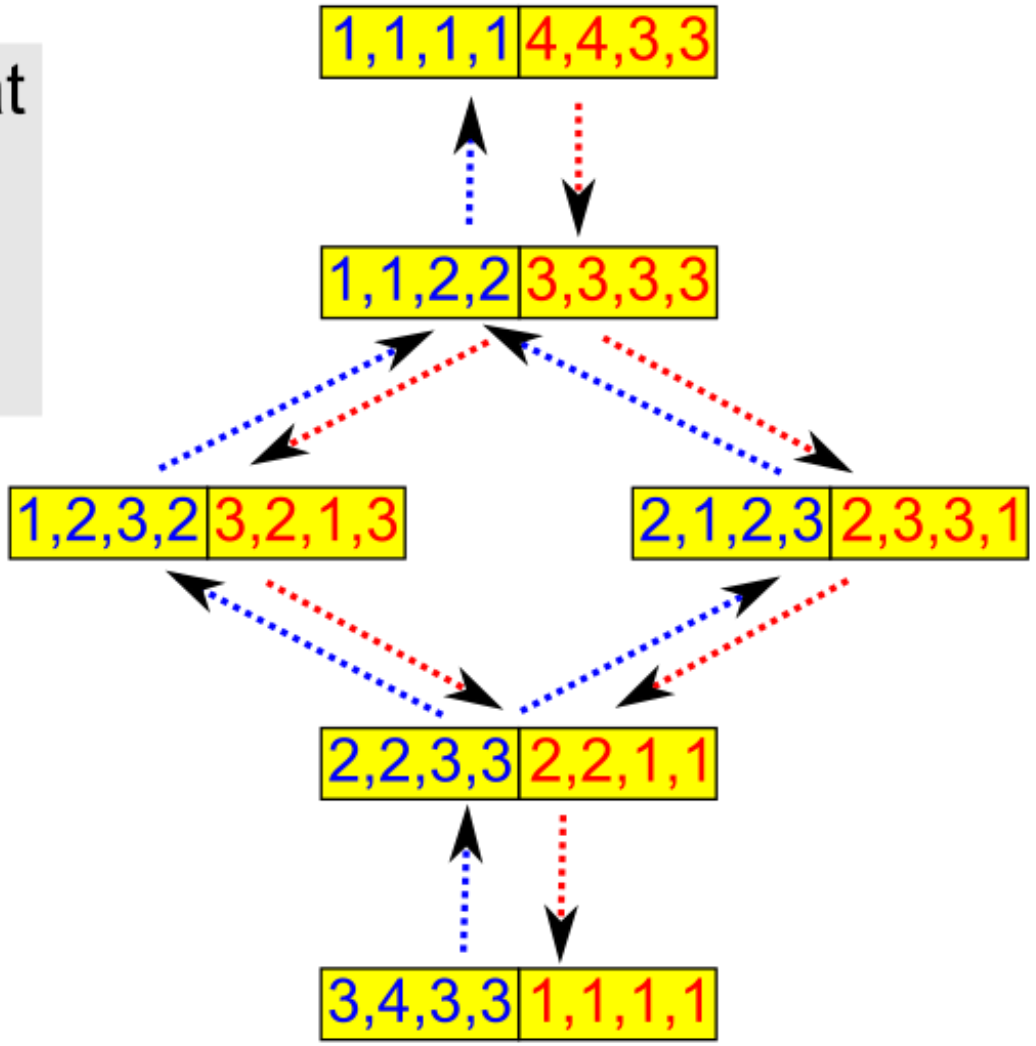

A man m and a woman w are **achievable** for each other if there is some stable matching in which they are matched with each other.

