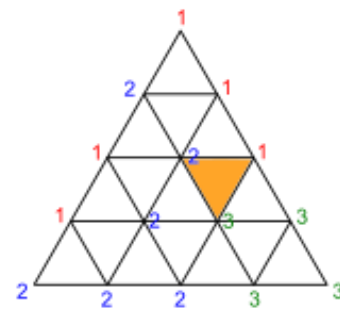


Cake cutting, rental harmony and triangles



Rohit Vaish

Fair Division

Fair Division



Fair Division



Fair Division



Fair Division



Fair Division

Divisible



Fair Division

Divisible



Indivisible



Fair Division

Divisible



Indivisible

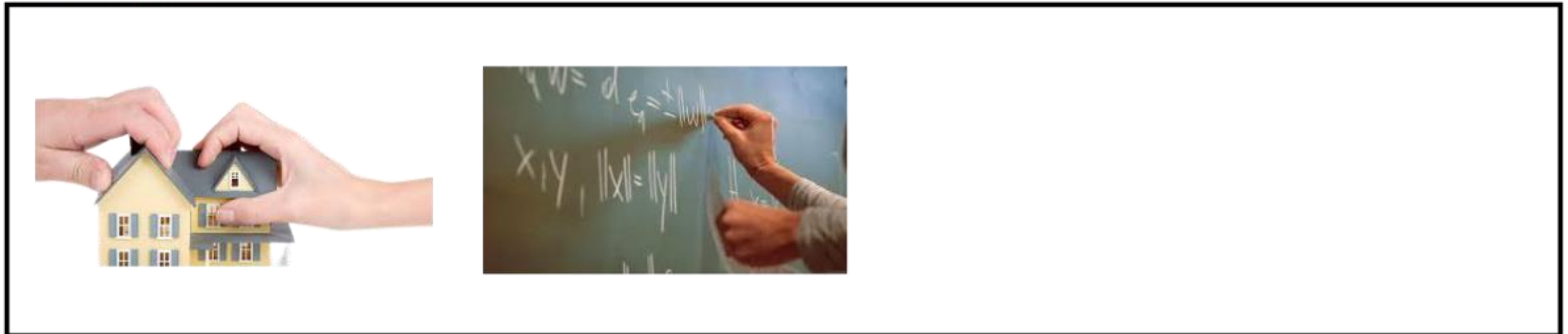


Fair Division

Divisible

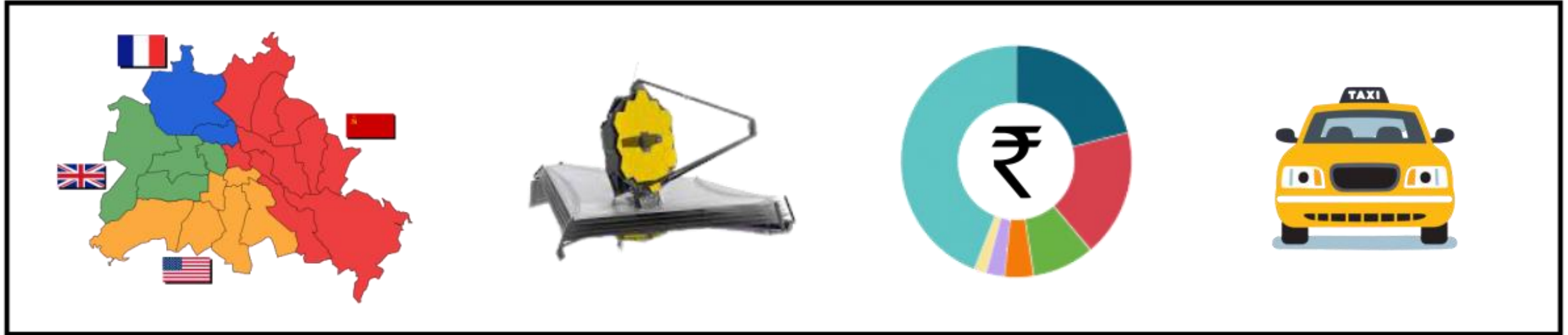


Indivisible



Fair Division

Divisible



Indivisible



Fair Division

Divisible



Indivisible

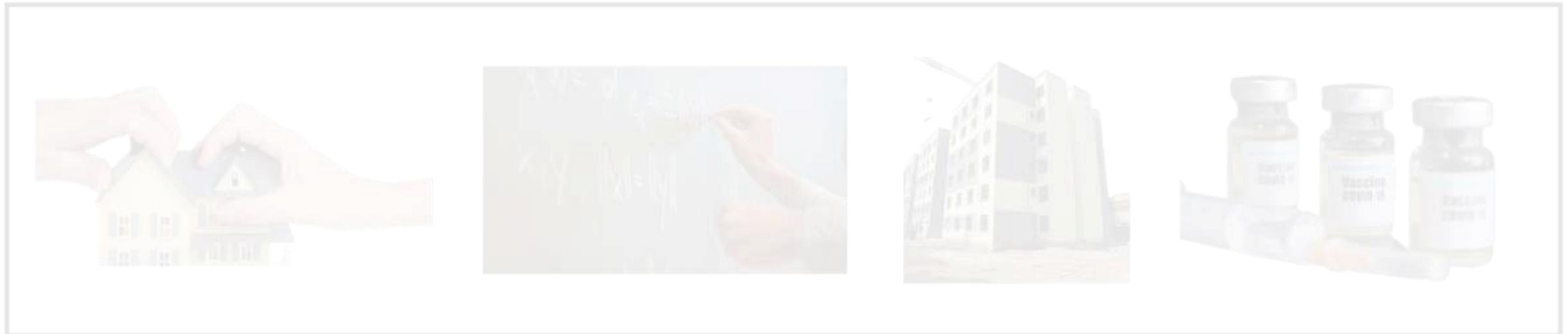


Fair Division

Divisible



Indivisible



Cake Cutting

Rent Division

Cake Cutting

Rent Division



Divide the cake such that



Divide the cake such that



*I prefer my piece
over anyone else's*

Divide the cake such that

*I prefer my piece
over anyone else's*



*I prefer my piece
over anyone else's*

Divide the cake such that

*I prefer my piece
over anyone else's*

*I prefer my piece
over anyone else's*



*I prefer my piece
over anyone else's*

*I prefer my piece
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Divide the cake such that

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*I prefer my piece
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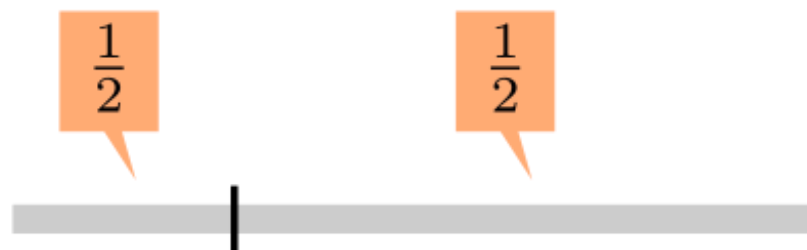
no kid envies another



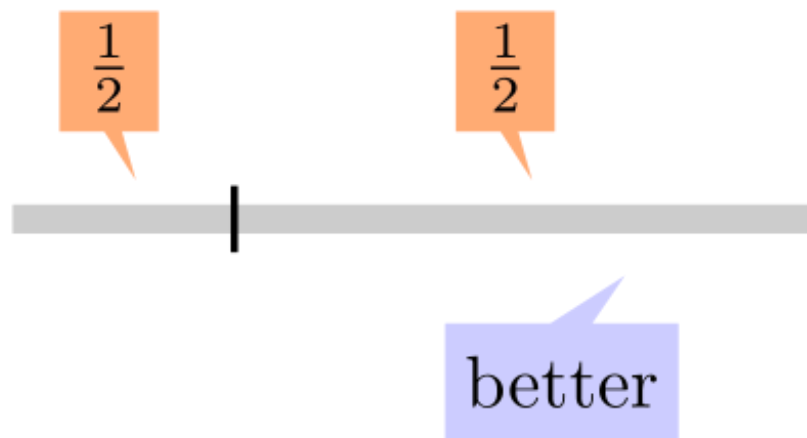
I Cut, You Choose



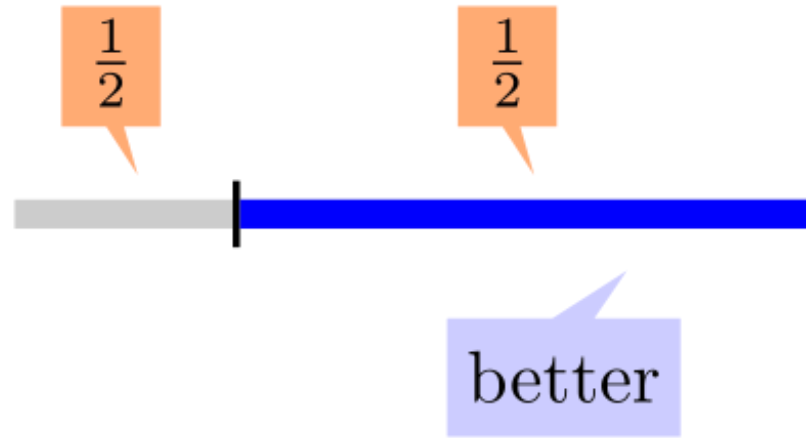
I Cut, You Choose



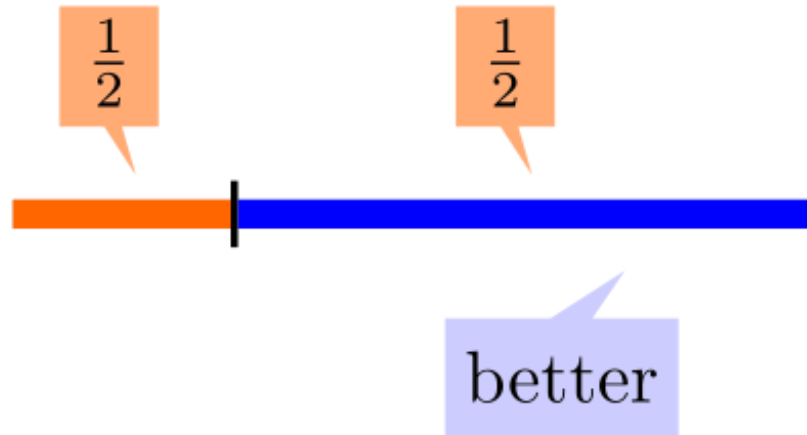
I Cut, You Choose



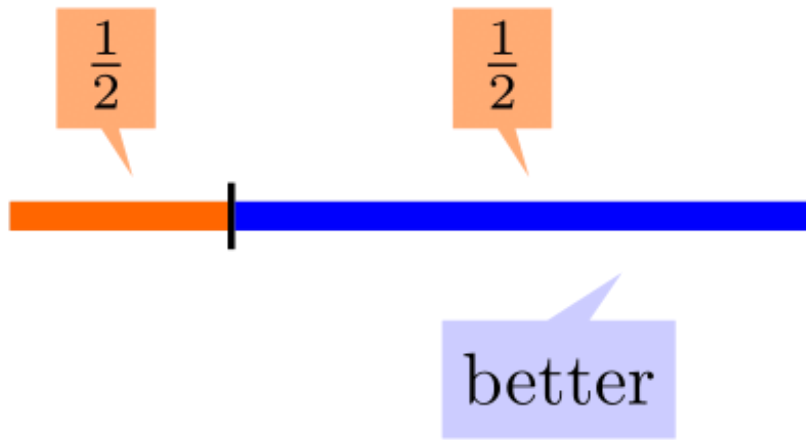
I Cut, You Choose



I Cut, You Choose



I Cut, You Choose



Not obvious for three or more kids

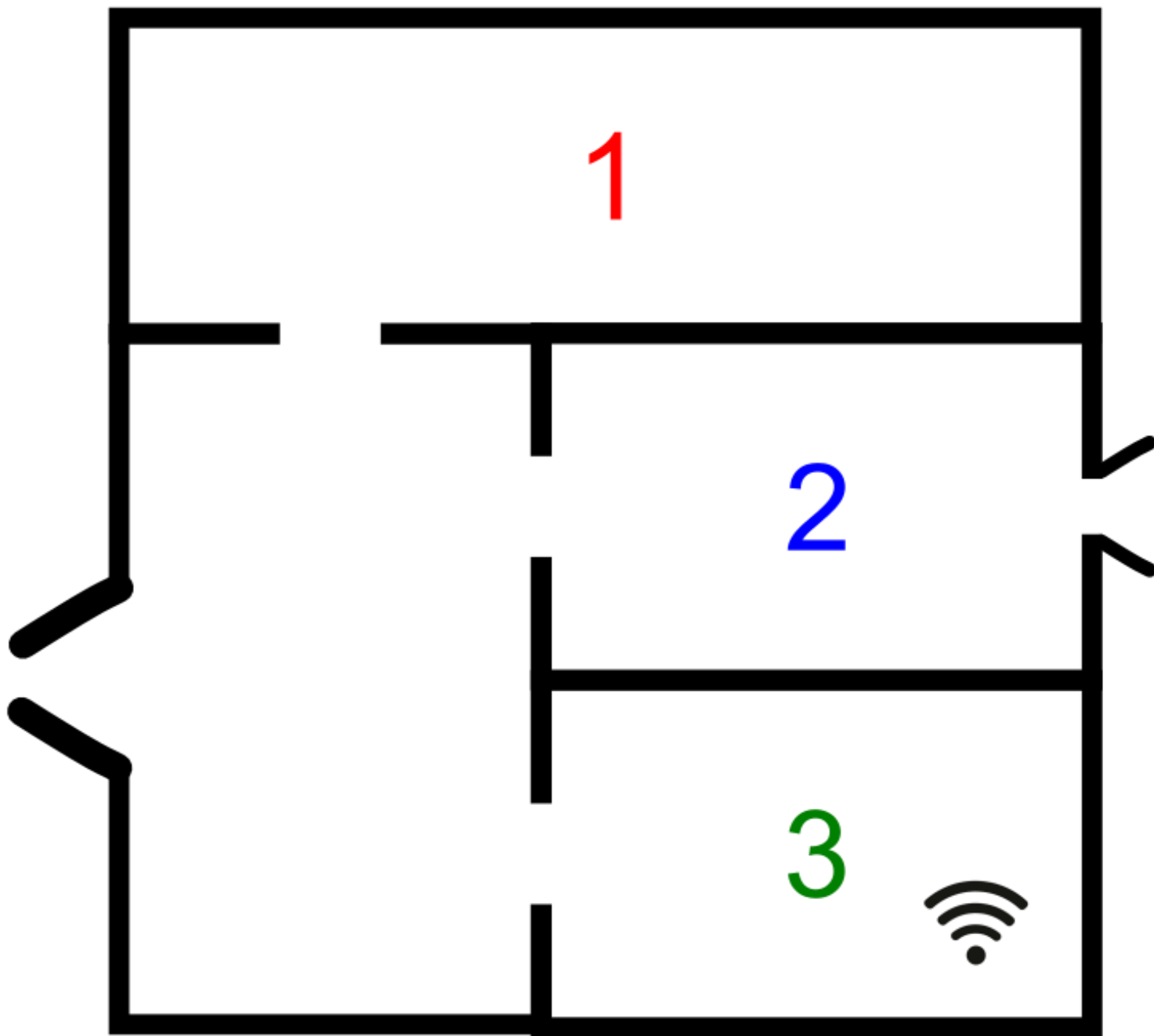
Cake Cutting

Rent Division

Cake Cutting

Rent Division



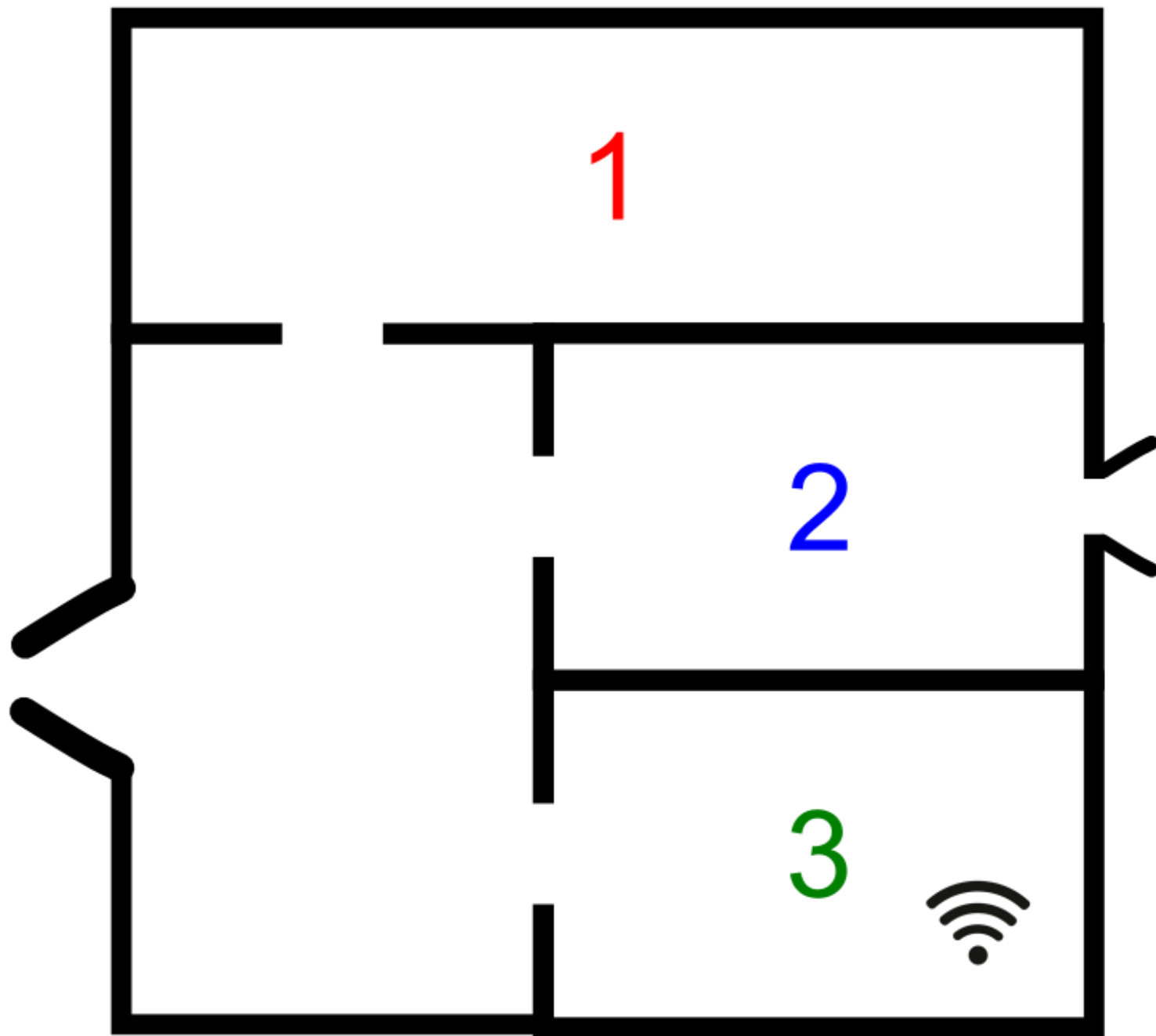


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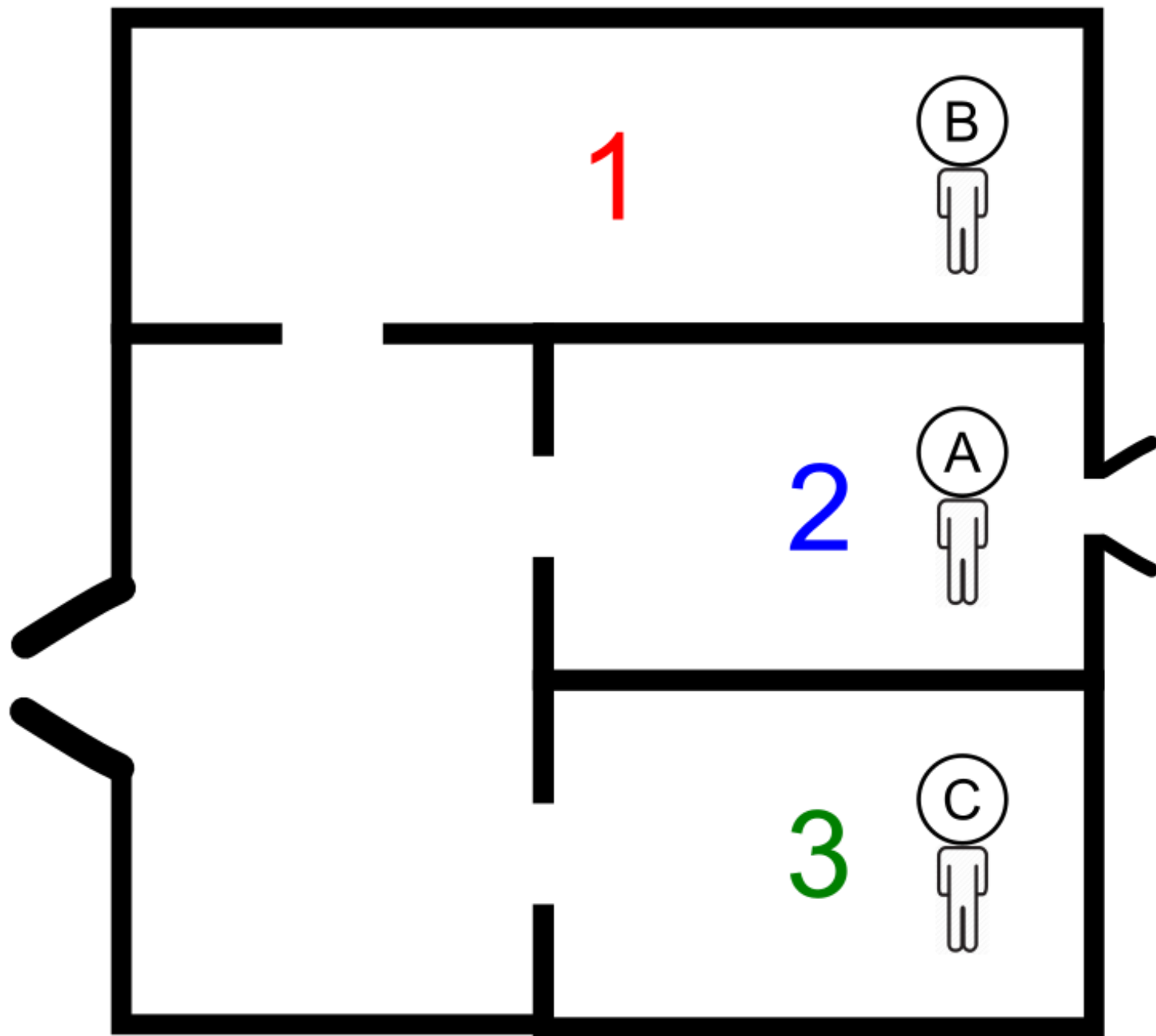
2

