Cake Cutting

How to fairly divide a cake

among agents with differing preferences?

Why is this problem interesting?

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Fair

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Is this division fair?

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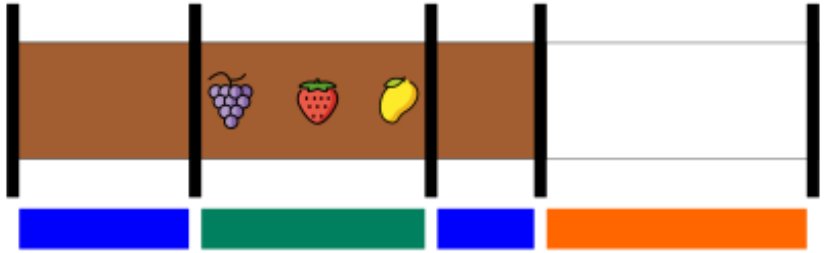

A fairer division

Why is this problem interesting?

Preferences matter!

The Model

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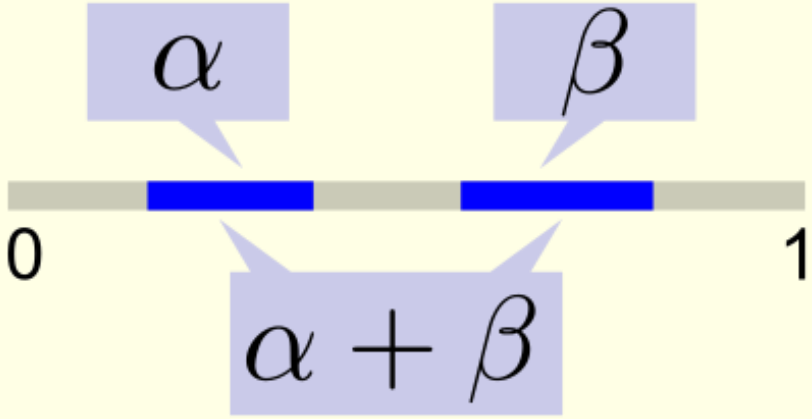

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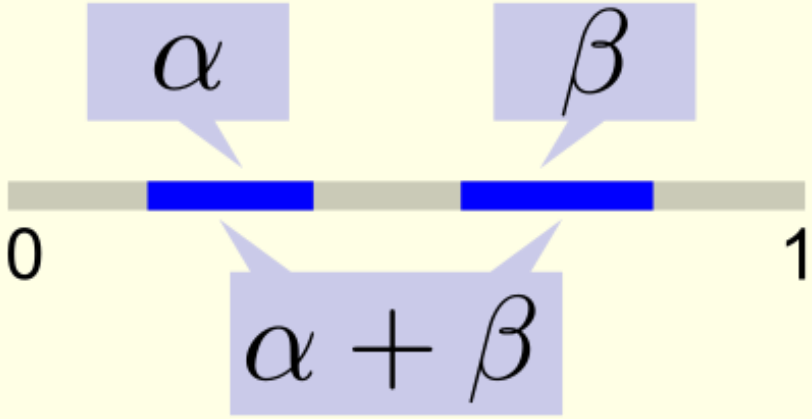

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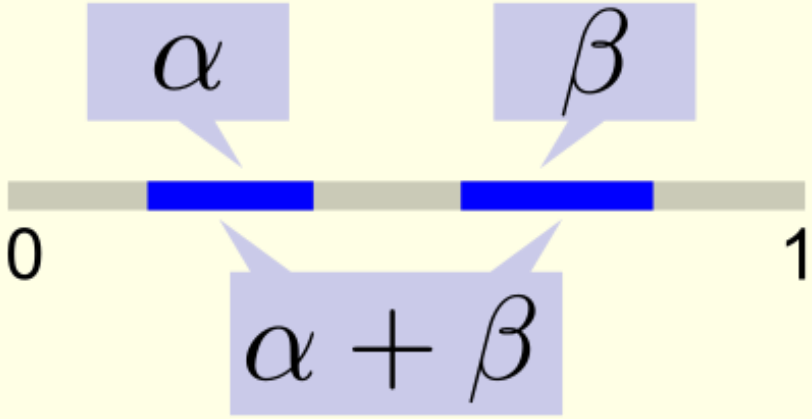

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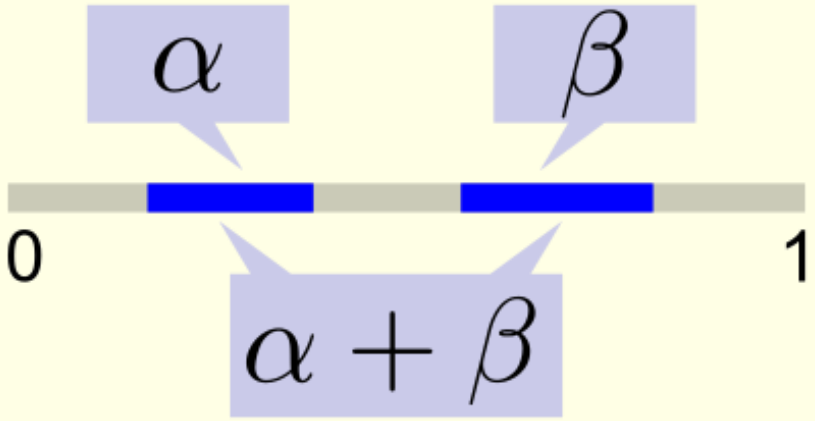

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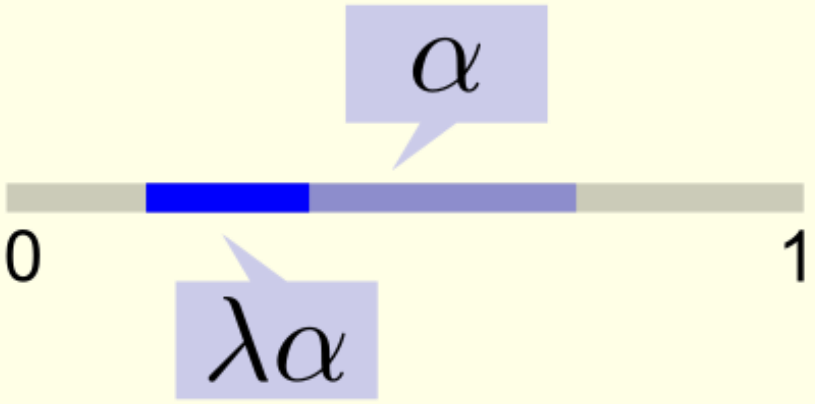

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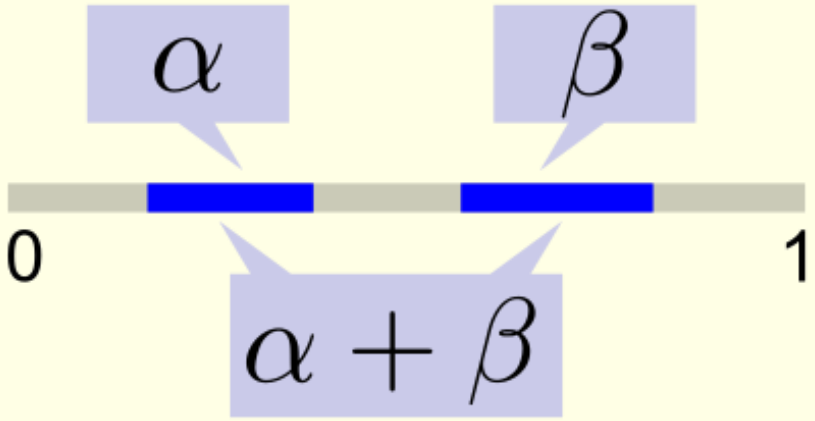

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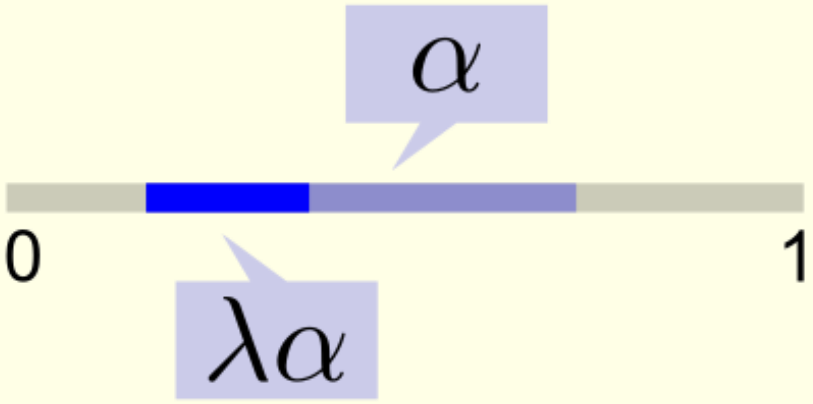

Normalization: for each agent i, $v_i([0,1]) = 1$.

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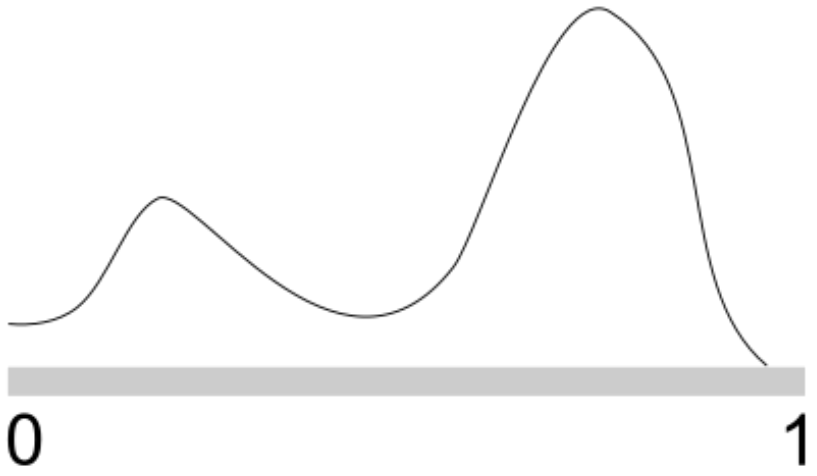
$$v_i(X) = \int_{x \in X} f_i(x) dx$$



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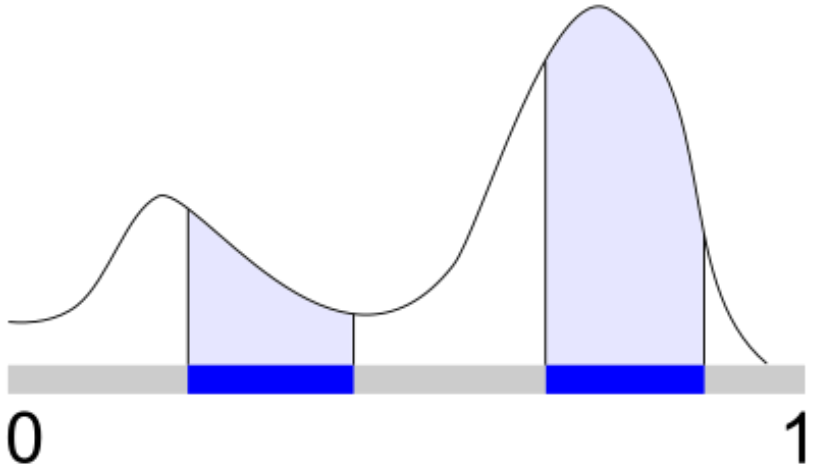
value density function



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- *Allocation/Division*: A partition $(A_1, A_2, \ldots, A_n)$ of the cake [0,1] where each A_i is a piece of cake.

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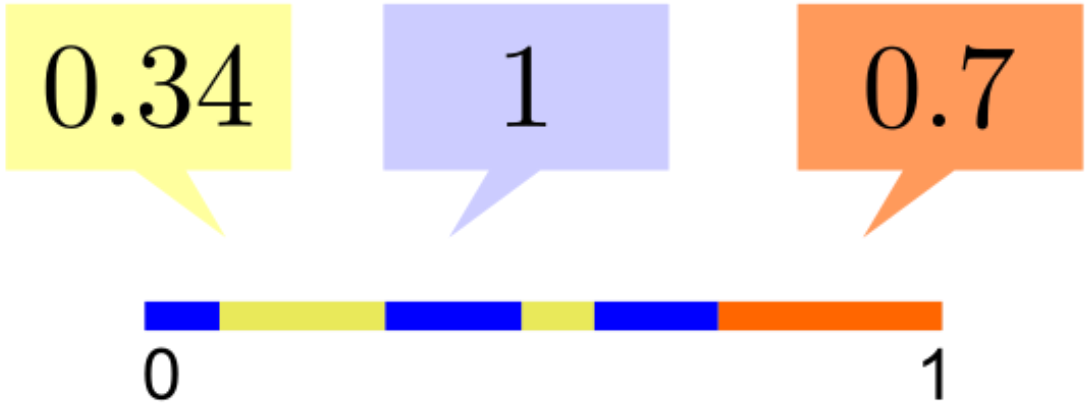