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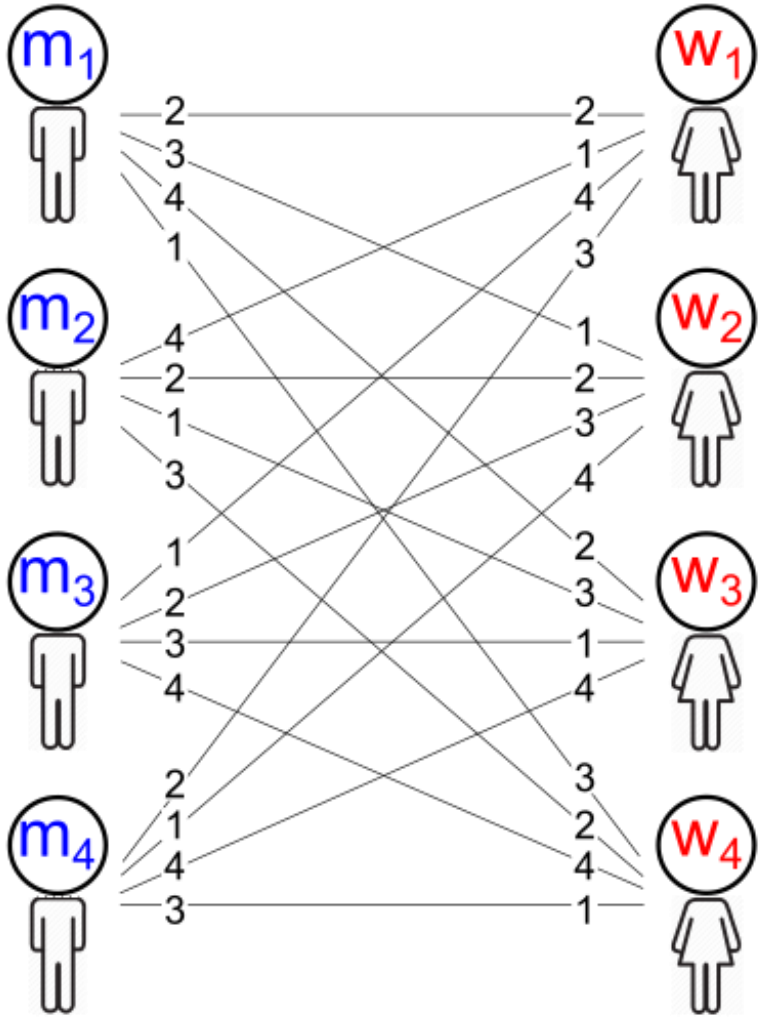

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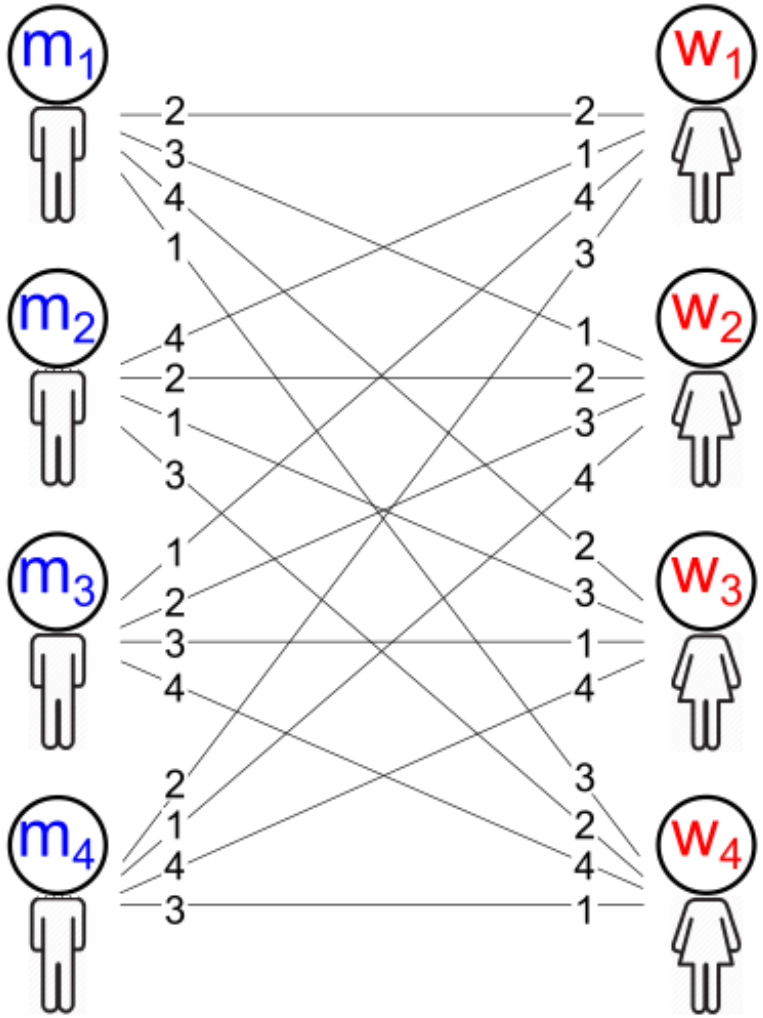