3

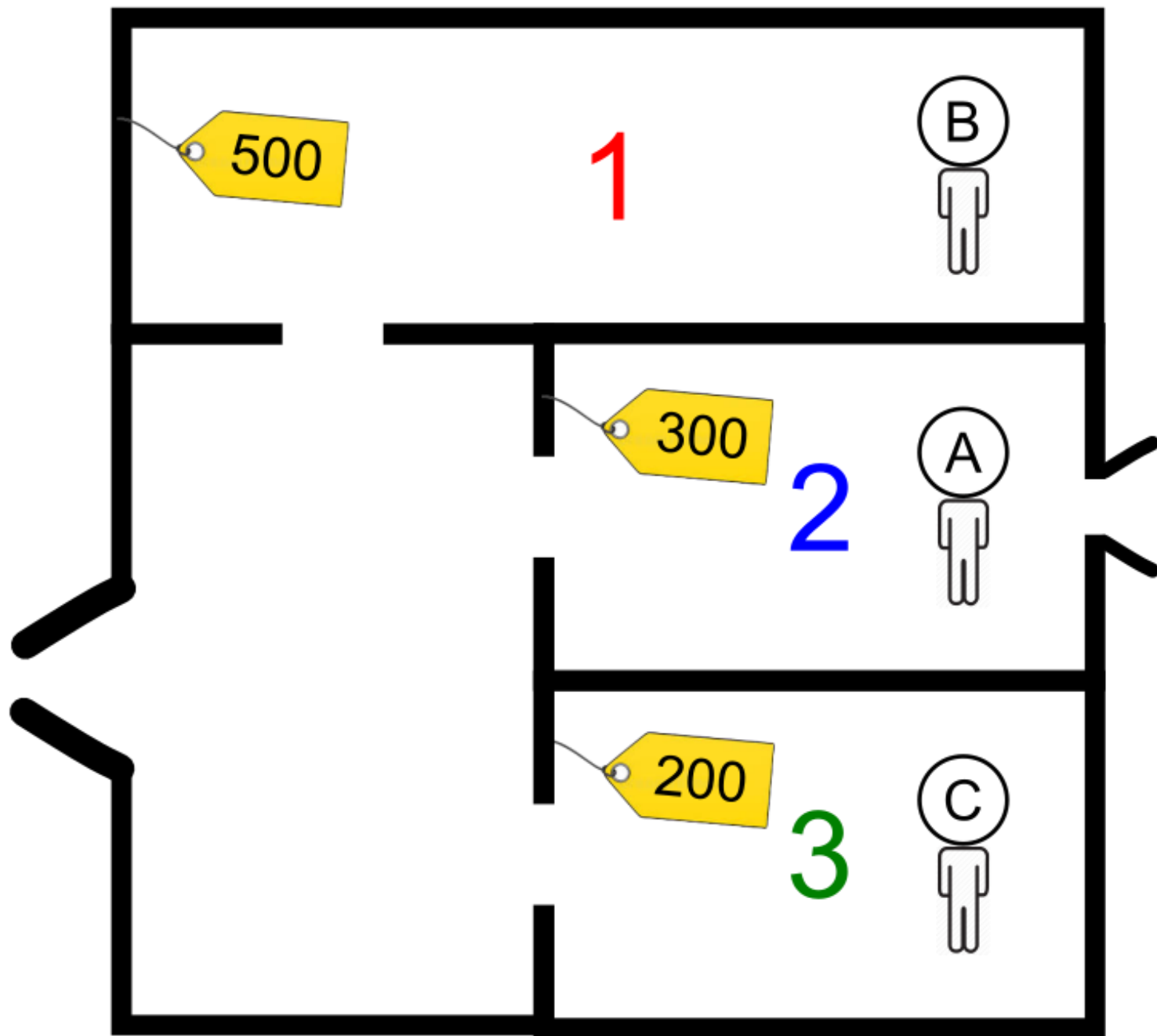




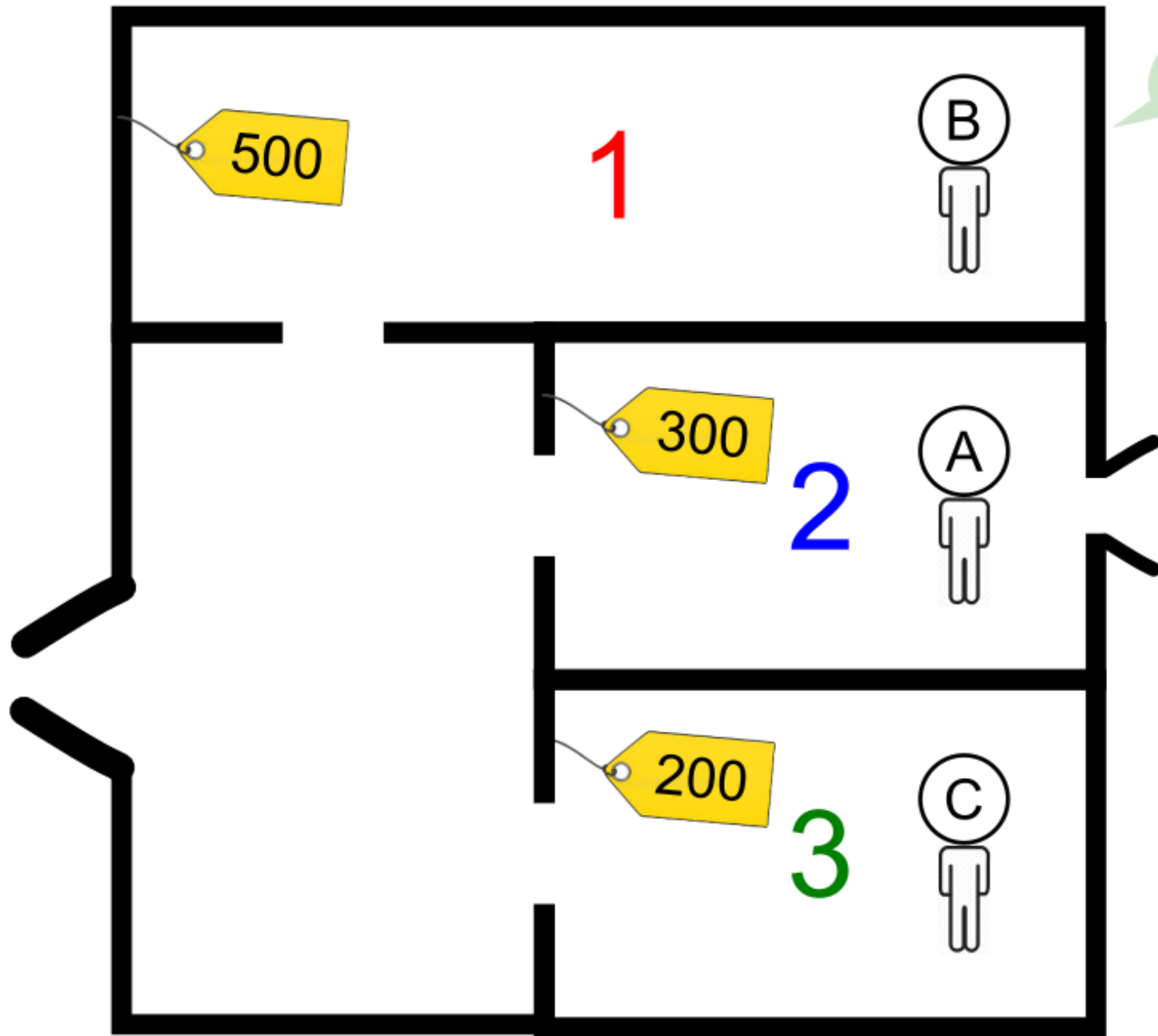
Rent
=
1000



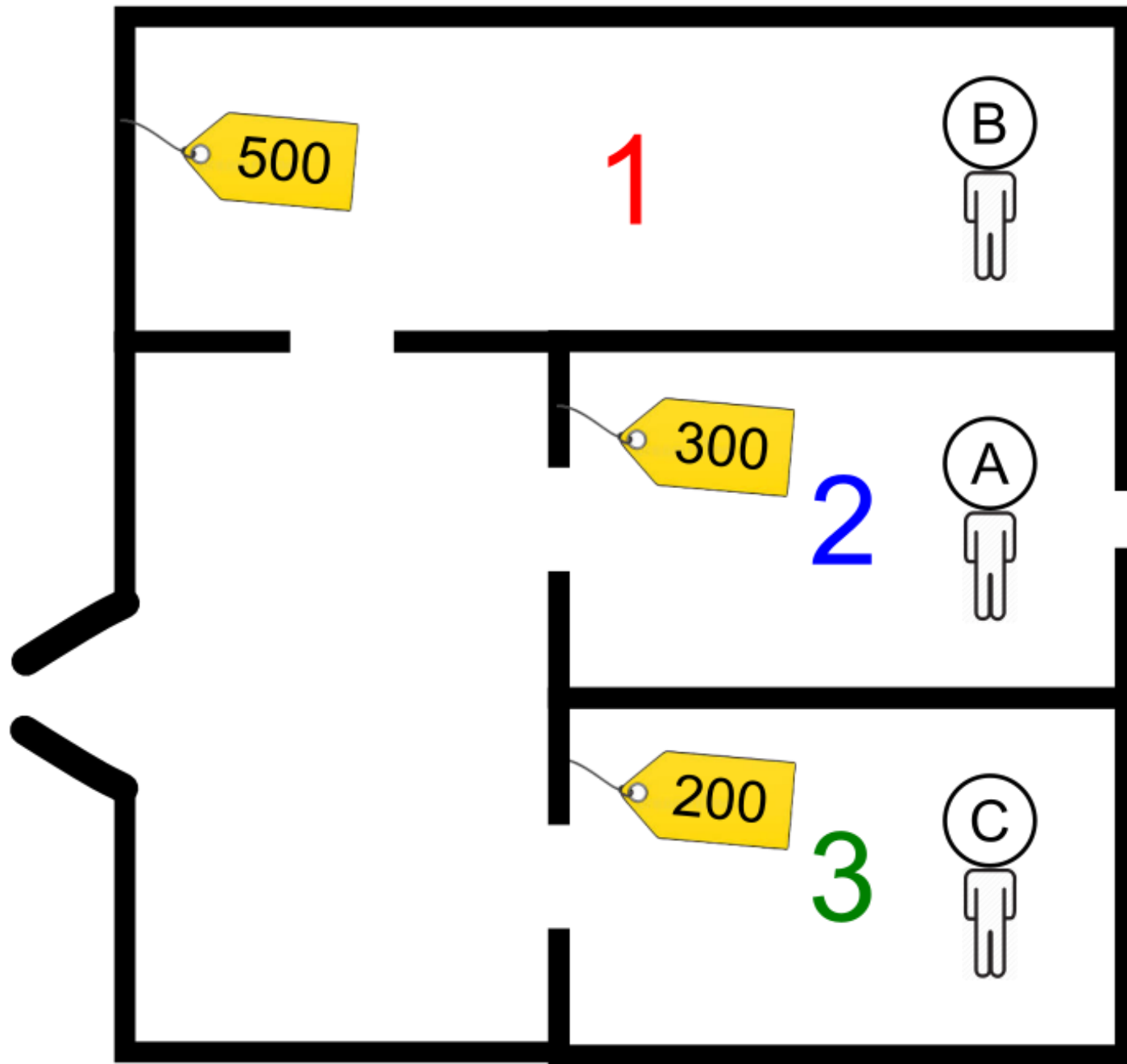
Rent
=
1000



Rent
=
1000

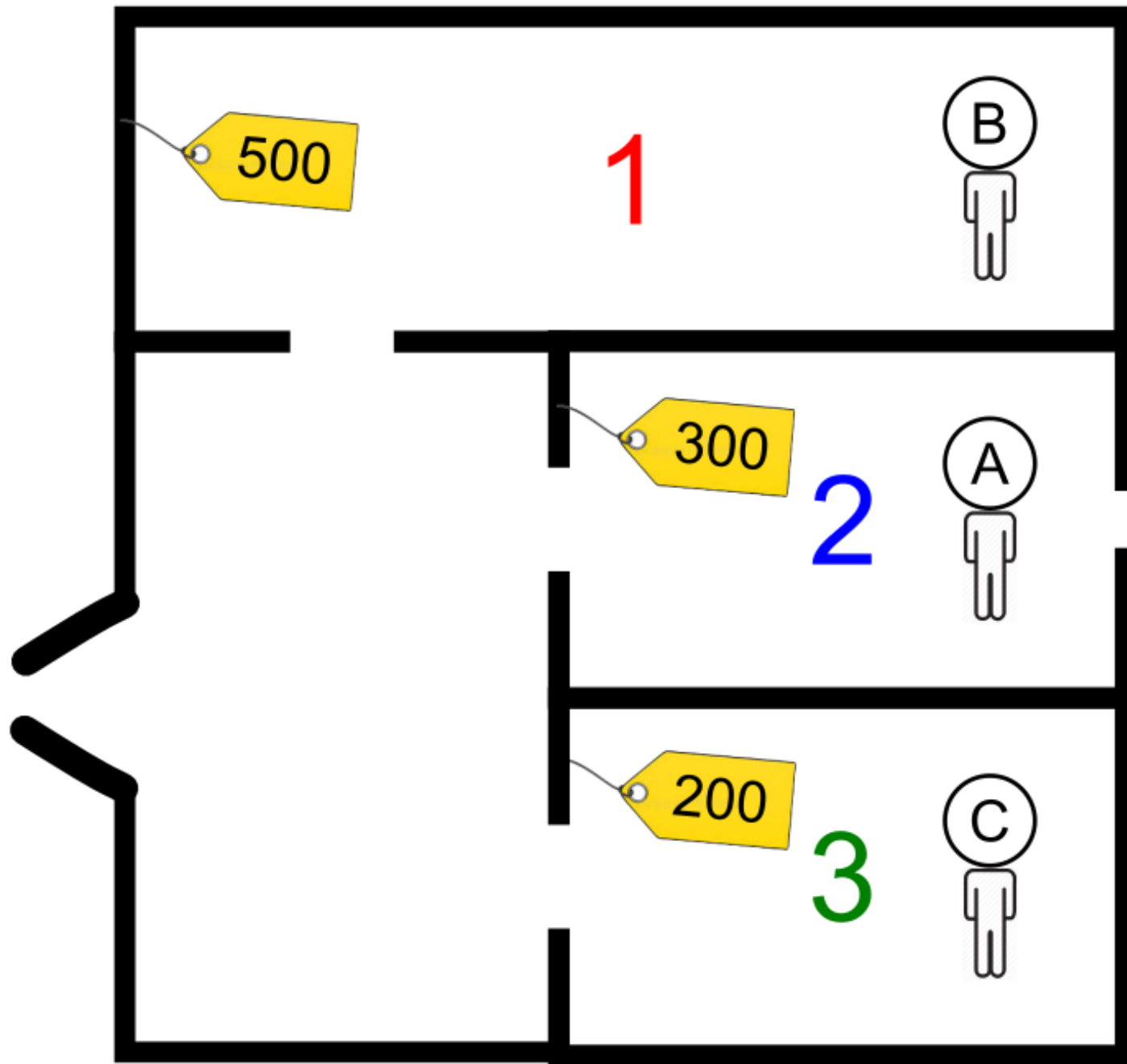


I like Room 1 for Rs. 500 over Room 2 for Rs. 300 and Room 3 for Rs. 200.



I like Room 1 for Rs. 500 over Room 2 for Rs. 300 and Room 3 for Rs. 200.

I like Room 2 for Rs. 300 over Room 1 for Rs. 500 and Room 3 for Rs. 200.

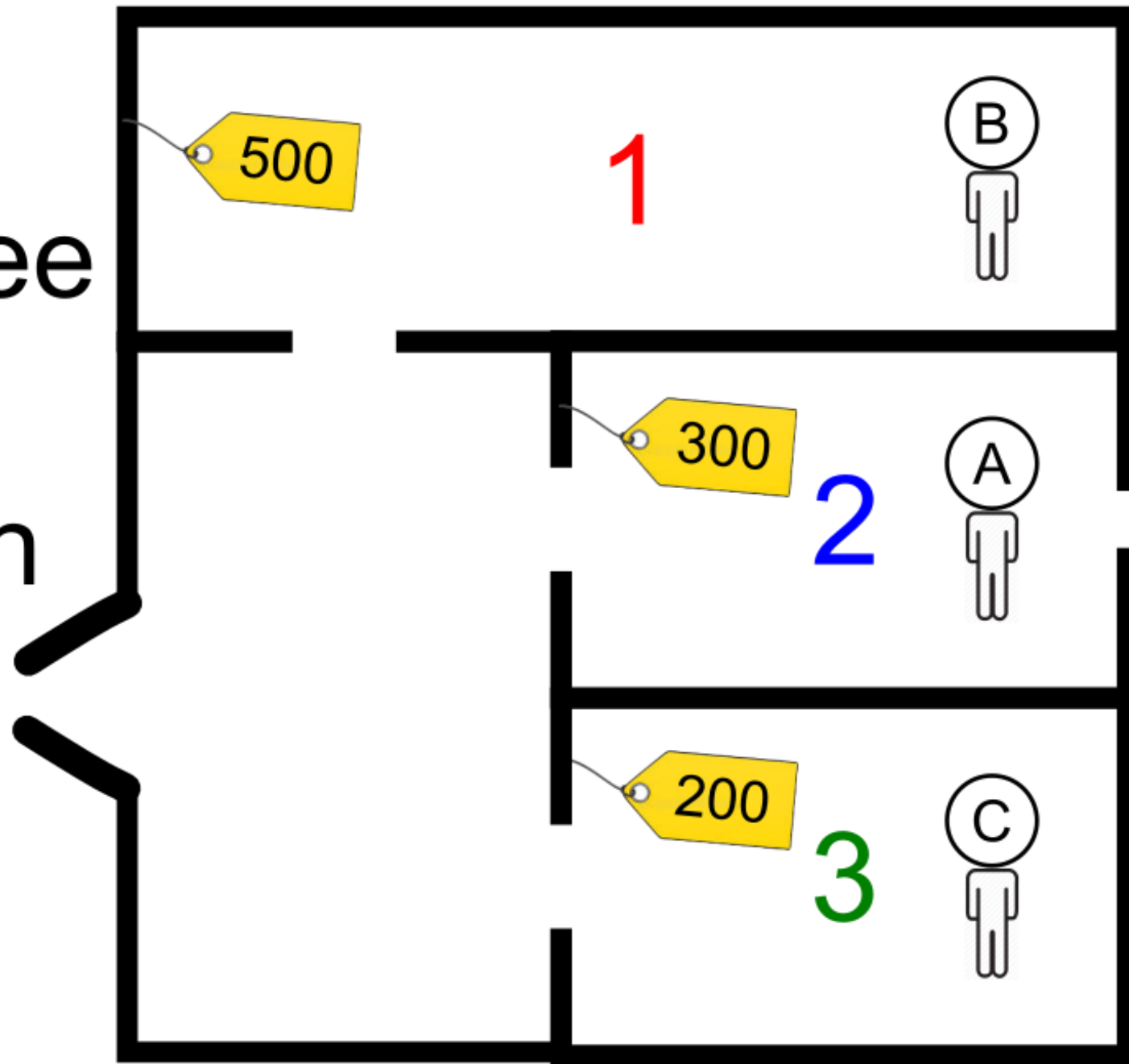


I like Room 1 for Rs. 500 over Room 2 for Rs. 300 and Room 3 for Rs. 200.

I like Room 2 for Rs. 300 over Room 1 for Rs. 500 and Room 3 for Rs. 200.

I like Room 3 for Rs. 200 over Room 1 for Rs. 500 and Room 2 for Rs. 200.

Envy-free rent division



I like Room 1 for Rs. 500 over Room 2 for Rs. 300 and Room 3 for Rs. 200.

I like Room 2 for Rs. 300 over Room 1 for Rs. 500 and Room 3 for Rs. 200.

I like Room 3 for Rs. 200 over Room 1 for Rs. 500 and Room 2 for Rs. 200.

Cake Cutting

Rent Division

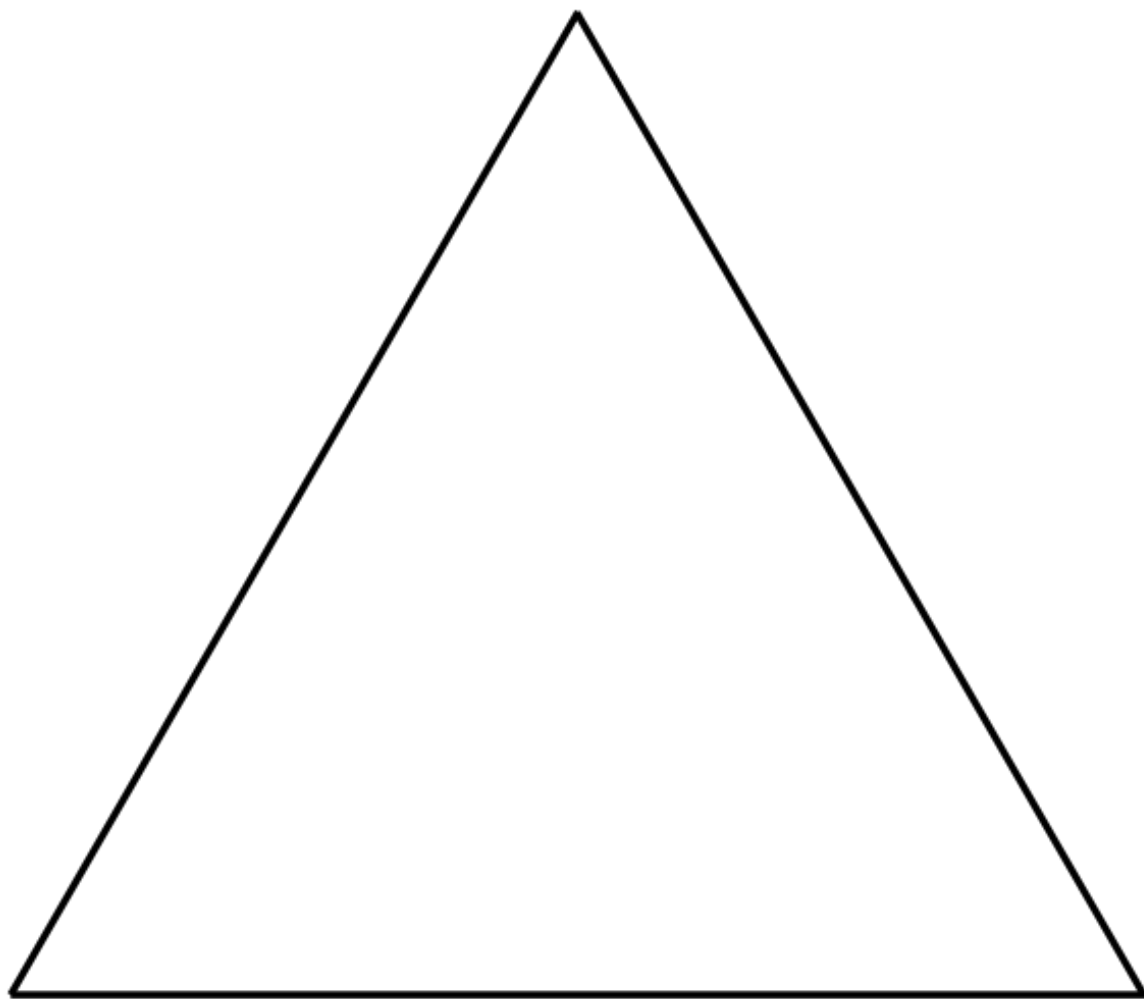
Triangles!

Cake Cutting

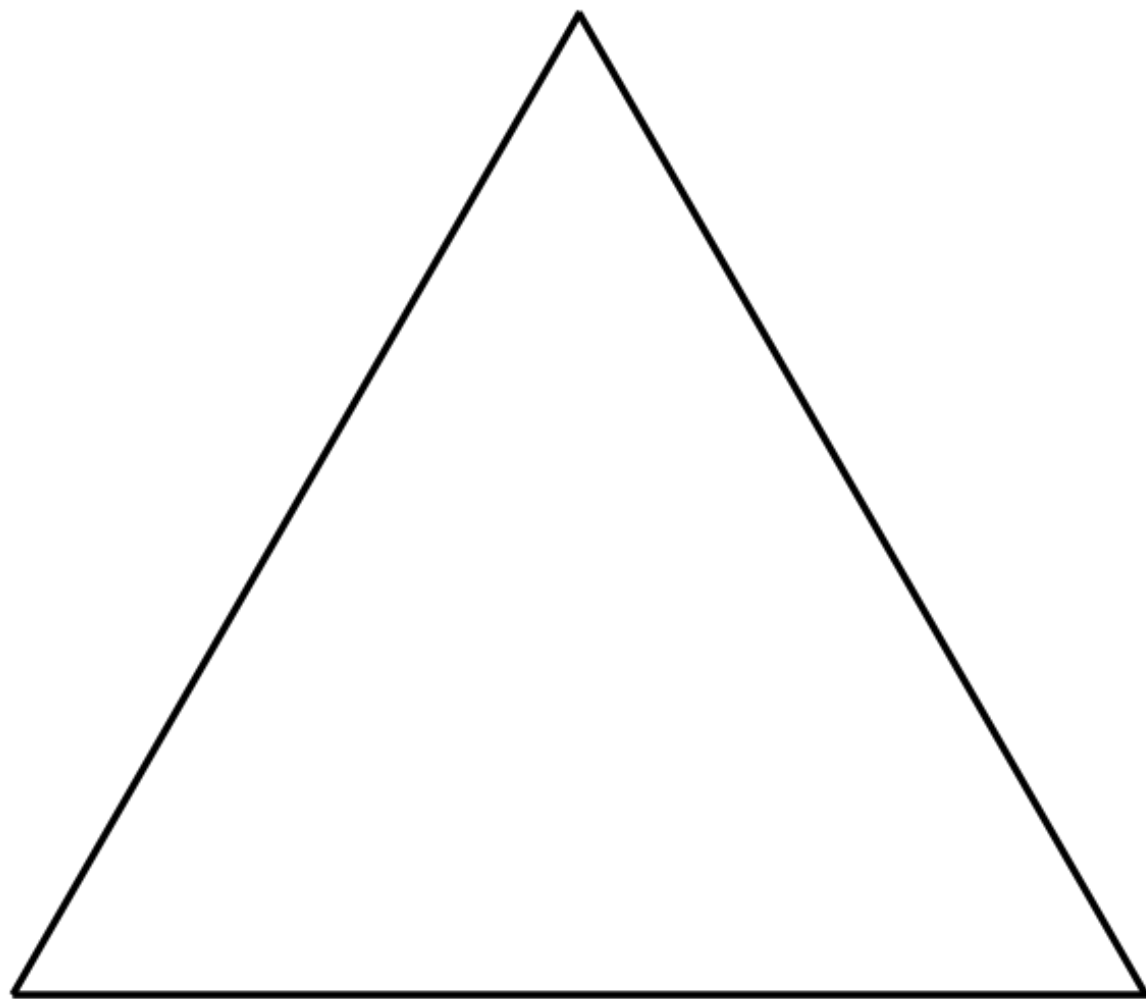
Rent Division

Sperner's Lemma

Sperner's Lemma



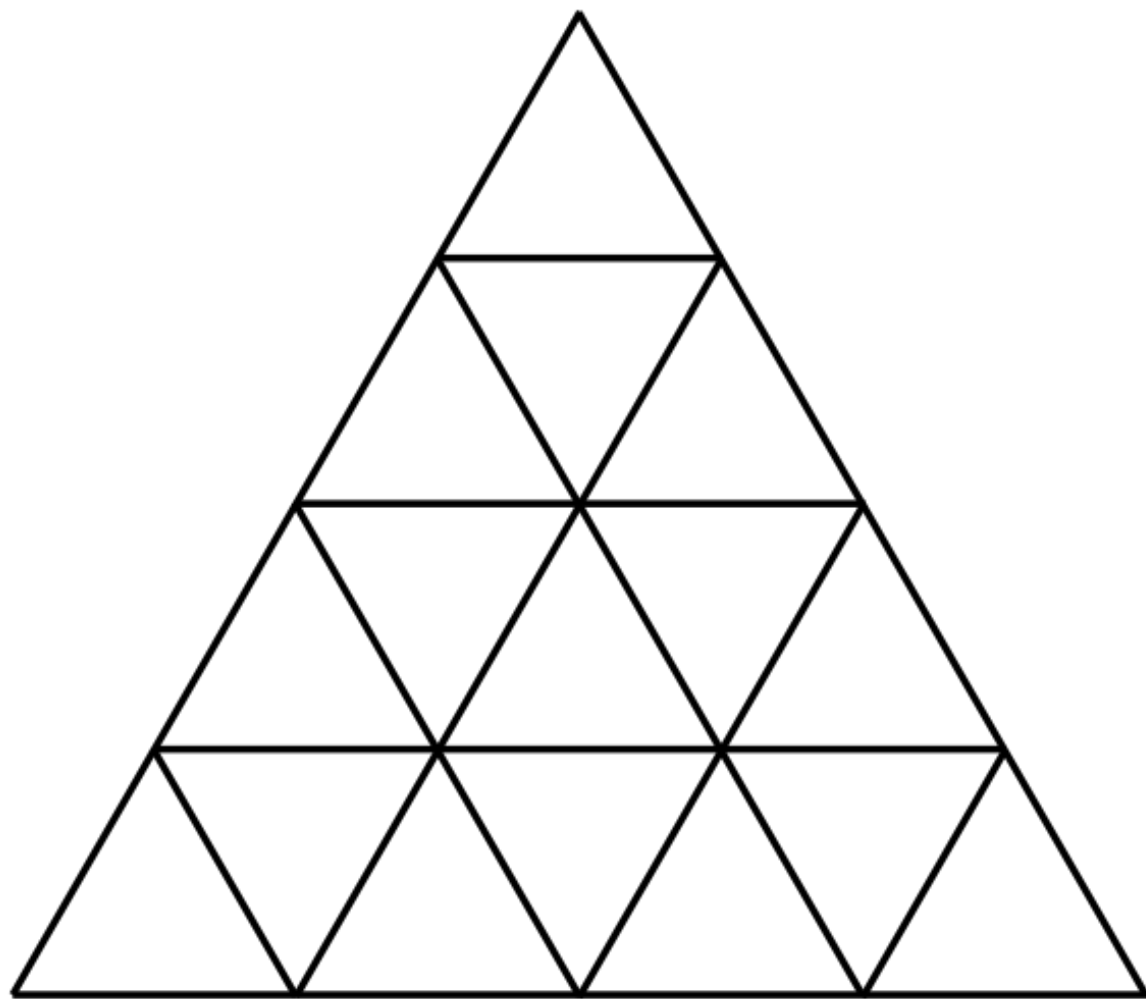
Sperner's Lemma



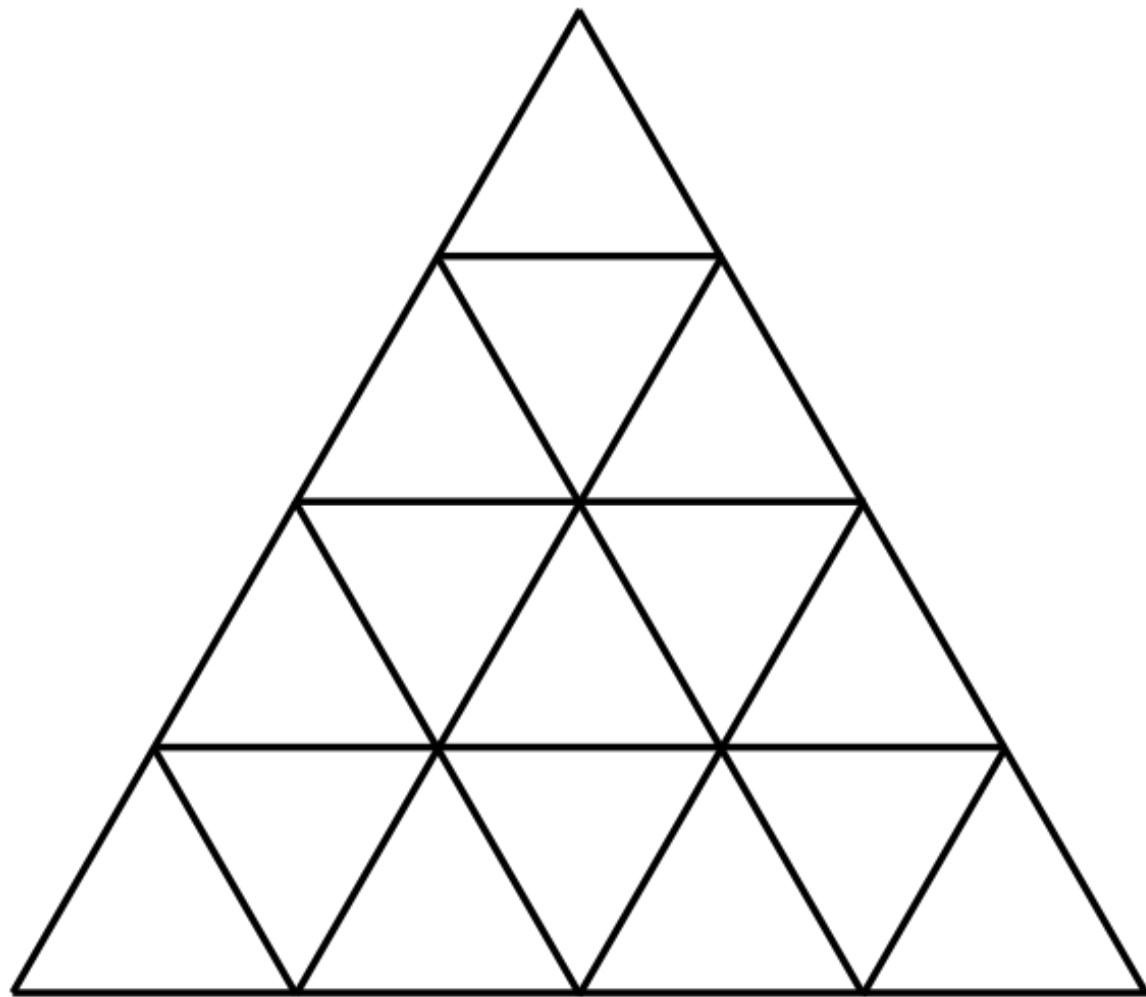
1. Triangulation into *elementary* triangles

