Lecture 14

Assignment 2 Discussion

Mar 23, 2026 | Rohit Vaish

PROBLEM 1

Problem 1 [25 points]

Prove or disprove: No divisor method simultaneously violates upper quota for one state and lower quota for another state on a given instance.

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Solution by Manya and Harshin

Let $f(\cdot)$ denote the divisor method.

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Given any instance $I = (h; p_1, p_2, \dots, p_n)$, let the associated seat assignment be $(s_1, s_2 \dots s_n)$ and the divisor be D.

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Given any instance $I = (h; p_1, p_2, \dots, p_n)$, let the associated seat assignment be $(s_1, s_2, \dots, s_n)$ and the divisor be D.

Suppose, for contradiction, that upper quota for state i and lower quota for state j are violated.

That is, $s_i > \lceil q_i \rceil$ and $s_j < \lfloor q_j \rfloor$.

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$$\Rightarrow \quad \frac{p_i}{D} + 1 > \frac{p_i}{P} \cdot h + 1$$ since $x+1 > \lceil x \rceil$

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$$\Rightarrow \quad \frac{P}{h} > D$$

Lower quota violation for state j:

$$\Longrightarrow \quad s_j = f\left(\frac{p_j}{D}\right) \leq \lfloor q_j \rfloor - 1$$

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$$\Rightarrow \quad \left\lceil \frac{p_j}{D} \right\rceil \leq \frac{p_j}{P} \cdot h$$

since $\lfloor x \rfloor + 1 \leq f(x) + 1$
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$$\Rightarrow \quad \frac{P}{h} \leq D$$

which contradicts ☺.

∎

PROBLEM 2(a)

(a) [**15 points**] Prove or disprove: For Webster's method, if state i receives more seats than its upper quota, then each state j that receives its lower quota must be smaller, that is, $p_j \leqslant p_i$.

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$s_j = \lfloor q_j \rfloor$

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$$\Rightarrow \quad \frac{p_i}{D} + \frac{1}{2} \geqslant q_i + 1 \qquad \text{since } [x] \leq x + 1/2$$

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since $[x] \geq x - 1/2$

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$$\Rightarrow \frac{1}{D} - \frac{h}{P} \le \frac{1}{2p_j}$$

☺ ☺

Combining ☺ and ☺☺

$$\frac{1}{2p_i} \leq \frac{1}{D} - \frac{h}{P} \leq \frac{1}{2p_j}$$

or $$p_j \leq p_i \,. \qquad \blacksquare$$

PROBLEM 2(a)

(a) **[15 points]** Prove or disprove: For Webster's method, if state i receives more seats than its upper quota, then each state j that $\underbrace{\text{receives its lower quota}}$ must be smaller, that is, $p_j \leqslant p_i$.

True for $s_j \geqslant \lfloor q_j \rfloor$?

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True for $s_j \geqslant \lfloor q_j \rfloor$?

No! Counterexample due to Prakhar and Shivanshu.

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State	p_i	q_i	p_i/D	s_i
1	13			
2	14			
3 - 10	2			

Counterexample for $s_j \geq \lfloor q_j \rfloor$.

State	p_i	q_i	p_i/D $(D=5)$	s_i
1	13	1.81	2.6	3
2	14	1.95	2.8	3
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← state 1 violates upper quota

← a larger state receives at least its lower quota

PROBLEM 2(b)

(b) **[10 points]** Prove or disprove: For Adams' method, if state j receives fewer seats than its lower quota, then each state i that received its upper quota must be smaller, that is, $p_i \leqslant p_j$.

Proved analogously.

PROBLEM 3

Problem 3 [25 points]

A seat division $s_1, \ldots, s_n$ is said to be *near quota* if it is not possible to take a seat from one state and give it to another state and simultaneously bring both of them nearer to their quotas. That is, there should be no states i and j such that $q_i - (s_i - 1) < s_i - q_i$ and $s_j + 1 - q_j < q_j - s_j$.

Prove or disprove: Webster's method is the unique divisor method that is near quota.

Lemma 1 : Webster's method satisfies "near quota".

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Proof : Violation of near quota implies that there exist states i, j such that

$$q_i - (s_i - 1) < s_i - q_i \quad \text{and} \quad s_j + 1 - q_j < q_j - s_j .$$

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$$q_i - (s_i - 1) < s_i - q_i \quad \text{and} \quad s_j + 1 - q_j < q_j - s_j .$$

$$\Longrightarrow \quad s_i - 1/2 > q_i \quad \text{and} \quad s_j + 1/2 < q_j .$$

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$$\Longrightarrow \quad \underbrace{s_i - \tfrac{1}{2} > q_i}_{1} \quad \text{and} \quad \underbrace{s_j + \tfrac{1}{2} < q_j}_{2} .$$

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$$q_i - (s_i - 1) < s_i - q_i \quad \text{and} \quad s_j + 1 - q_j < q_j - s_j .$$

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By defn of Webster :

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By defn of Webster : $s_j + \frac{1}{2} \geq \frac{p_j}{D} \xRightarrow{2} \frac{h}{P} > \frac{1}{D}$.

