COL 8184 : ALGORITHMS FOR FAIR REPRESENTATION

LECTURE 10

BIZARRE CORRELATIONS IN GRIMMETT'S OUTCOMES

FEB 16, 2026 | ROHIT VAISH

GRIMMETT'S METHOD

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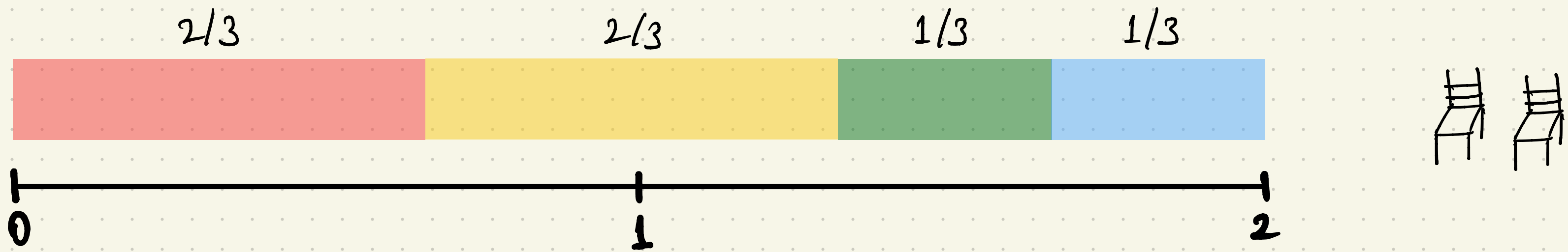
* Consider a unif. random permutation of the states (wolog, identity).

* Provisionally assign $\lfloor q_i \rfloor$ seats to state i. Let $r_i := q_i - \lfloor q_i \rfloor$.
  i.e., the residue

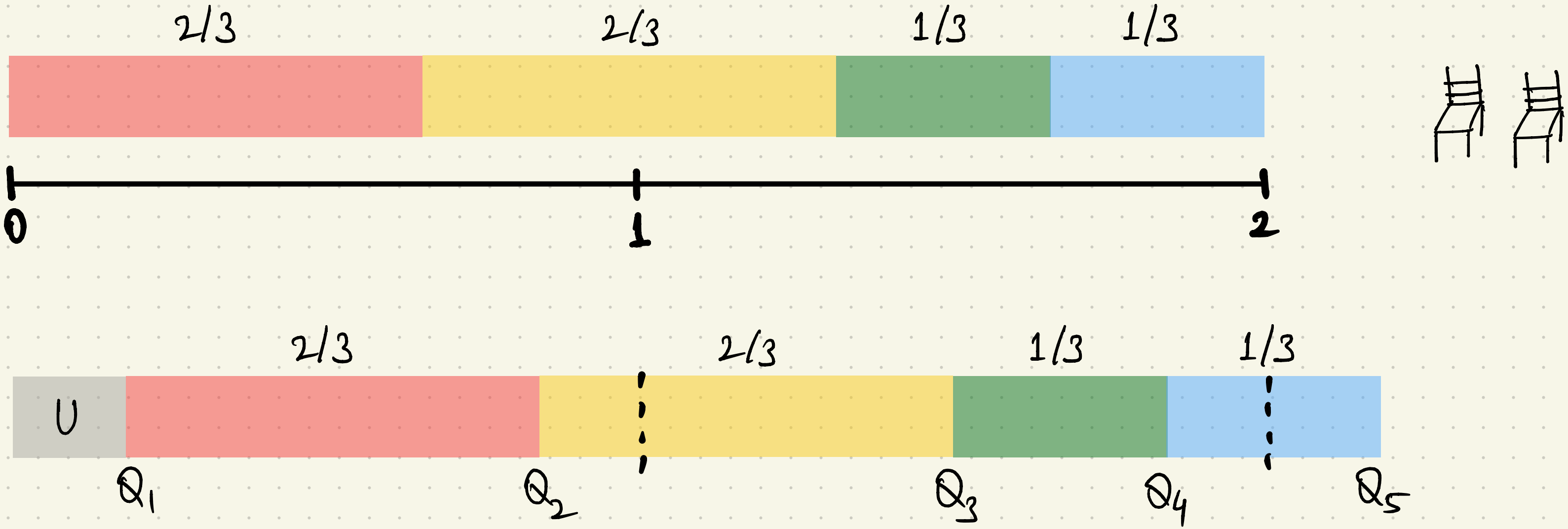
* Draw $U \sim \text{Unif}[0,1]$. Let $Q_i := U + \sum_{j=1}^{i-1} r_j$.

* For each state i, allocate an extra seat to state i if $[Q_i, Q_{i+1})$ contains an integer.