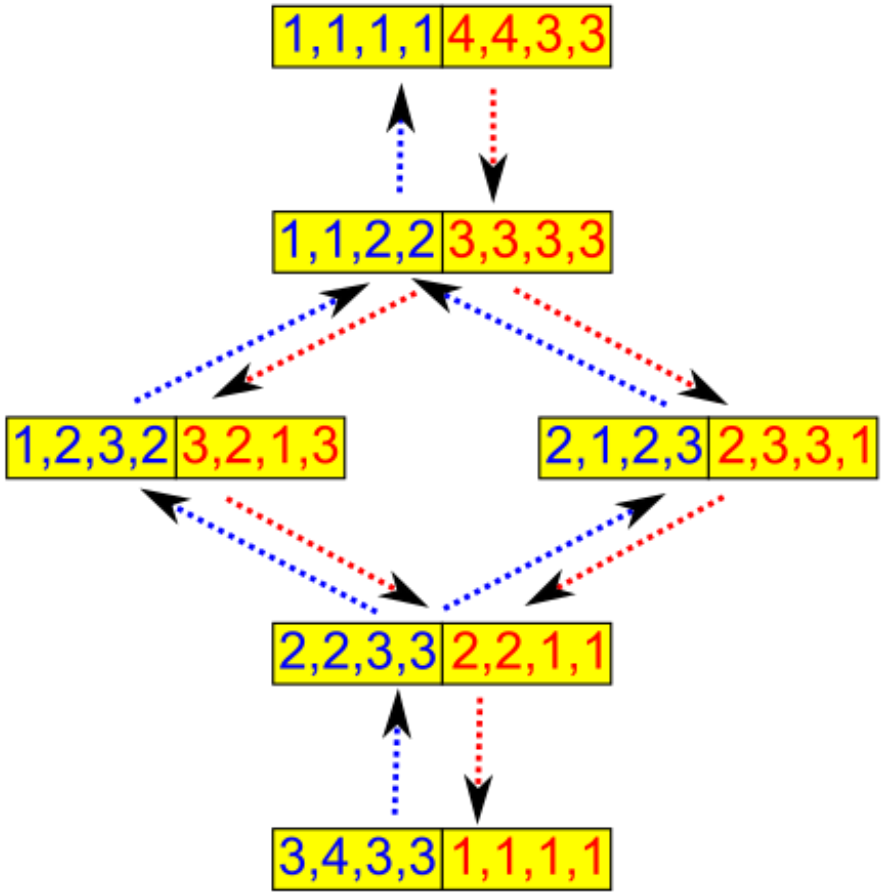

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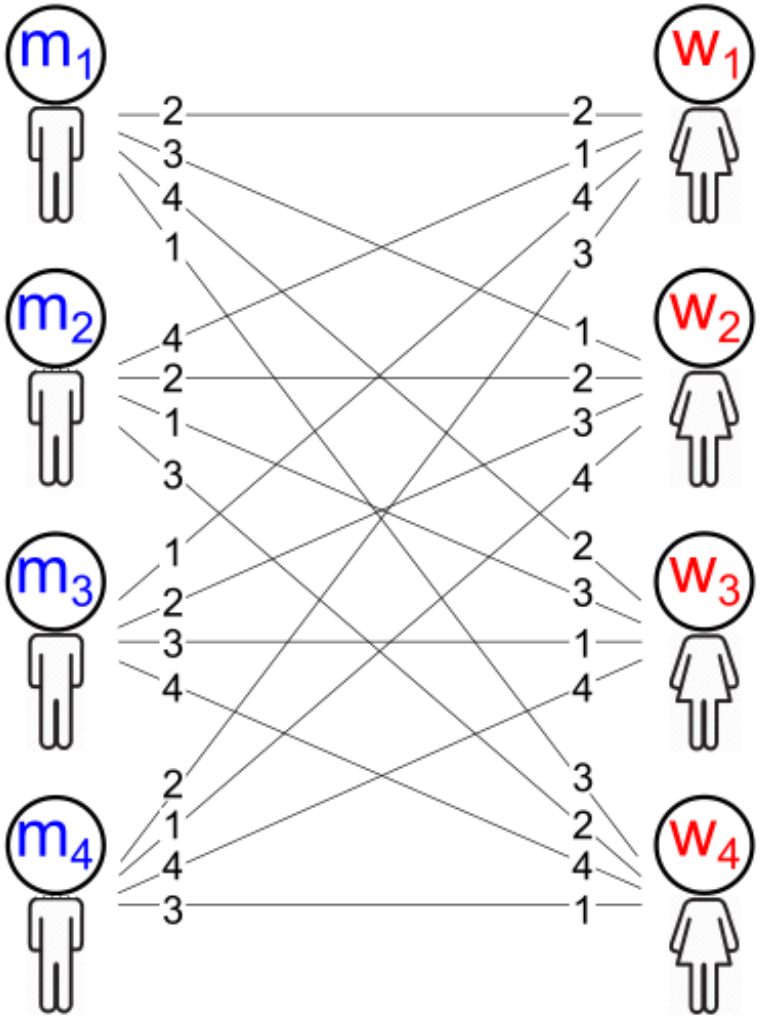