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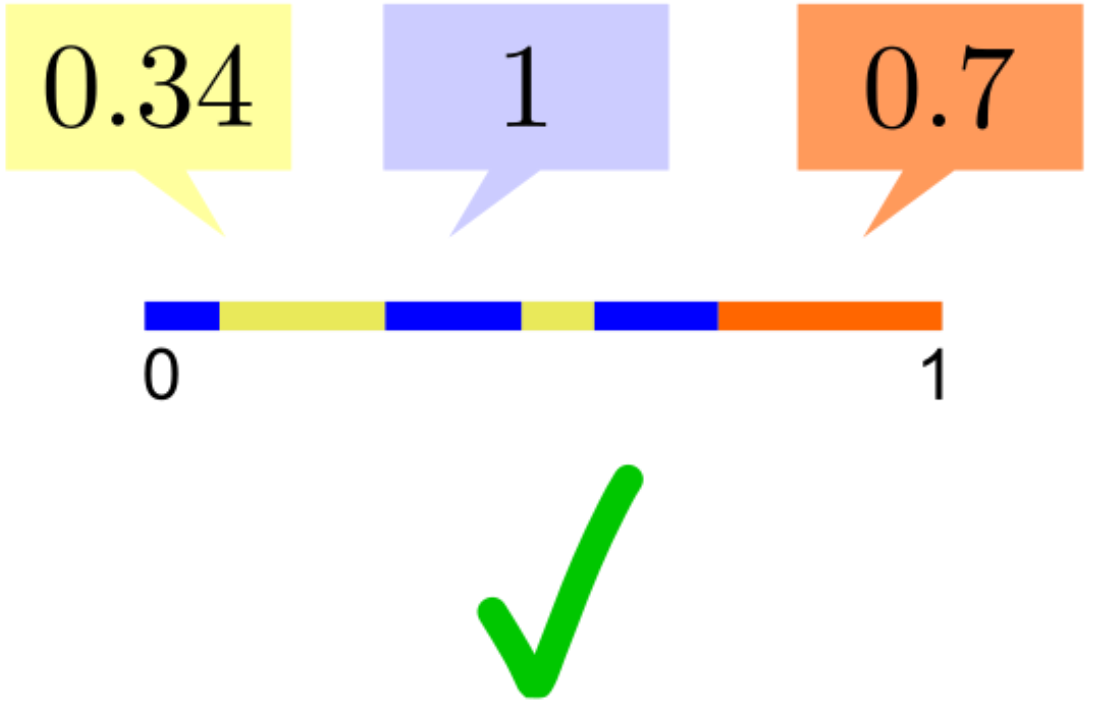

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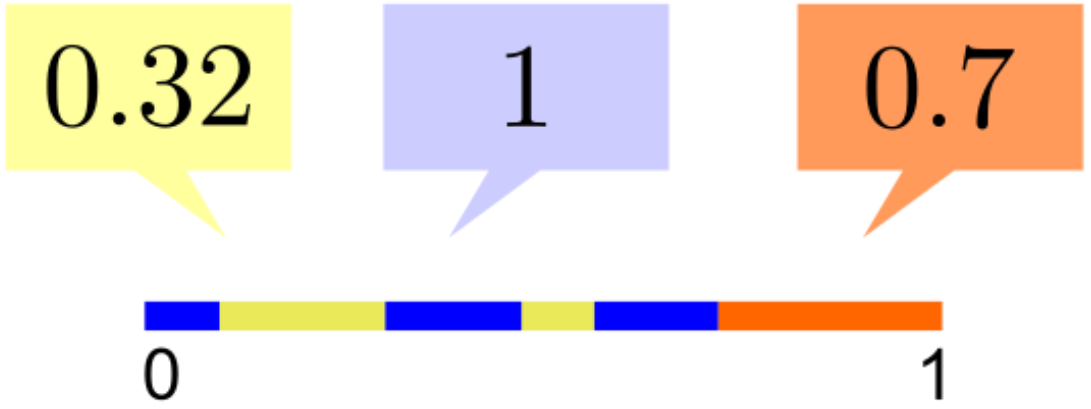

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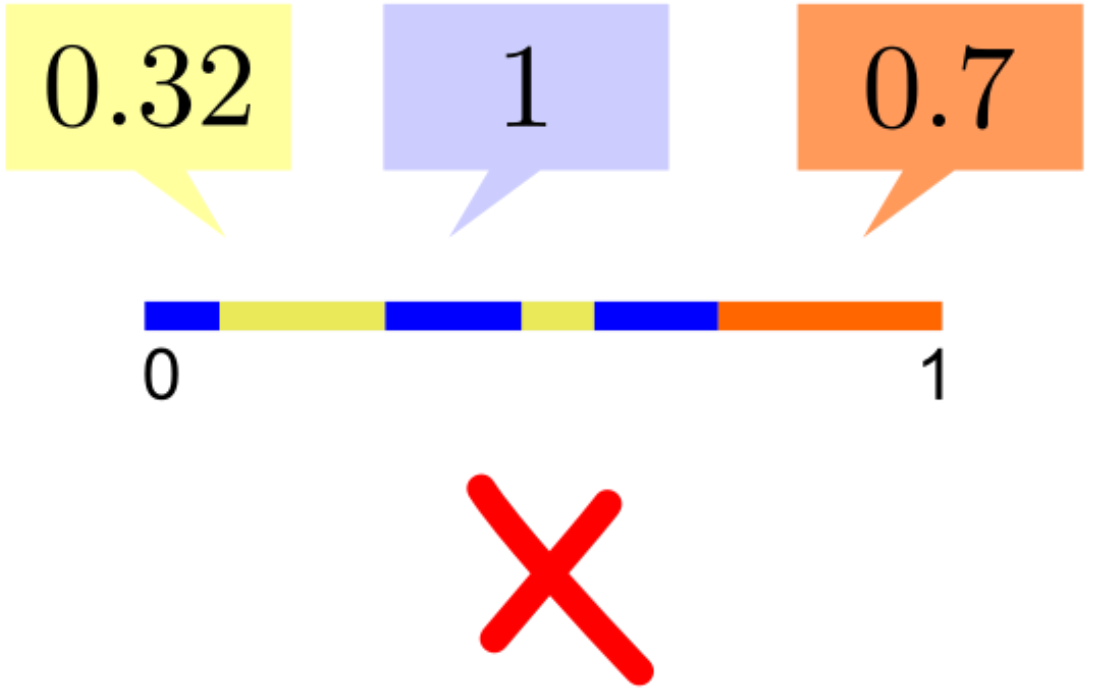

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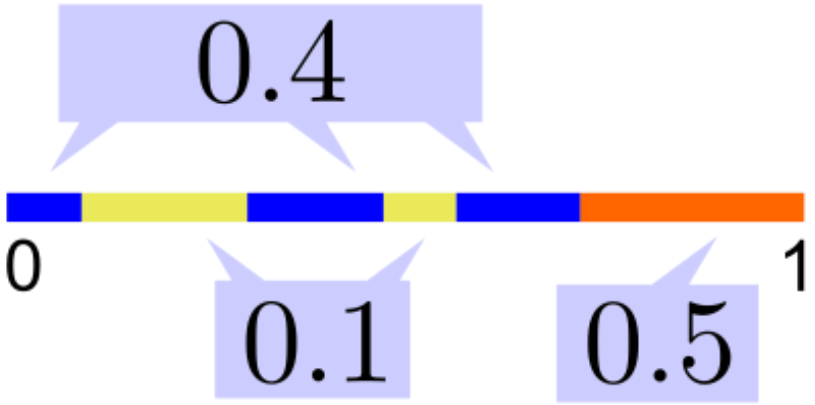

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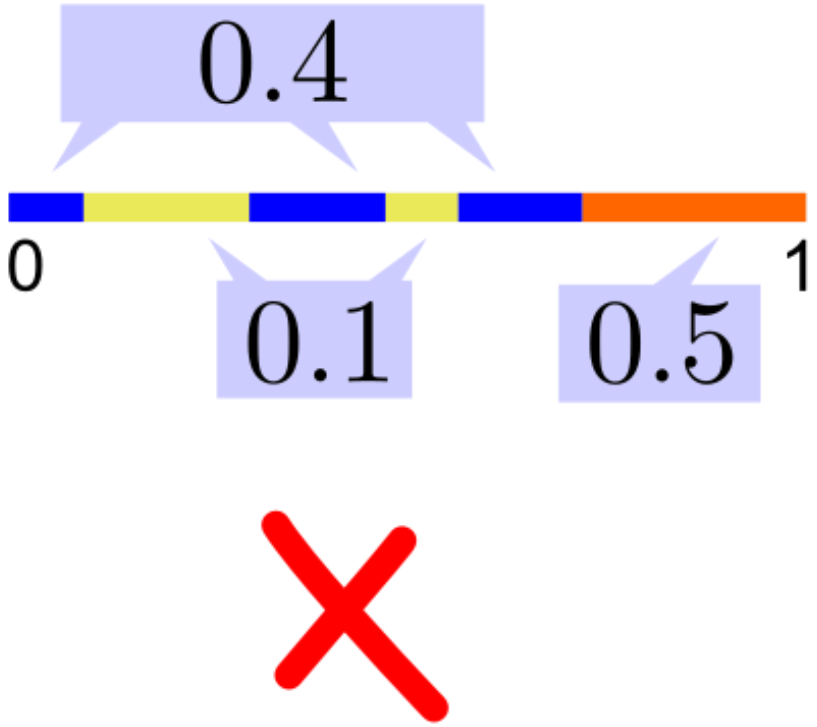

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For two agents (n=2), is one property stronger than the other?

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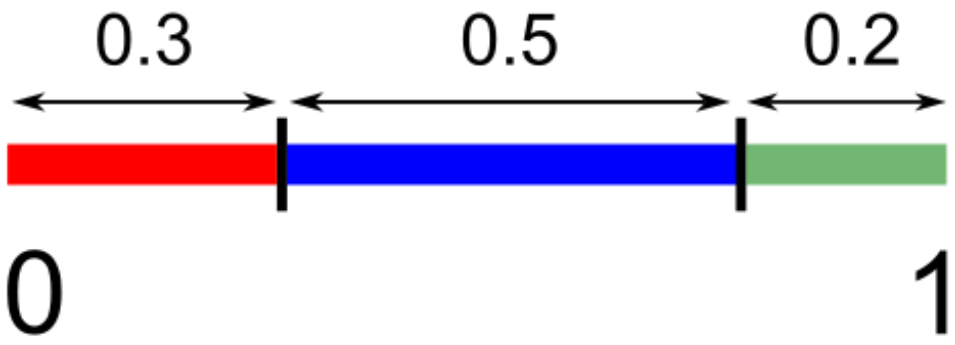
Prop implies EF for *two* agents (but no more)

An envy-free cake division always exists.