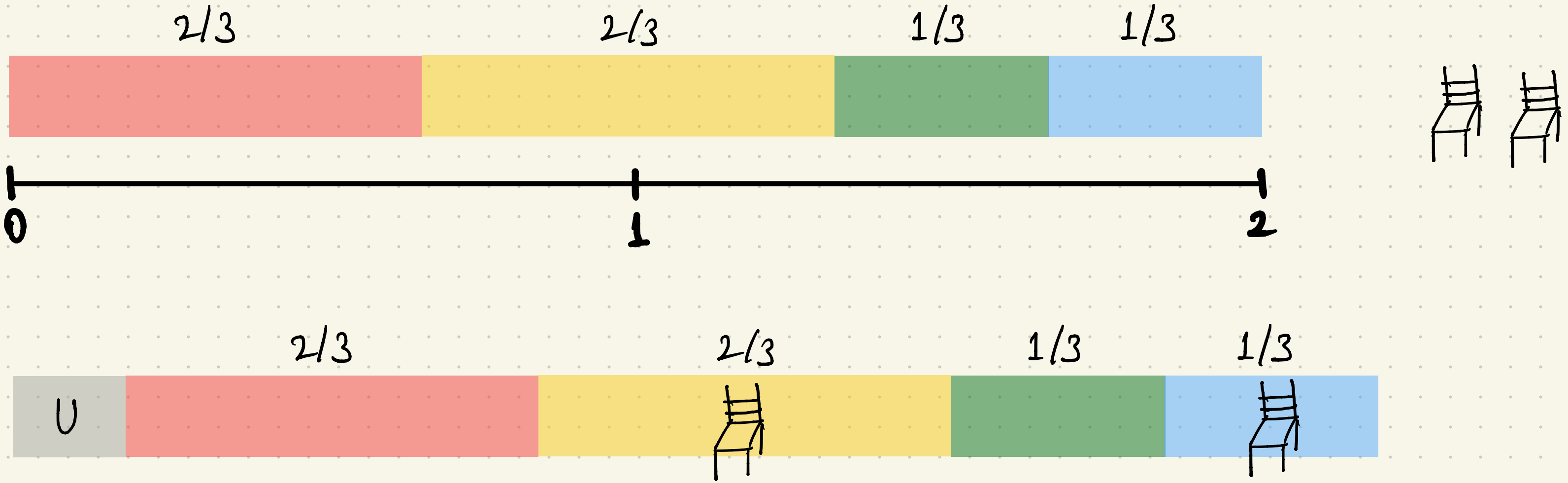
Grimmett's Method



Grimmett's Method



GRIMMETT'S METHOD



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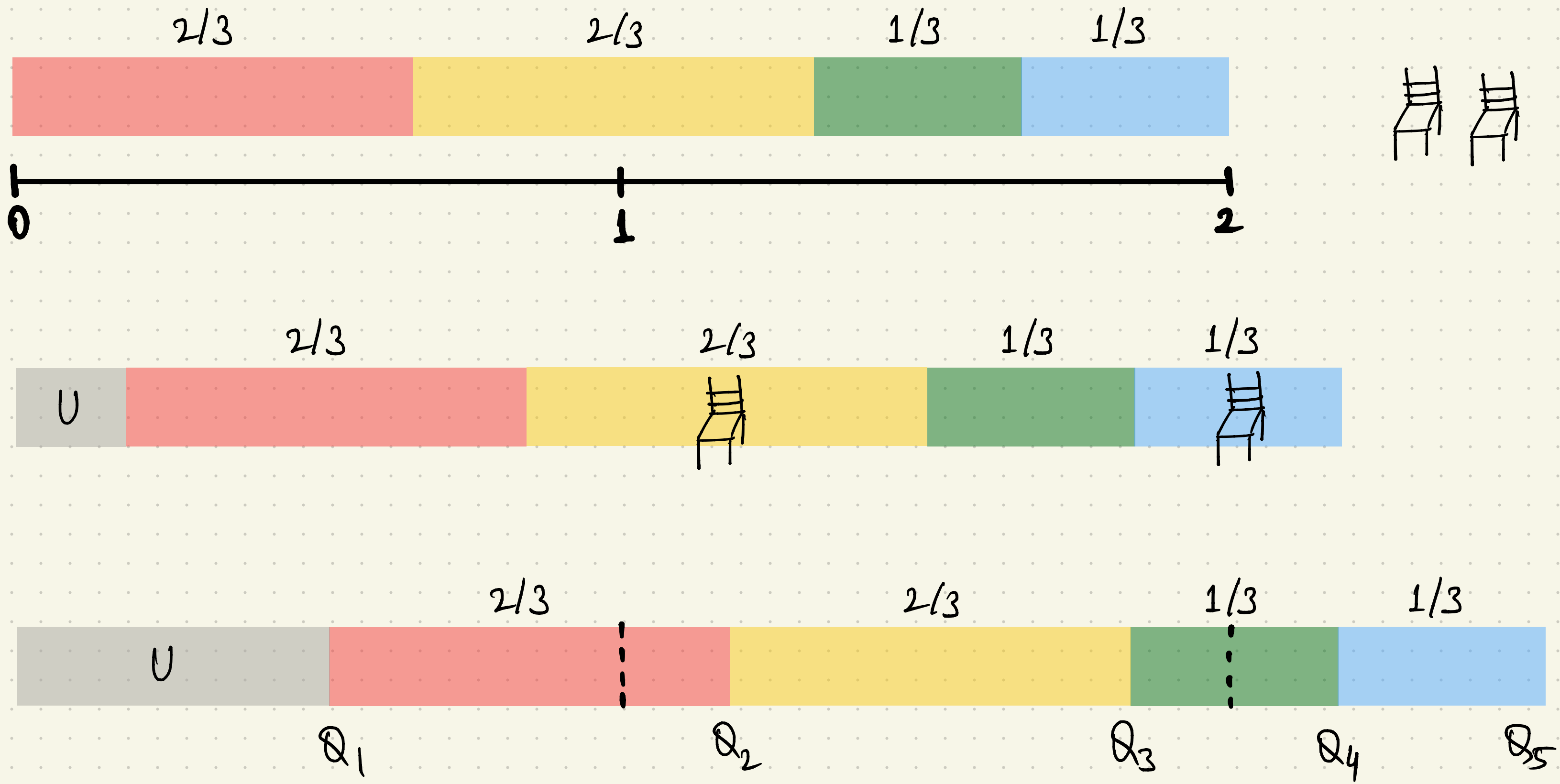
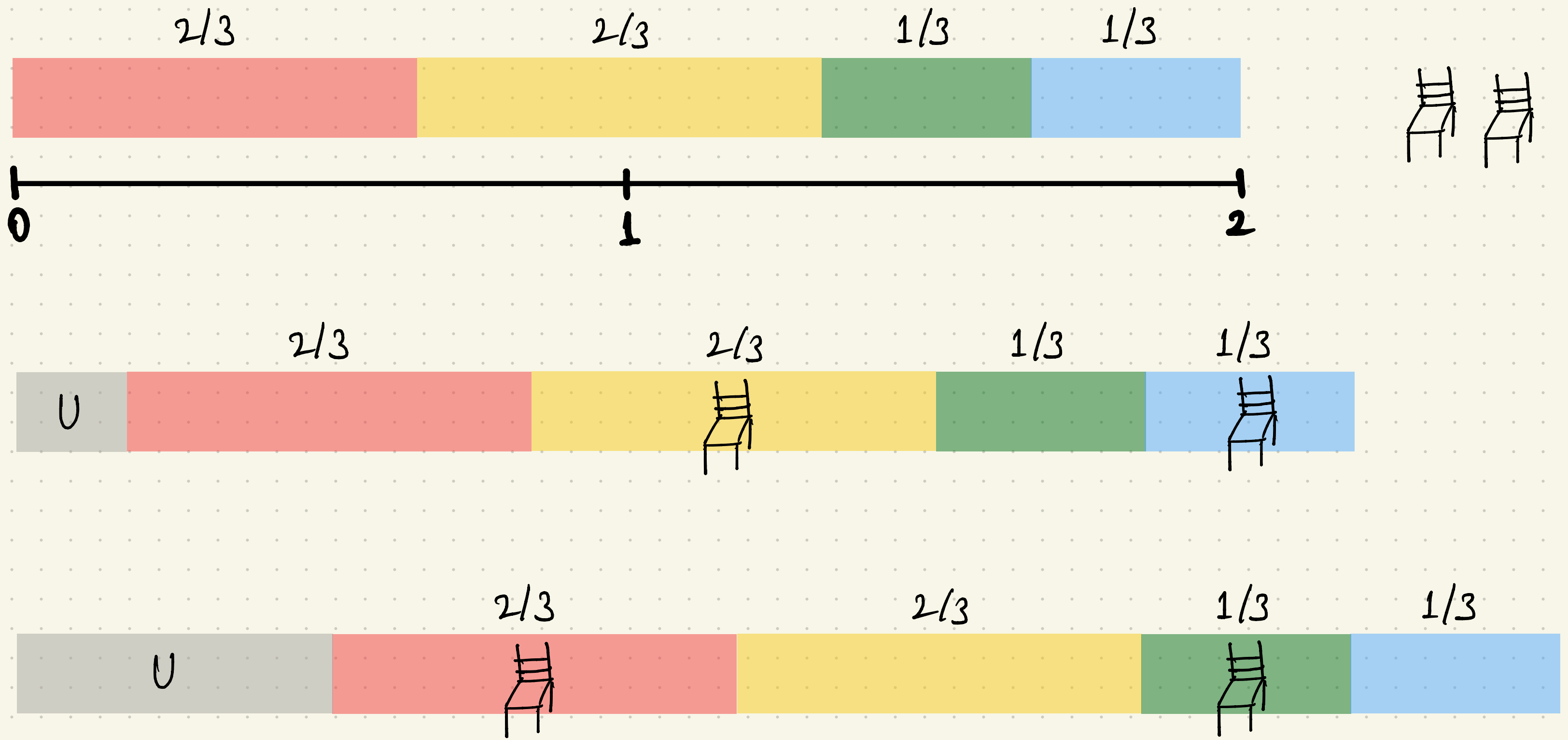
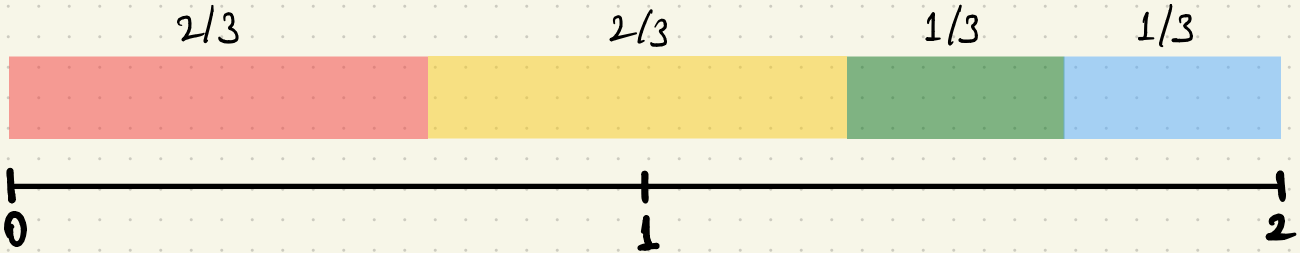
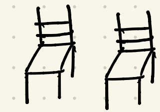