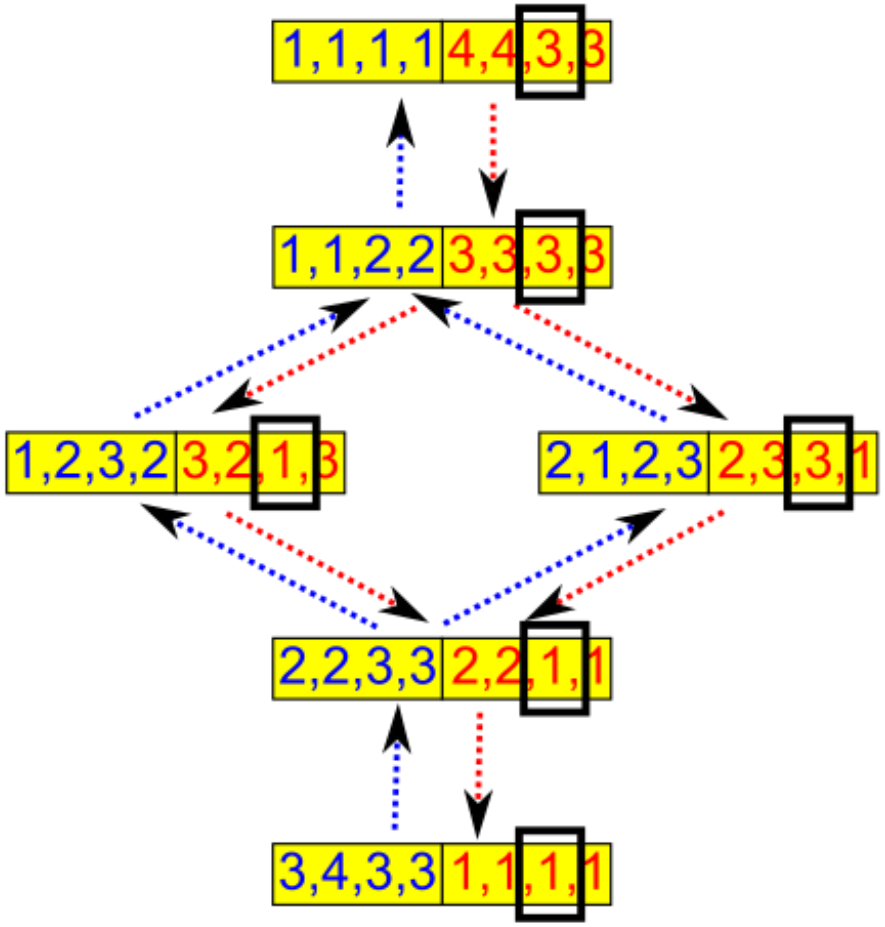

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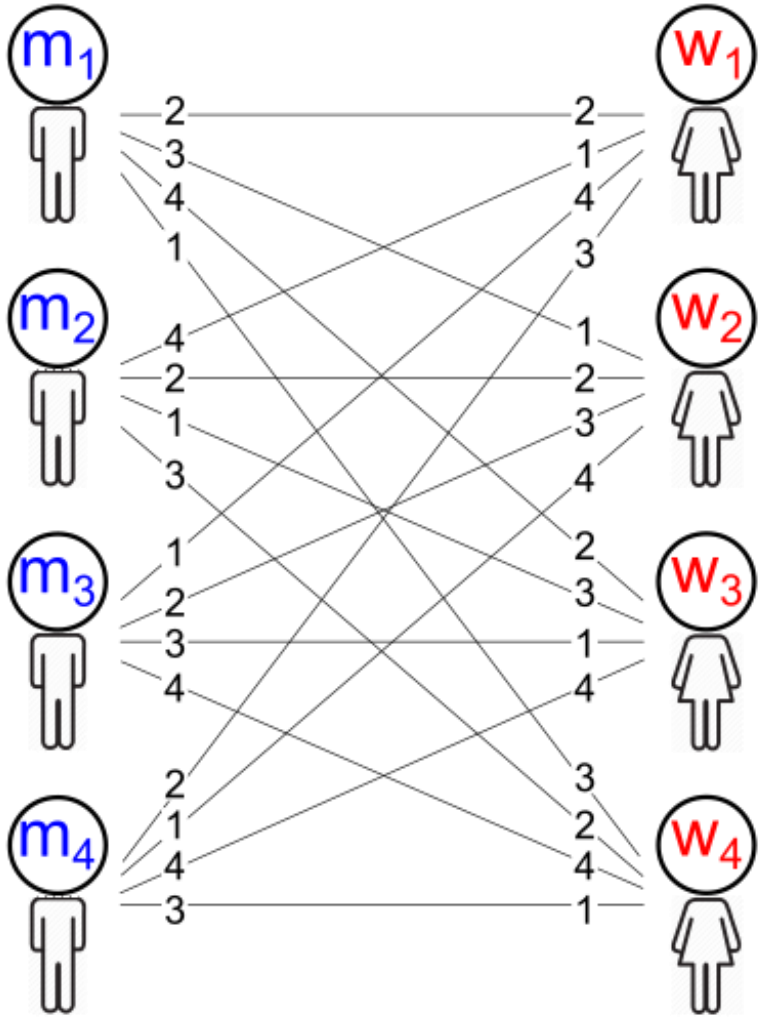

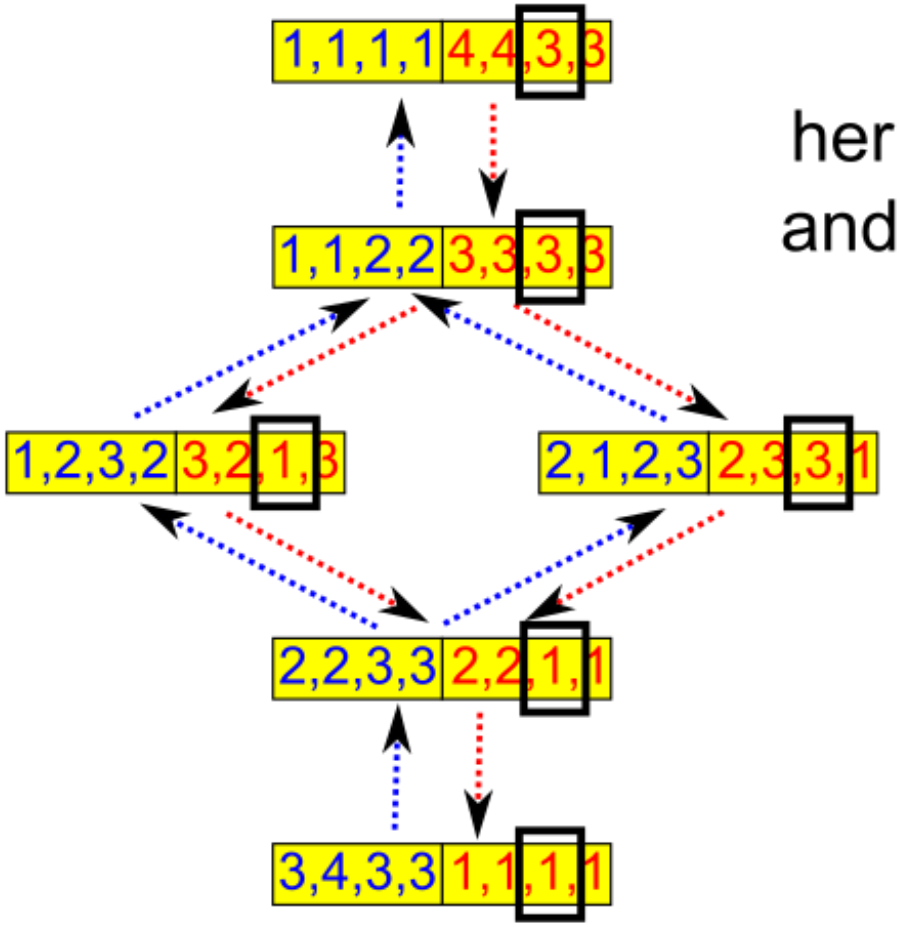

For w_3,
her first choice man m_3
and third choice man m_2
are **achievable**

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We will show that men/women-optimal mappings are *one-to-one*.

References

- Stability and the Deferred Acceptance Algorithm

David Gale and Lloyd Shapley
“College Admissions and the Stability of Marriage”
American Mathematical Monthly, 69(1), 1962 pg 9-15