Sperner's Lemma

1. Triangulation into *elementary* triangles

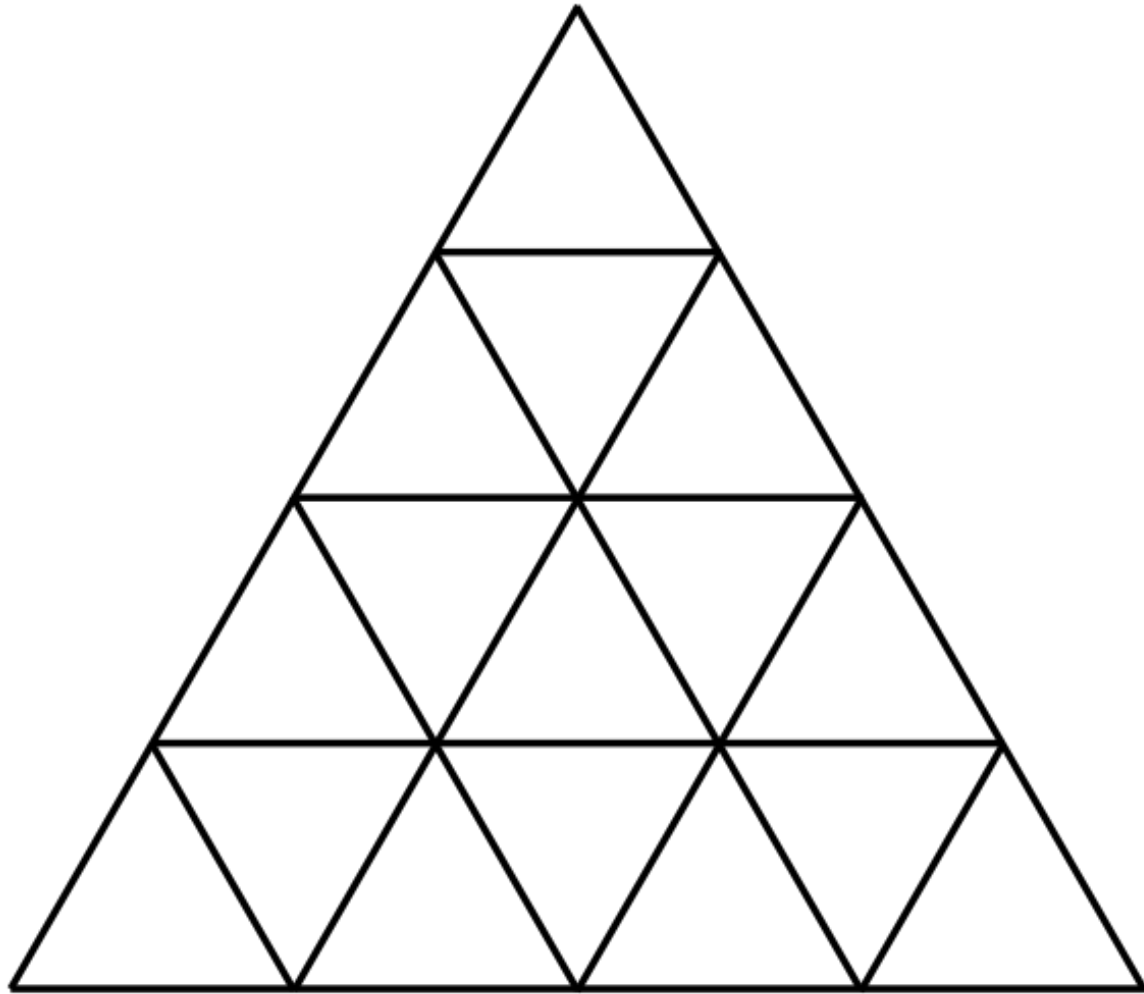


Sperner's Lemma



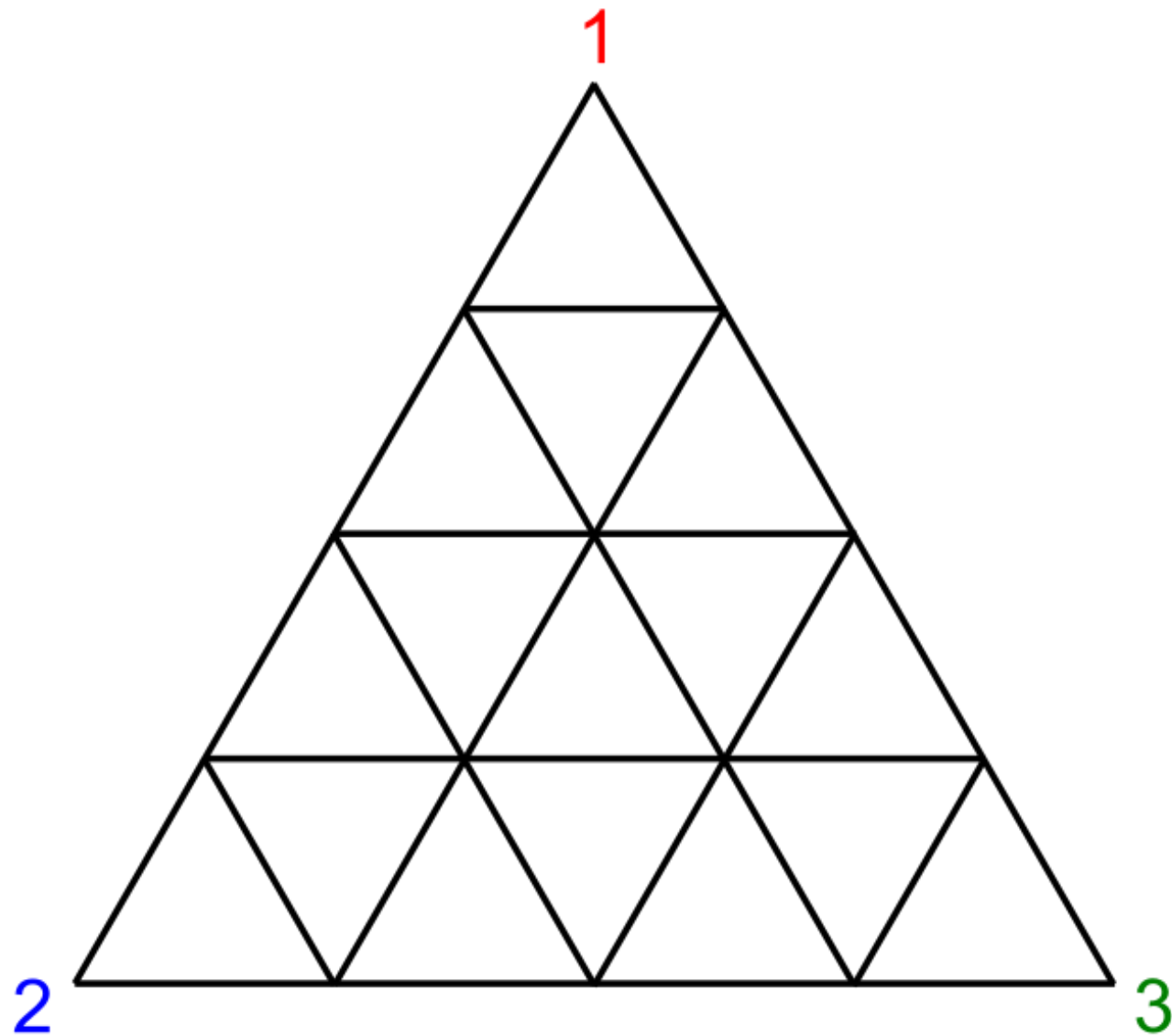
1. Triangulation into *elementary* triangles
2. Sperner labeling

Sperner's Lemma



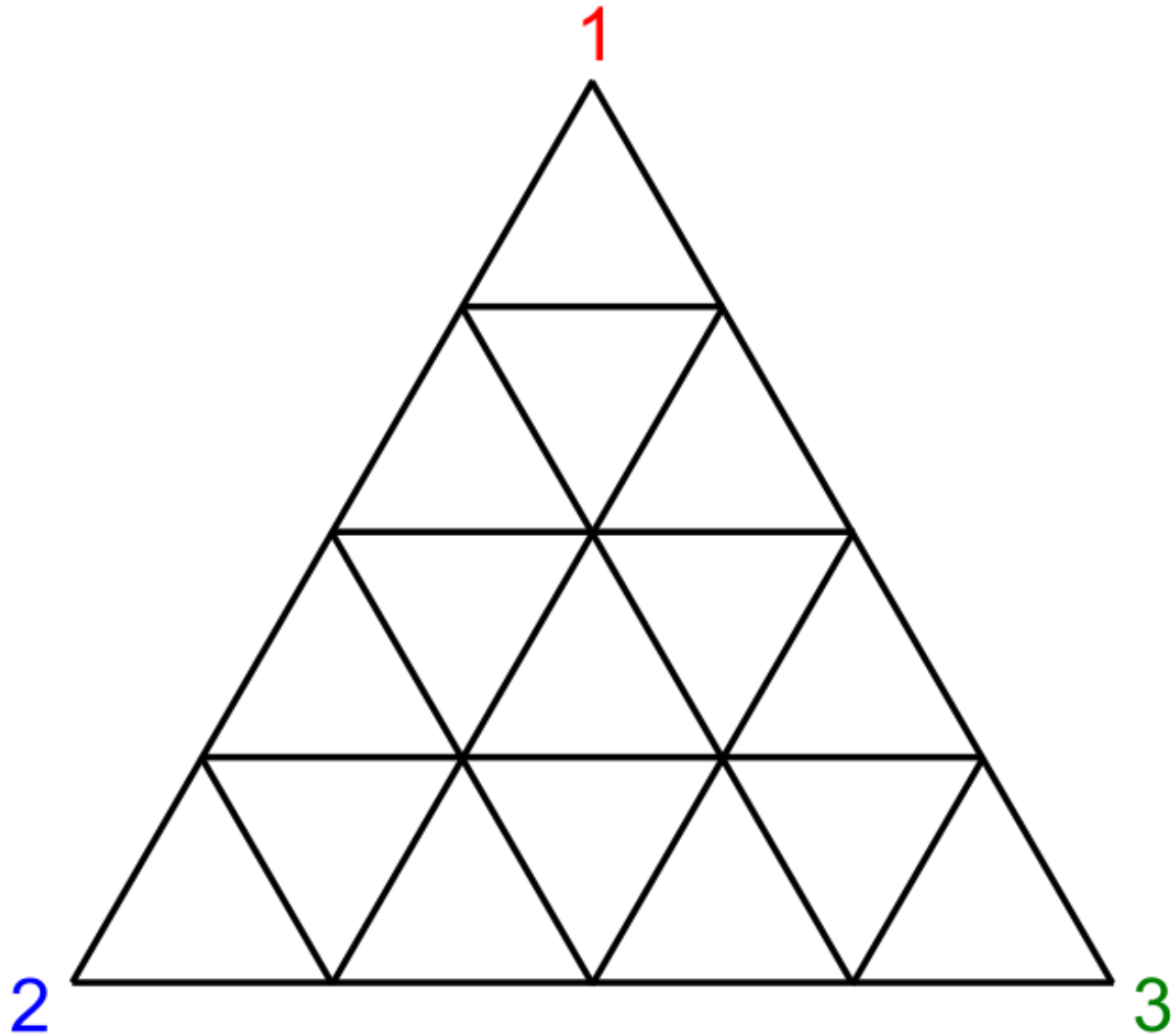
1. Triangulation into *elementary* triangles
2. Sperner labeling
 - Main vertices have distinct labels

Sperner's Lemma



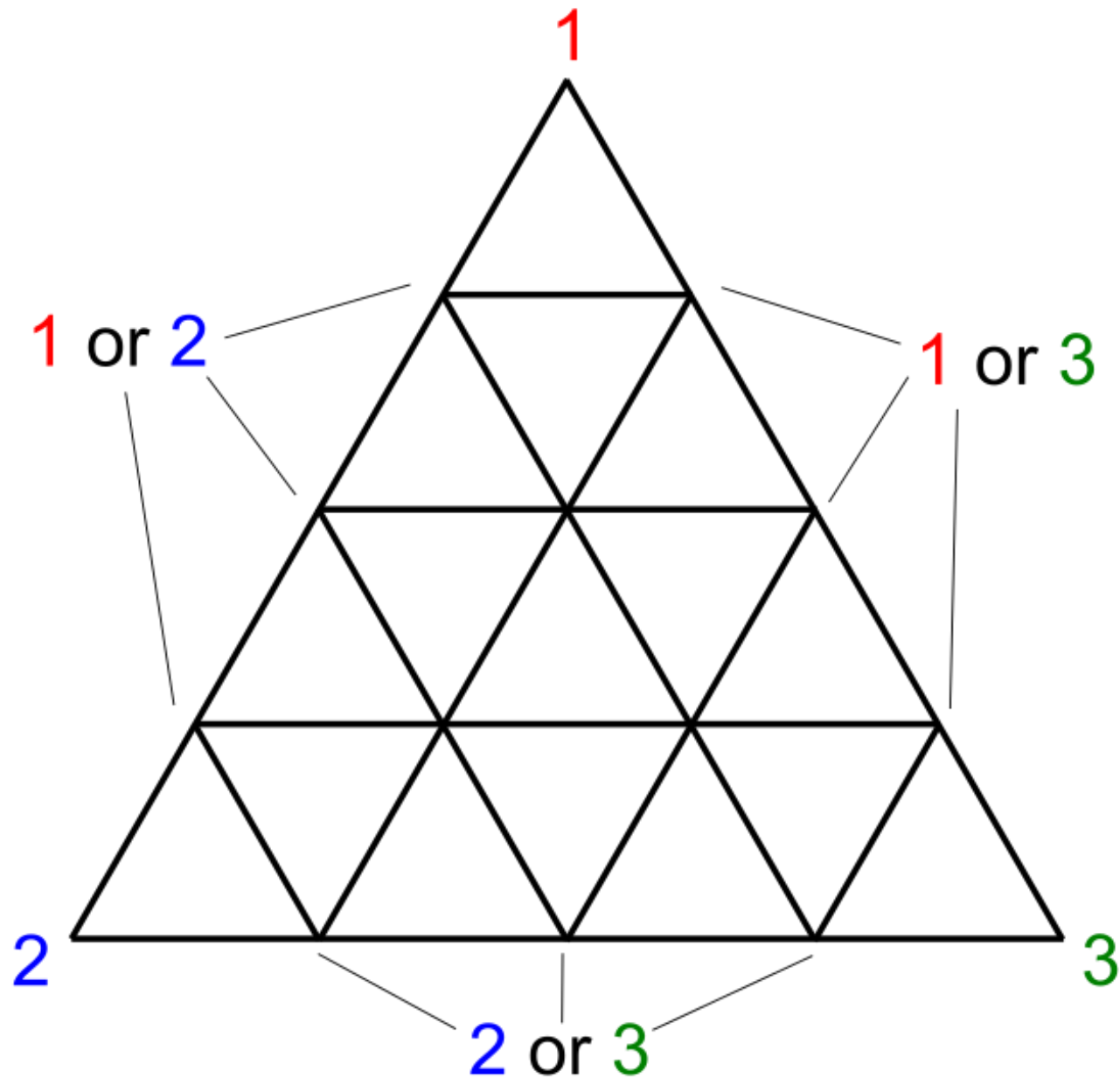
1. Triangulation into *elementary* triangles
2. Sperner labeling
 - Main vertices have distinct labels

Sperner's Lemma



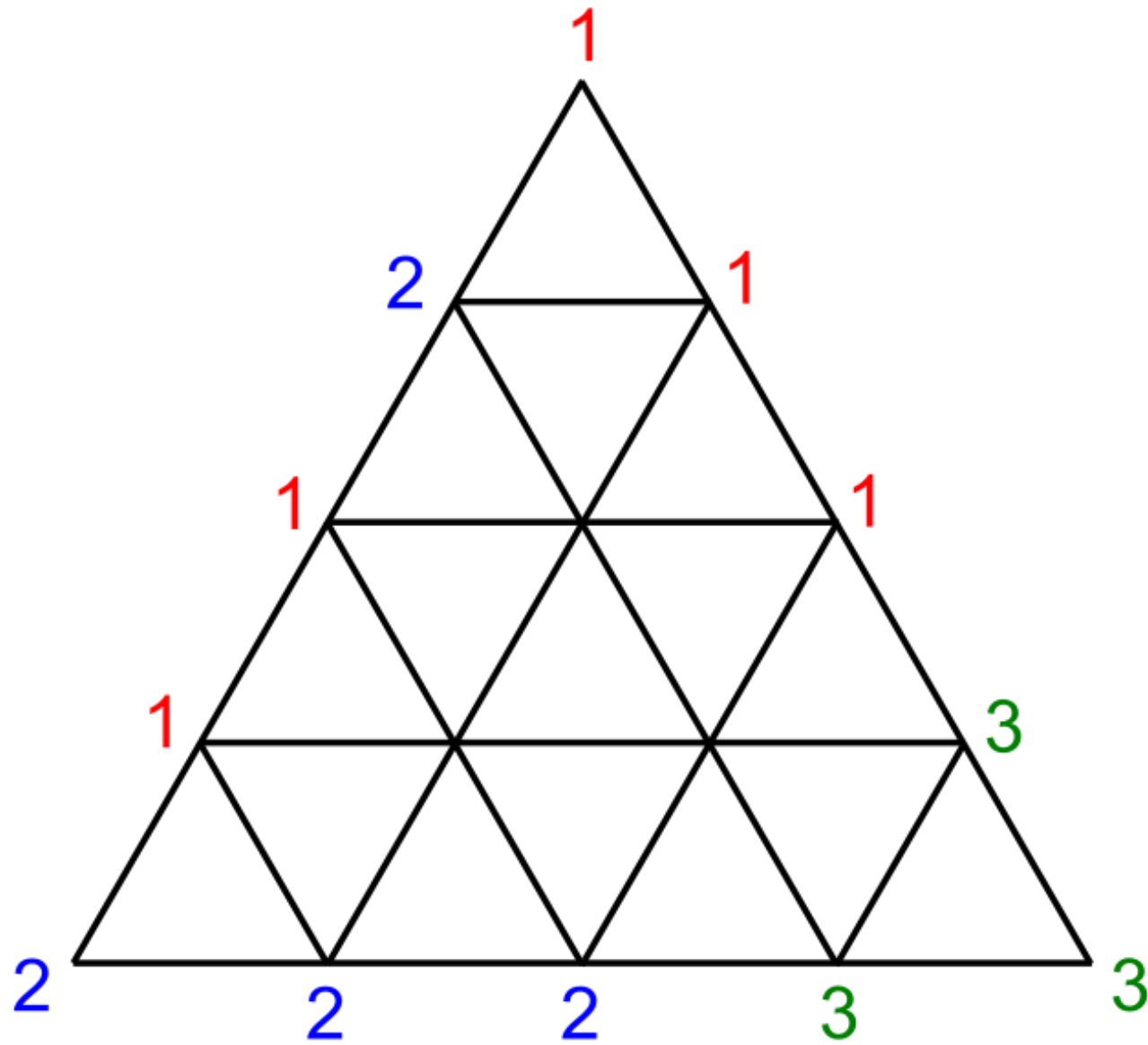
1. Triangulation into *elementary* triangles
2. Sperner labeling
 - Main vertices have distinct labels
 - Boundary vertices inherit the labels of the adjacent main vertices

Sperner's Lemma



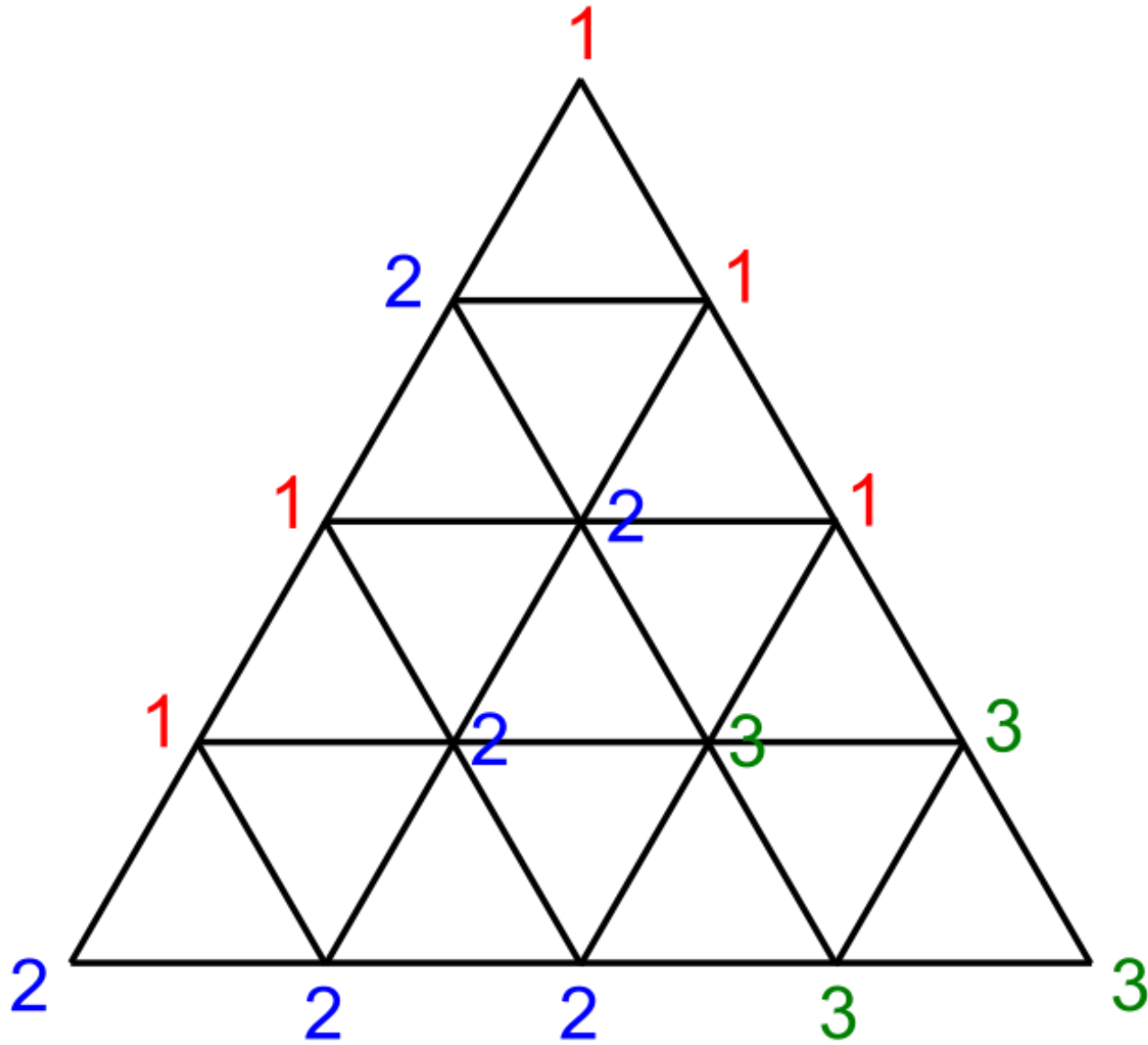
1. Triangulation into *elementary* triangles
2. Sperner labeling
 - Main vertices have distinct labels
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Sperner's Lemma



1. Triangulation into *elementary* triangles
2. Sperner labeling
 - Main vertices have distinct labels
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Sperner's Lemma

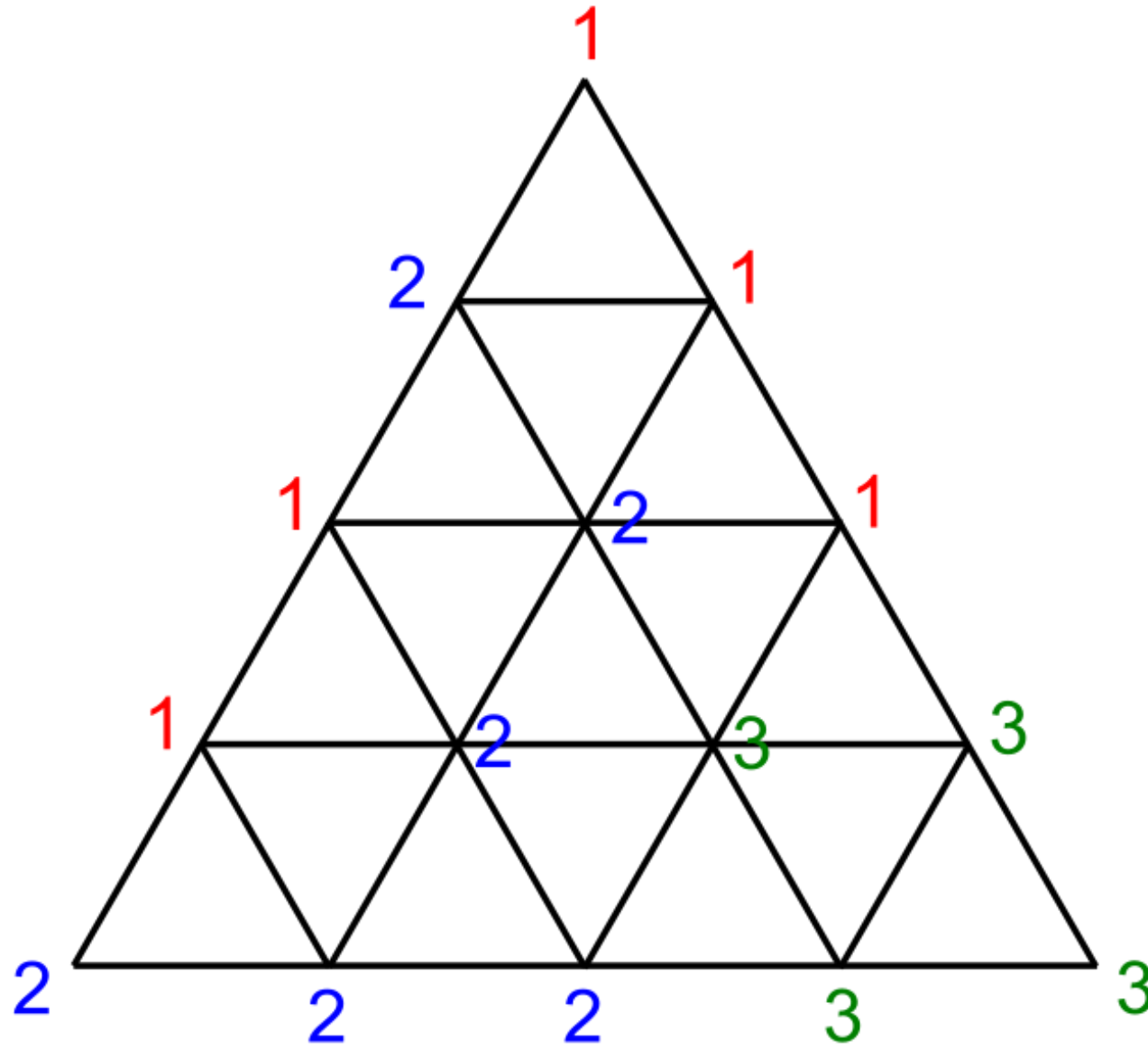


1. Triangulation into *elementary* triangles

2. Sperner labeling

- Main vertices have distinct labels
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Sperner's Lemma



1. Triangulation into *elementary* triangles

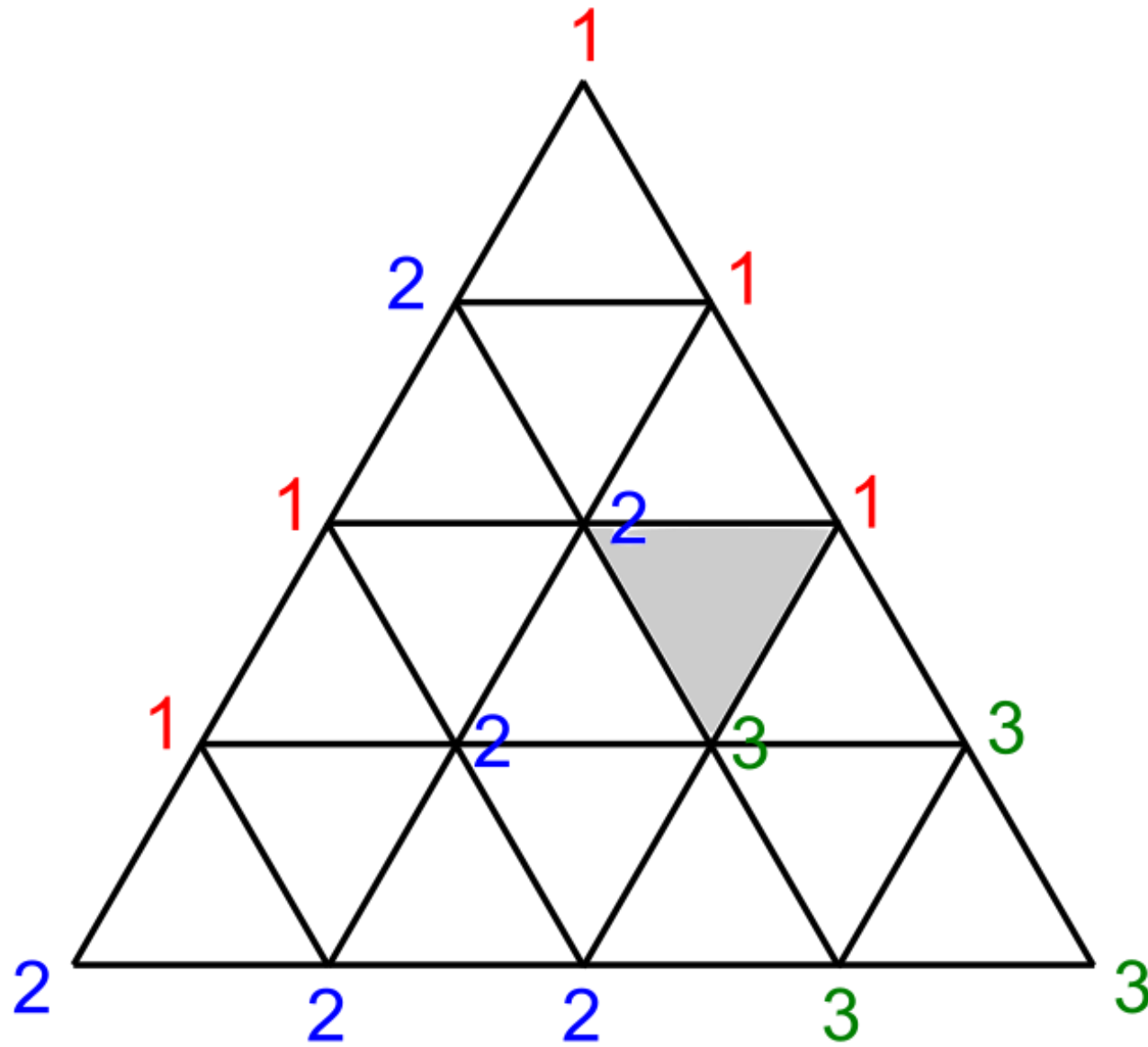
2. Sperner labeling

- Main vertices have distinct labels
- Boundary vertices inherit the labels of the adjacent main vertices

[Sperner, 1928]

Any Sperner labeled triangulation has at least one fully labeled elementary triangle.