0
1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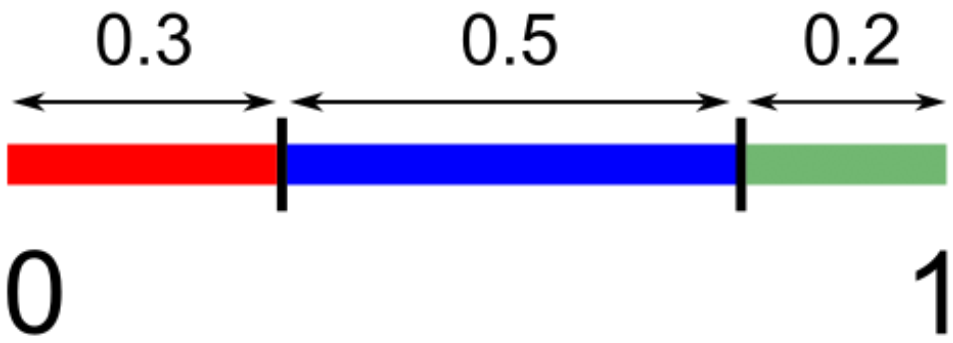
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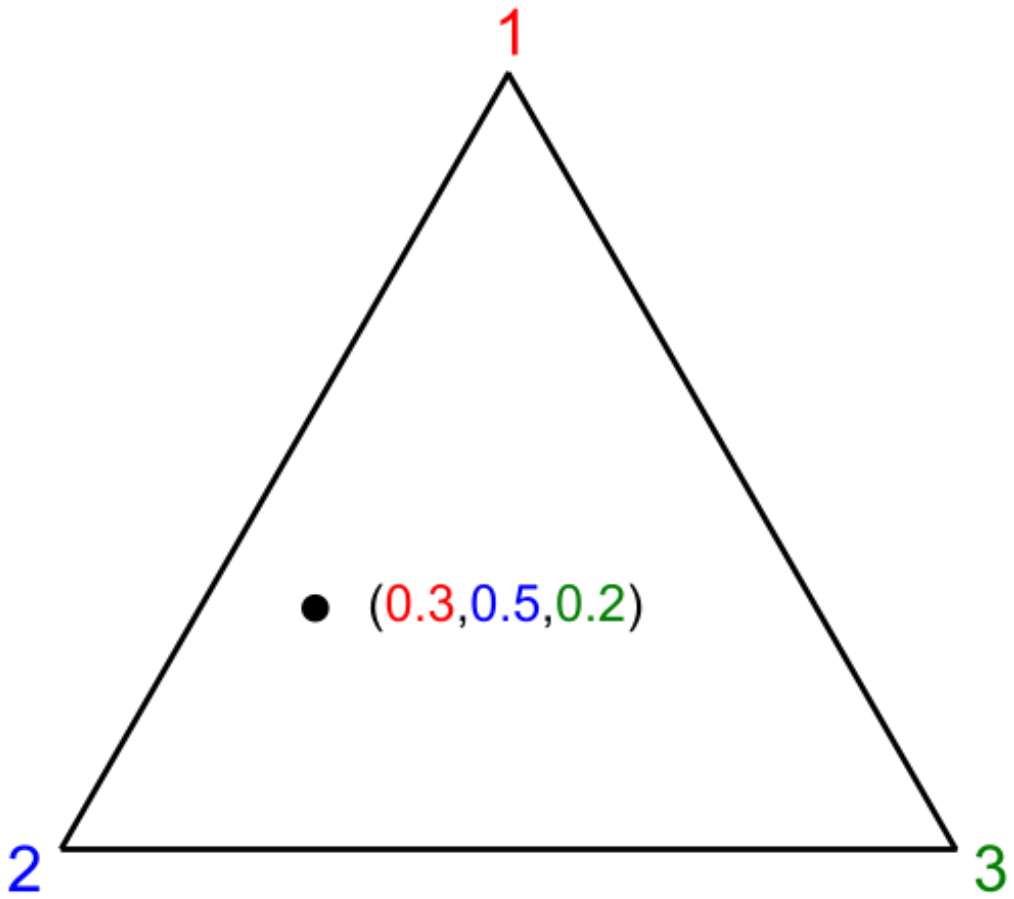
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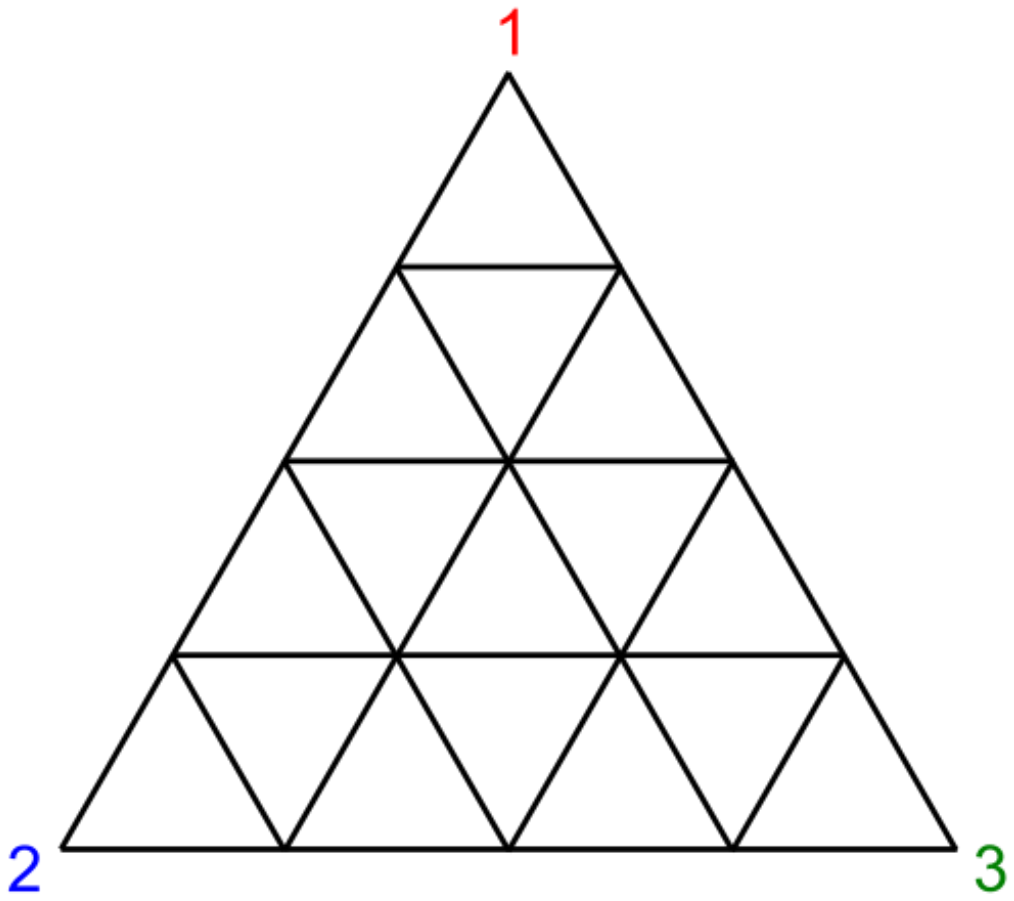
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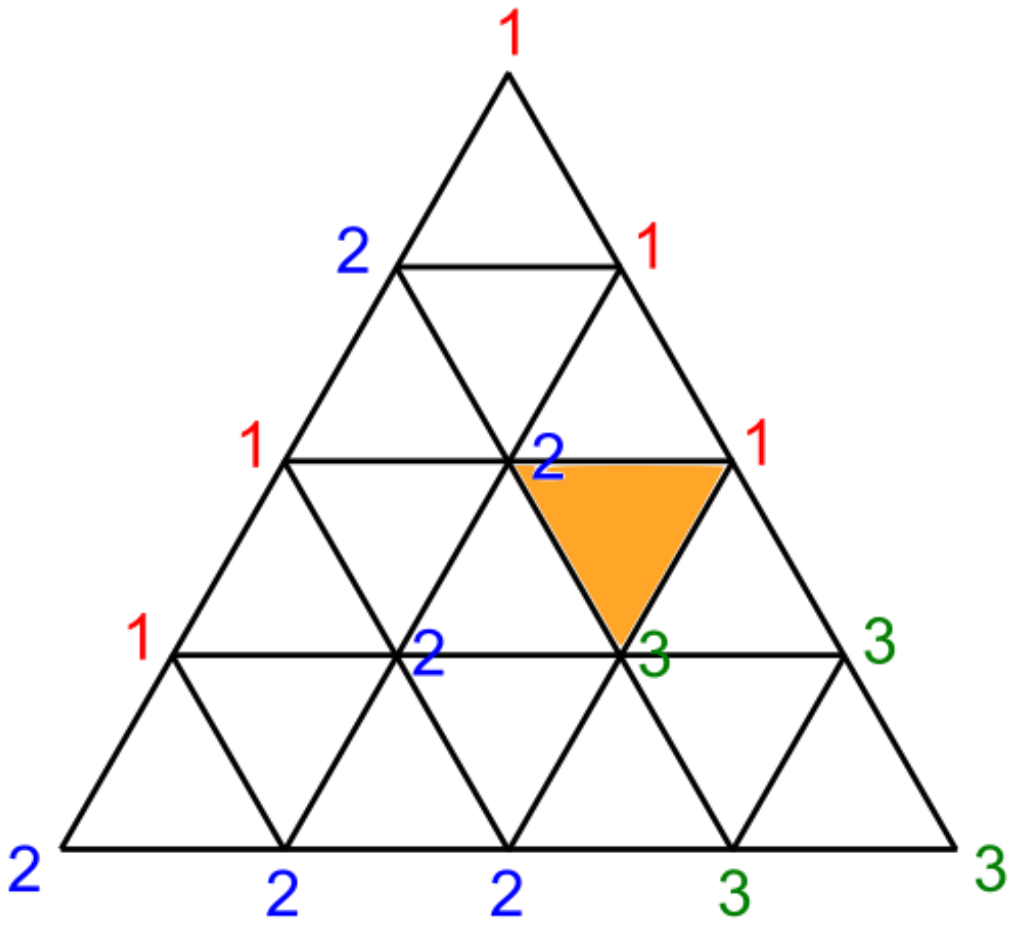
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Sperner's Lemma: Existential

What about algorithms for cake-cutting?

Robertson-Webb Query Model

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Types of queries that can be used to access the valuation functions

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$\texttt{eval}_i(x, y)$: returns $v_i([x, y])$

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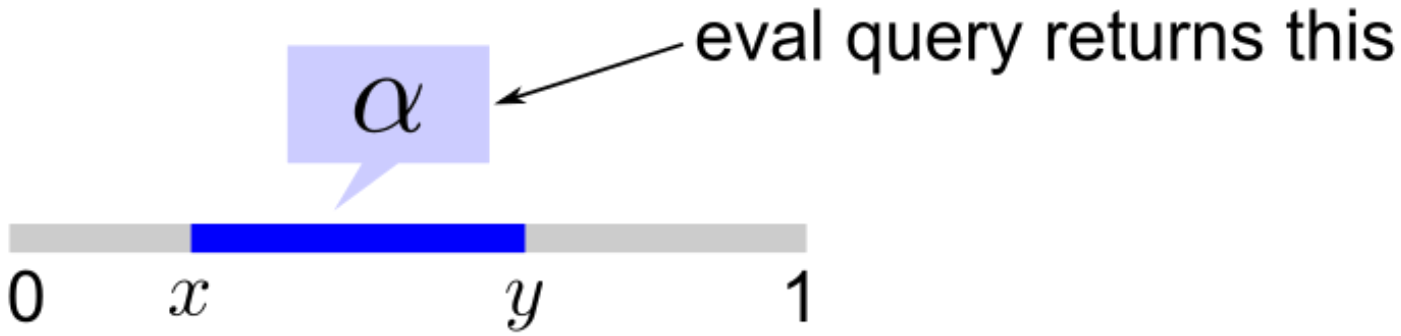

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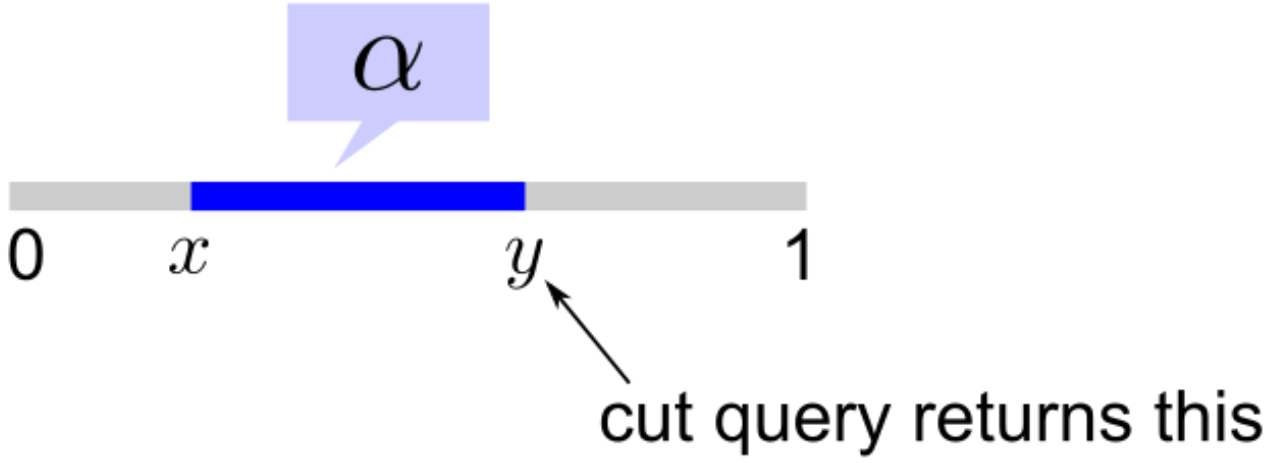

Cake-Cutting Algorithms

Let's start by thinking about proportionality for two agents.

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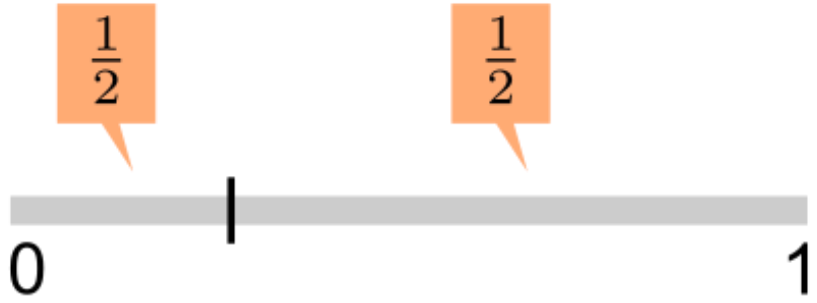

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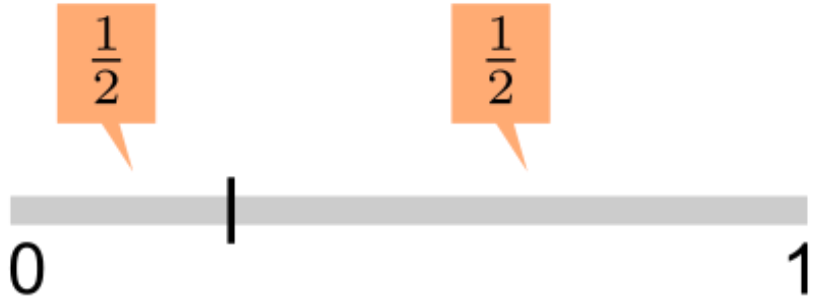

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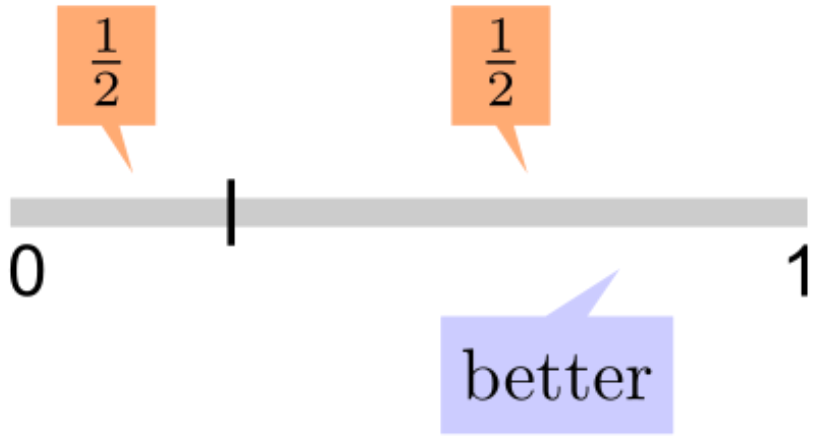

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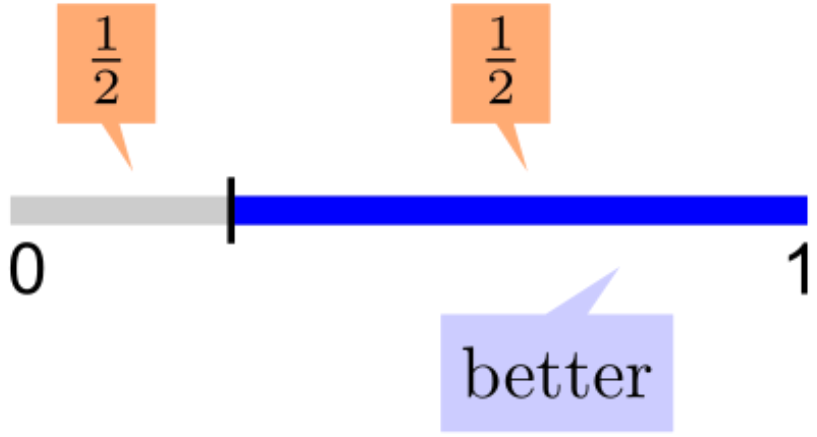