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By defn of Webster : $s_j + \frac{1}{2} \geq \frac{p_j}{D} \xrightarrow{2} \frac{h}{P} > \frac{1}{D}$.

$s_i - \frac{1}{2} \leq \frac{p_i}{D} \xrightarrow{1} \frac{h}{P} \leq \frac{1}{D}$.

Contradiction!

∎

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[Corollary of Problem 1]

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[Corollary of Problem 1]

Fact 2 : Any divisor method other than Webster differs from Webster on some two-state problem.

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Proof : We will use the following facts about two-state problems :

Fact 1 : All divisor methods satisfy quota for any two-state problem.
[Corollary of Problem 1]

Fact 2 : Any divisor method other than Webster differs from Webster on some two-state problem.
Quiz!

QUIZ

Prove or disprove:

Any divisor method other than Webster differs from Webster on some two-state problem.

Let f be a divisor method different from Webster.

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Fact 1 + Fact 2 $\Rightarrow$ There exists a two-state instance such that

Webster : $(x, y+1)$ $\qquad\qquad$ f : $(x+1, y)$

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Webster : $(x, y+1)$ $\quad\quad$ f: $(x+1, y)$

$$\frac{p_1}{x+\frac{1}{2}} < \frac{p_2}{y+\frac{1}{2}}$$

"last" seat assigned as per Webster rounding

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Webster : $(x, y+1)$ $\qquad$ f: $(x+1, y)$

$$\frac{p_1}{x+\frac{1}{2}} < \frac{p_2}{y+\frac{1}{2}}$$ "last" seat assigned as per Webster rounding

At $h = x+y+1$, state 1's quota $q_1 < x + \frac{1}{2}$ (check!)

state 2's quota $q_2 > y + \frac{1}{2}$ (check!)

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At $h = x+y+1$, state 1's quota $q_1 < x + \frac{1}{2}$ (check!)

state 2's quota $q_2 > y + \frac{1}{2}$ (check!)

$\Rightarrow$ f is not near-quota. $\blacksquare$

PROBLEM 4

Problem 4 [25 points]

Prove or disprove: Grimmett's randomized apportionment method (including its initial random permutation and random offset) is selection monotone.

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Solution by Anish and Amaan

Selection Monotonicity

[Correa, Gölz, Schmidt-Kraepelin, Tucker-Foltz, and Verdugo ; EC 2024]

Let $\vec{r}, \vec{r}' \in [0,1)^n$ be two residue vectors summing up to the same integer k. Let T be a set of k states such that

$$r_i' \geq r_i \quad \forall\ i \in T \qquad \text{and} \qquad r_i' \leq r_i \quad \forall\ i \notin T.$$

A randomized apportionment method f satisfies selection monotonicity if

$$\Pr_{S \sim f(\vec{r}')}[S = T] \geq \Pr_{S \sim f(\vec{r})}[S = T].$$

Consider a six-state problem with

$$n = 3 \qquad p_1 = p_2 = p_3 = 9 \qquad p_4 = p_5 = p_6 = 1$$

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Let $\vec{r} = (0.9,\ 0.9,\ 0.9,\ 0.1,\ 0.1,\ 0.1)$

and $\vec{r}' = (0.9,\ 0.9,\ 0.8,\ 0.1,\ 0.1,\ 0.2)$

Consider a six-state problem with

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and $\vec{r}' = (0.9,\ 0.9,\ 0.8,\ \underbrace{0.1,\ 0.1,\ 0.2}_{T})$

Let T = states 4, 5, and 6.

Consider a six-state problem with

$$h = 3 \qquad p_1 = p_2 = p_3 = 9 \qquad p_4 = p_5 = p_6 = 1$$

Let $\vec{r} = (0.9, 0.9, 0.9, 0.1, 0.1, 0.1)$

and $\vec{r}\,' = (0.9, 0.9, 0.8, \underbrace{0.1, 0.1, 0.2}_{T})$

Let T = states 4, 5, and 6.