Sperner's Lemma



1. Triangulation into *elementary* triangles

2. Sperner labeling

- Main vertices have distinct labels
- Boundary vertices inherit the labels of the adjacent main vertices

[Sperner, 1928]

Any Sperner labeled triangulation has at least one fully labeled elementary triangle.

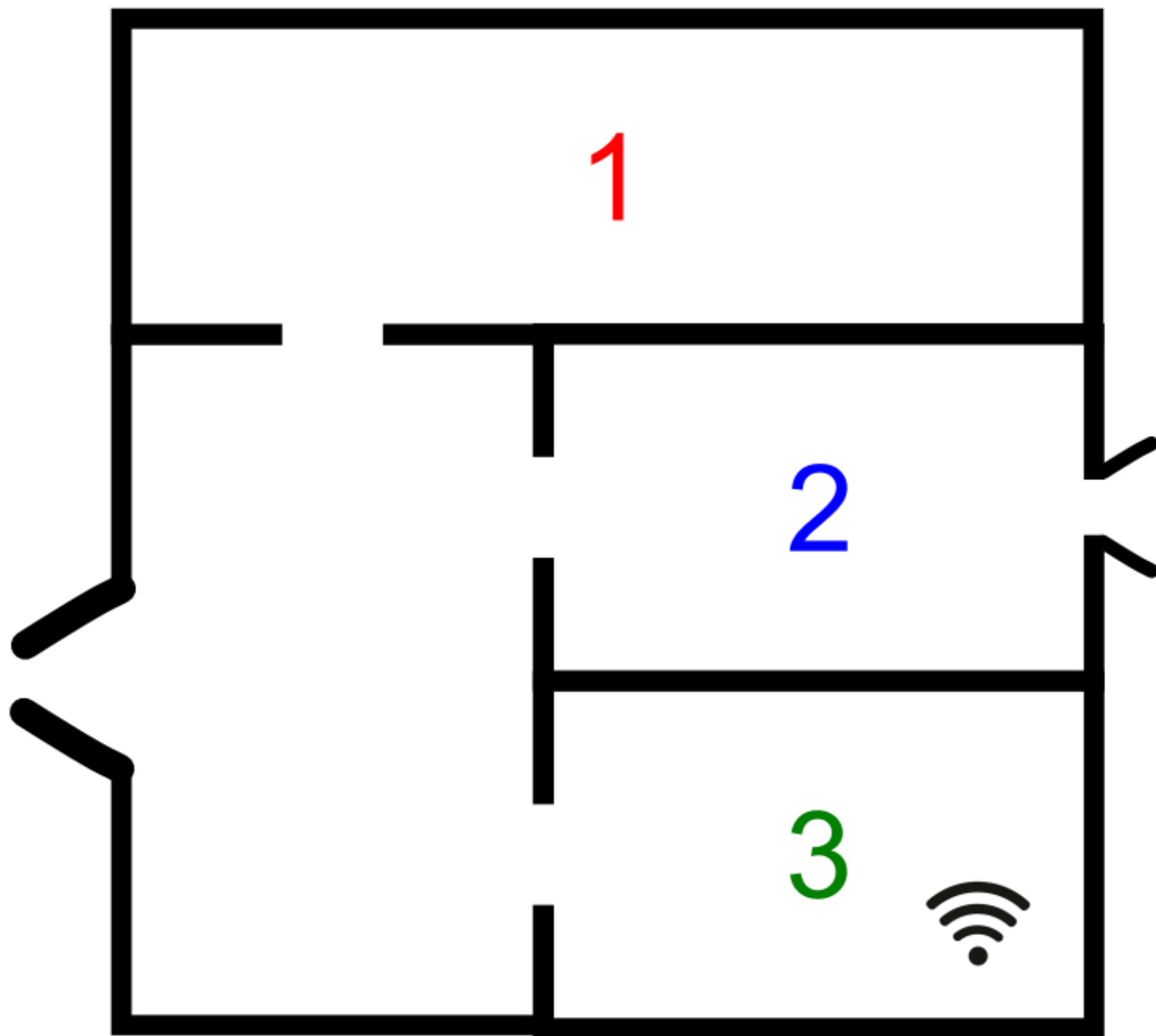
Triangles!

Cake Cutting

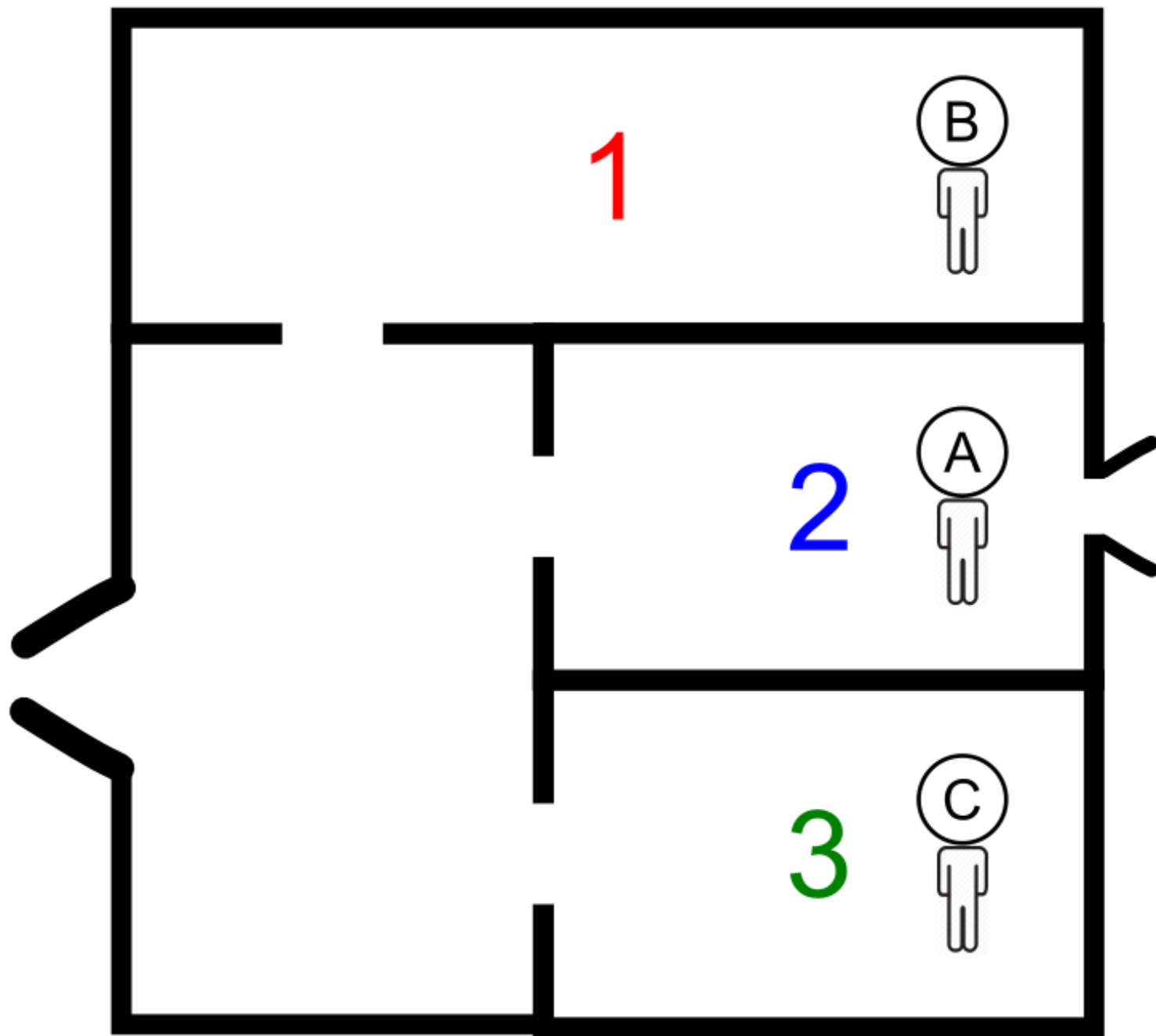
Rent Division

Cake Cutting

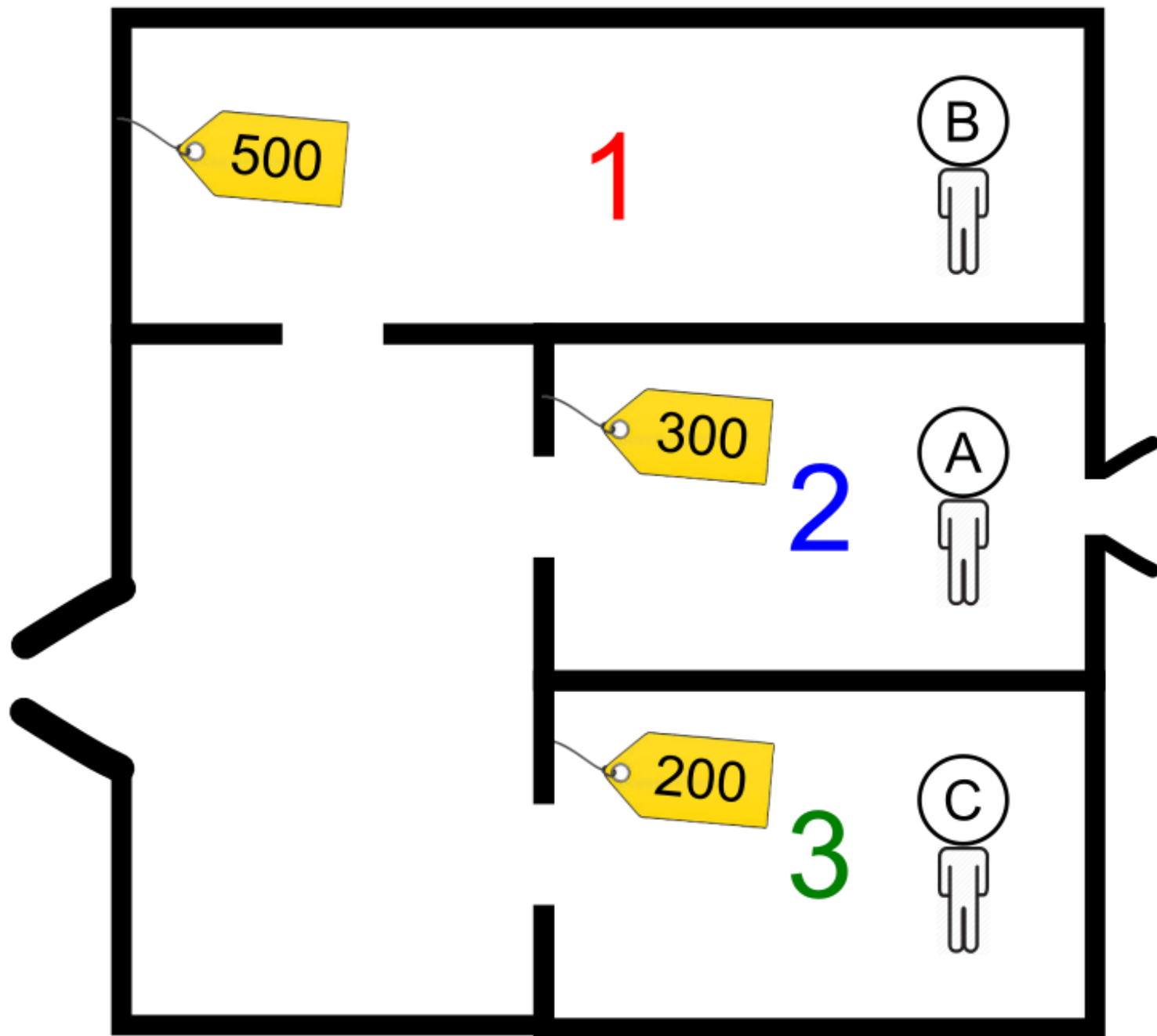
Rental harmony
via
Sperner's lemma



Rent
=
1000

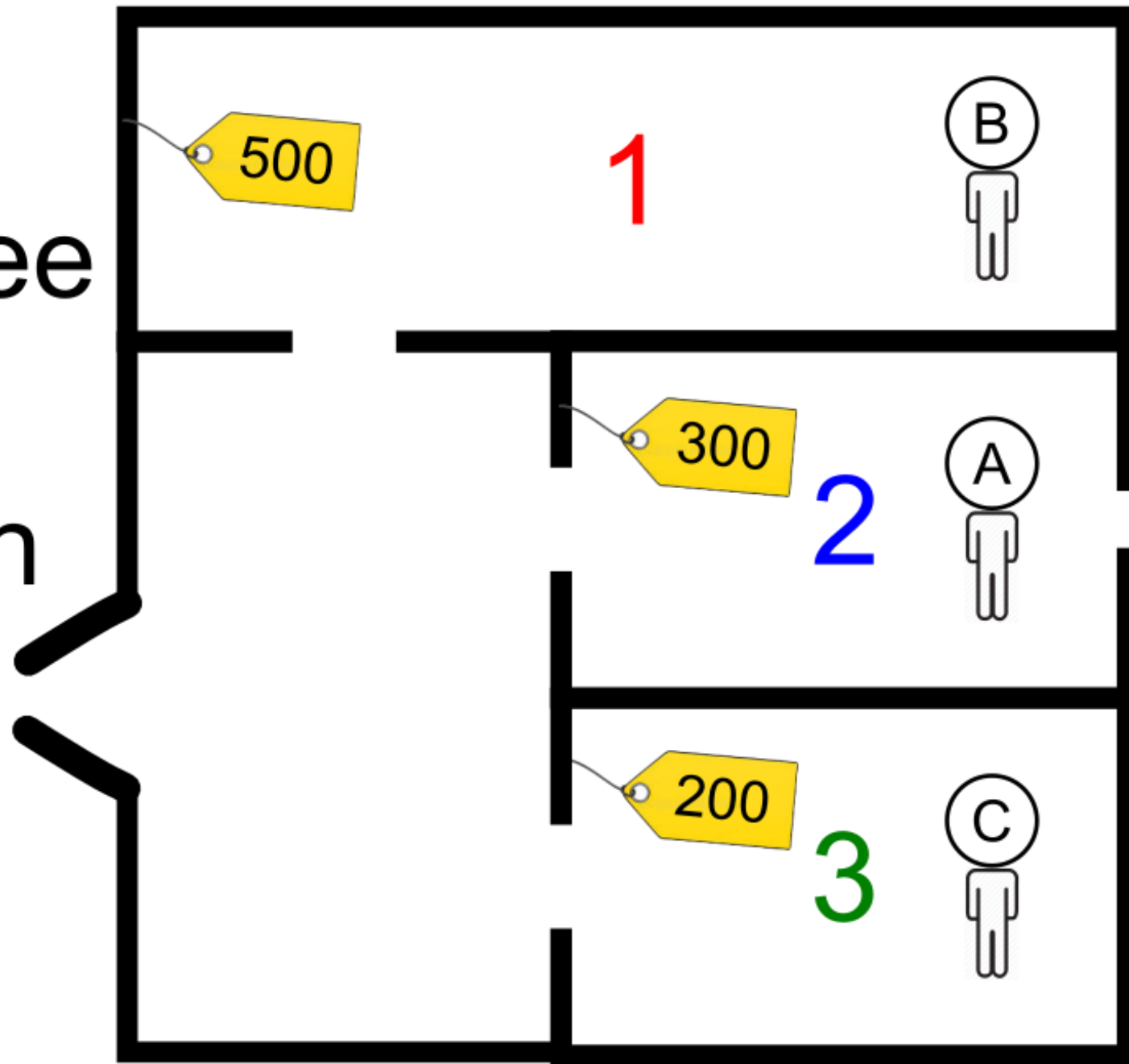


Rent
=
1000



Rent
=
1000

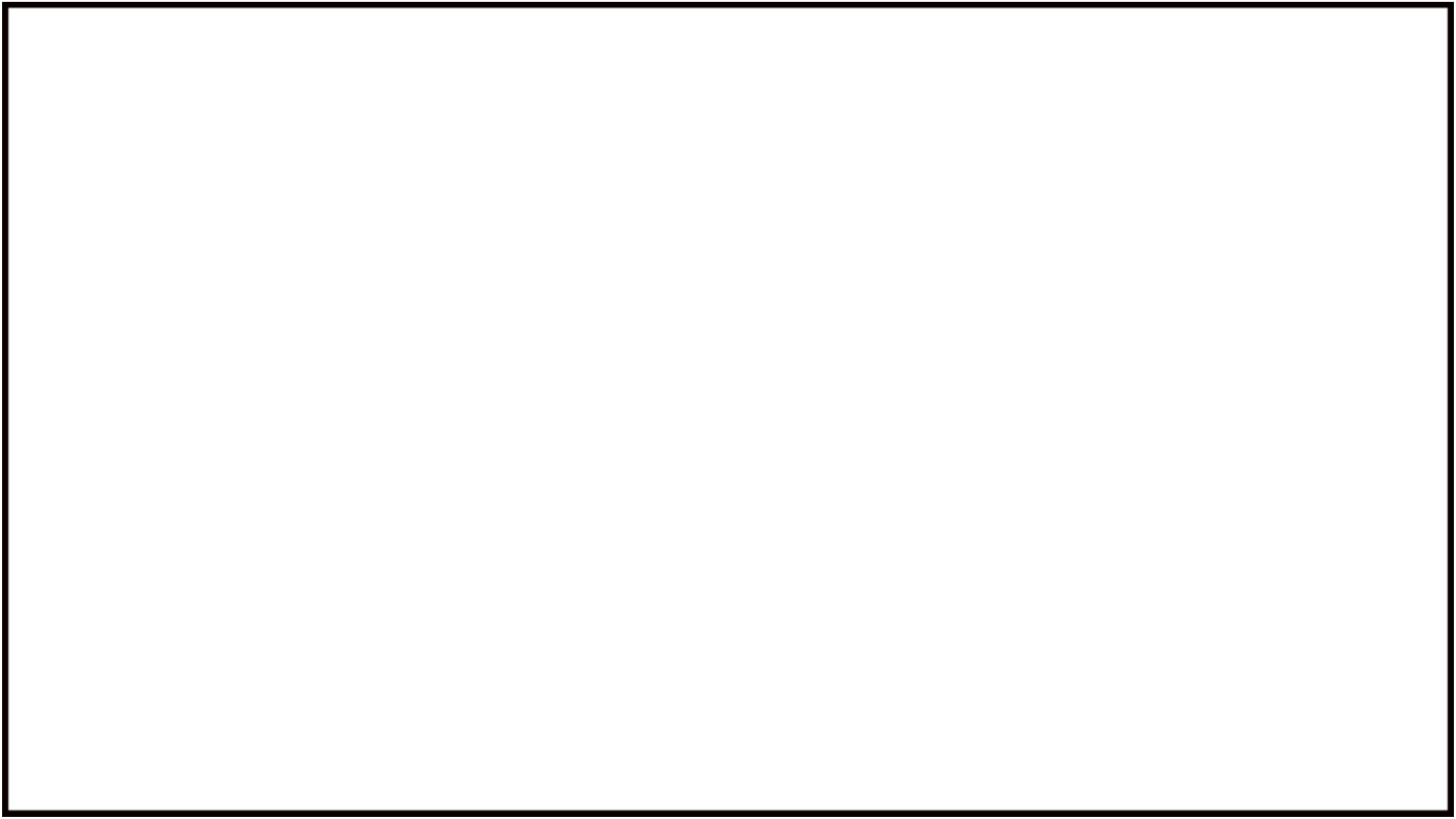
Envy-free rent division



I like Room 1 for Rs. 500 over Room 2 for Rs. 300 and Room 3 for Rs. 200.

I like Room 2 for Rs. 300 over Room 1 for Rs. 500 and Room 3 for Rs. 200.

I like Room 3 for Rs. 200 over Room 1 for Rs. 500 and Room 2 for Rs. 300.