Grimmett's Method

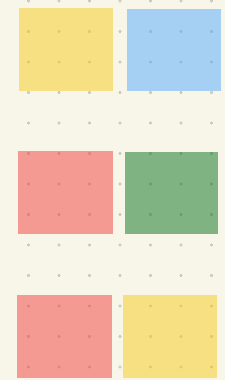


Grimmett's Method





Resulting probability distribution is



w.p. 1/3

w.p. 1/3

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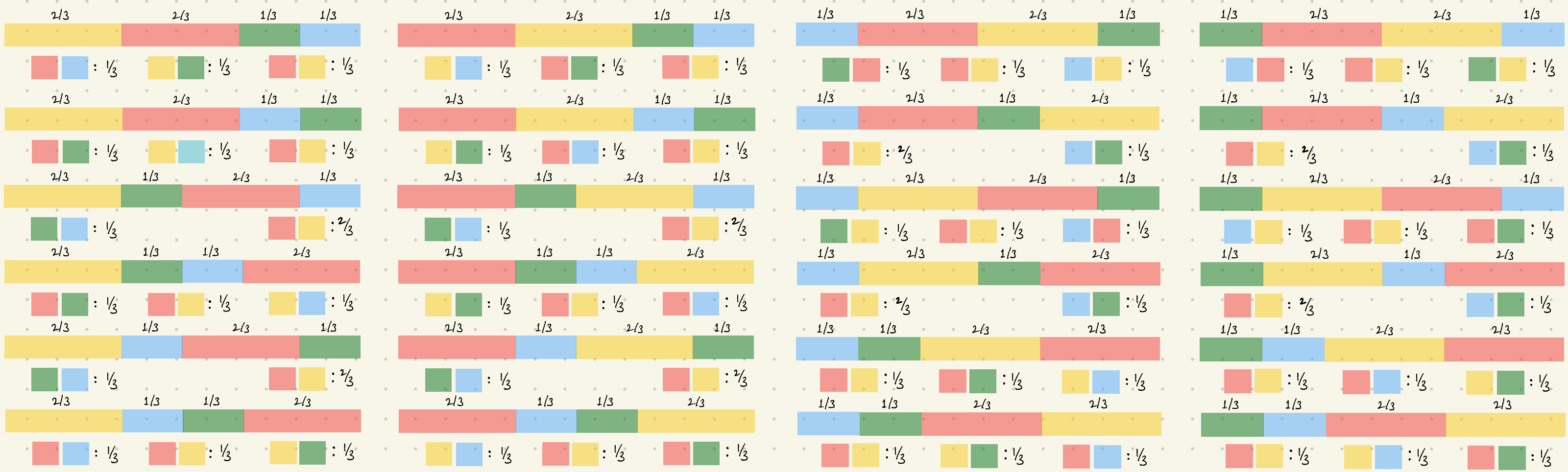
GRIMMETT'S METHOD

4! permutations

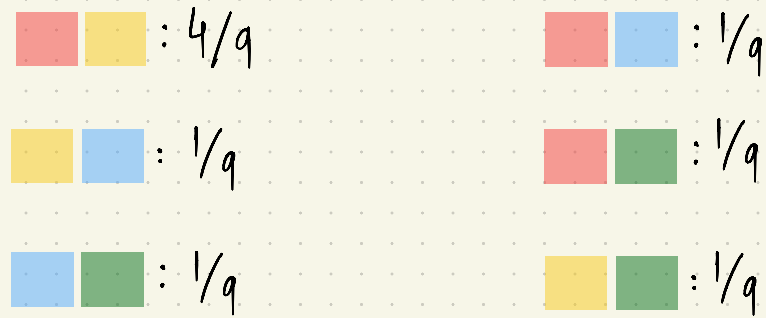
GRIMMETT'S METHOD



4! permutations and the resulting distributions

GRIMMETT'S METHOD

Overall distribution :