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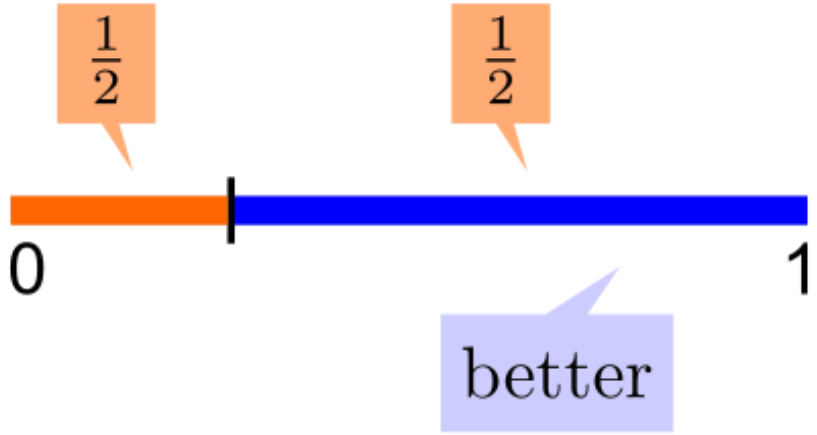

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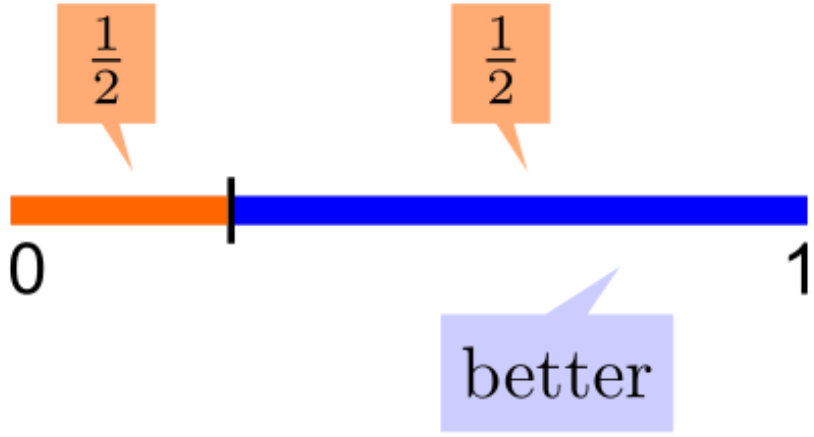

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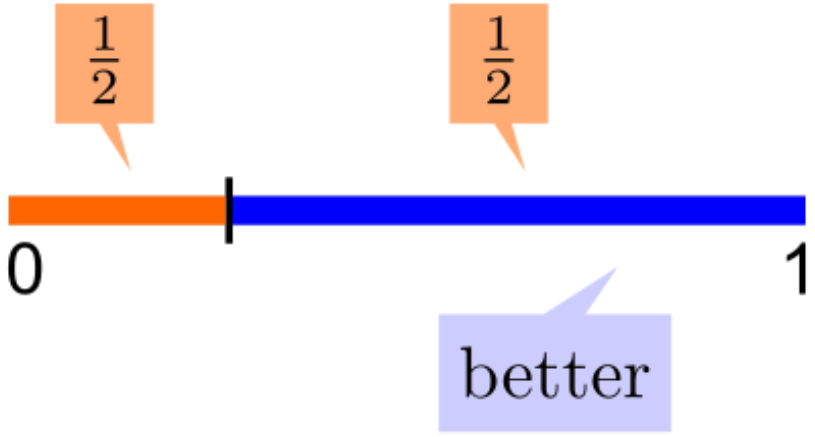

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Yes! Agent 2's value is at least 1/2. Agent 1's value is exactly 1/2.

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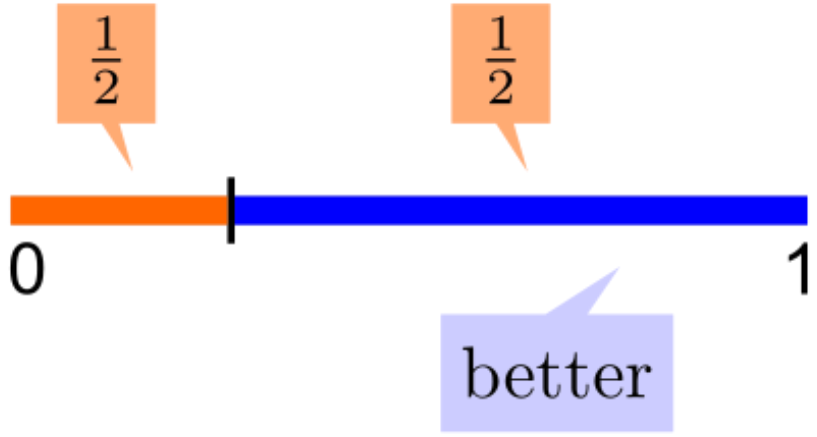

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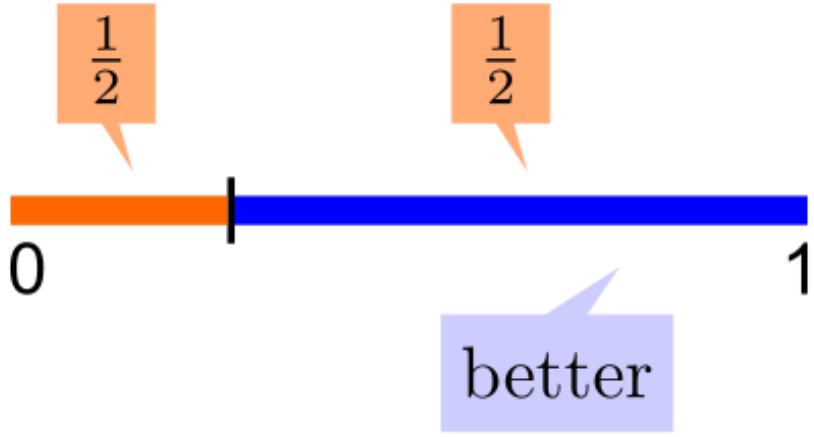

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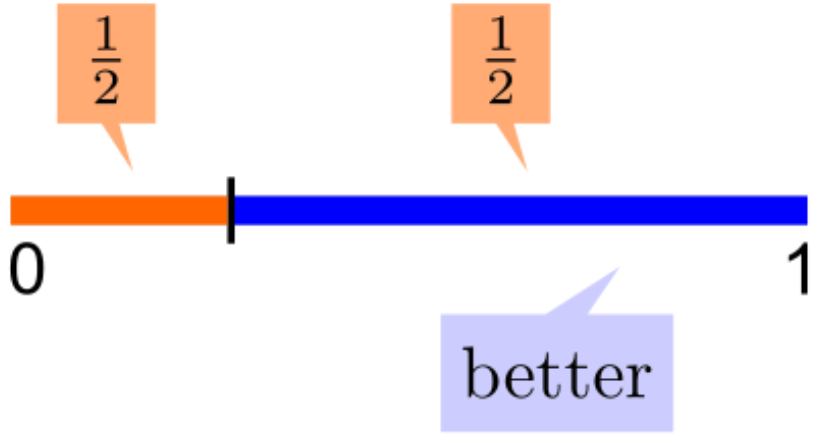

Can cut-and-choose be implemented in the Robertson-Webb model?

Cut and Choose

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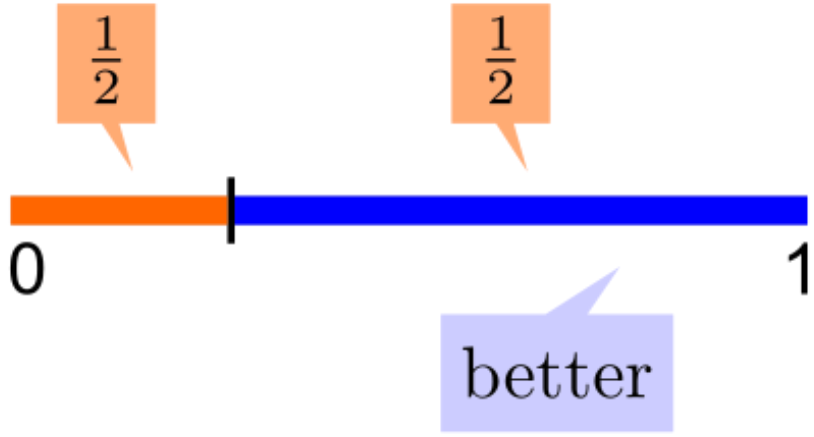

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$$y = \mathtt{cut}_1(0, 1/2)$$

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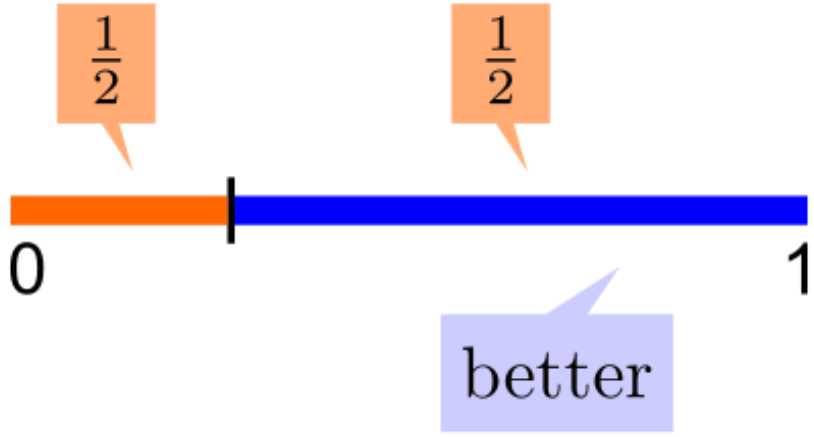

Can cut-and-choose be implemented in the Robertson-Webb model?

$$y = \texttt{cut}_1(0, 1/2)$$

$$\texttt{eval}_2(0, y)$$

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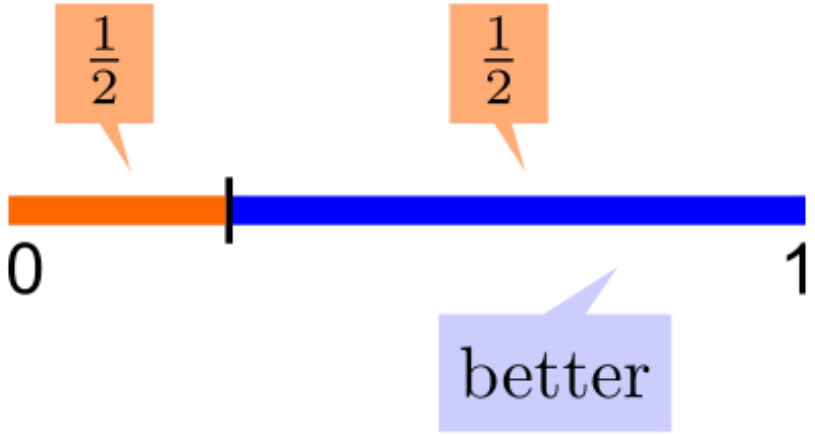

For two agents, an envy-free/proportional cake division can be computed using two queries.

Dubins-Spanier Procedure

A proportional cake division protocol for any number of agents

Dubins-Spanier Procedure

1. A referee gradually moves a knife from left to right.

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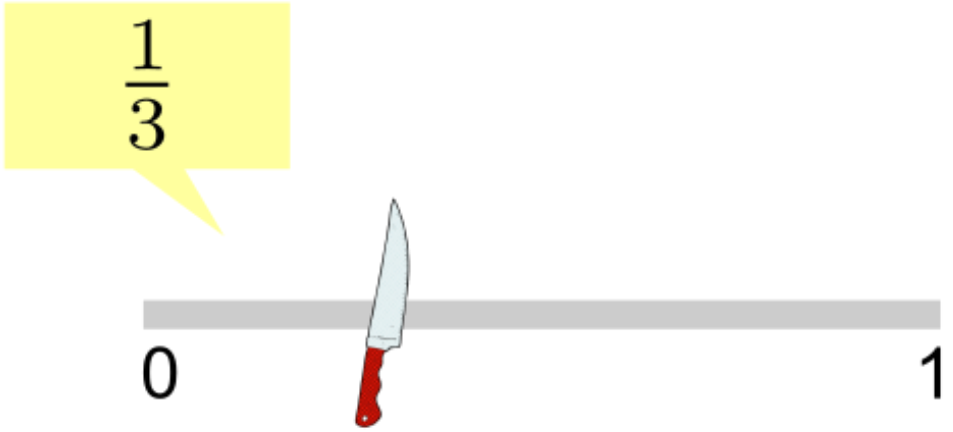

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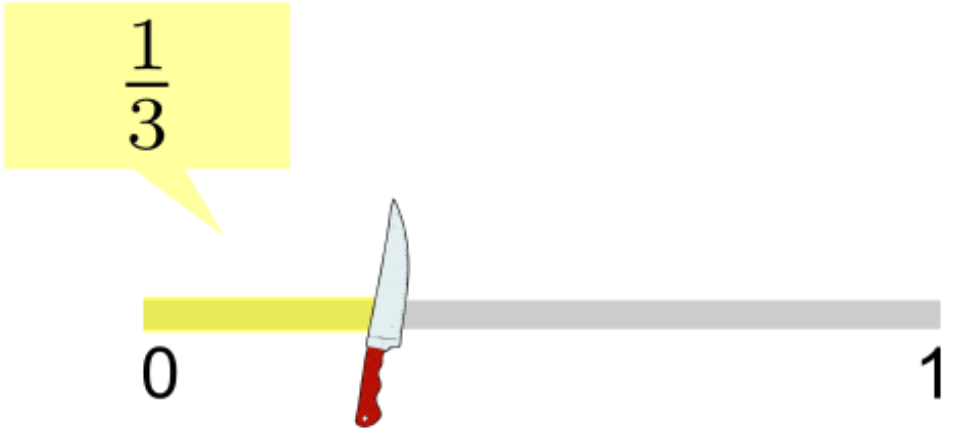