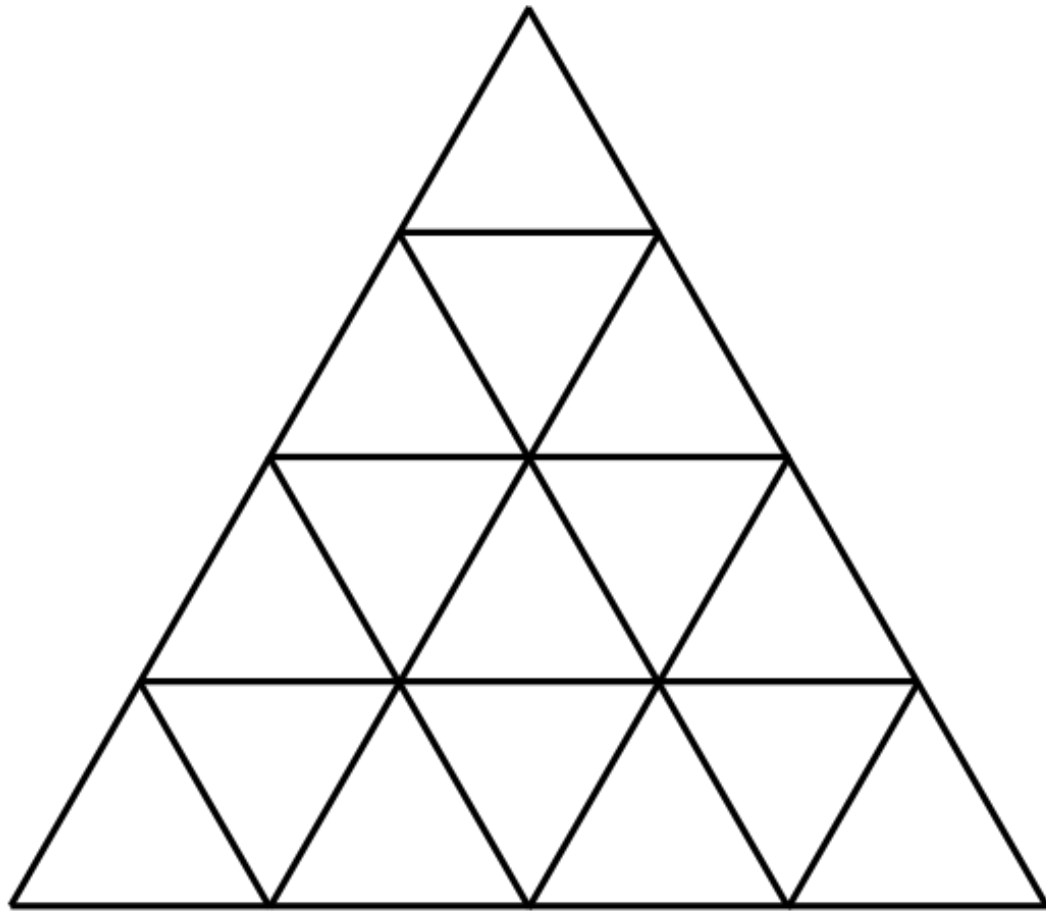


Ownership labeling

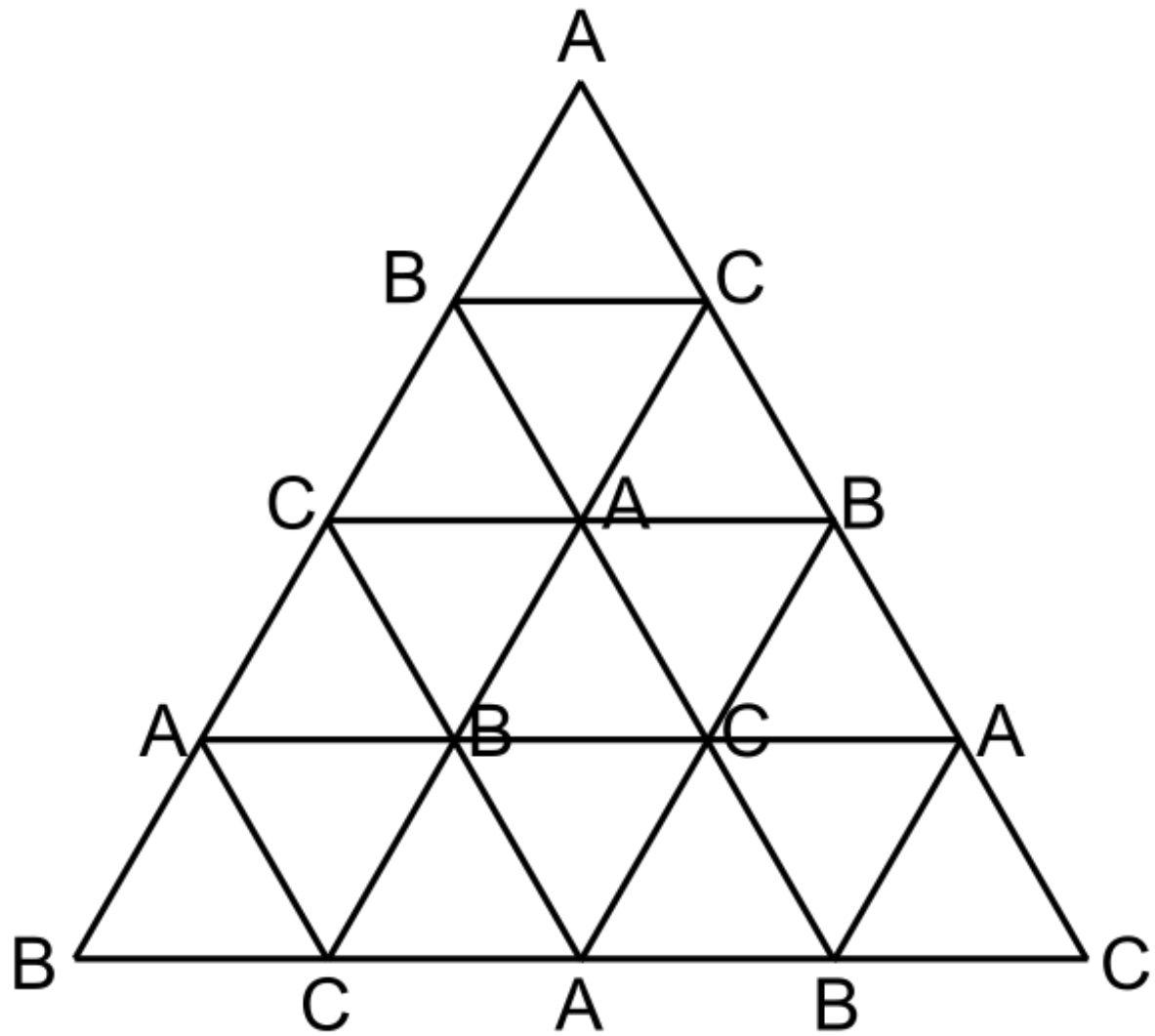
Preference labeling

Ownership labeling

Preference labeling

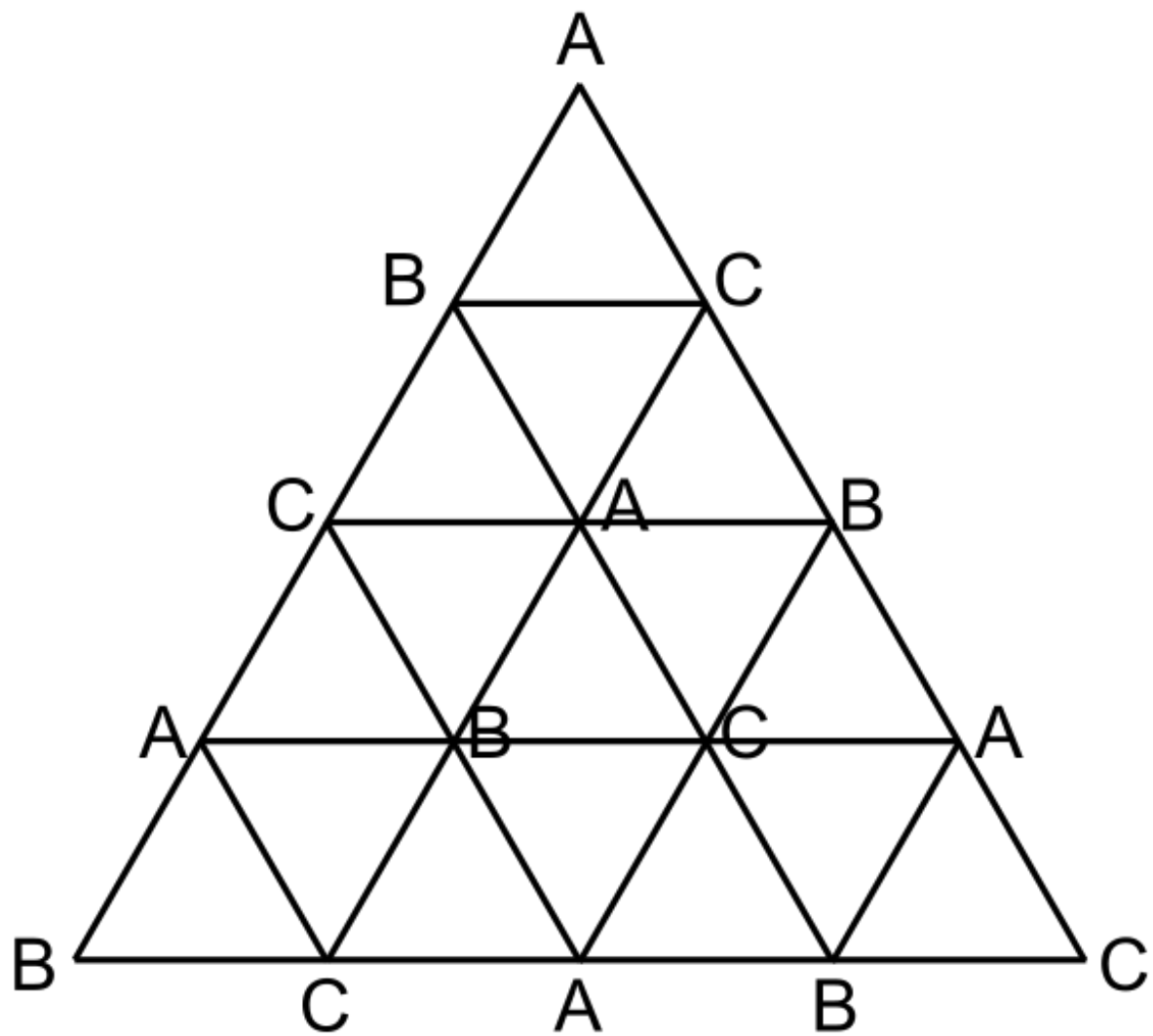


Ownership labeling

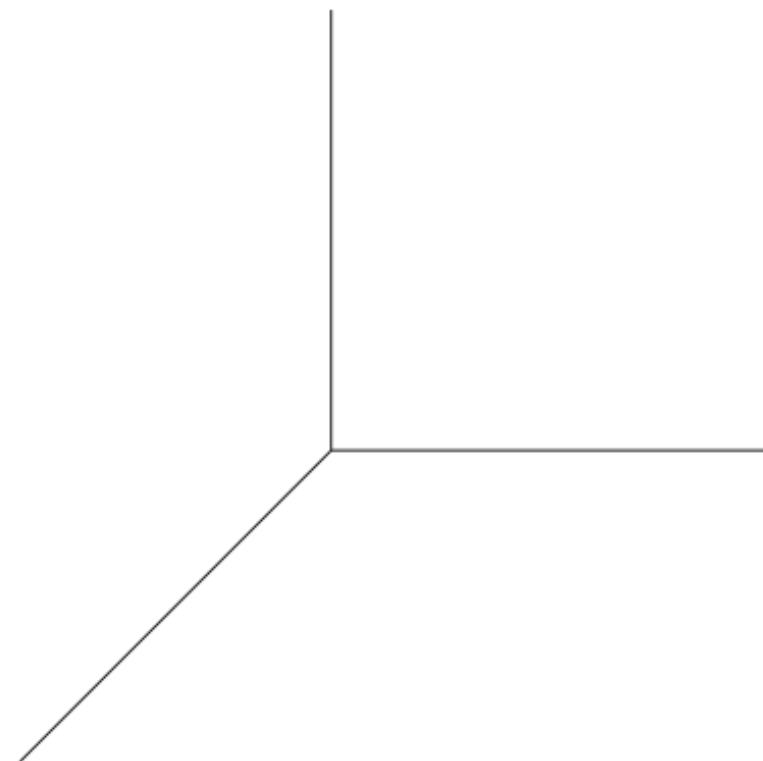


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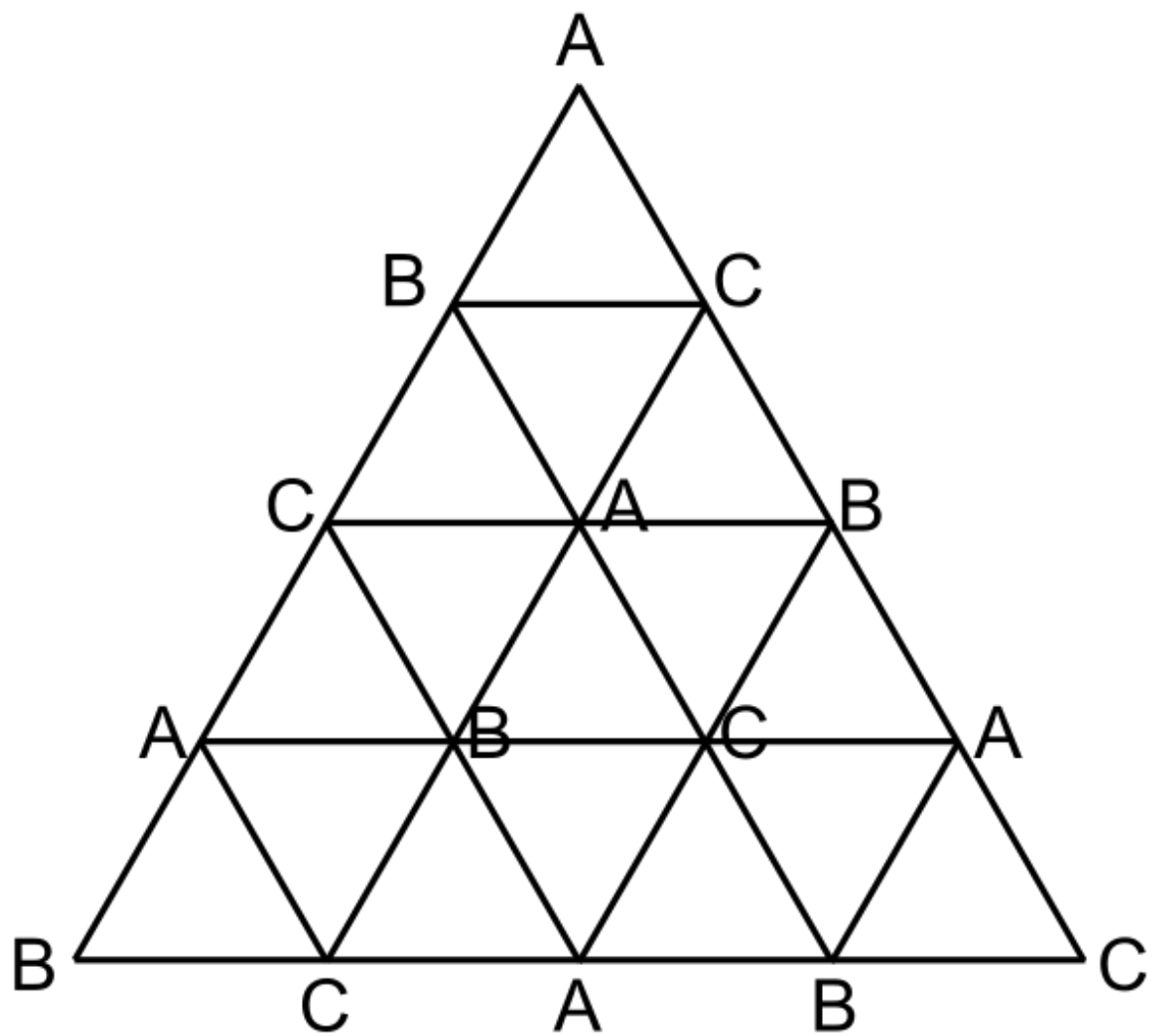
Ownership labeling



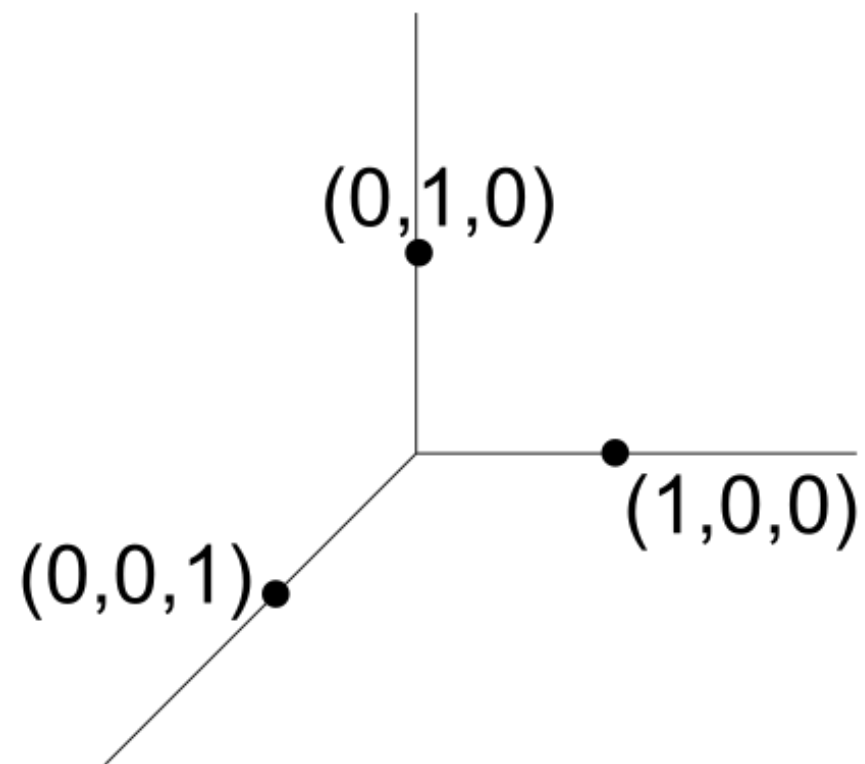
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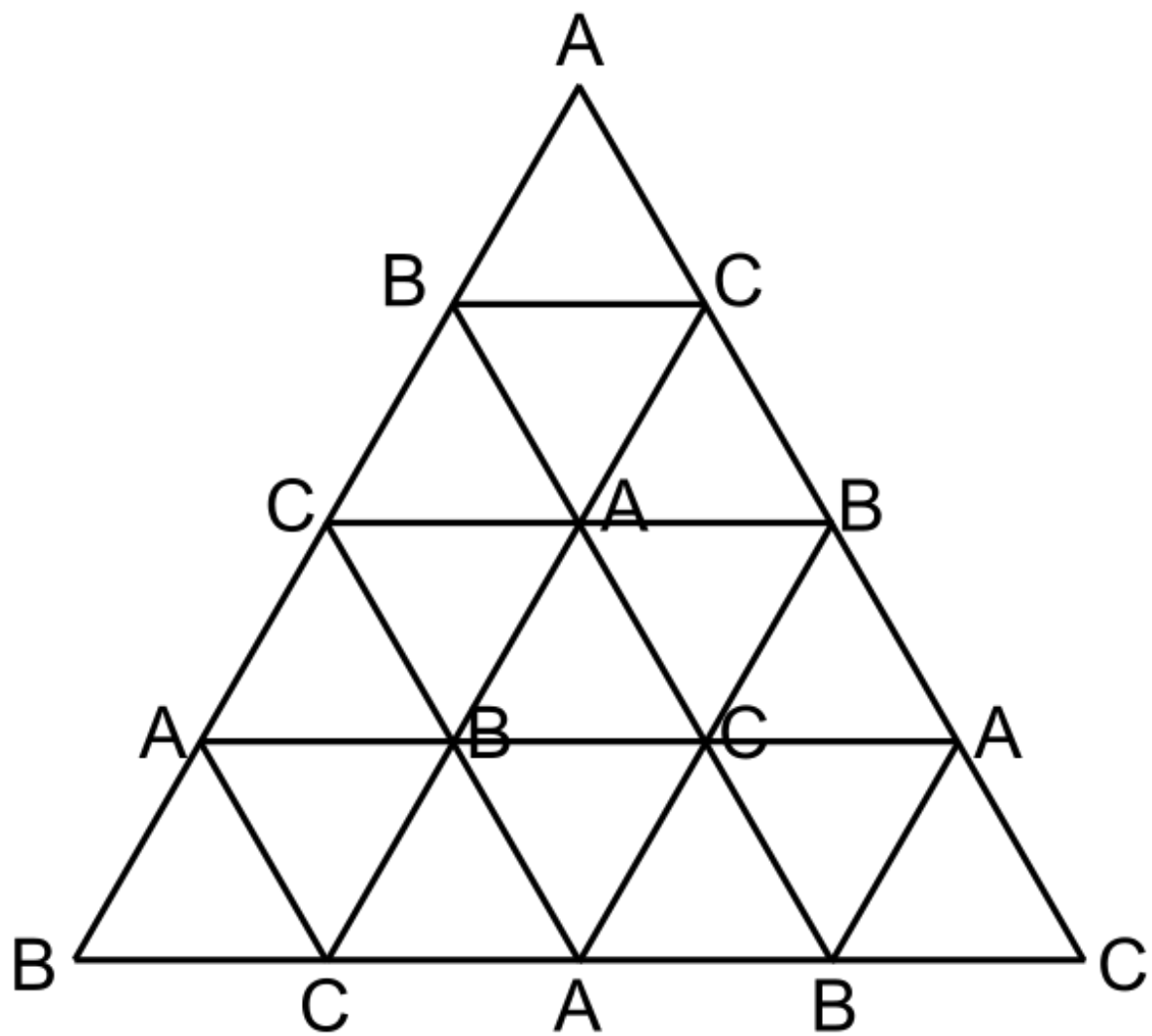
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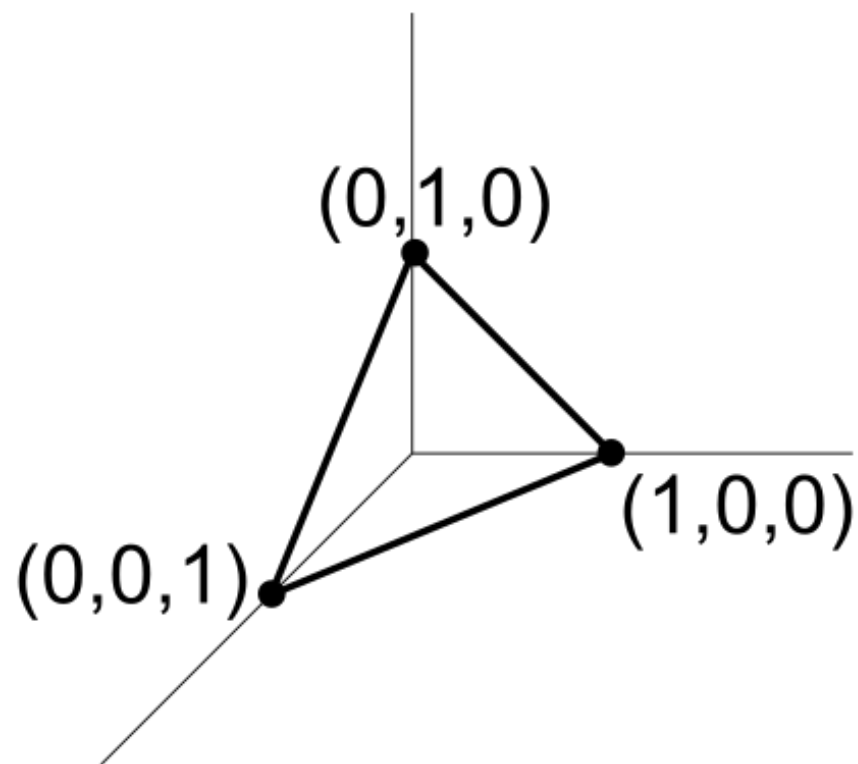
Preference labeling



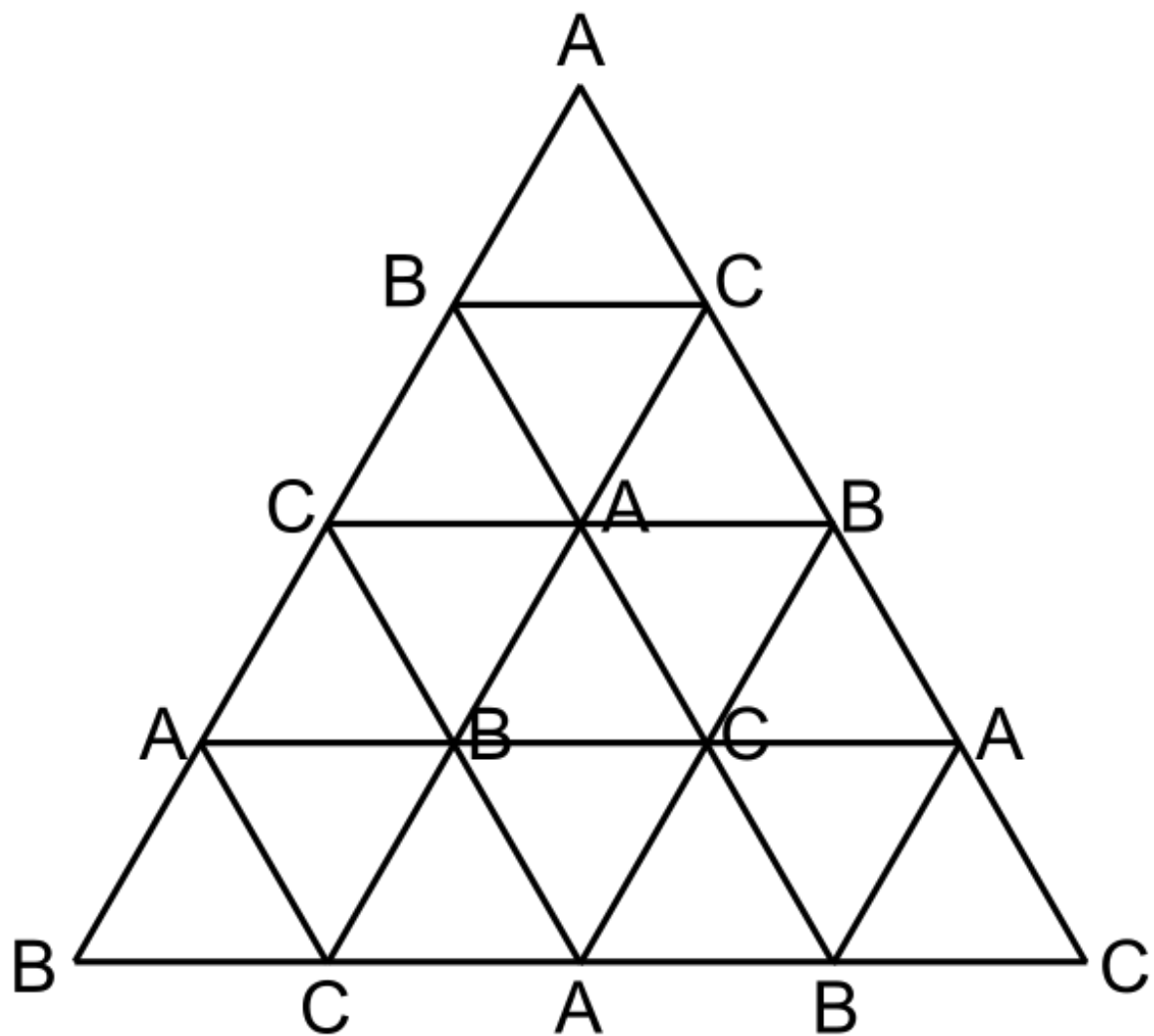
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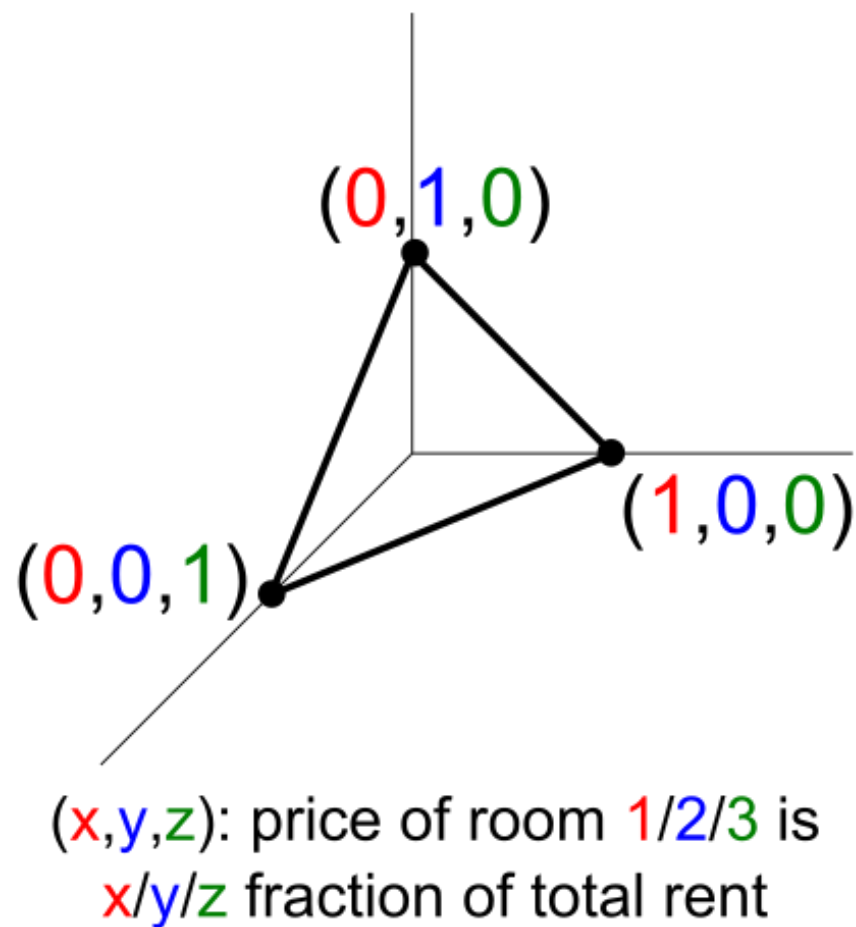
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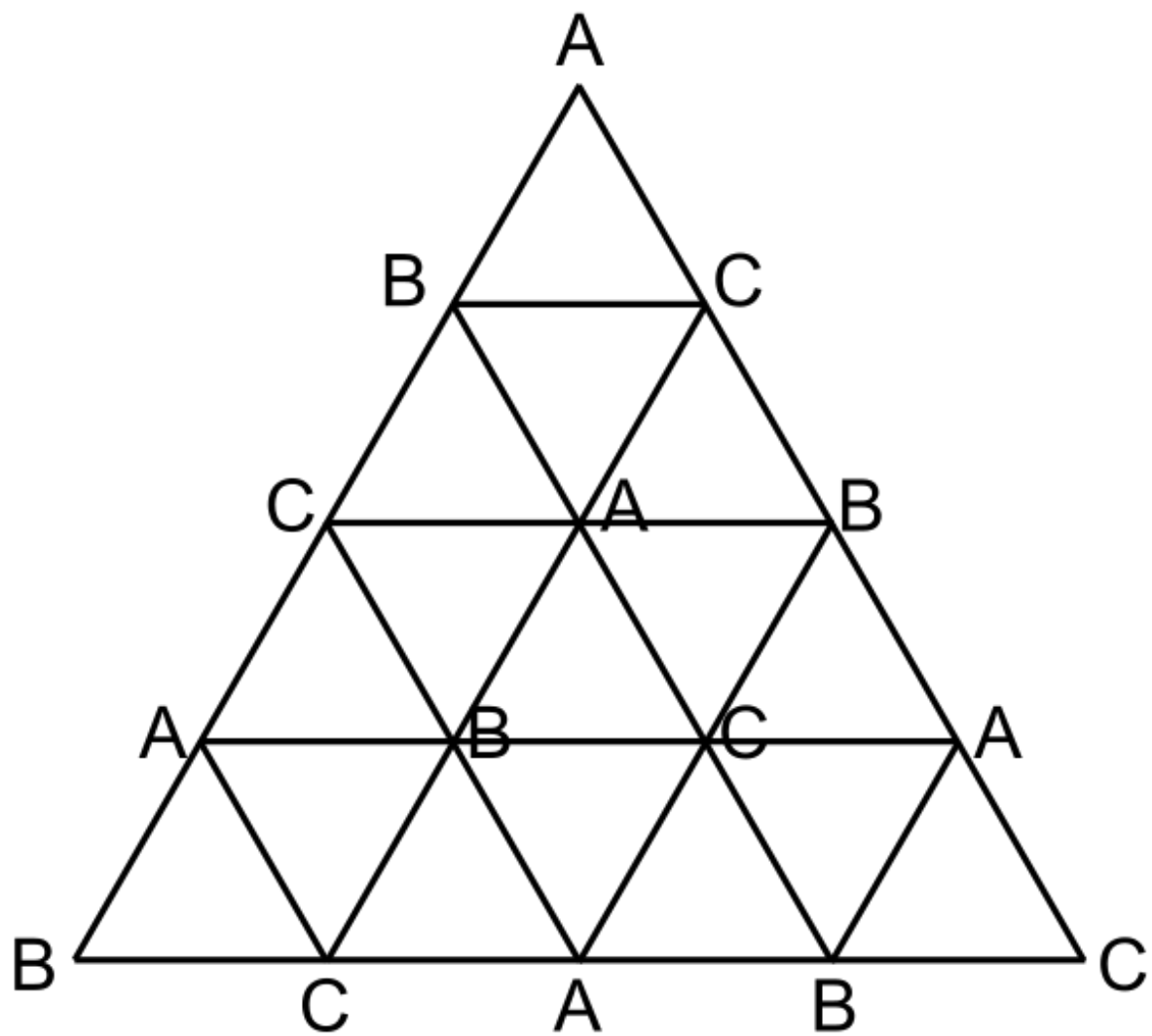
Ownership labeling