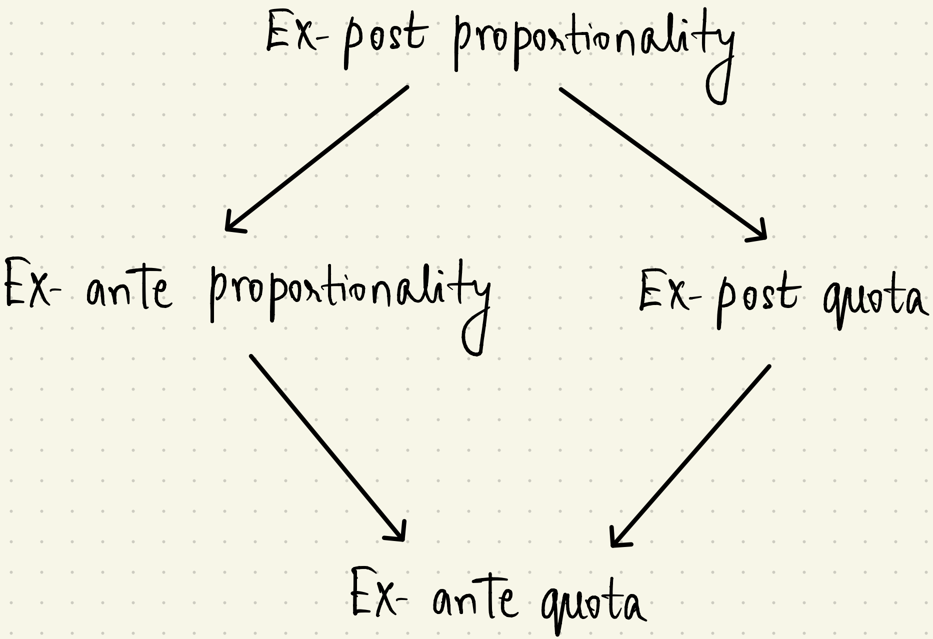
PROPERTIES OF GRIMMETT'S METHOD

Theorem : Grimmett's method satisfies :