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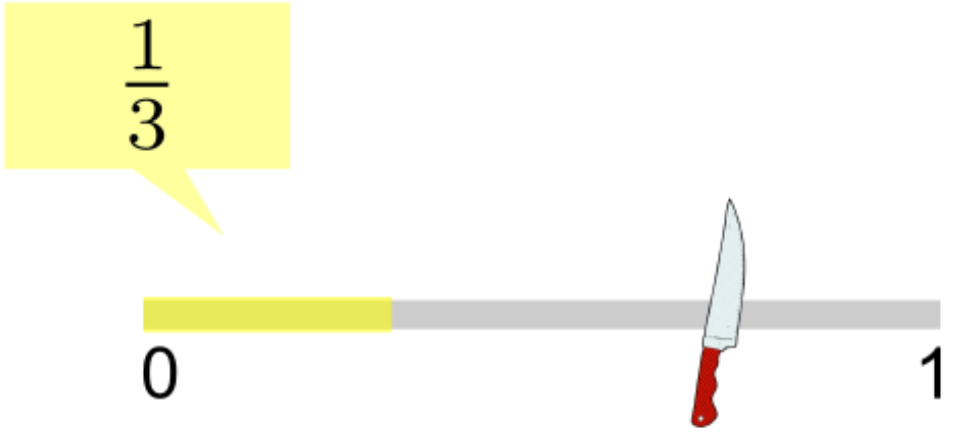

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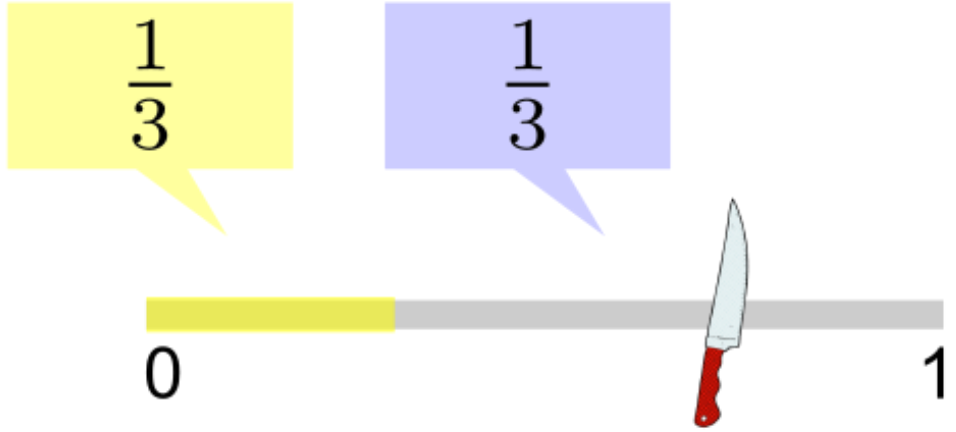

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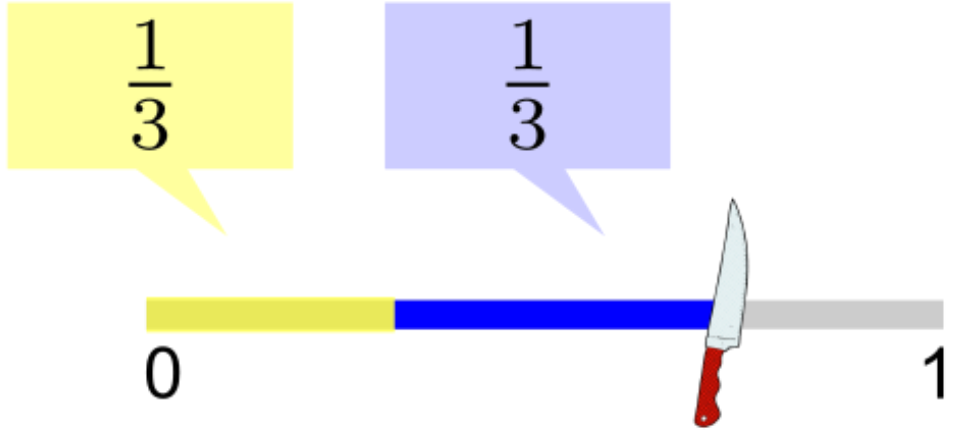

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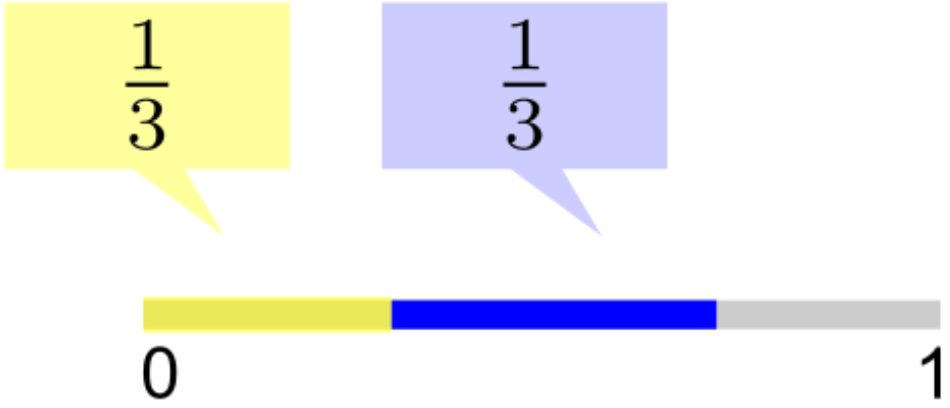

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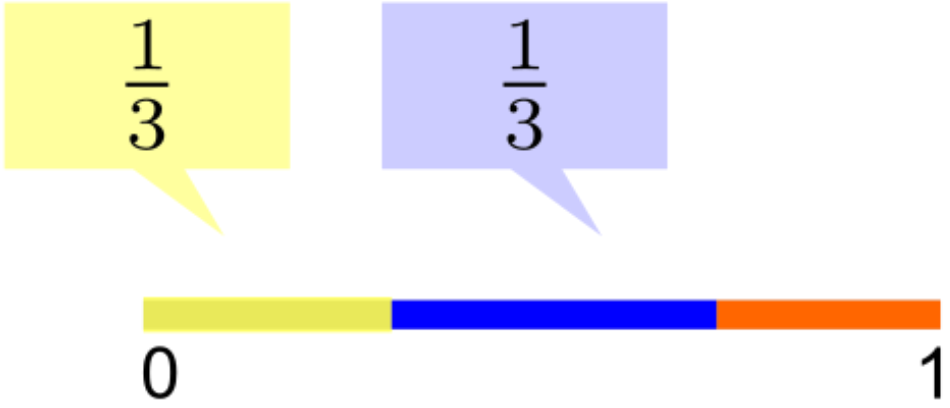

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Why is the resulting allocation proportional?

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Why is the resulting allocation proportional?

Every agent except for the last one gets *exactly* 1/n.
The last agent gets *at least* 1/n.

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Can this procedure be implemented in the Robertson-Webb model?

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Can this procedure be implemented in the Robertson-Webb model?

Yes!

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Query complexity in the Robertson-Webb model?

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Query complexity in the Robertson-Webb model?

$\mathcal{O}(n^2)$ queries (Exercise)

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4. The procedure repeats with the remaining agents.

For n agents, a proportional cake division can be computed using $O(n^2)$ queries.

The Story of Proportionality

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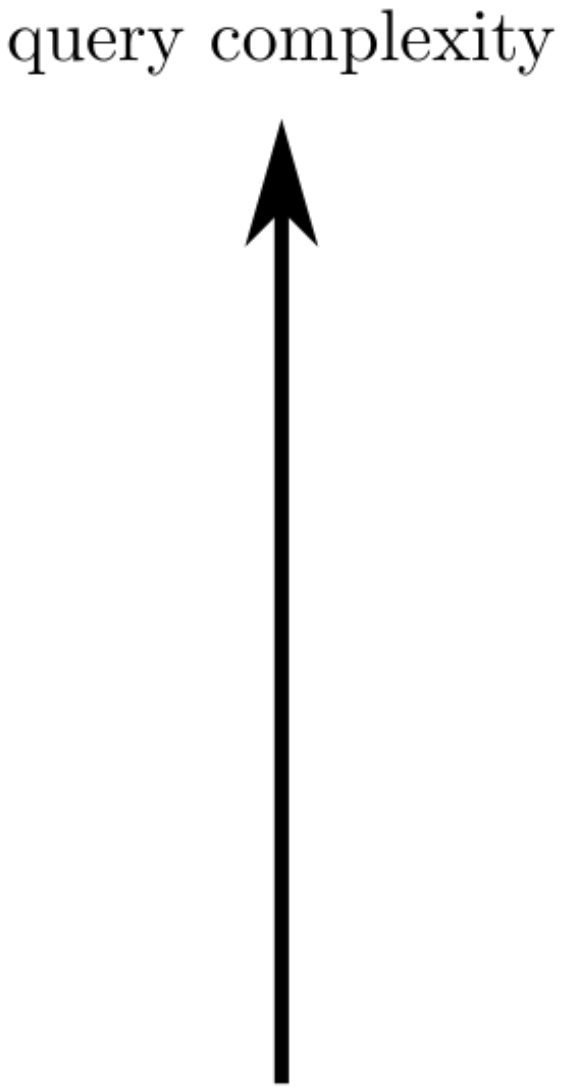

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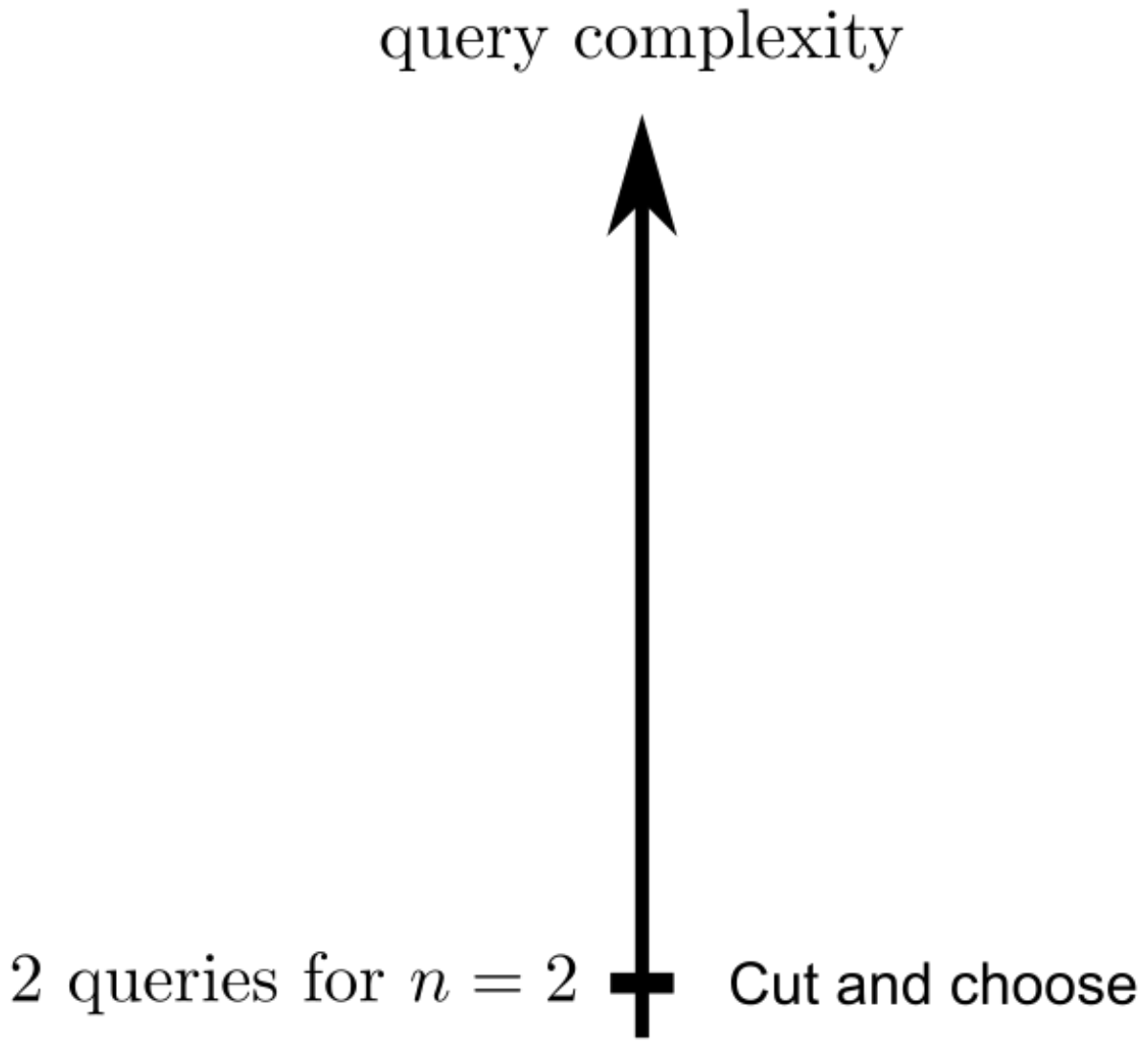