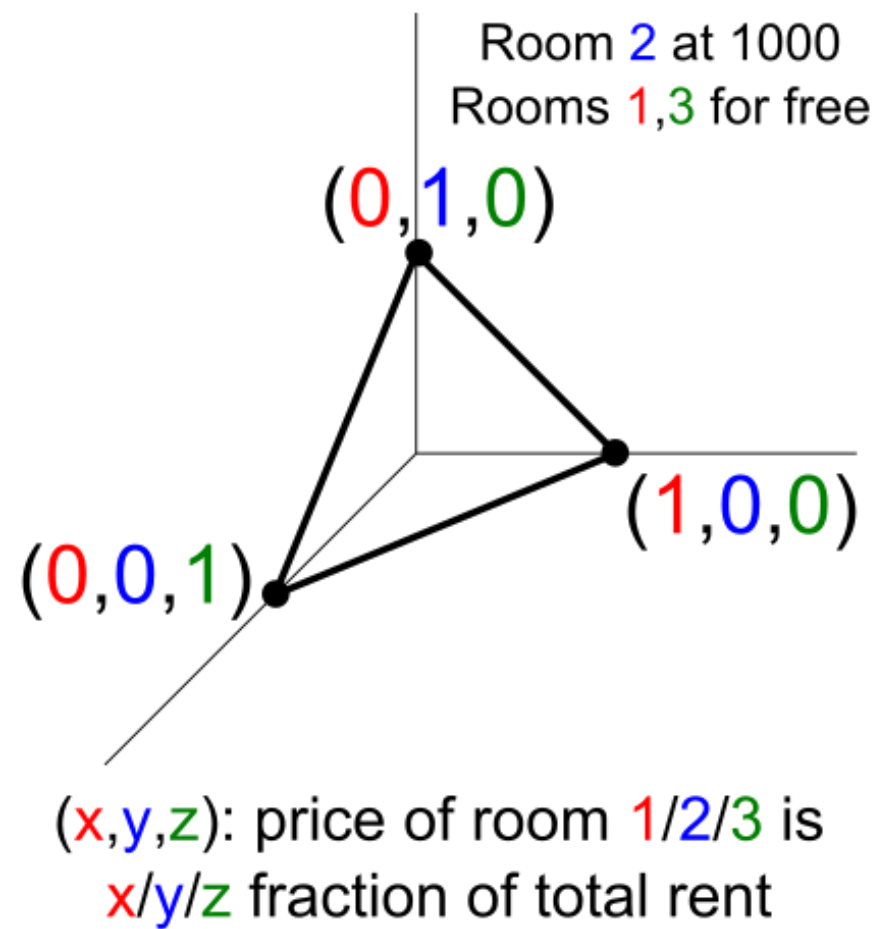
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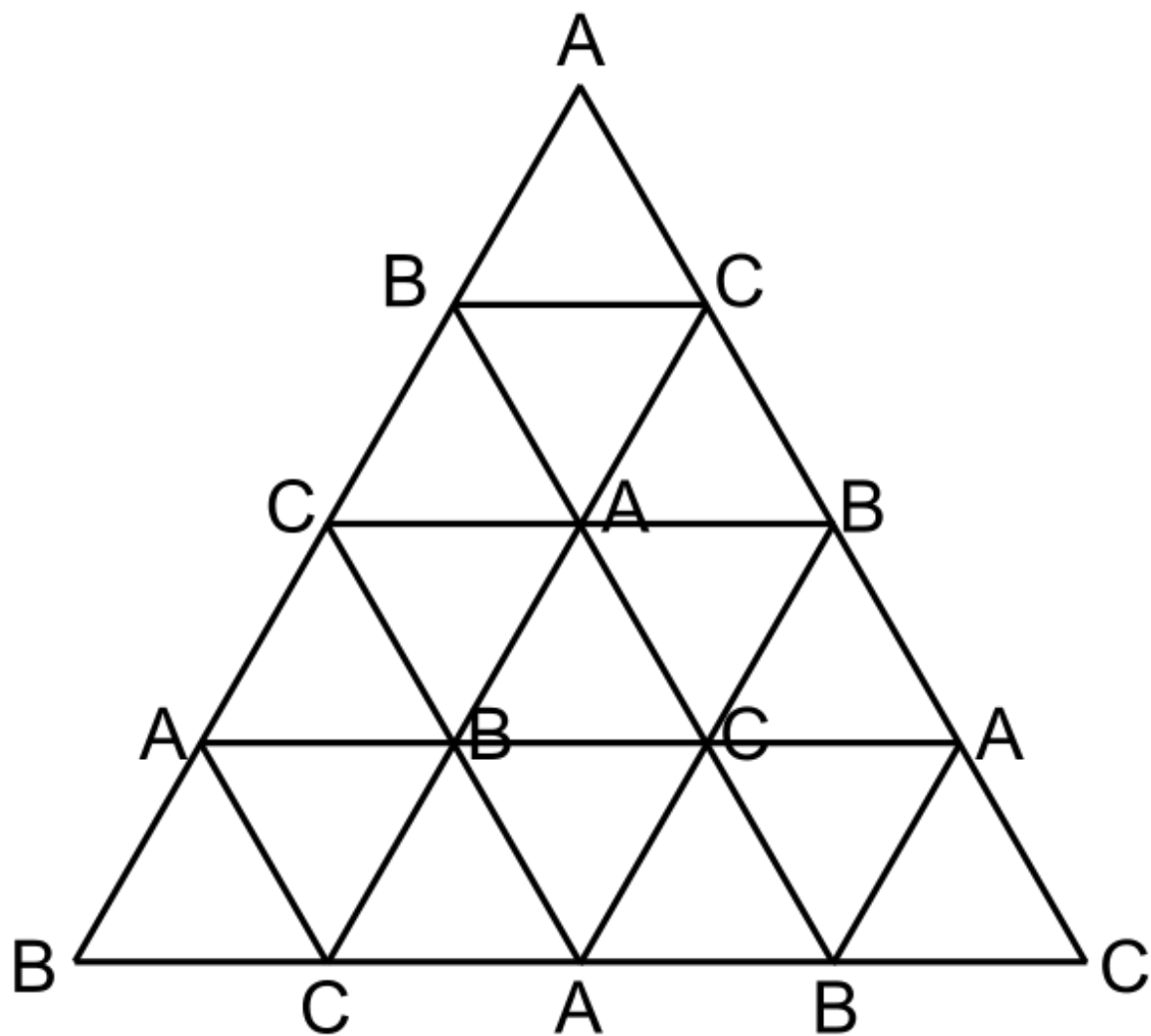
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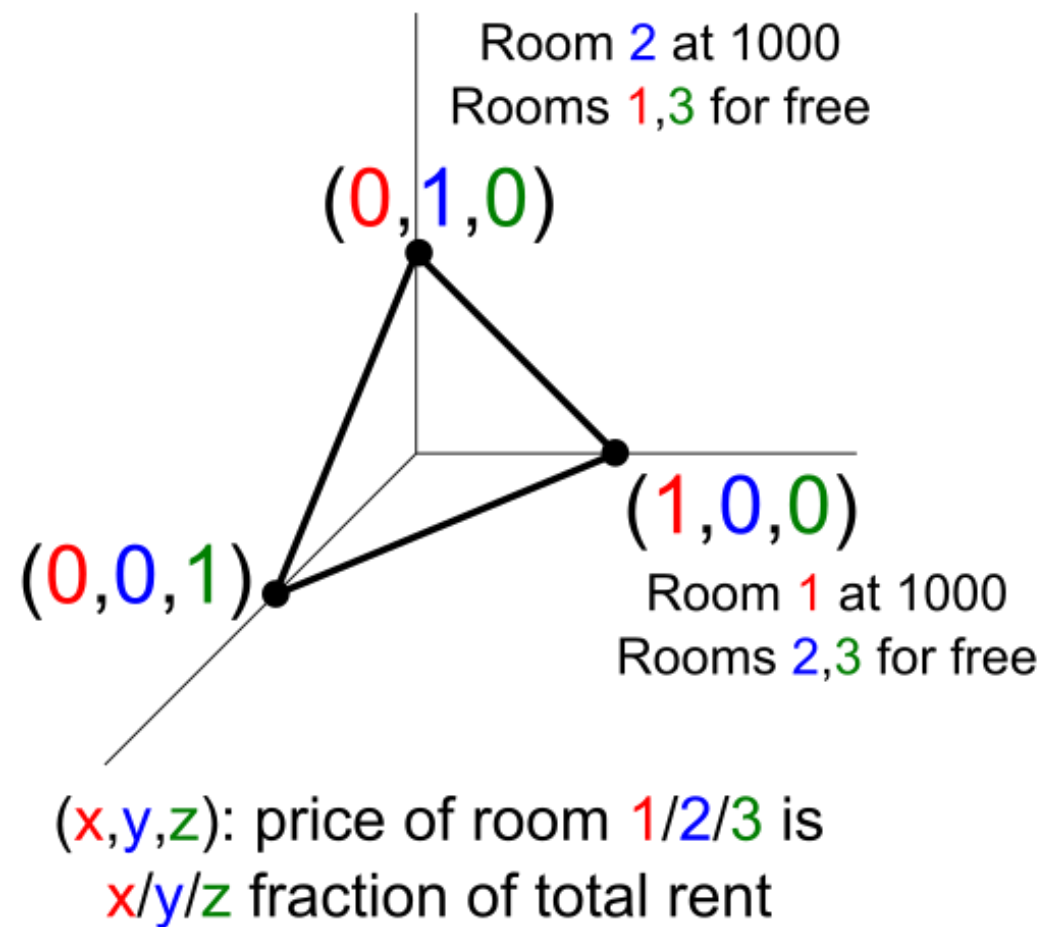
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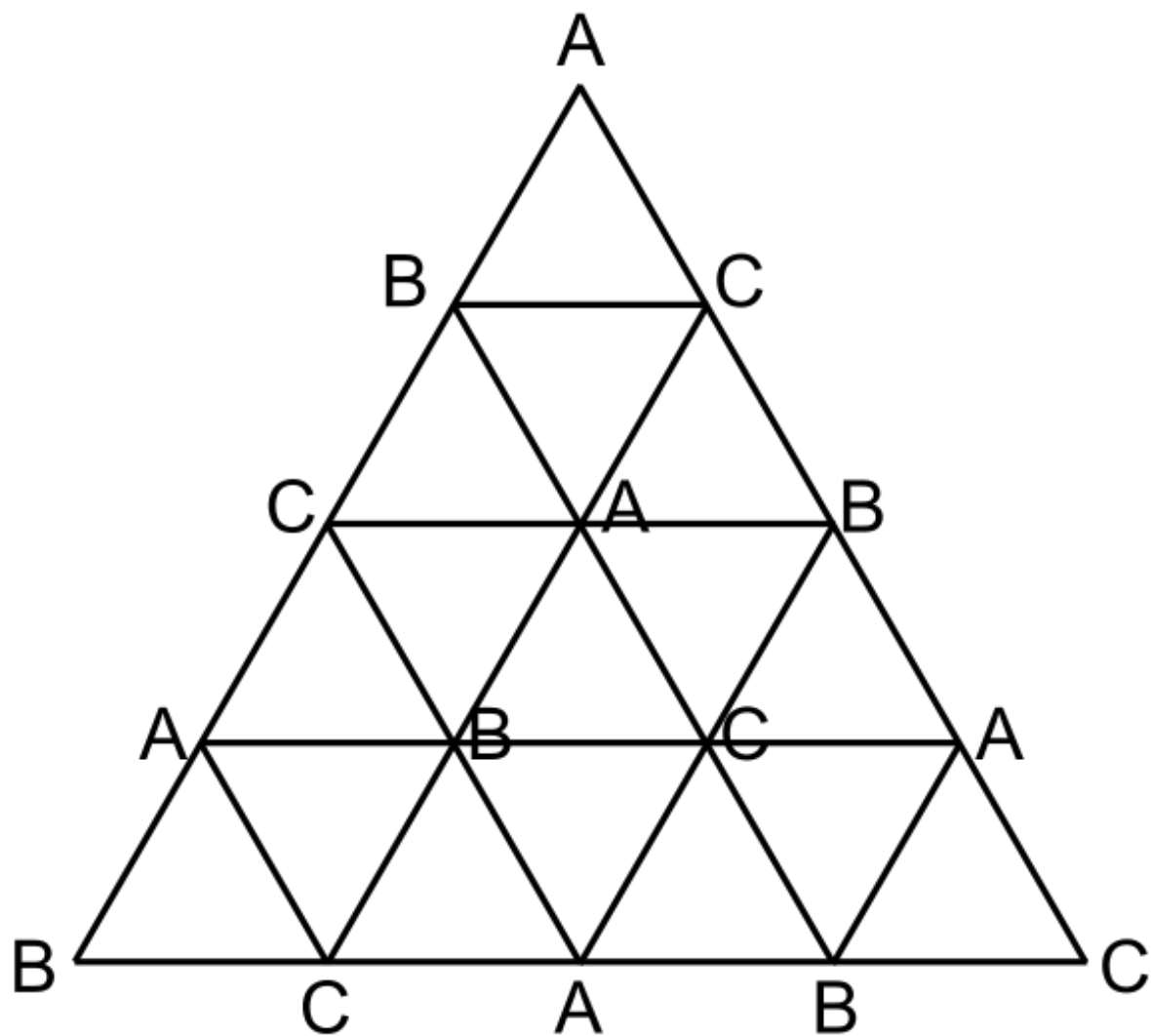
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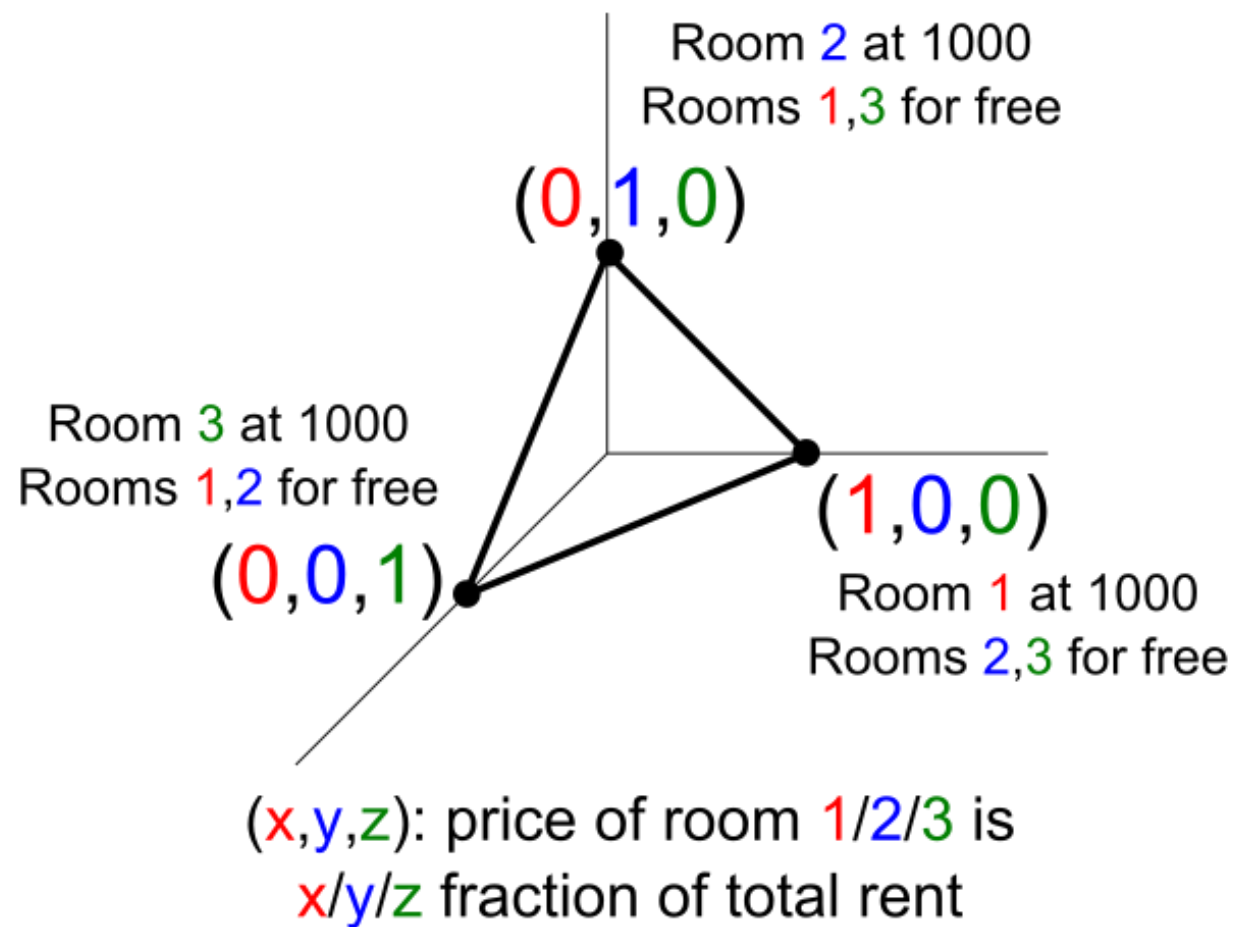
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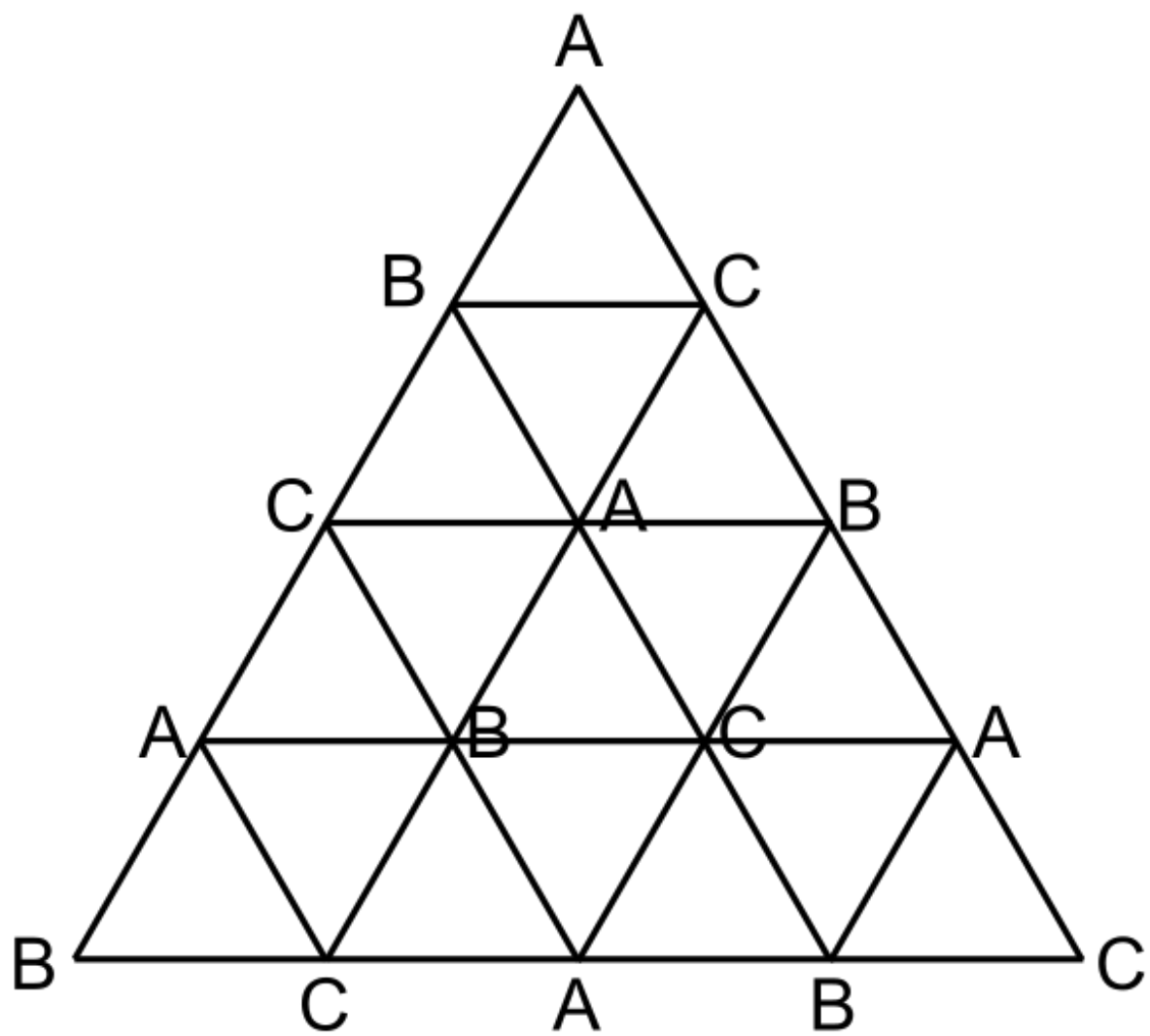
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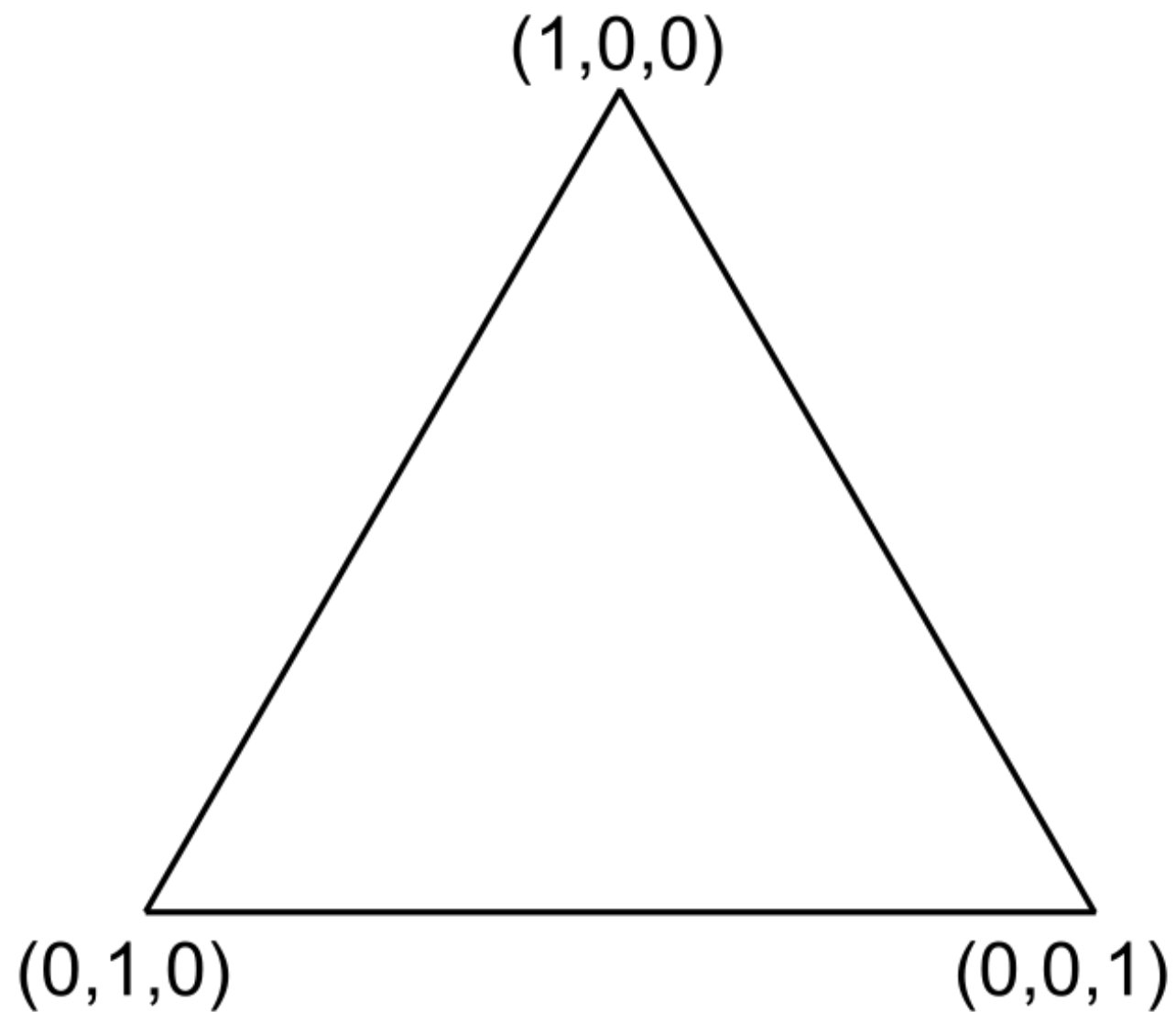
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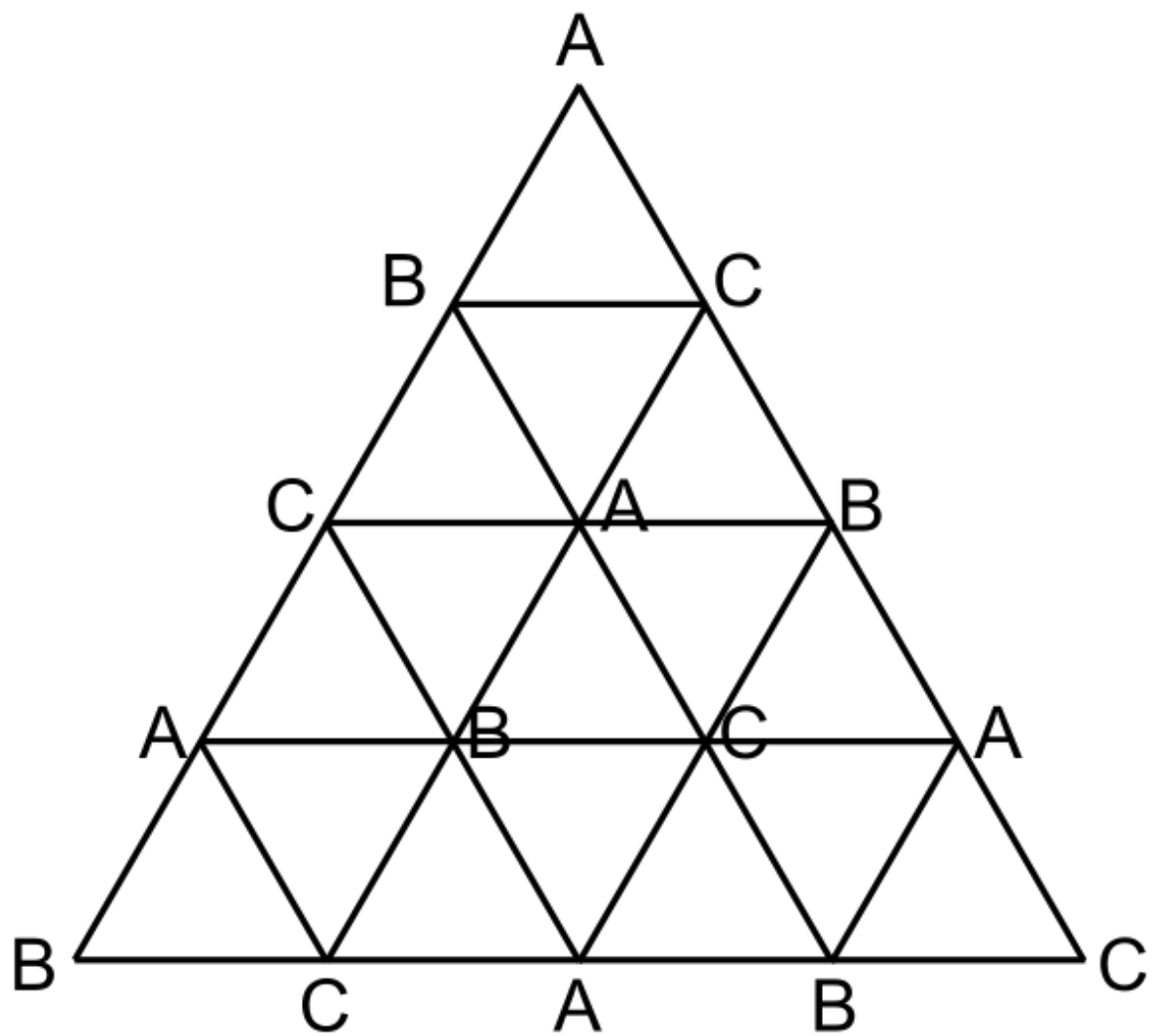
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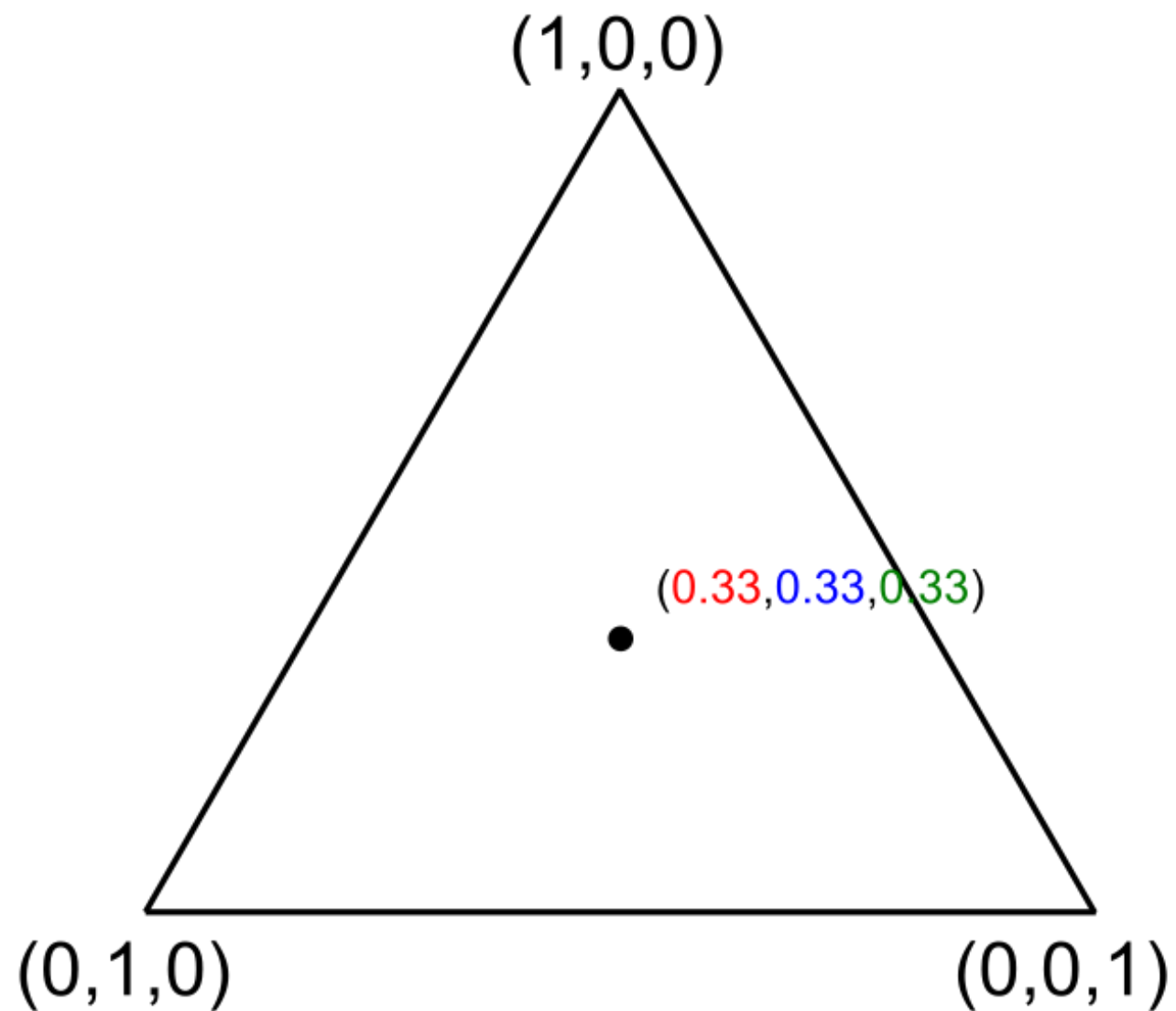
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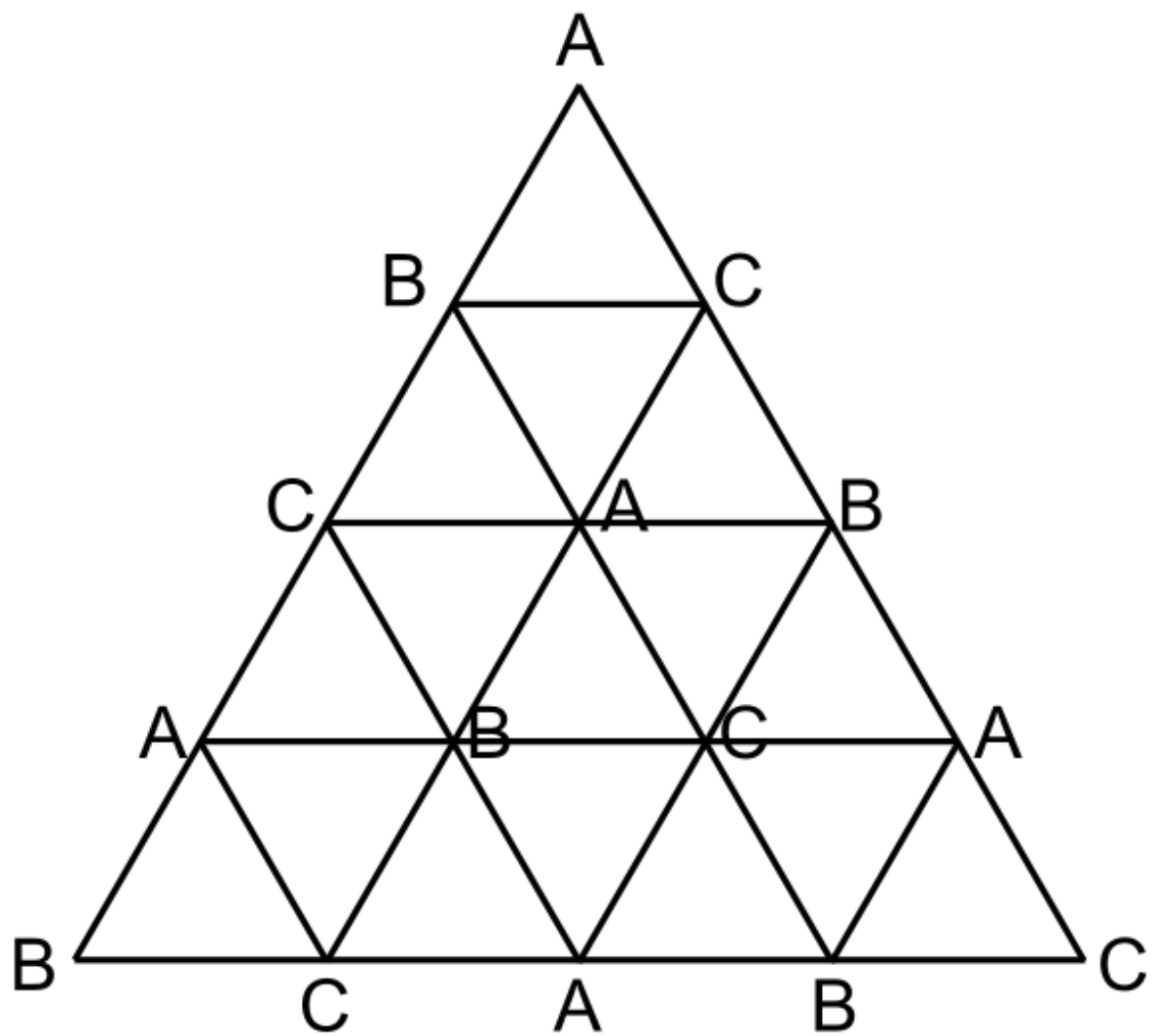
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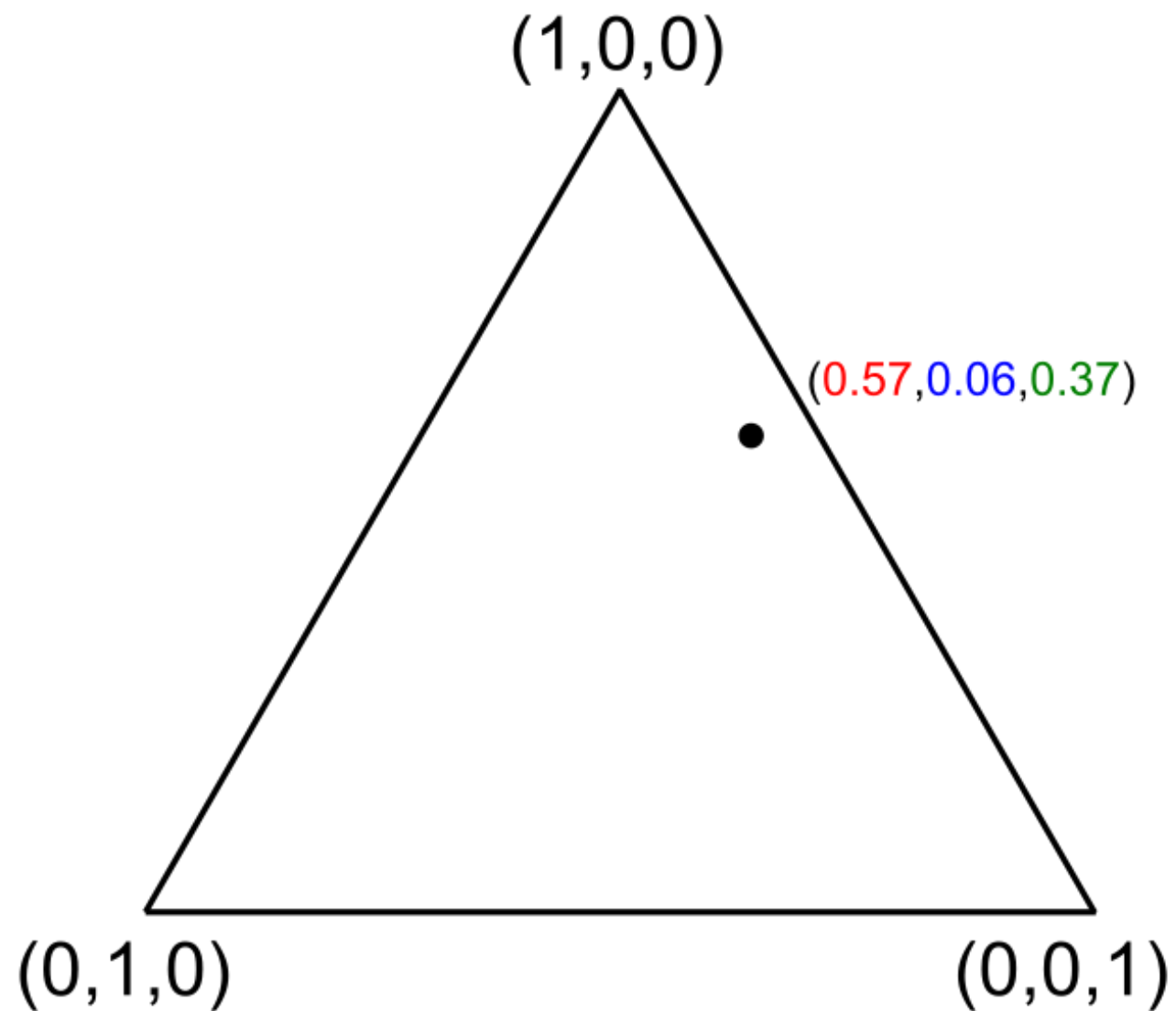
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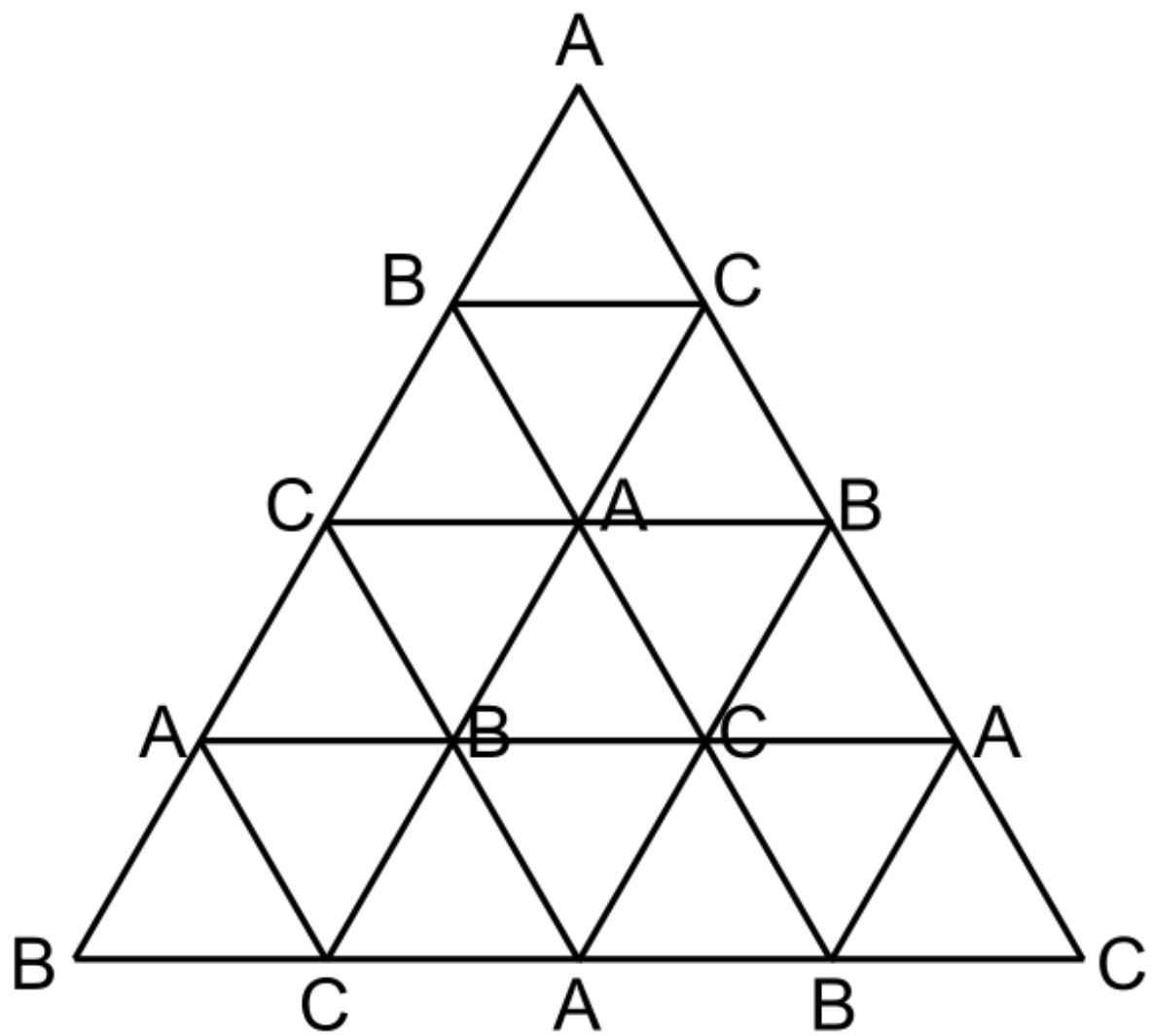
Ownership labeling