* ex-ante proportionality
* ex-ante population monotonicity
* ex-post quota

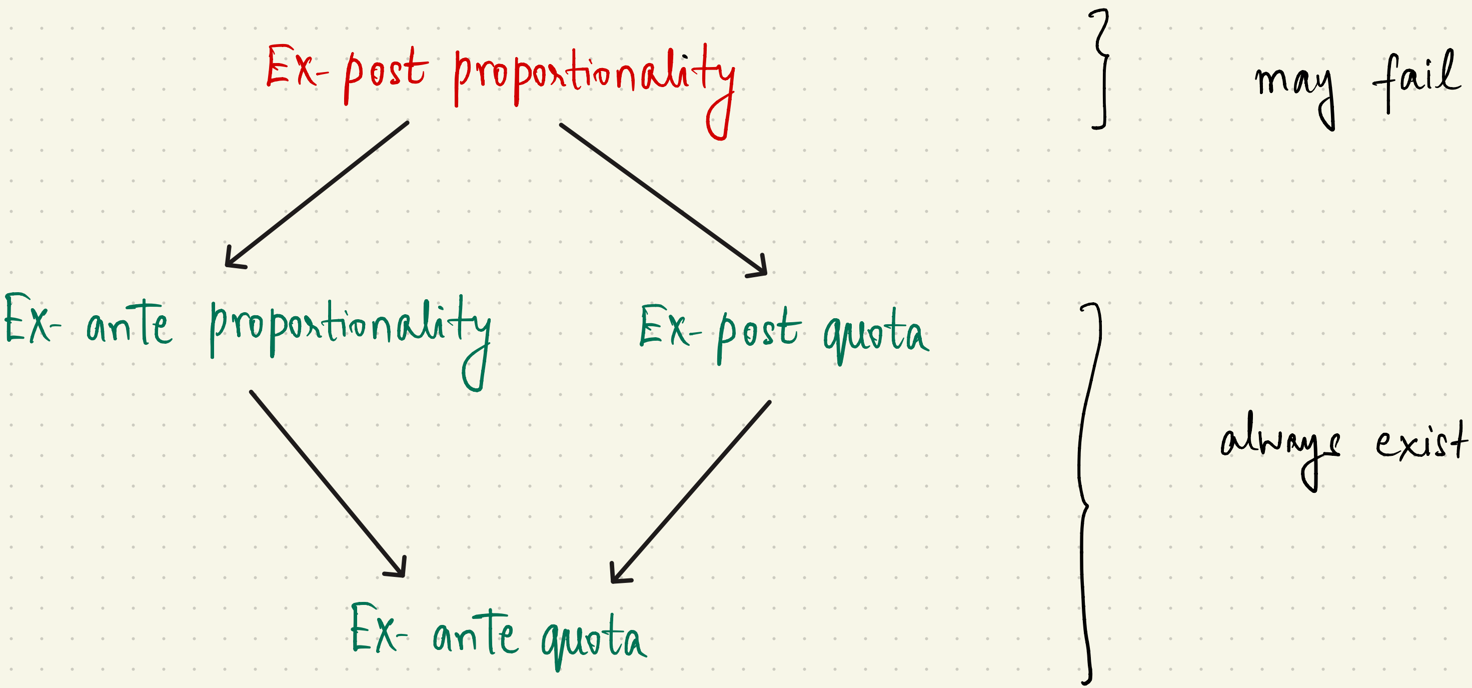
LANDSCAPE

LANDSCAPE

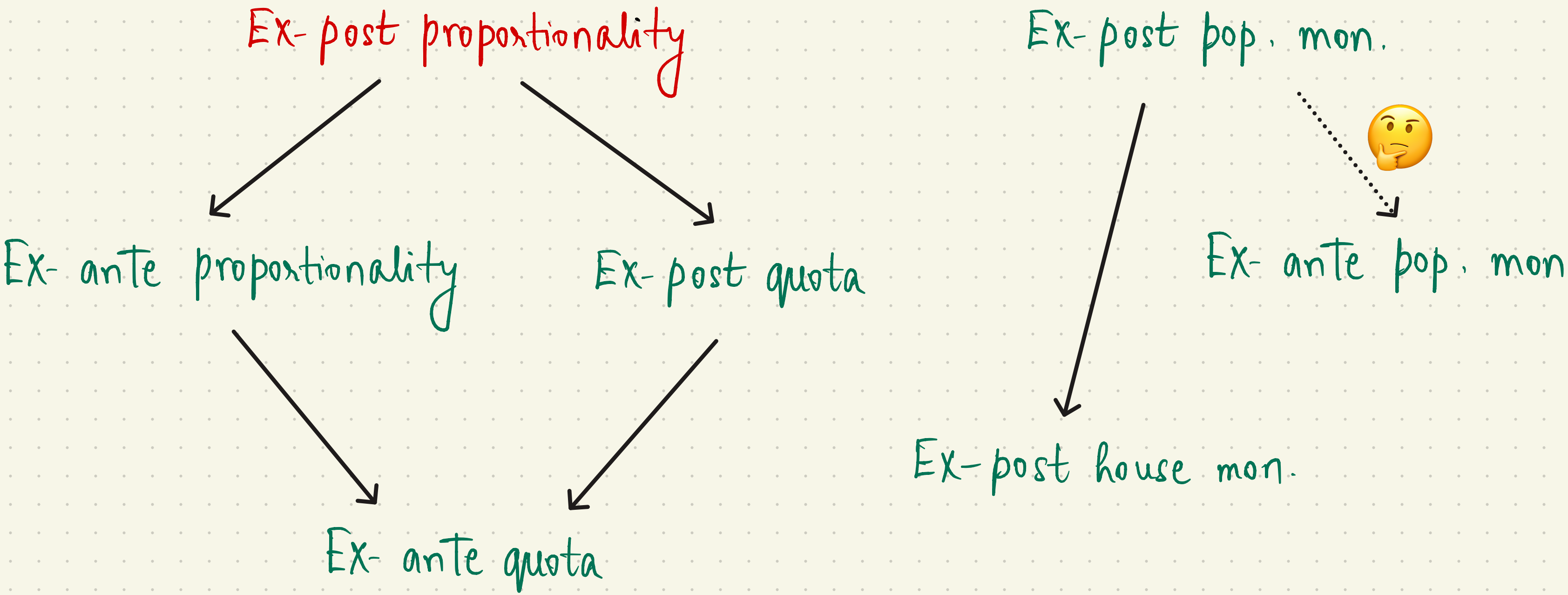


Ex-post proportionality
Ex-ante proportionality
Ex-post quota
Ex-ante quota

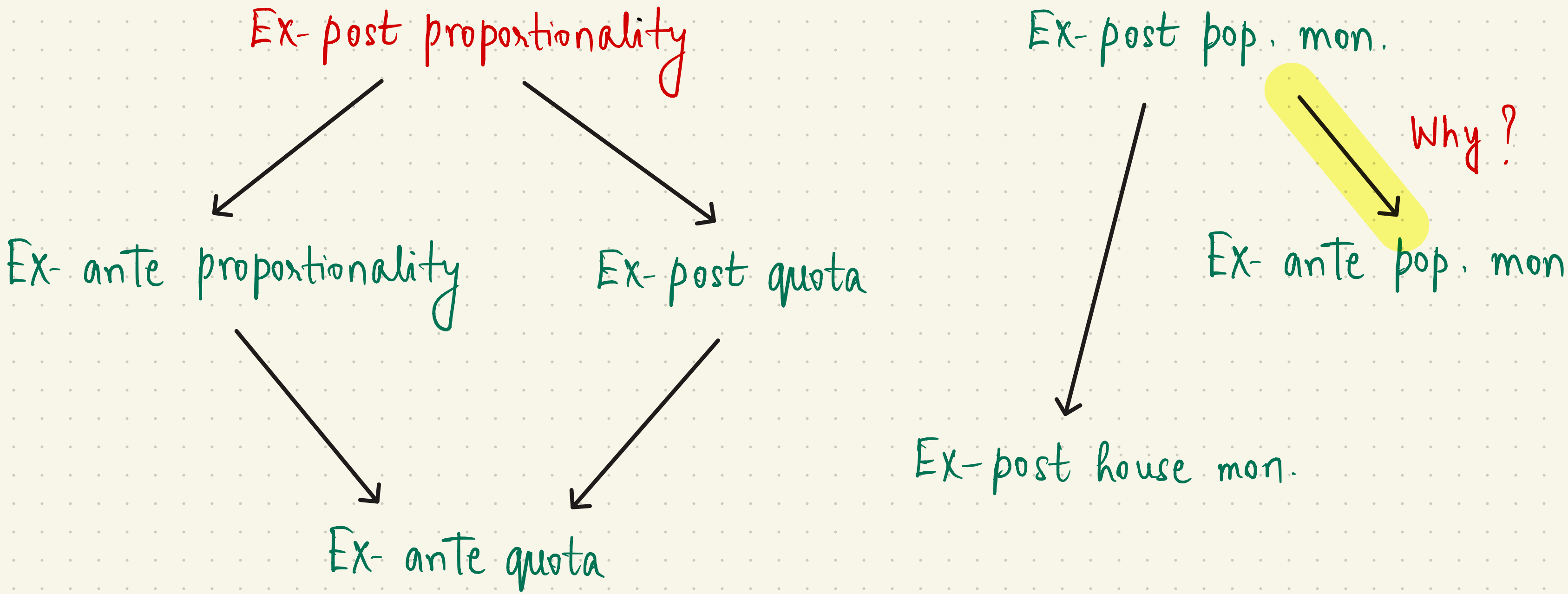
LANDSCAPE


Ex-post proportionality
Ex-ante proportionality
Ex-post quota
Ex-ante quota
may fail
always exist

LANDSCAPE


Ex-post proportionality
Ex-ante proportionality
Ex-post quota
Ex-ante quota
Ex-post pop. mon.
Ex-ante pop. mon.
Ex-post house mon.

LANDSCAPE


Ex-post proportionality
Ex-ante proportionality
Ex-post quota
Ex-ante quota
Ex-post pop. mon.
Why ?
Ex-ante pop. mon.
Ex-post house mon.

LANDSCAPE

Ex-post population monotonicity $\Rightarrow$ Ex-ante population monotonicity

Any ex-post population monotone method must be deterministic.