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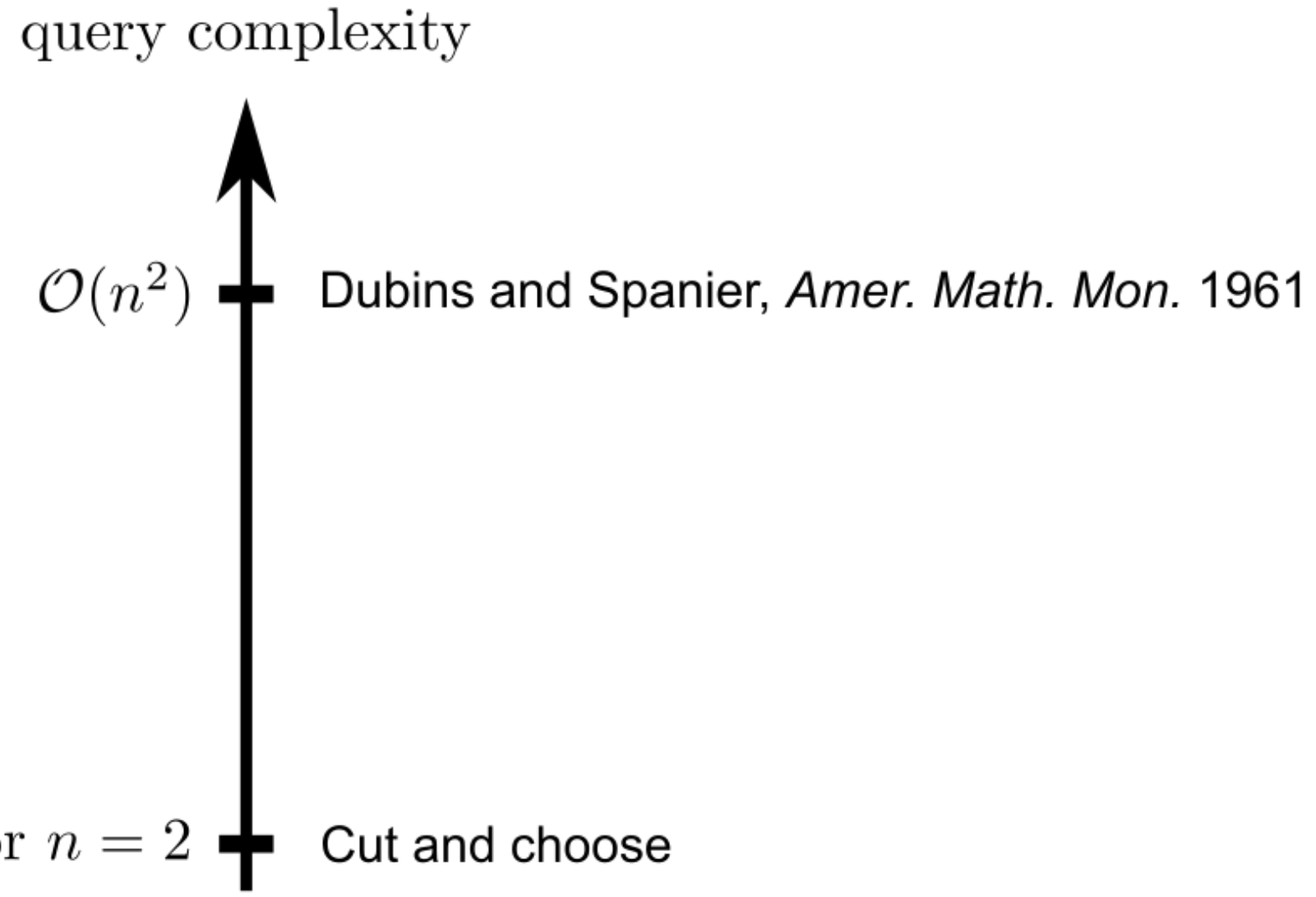

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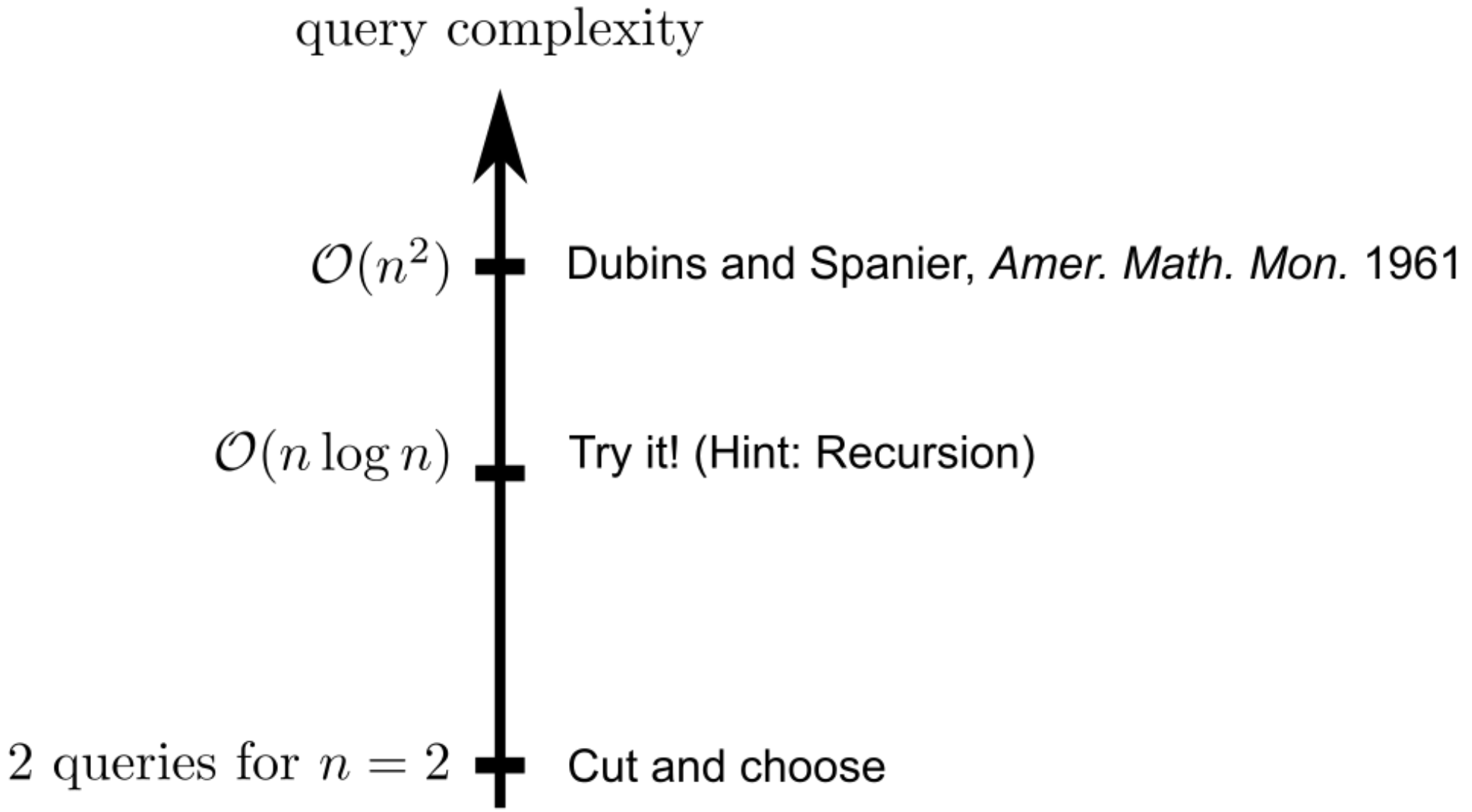

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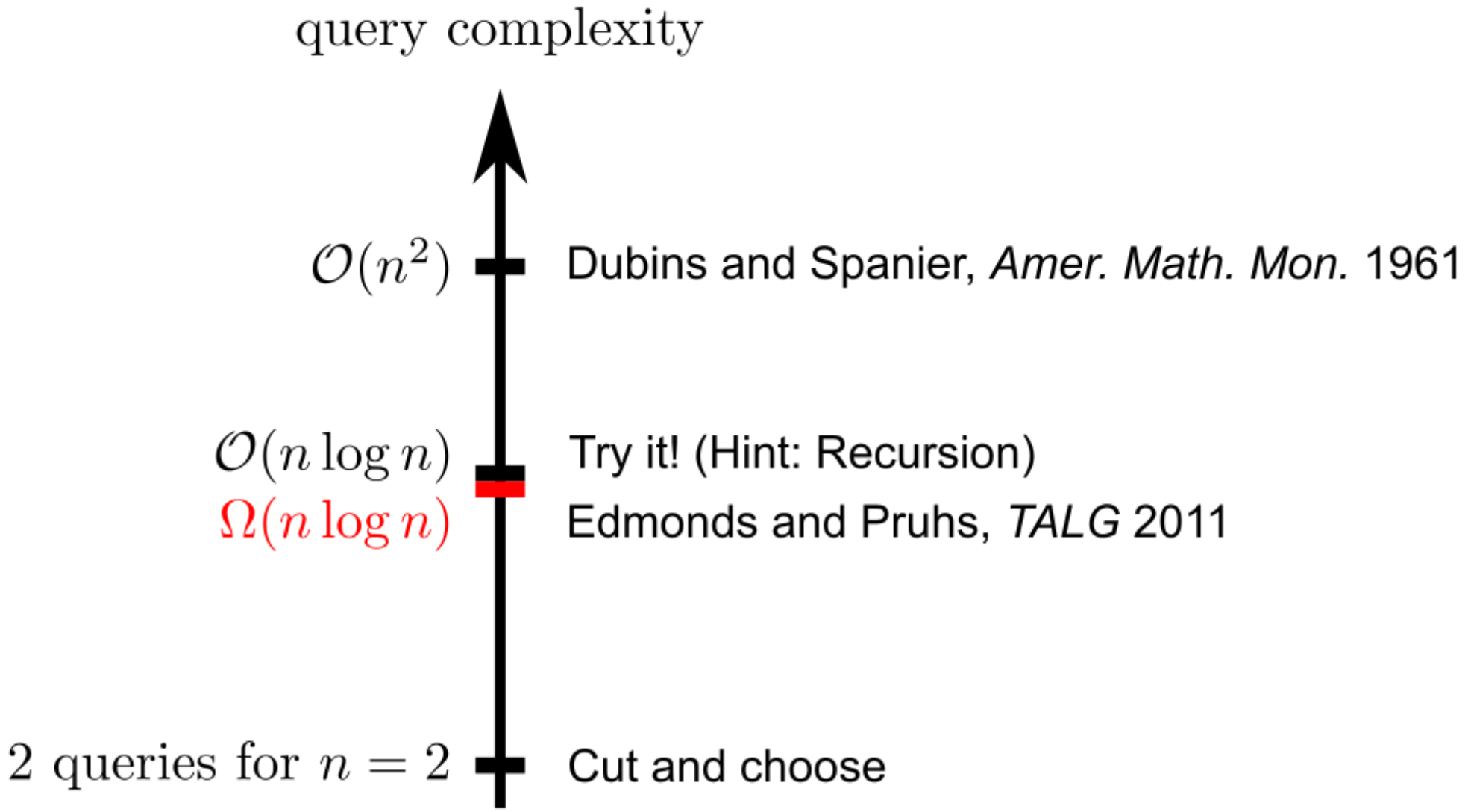