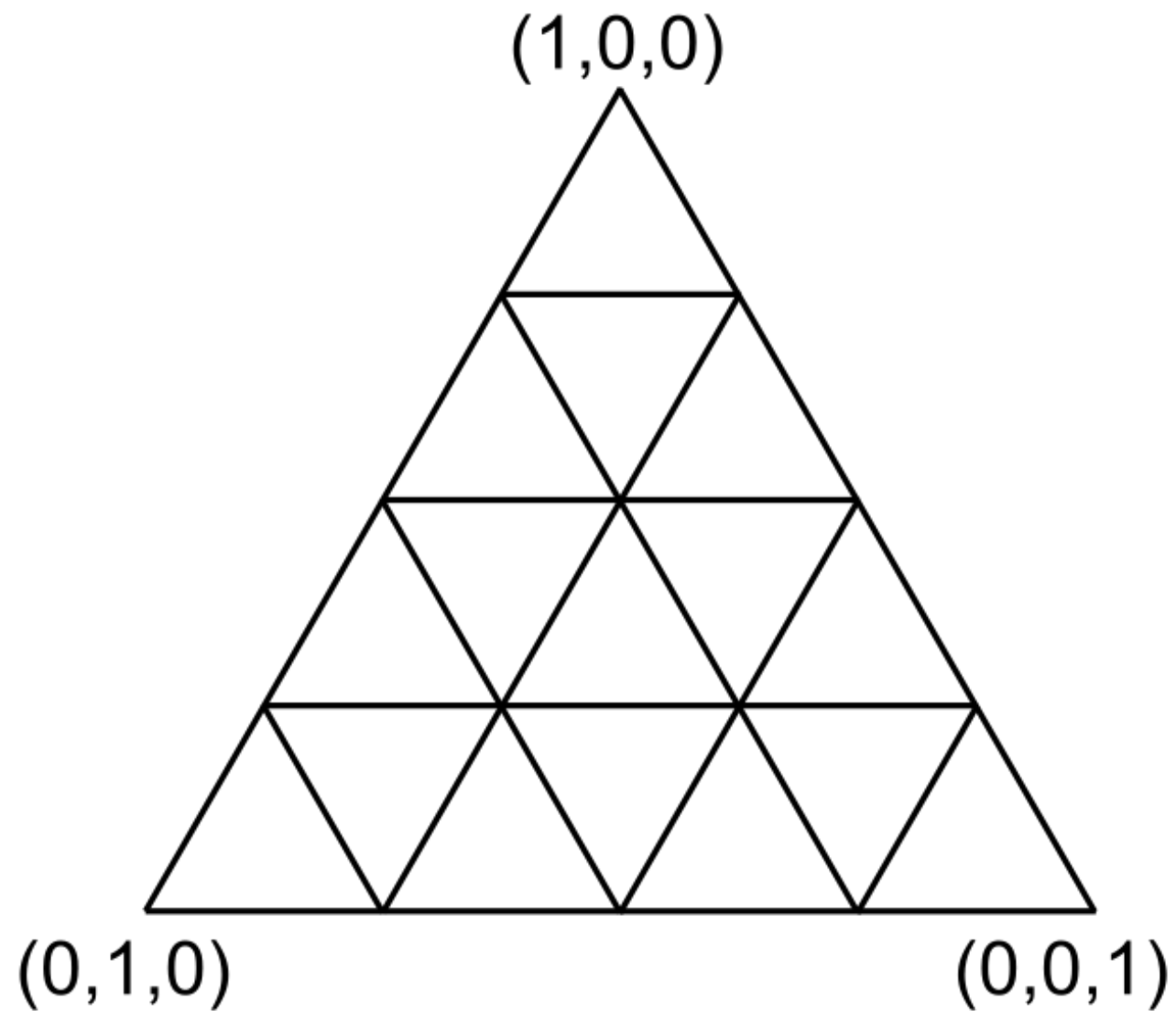
Preference labeling



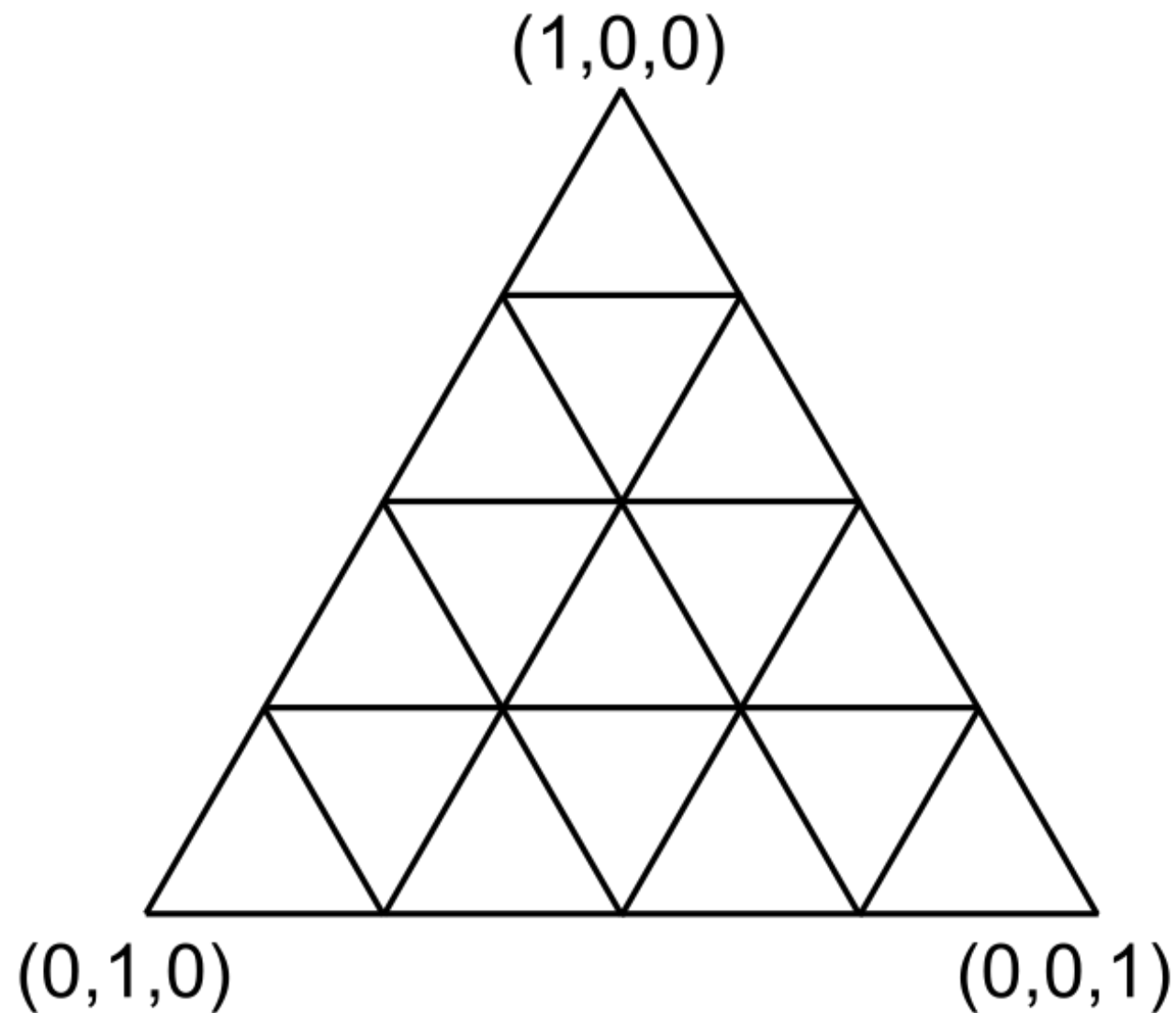
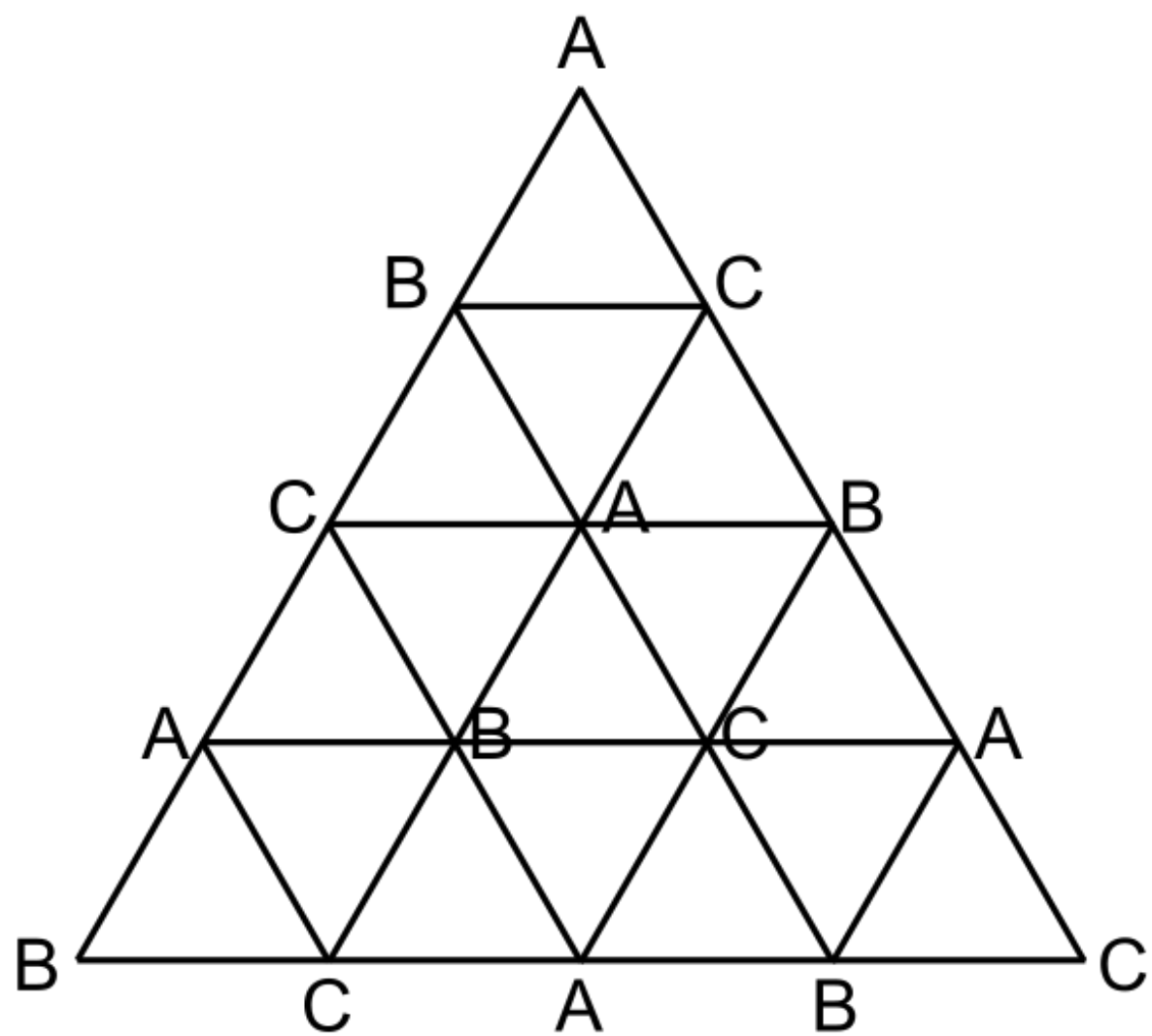
Ownership labeling