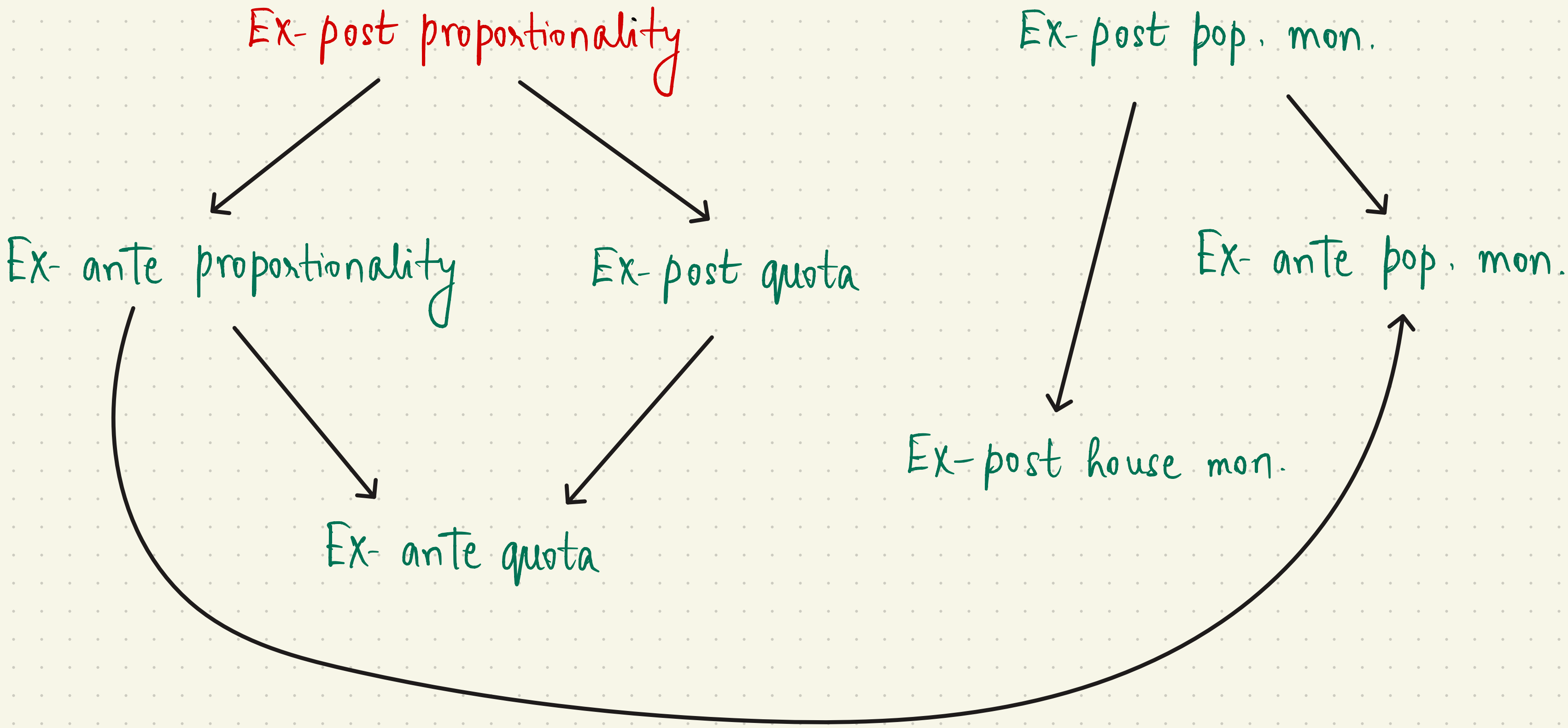
Indeed, if not, then for some instance I, there must exist two distinct seat assignments $(s_1, \ldots, s_n)$ and $(s_1', \ldots, s_n')$ in the support.

Then, there must exist a pair of states i and j such that $s_i < s_i'$ and $s_j > s_j'$. However, the populations of i and j haven't changed. ∎

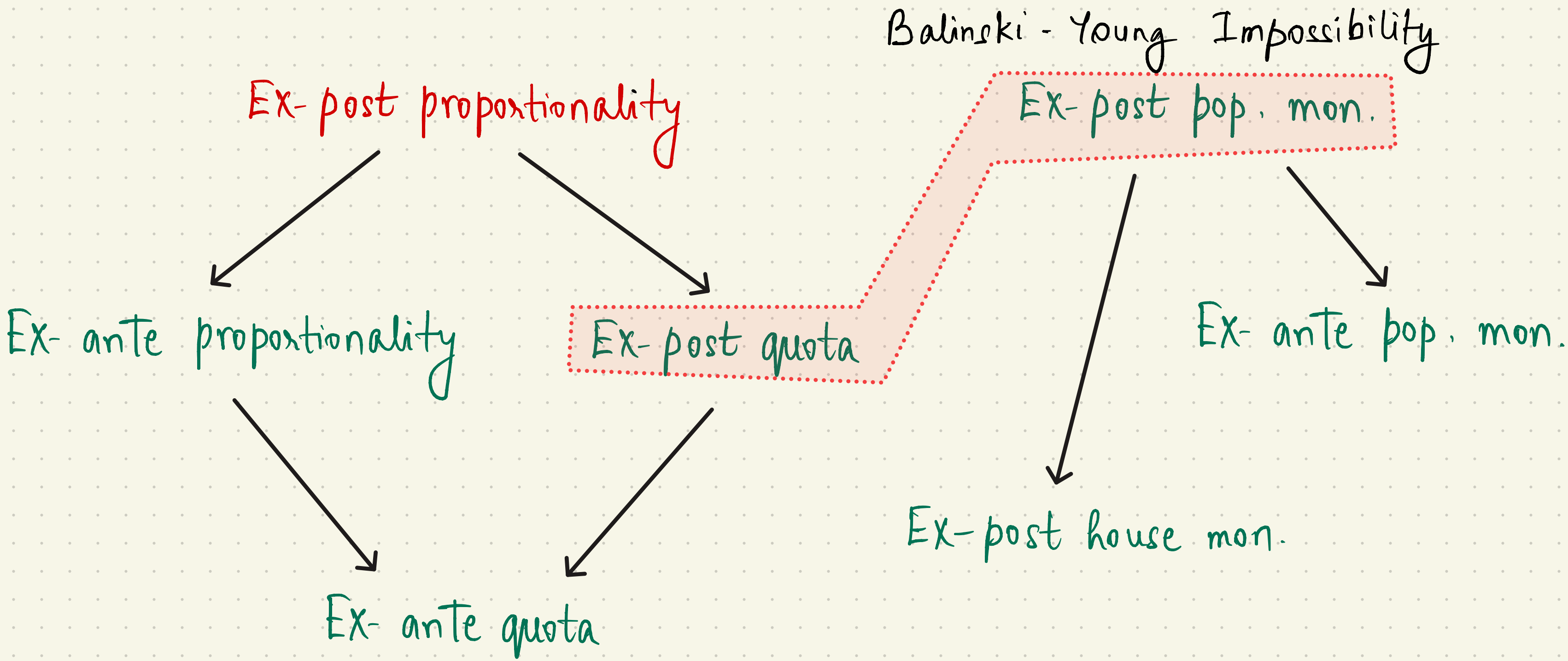
Ex-post population monotonicity $\not\Leftarrow$ Ex-ante population monotonicity