The Story of Envy-freeness

Selfridge-Conway Procedure

An envy-free cake division protocol for three agents

Envy-free for three

Envy-free for three

A 

B 

C 

Envy-free for three

A

B

C

Envy-free for three

A 

B 

C 

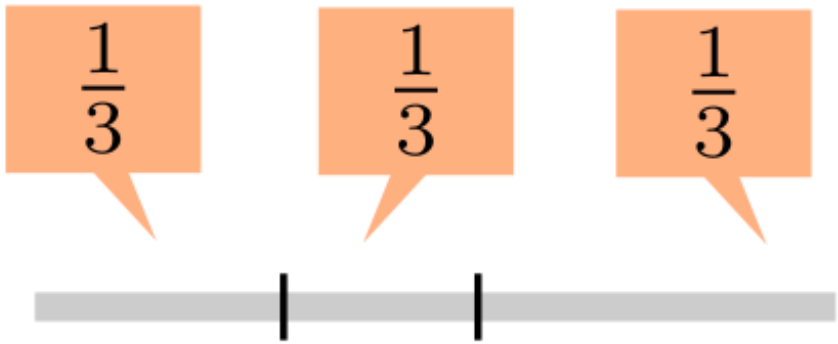
Envy-free for three

A

equal in my view

B

C



Envy-free for three

A 

B 

C 

Envy-free for three

A

B

C

Envy-free for three

A

B

two-way tie

C

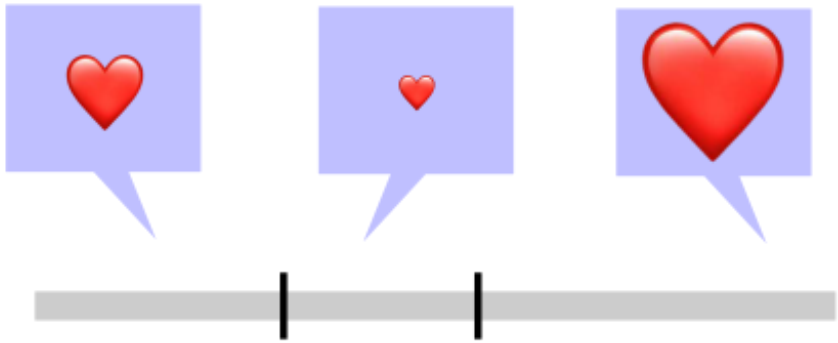
Envy-free for three

A

B

two-way tie

C

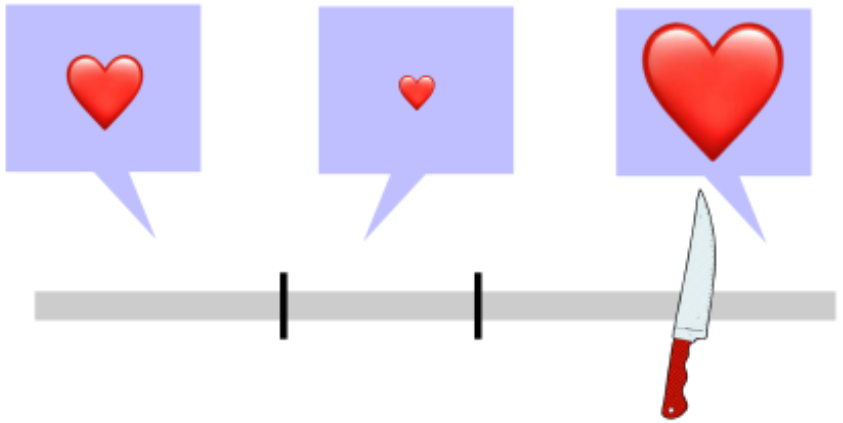


Envy-free for three

A

B

two-way tie



C

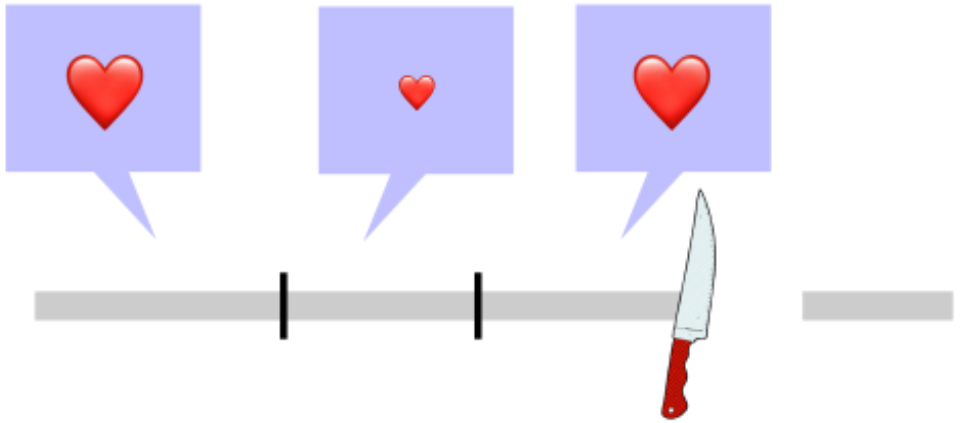
Envy-free for three

A

B

two-way tie

C

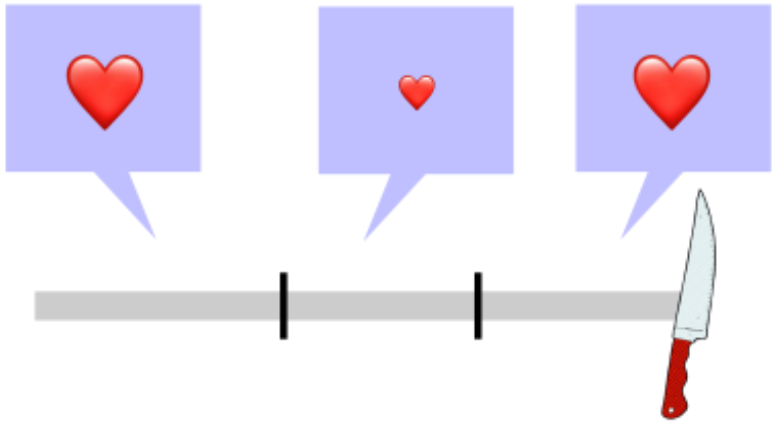


Envy-free for three

A

B

two-way tie



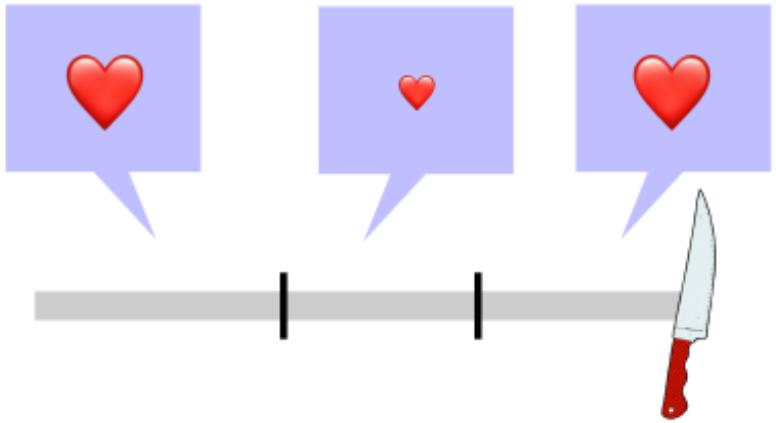
C

Envy-free for three

A

B

two-way tie



C

Trimmings

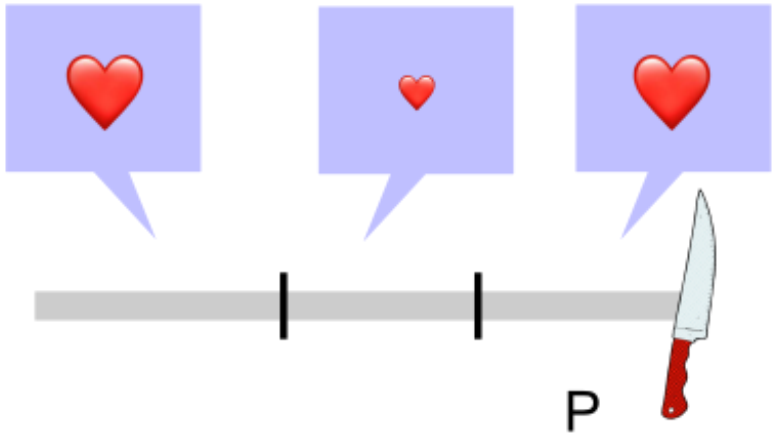
Envy-free for three

A

B

two-way tie

C



Trimmings

Envy-free for three

A

B

P

C

Trimmings

Envy-free for three

A

B

C

1st

Trimmings

Envy-free for three

A

B

2nd

P

C

1st

Trimmings

Envy-free for three

A

3rd

B

2nd

C

1st

Trimmings

Envy-free for three

A

3rd

B

2nd

I pick P if C doesn't

C

1st

P

Trimmings

Envy-free for three

A

3rd

B

2nd

I pick P if C doesn't

C

1st

Trimmings

Envy-free for three

A

3rd

B

2nd

I pick P if C doesn't

P

C

1st

Trimmings

Envy-free for three

A

3rd

B

2nd

I pick P if C doesn't

P

C

1st

Trimmings

Envy-free for three

A

EF because of equal cuts

B

EF because of two-way tie

C

EF because I picked first

Trimmings

Envy-free for three

A

B

C

Envy-free for three

A

B

C

I equidivide

P

Trimmings

Envy-free for three

A

B

C

I equidivide

P

Trimmings

equally good

Envy-free for three

A

B

C

I equidivide

P

Trimmings

Envy-free for three

A

B

I pick first yay!

C

I equidivide

Trimmings

Envy-free for three

A

I can pick after B does

B

I pick first yay!

C

I equidivide

Trimmings

Envy-free for three

A

I can pick after B does

B

I pick first yay!

C

I equidivide and pick last

Trimmings

Envy-free for three

A

I can pick after B does

B

I pick first yay!

C

I equidivide and pick last

Trimmings

Envy-free for three

A

I can pick after B does

B

I pick first yay!

C

I equidivide and pick last

Trimmings

Envy-free for three

A

I can pick after B does

B

I pick first yay!

C

I equidivide and pick last

Trimmings

Envy-free for three

A

Irrevocable advantage

B

EF because I picked first

C

EF because I equidivided

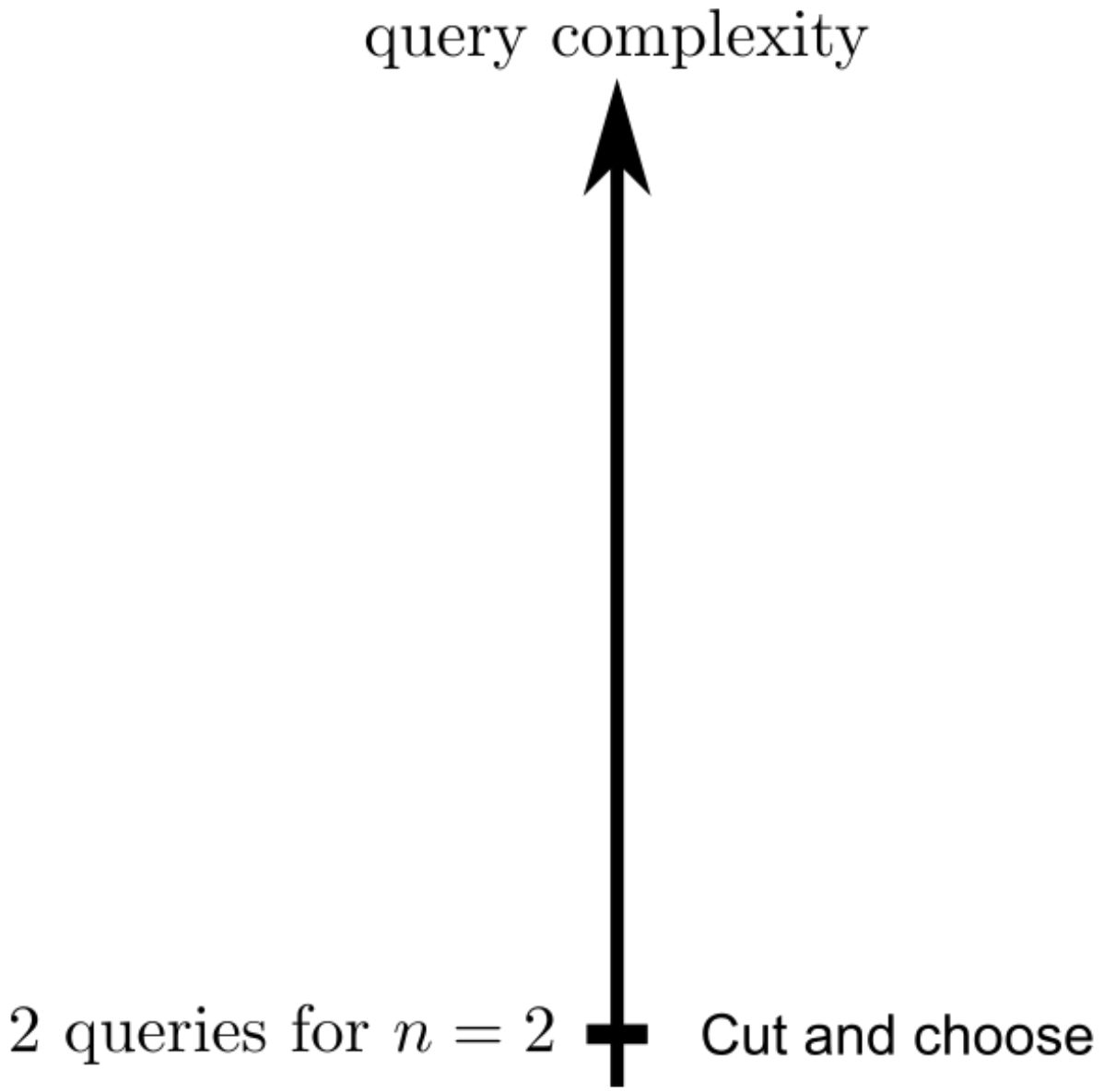
Trimmings

Exercise

How many queries does the three-person EF protocol require?