Observe that

$$r_i' \geq r_i \quad \forall\ i \in T \qquad \text{and} \qquad r_i' \leq r_i \quad \forall\ i \notin T.$$

Consider a six-state problem with

$$k=3 \qquad p_1=p_2=p_3=9 \qquad p_4=p_5=p_6=1$$

Let $\vec{r} = (0.9,\ 0.9,\ 0.9,\ 0.1,\ 0.1,\ 0.1)$

and $\vec{r}\,' = (0.9,\ 0.9,\ 0.8,\ \underbrace{0.1,\ 0.1,\ 0.2}_{T})$

To show failure of selection monotonicity:

$$\Pr_{S\sim f(\vec{r}\,')}[S=T] < \Pr_{S\sim f(\vec{r})}[S=T].$$

Analysis for $\vec{r} = (0.9, 0.9, 0.9, \underbrace{0.1, 0.1, 0.1}_{T})$

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* Disregard any permutation where T and $\overline{T}$ are not alternating

Analysis for $\vec{r} = (0.9,\ 0.9,\ 0.9,\ \underbrace{0.1,\ 0.1,\ 0.1}_{T})$

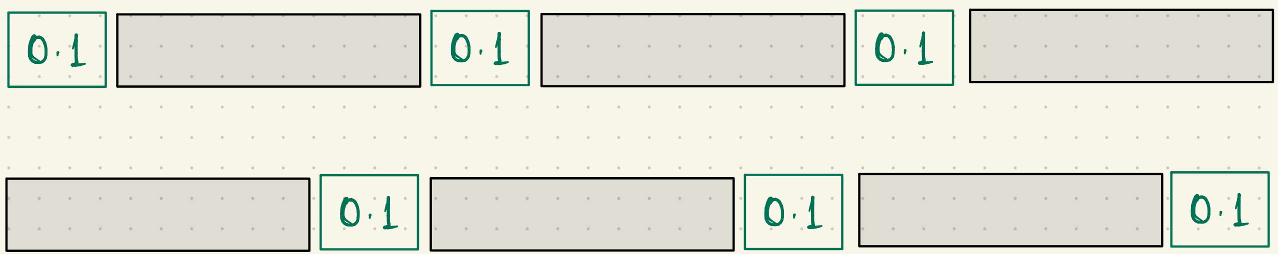
* Disregard any permutation where T and $\overline{T}$ are not alternating

at least two states in T are either "too close" or "too far".

Analysis for $\vec{r} = (0.9,\ 0.9,\ 0.9,\ \underbrace{0.1,\ 0.1,\ 0.1}_{T})$

* Disregard any permutation where T and $\overline{T}$ are not alternating

* For alternating permutations



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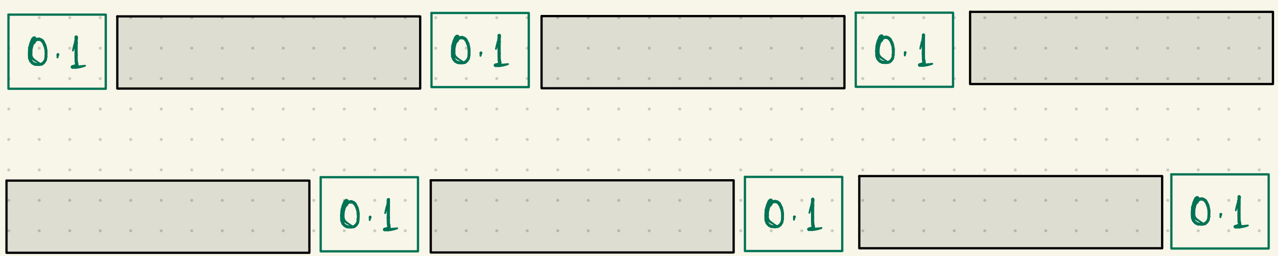

$$\Pr(\text{all states in } T \text{ receive seats}) = \frac{3! \times 3! \times 2}{6!} \times 0.1$$

Analysis for $\vec{r'} = (0.9,\ 0.9,\ 0.8,\ \underbrace{0.1,\ 0.1,\ 0.2}_{T})$

Analysis for $\vec{r'} = (0.9, 0.9, 0.8, \underbrace{0.1, 0.1, 0.2}_{T})$

* Disregard any permutation where T and $\overline{T}$ are not alternating

Similar reasoning

Analysis for $\vec{r'} = (0.9,\ 0.9,\ 0.8,\ \underbrace{0.1,\ 0.1,\ 0.2}_{T})$

* Disregard any permutation where T and $\overline{T}$ are not alternating

* $\exists$ an alternating permutation where, for any value of the offset, states in T fail to receive all seats.

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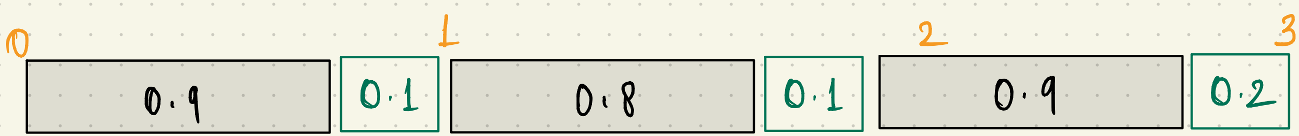

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* Disregard any permutation where T and $\overline{T}$ are not alternating

* $\exists$ an alternating permutation where, for any value of the offset, states in T fail to receive all seats.

$$\Pr(\text{all states in } T \text{ receive seats}) < \frac{3! \times 3! \times 2}{6!} \times 0.1 \qquad \square$$