Grimmett's method is ex-ante but not ex-post population monotone. ∎

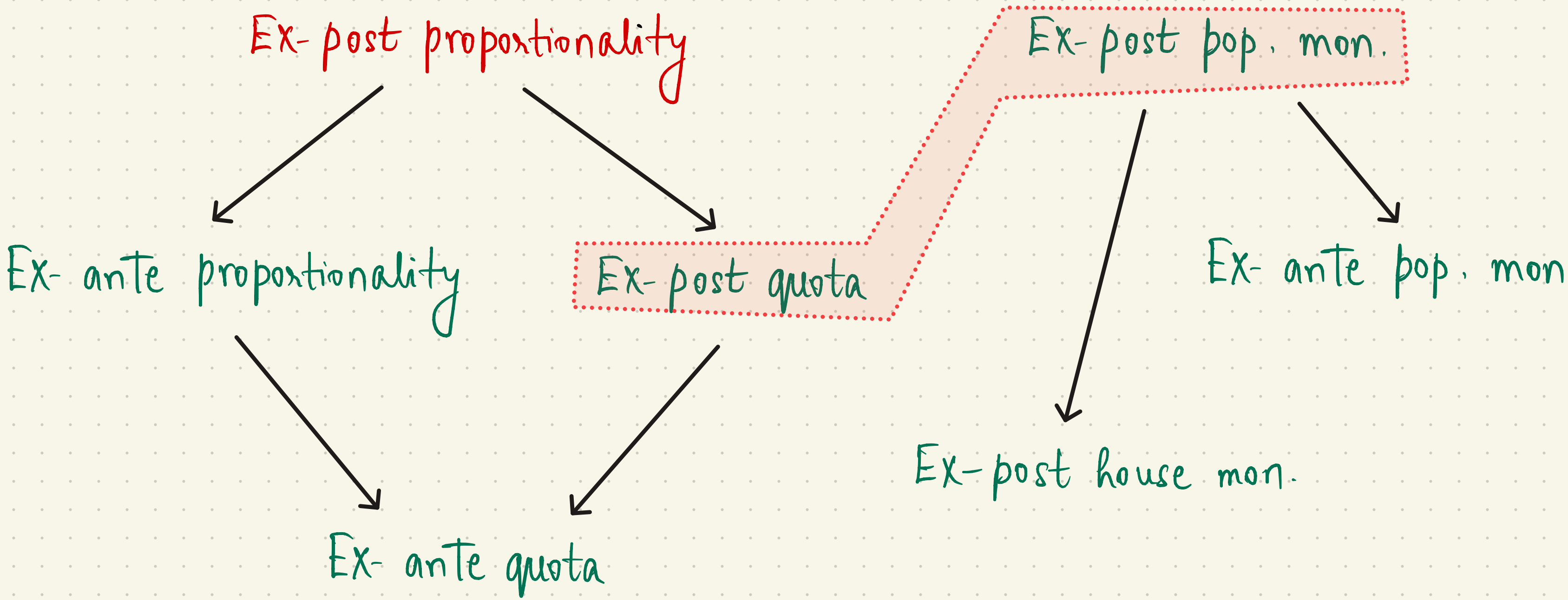
LANDSCAPE


Ex-post proportionality
Ex-post pop. mon.
Ex-ante proportionality
Ex-post quota
Ex-ante pop. mon.
Ex-post house mon.
Ex-ante quota

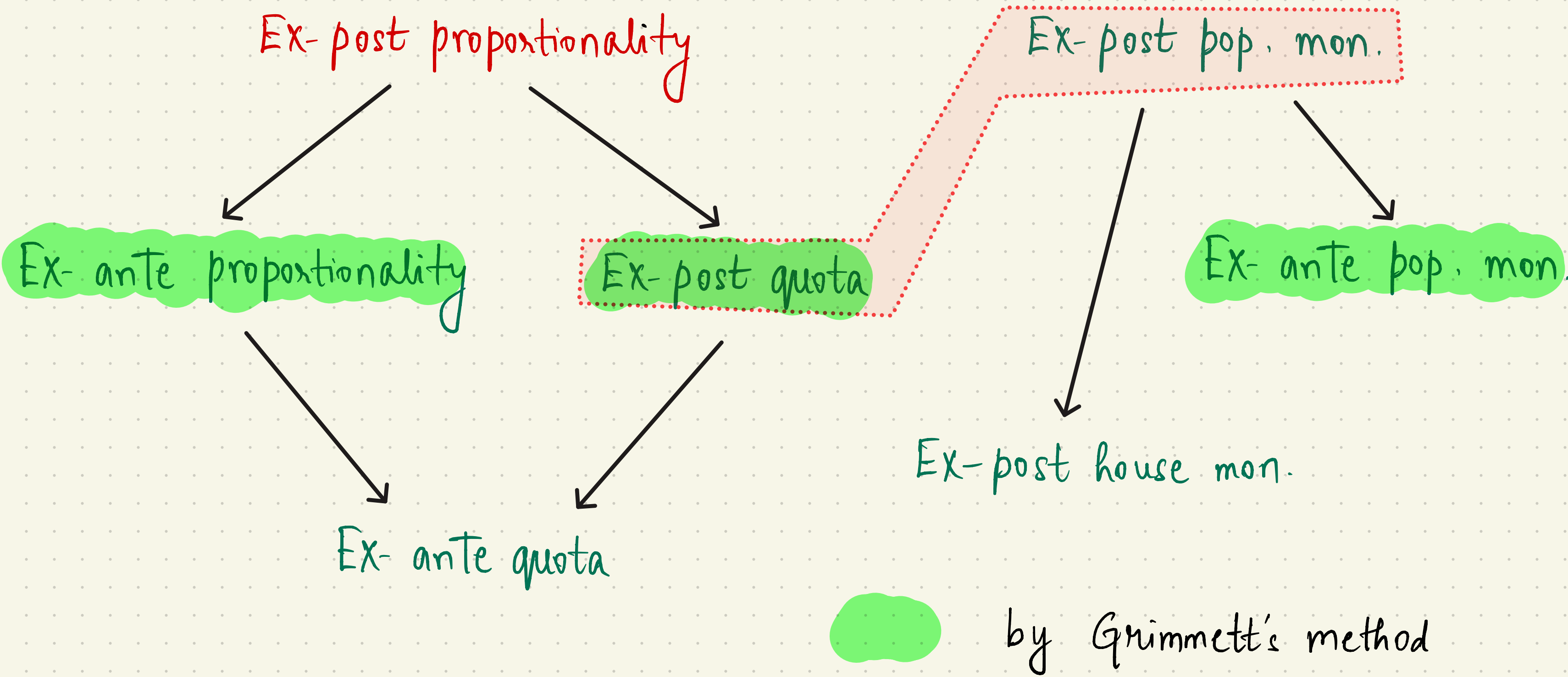
LANDSCAPE


Balinski - Young Impossibility
Ex-post proportionality
Ex-post pop. mon.
Ex-ante proportionality
Ex-post quota
Ex-ante pop. mon.
Ex-post house mon.
Ex-ante quota