The Story of Envy-freeness

The Story of Envy-freeness



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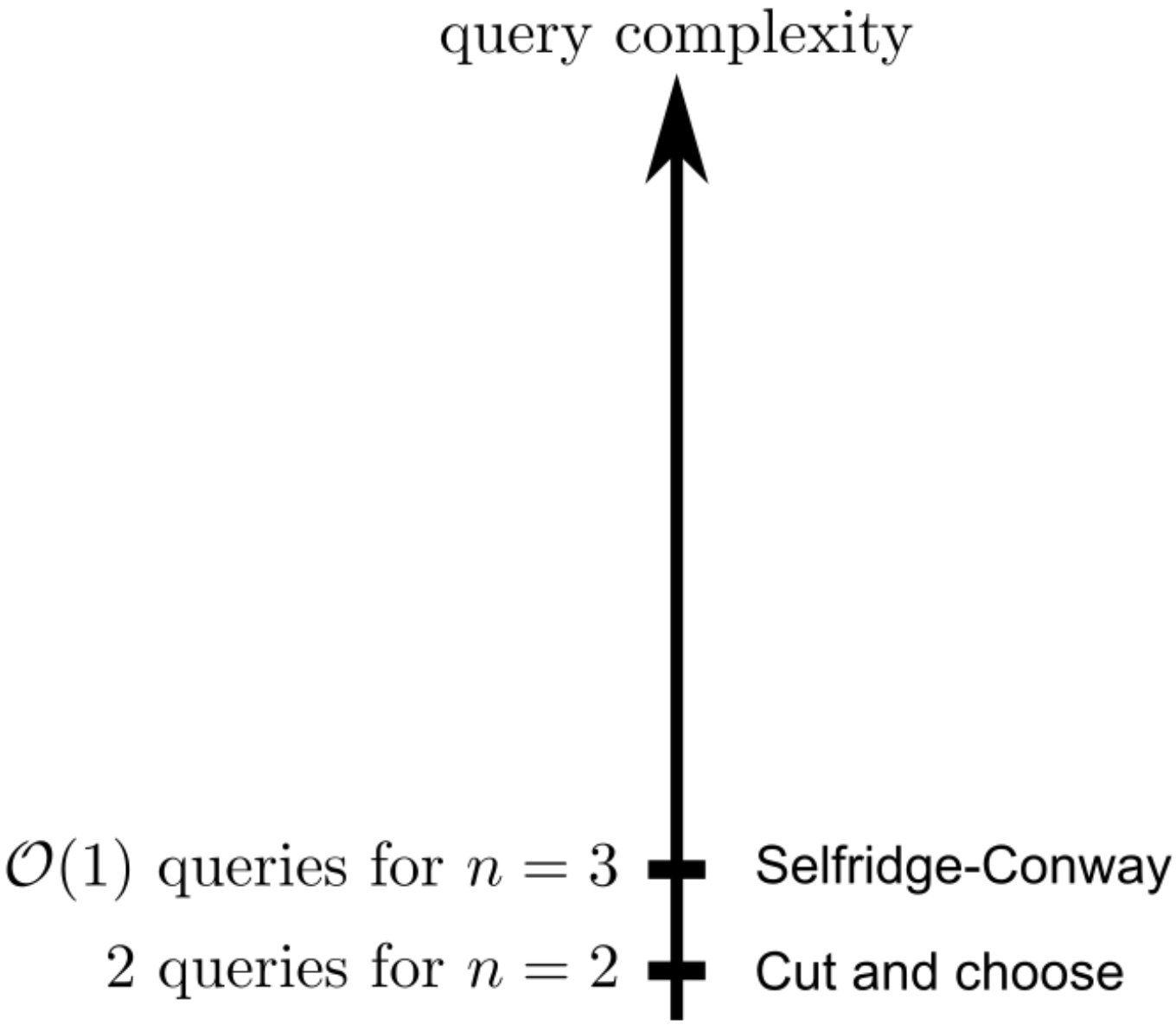

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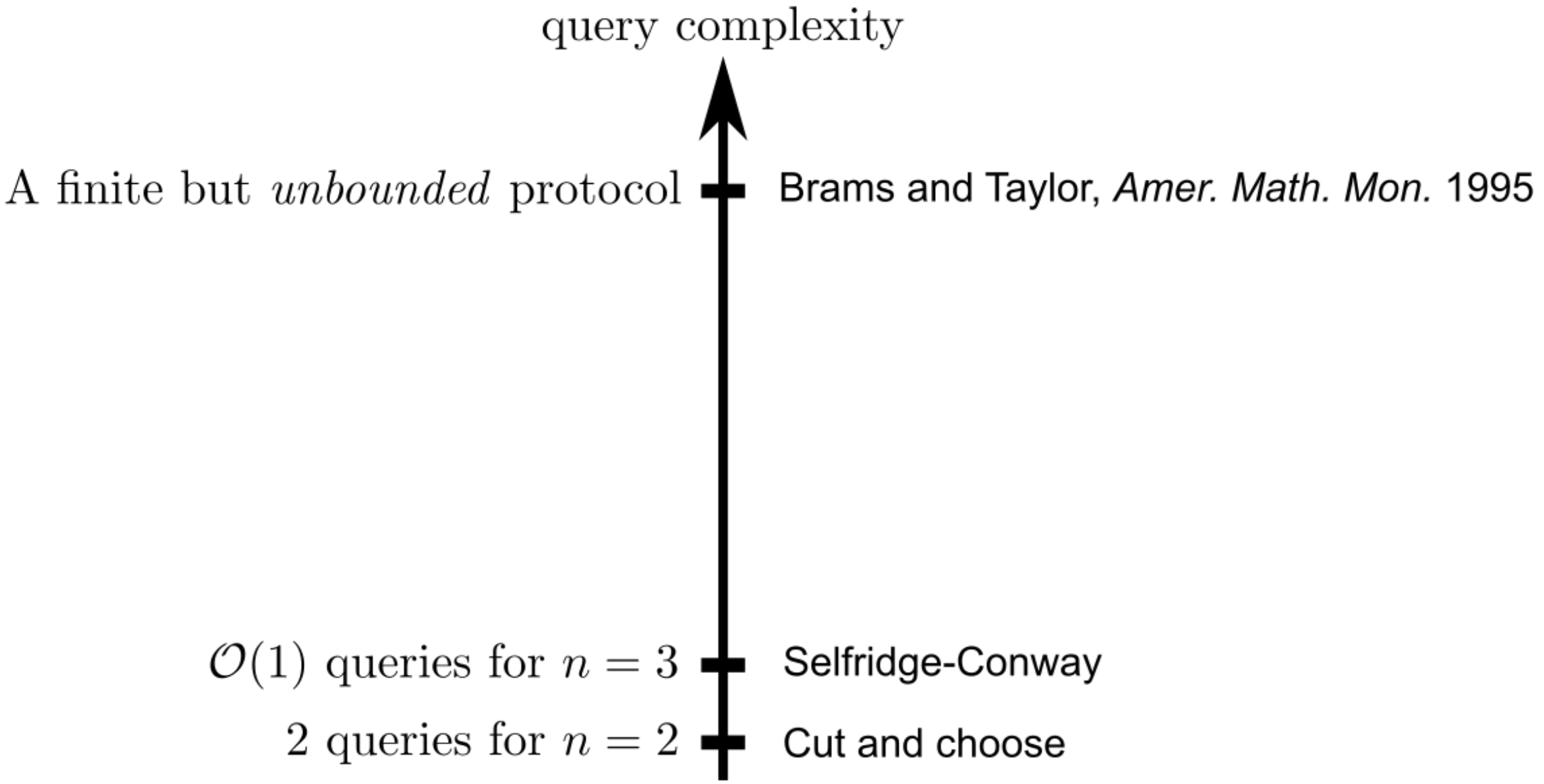

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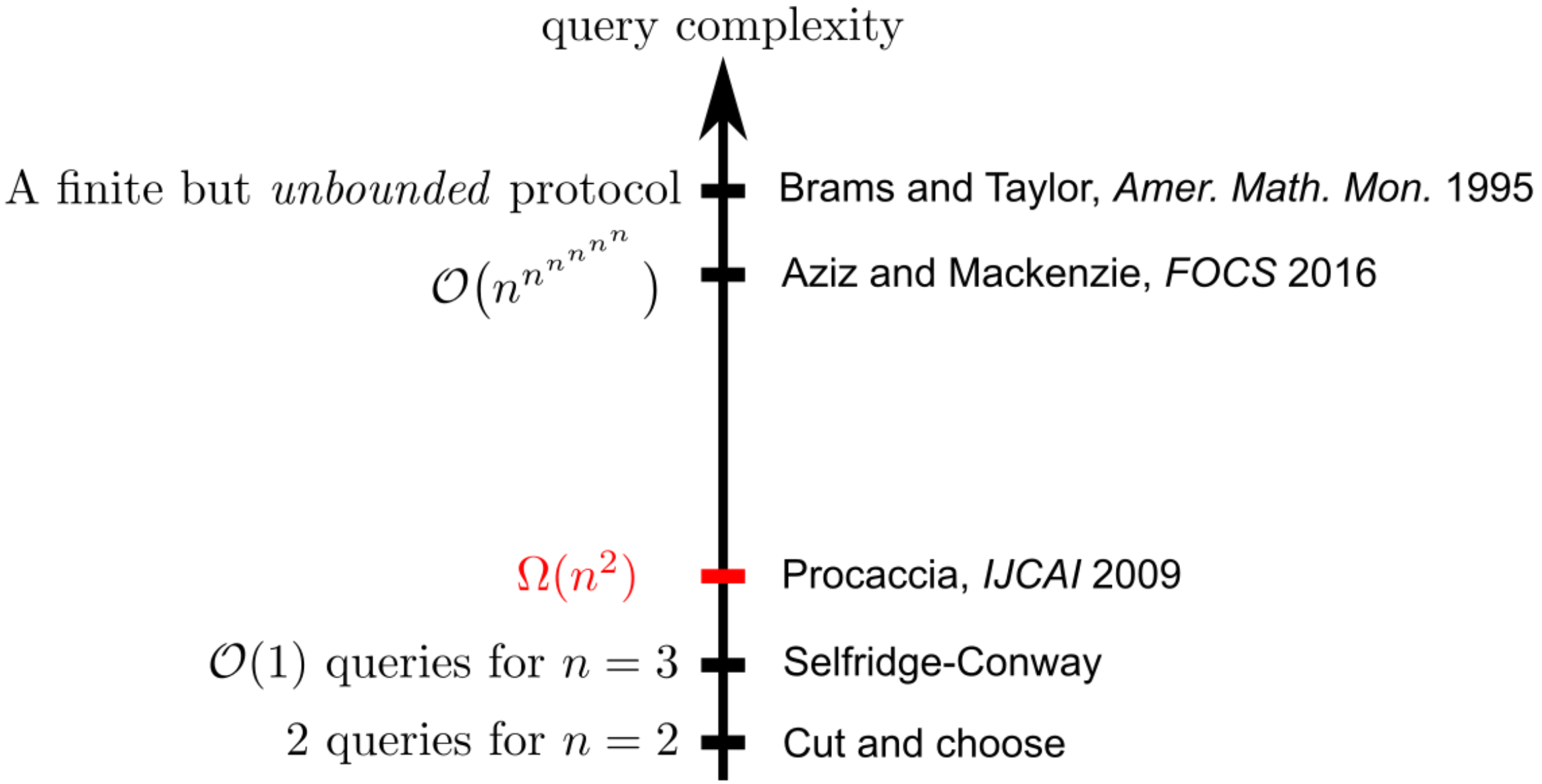

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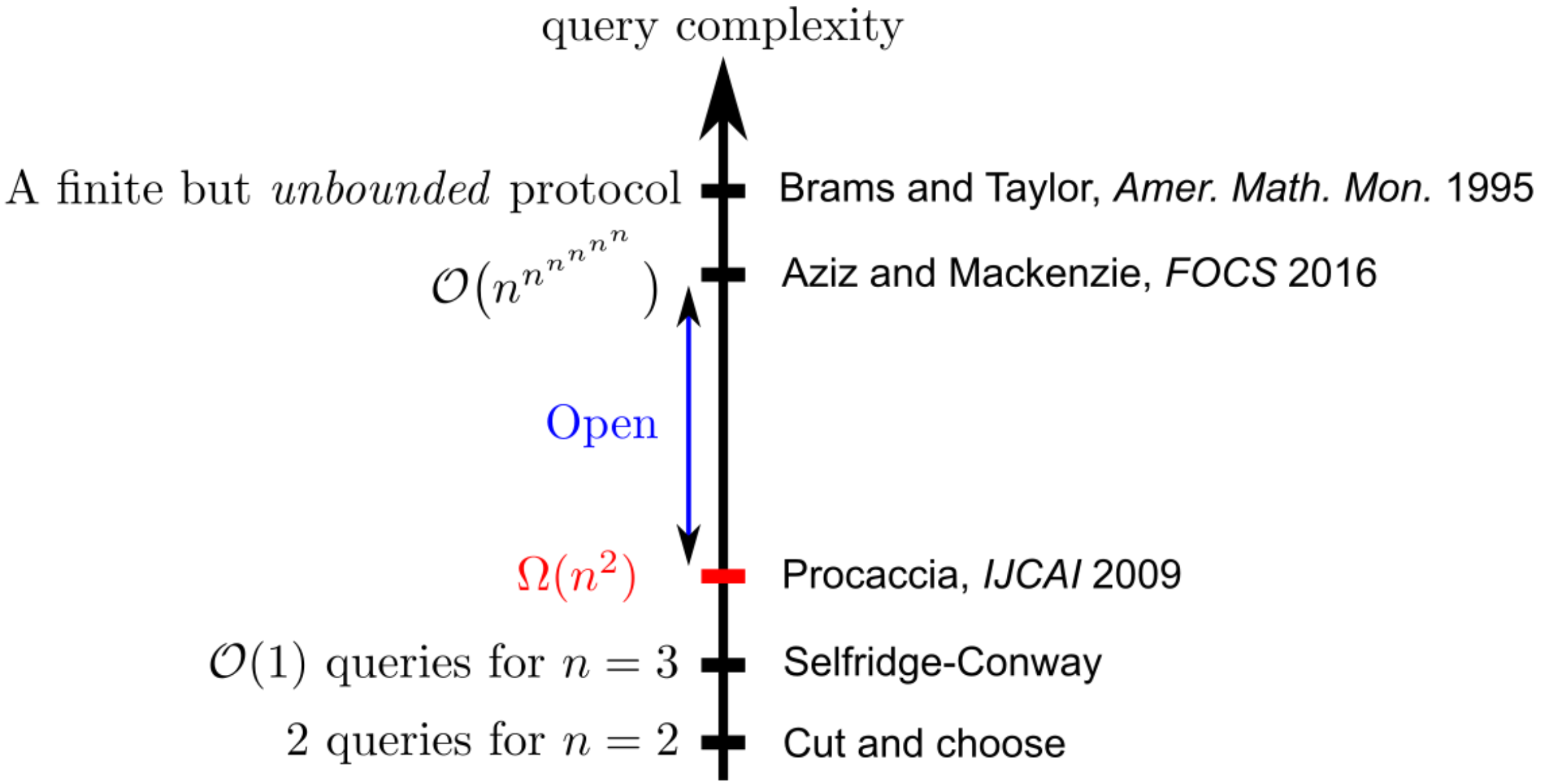

References

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  Ariel Procaccia
  "*Cake Cutting Algorithms*"
  Chapter 13 in Handbook of Computational Social Choice

- Lecture by Ariel Procaccia on "Cake cutting" in the *Optimized Democracy* course.
  https://sites.google.com/view/optdemocracy/schedule