LANDSCAPE

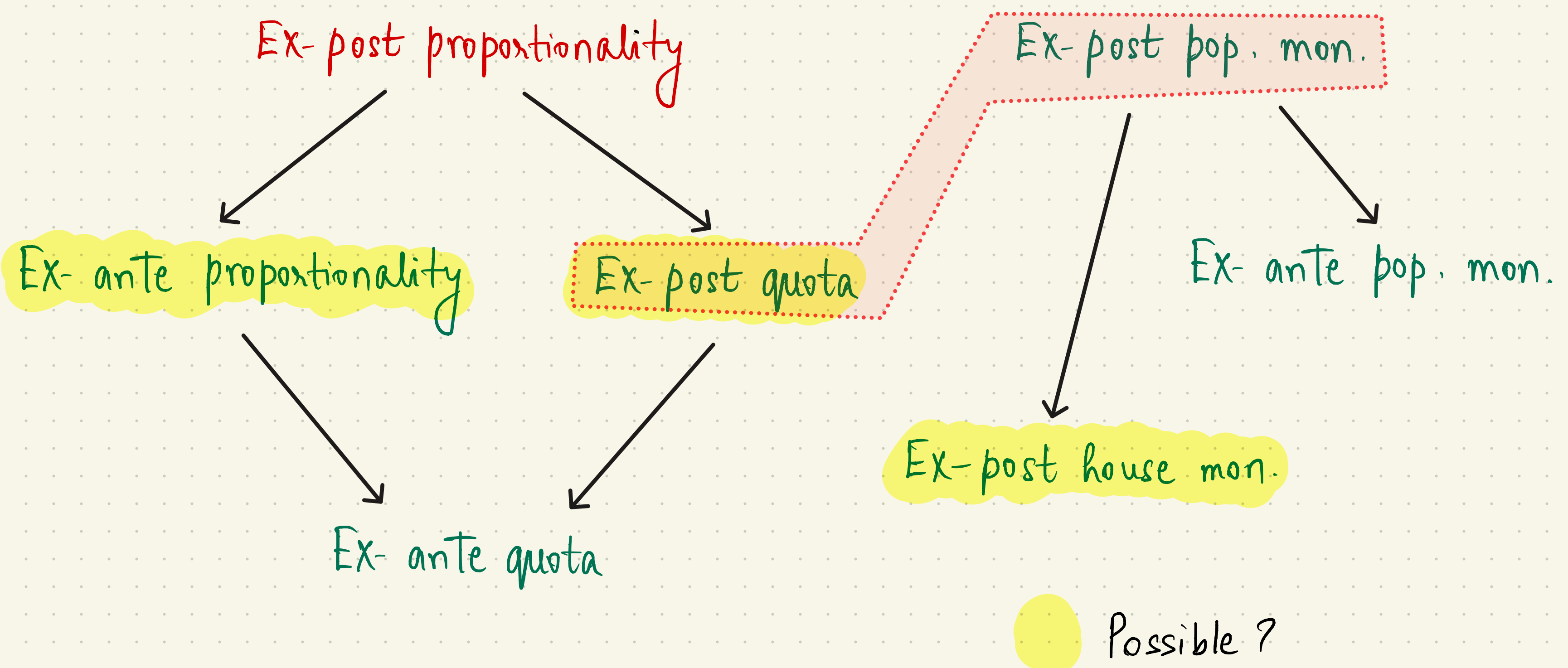

Ex-post proportionality
Ex-ante proportionality
Ex-post quota
Ex-ante quota
Ex-post pop. mon.
Ex-ante pop. mon.
Ex-post house mon.

LANDSCAPE



Ex-post proportionality
Ex-post pop. mon.
Ex-ante proportionality
Ex-post quota
Ex-ante pop. mon.
Ex-post house mon.
Ex-ante quota
by Grimmett's method

LANDSCAPE


Ex-post proportionality
Ex-ante proportionality
Ex-post quota
Ex-ante quota
Ex-post pop. mon.
Ex-ante pop. mon.
Ex-post house mon.
Possible ?

GRIMMETT'S METHOD FAILS HOUSE MONOTONICITY

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$h = 2$ Four states with populations $(1, 2, 1, 2)$.

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Identity permutation and $U > \frac{2}{3} \Rightarrow$

Grimmett returns the seat allocation $(1, 0, 1, 0)$.

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at least one out of state 2 or state 4 still receives 0 seats.

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GRIMMETT'S METHOD FAILS HOUSE MONOTONICITY

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Identity permutation and $U > \frac{2}{3} \Rightarrow$ "toxic" assignment

Grimmett returns the seat allocation $(1,0,1,0)$.

Ex-post house monotonicity requires that for $h=3$,

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But quota compliance requires both states to receive exactly

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Quiz

Quiz I

QUIZ I

Calculate the probability distribution returned by Grimmett's method when the identity permutation of states is sampled in step I.

$h = 11$

State	1	110
State	2	270
State	3	210
State	4	160
State	5	70
State	6	280

QUIZ II

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Now suppose the odd-numbered states form a RED coalition, and even-numbered states form a BLUE coalition.

Under Grimmett's outcome with identity permutation, what is the probability that the RED coalition enjoys an ex-post majority in the house?

$h = 11$

State	1	110
State	2	270
State	3	210
State	4	160
State	5	70
State	6	280

QUIZ III

QUIZ III

Now suppose the populations of the states changes as shown.

Under Grimmett's outcome with identity permutation, what is the probability that the RED coalition enjoys an ex-post majority in the house?

$h = 11$

State	1	110	
State	2	270	(+20)
State	3	210	
State	4	160	(+30)
State	5	70	(−60)
State	6	280	(+10)