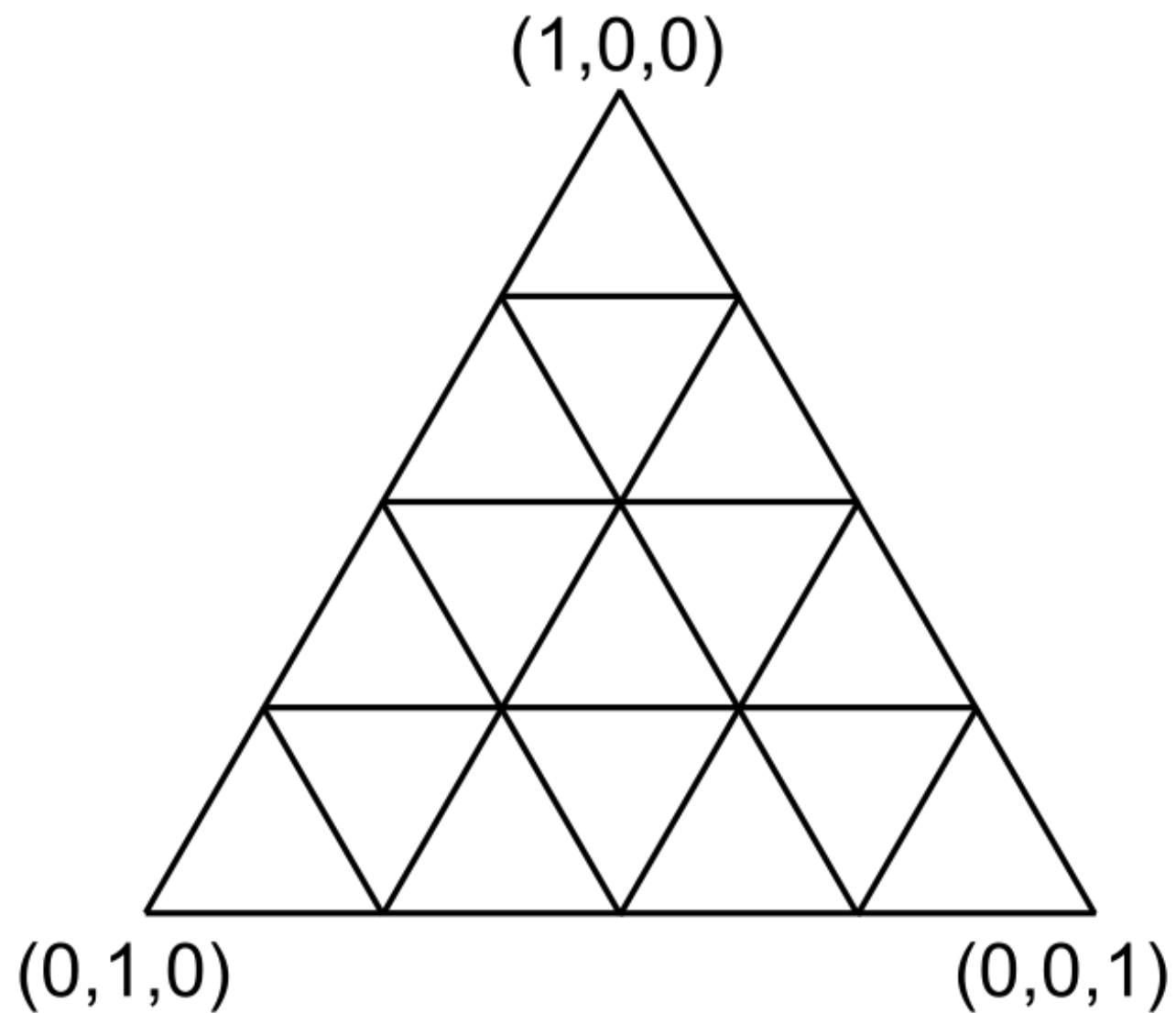
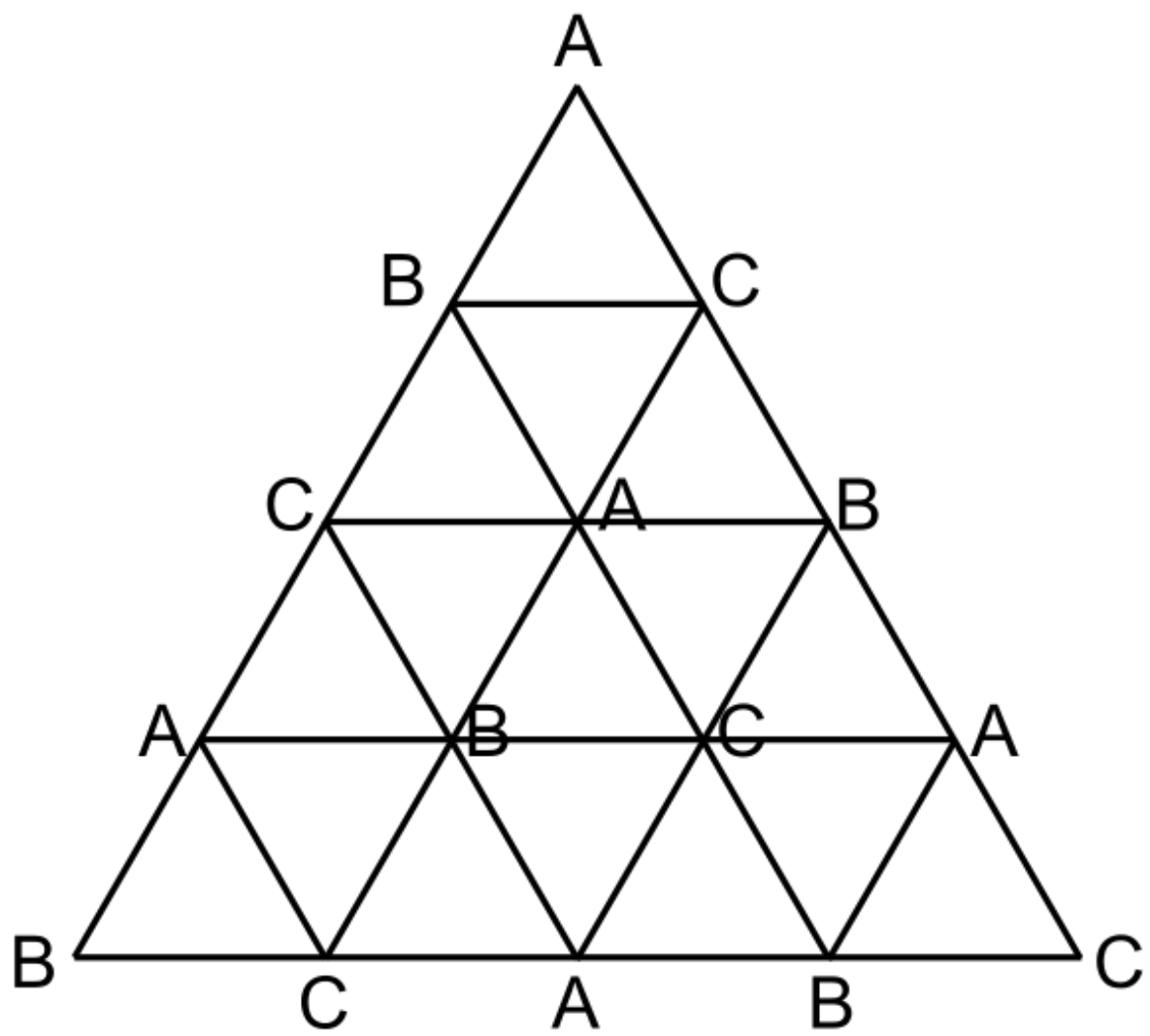


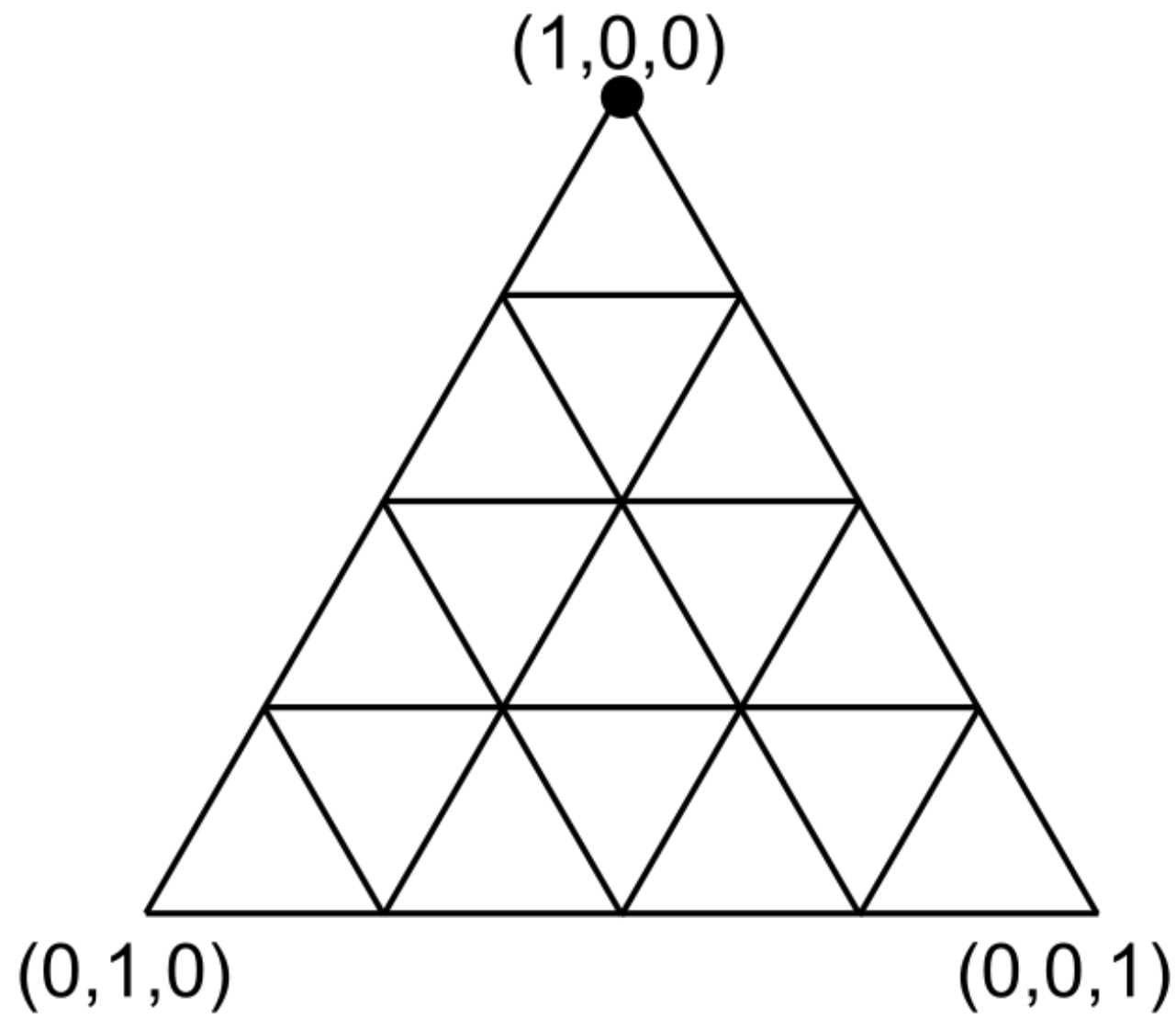
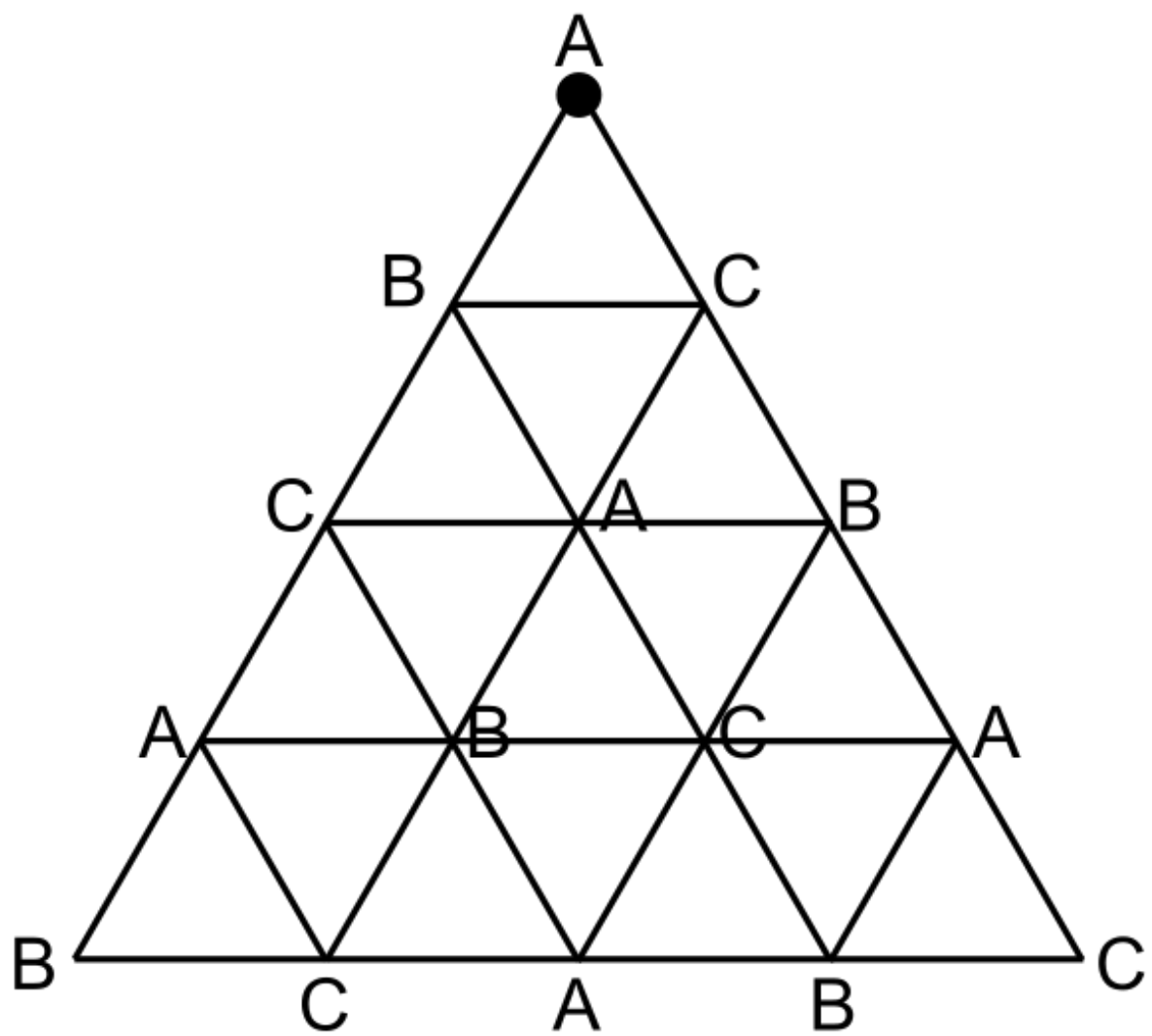
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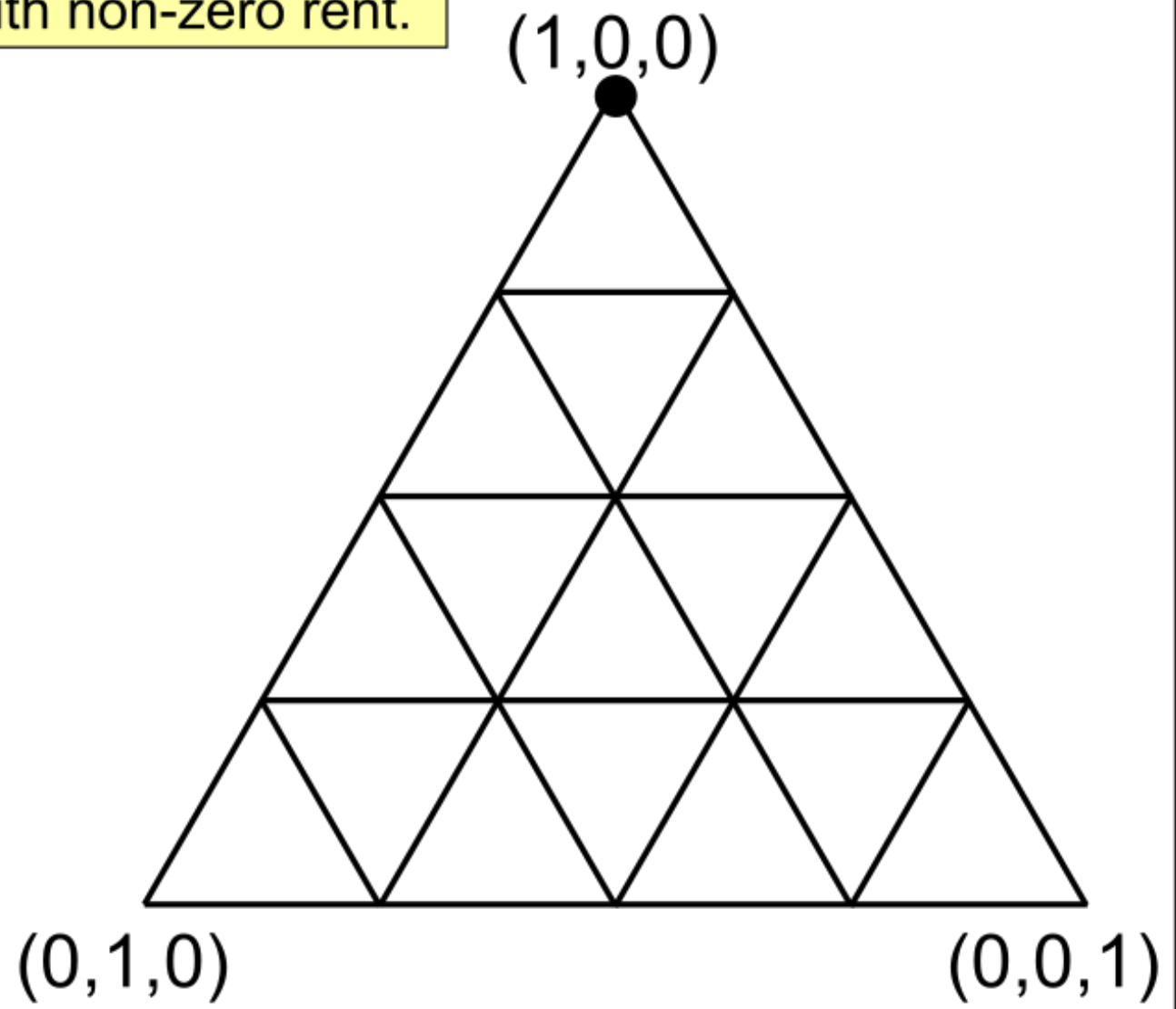
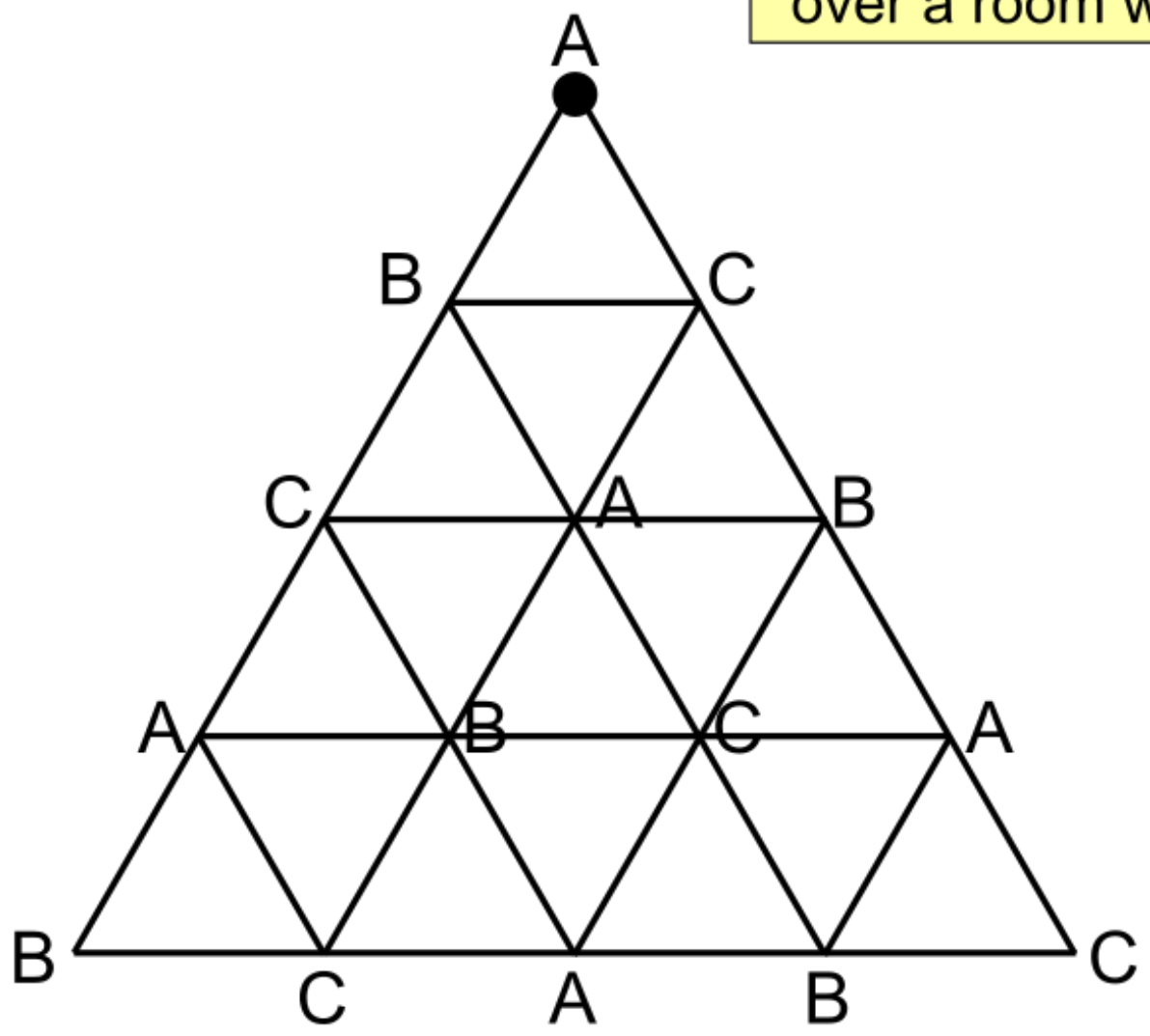
For each vertex in the ownership triangle, ask the owner its favorite room at the pricing given in the preference triangle.

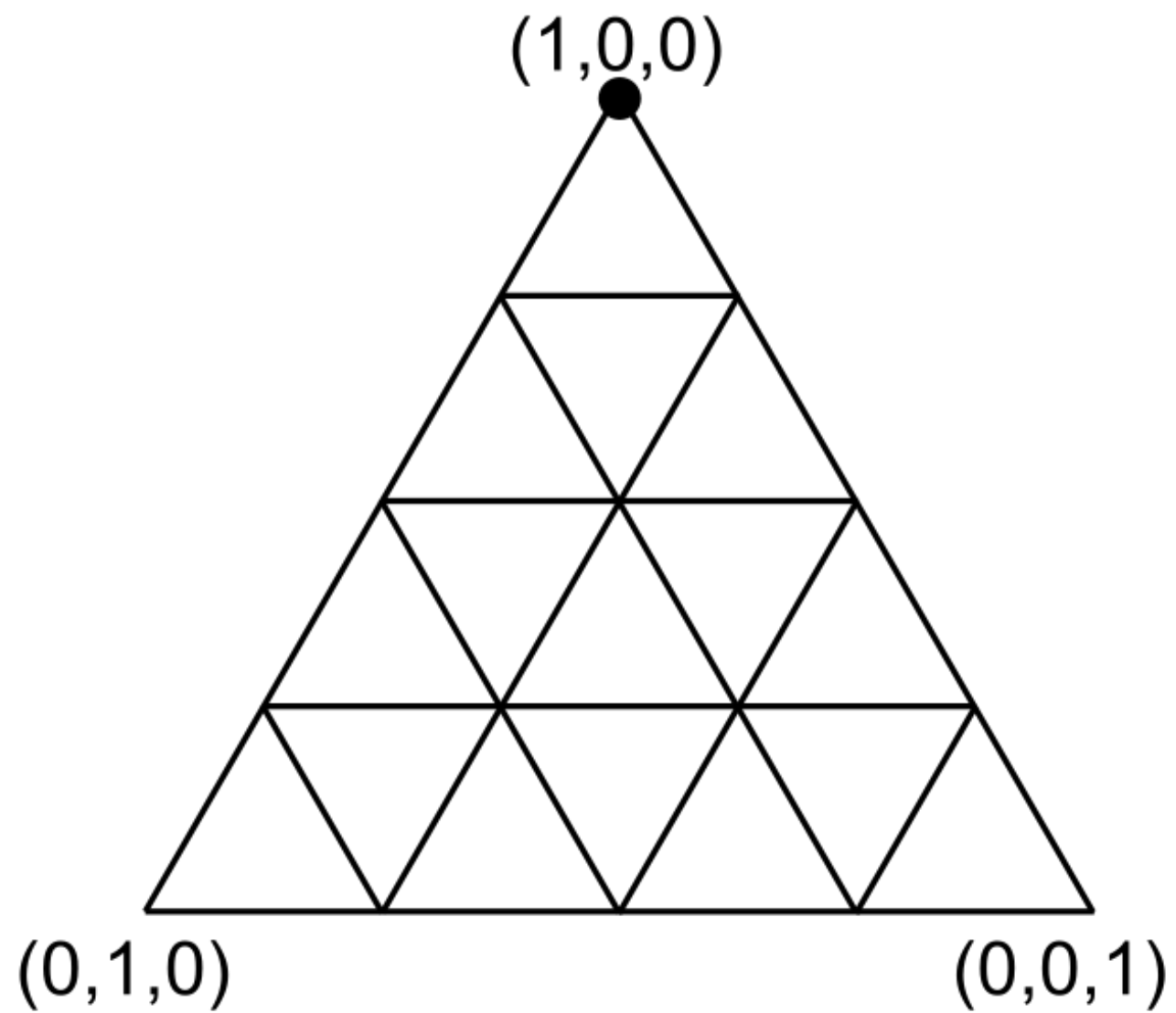
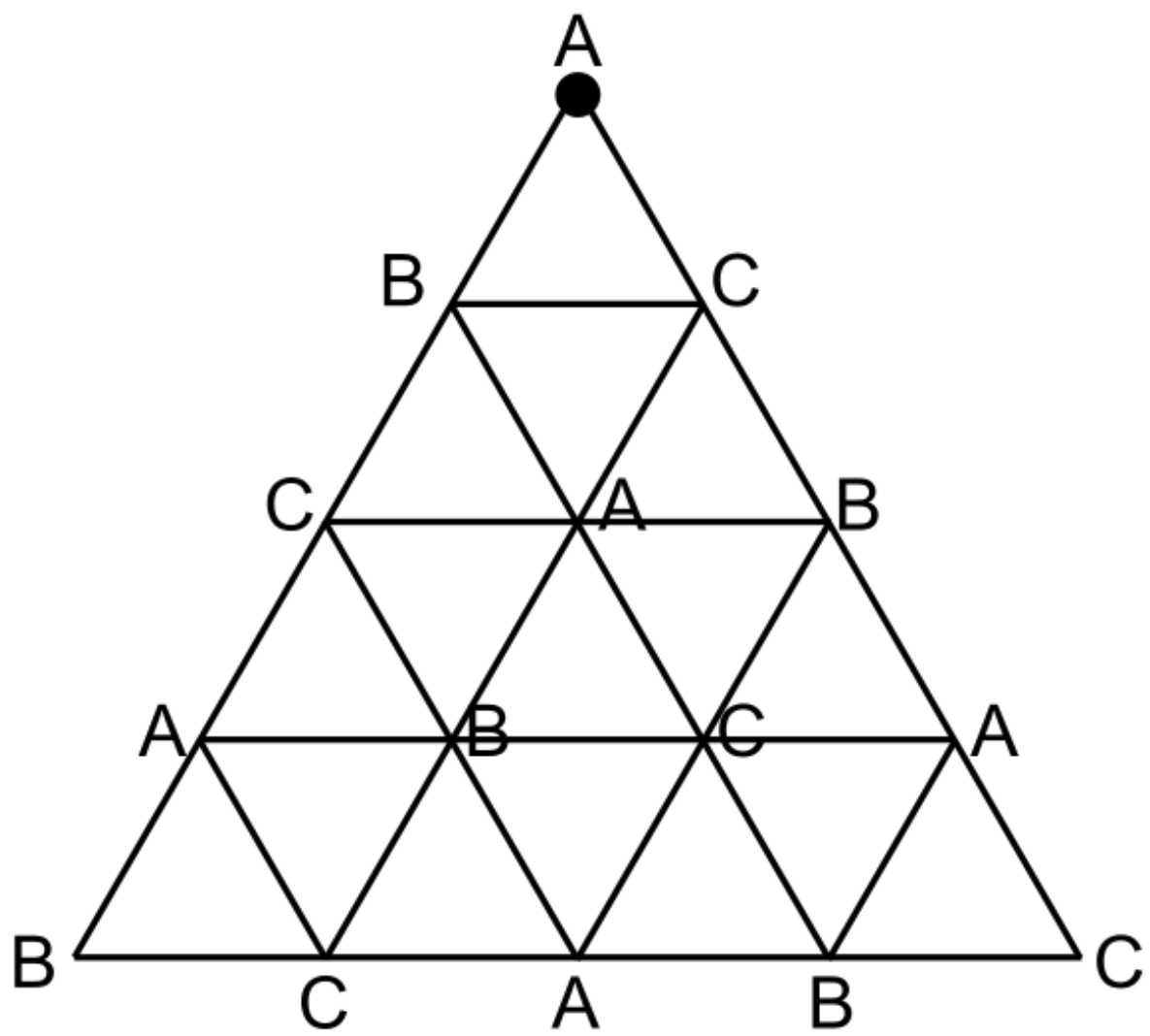


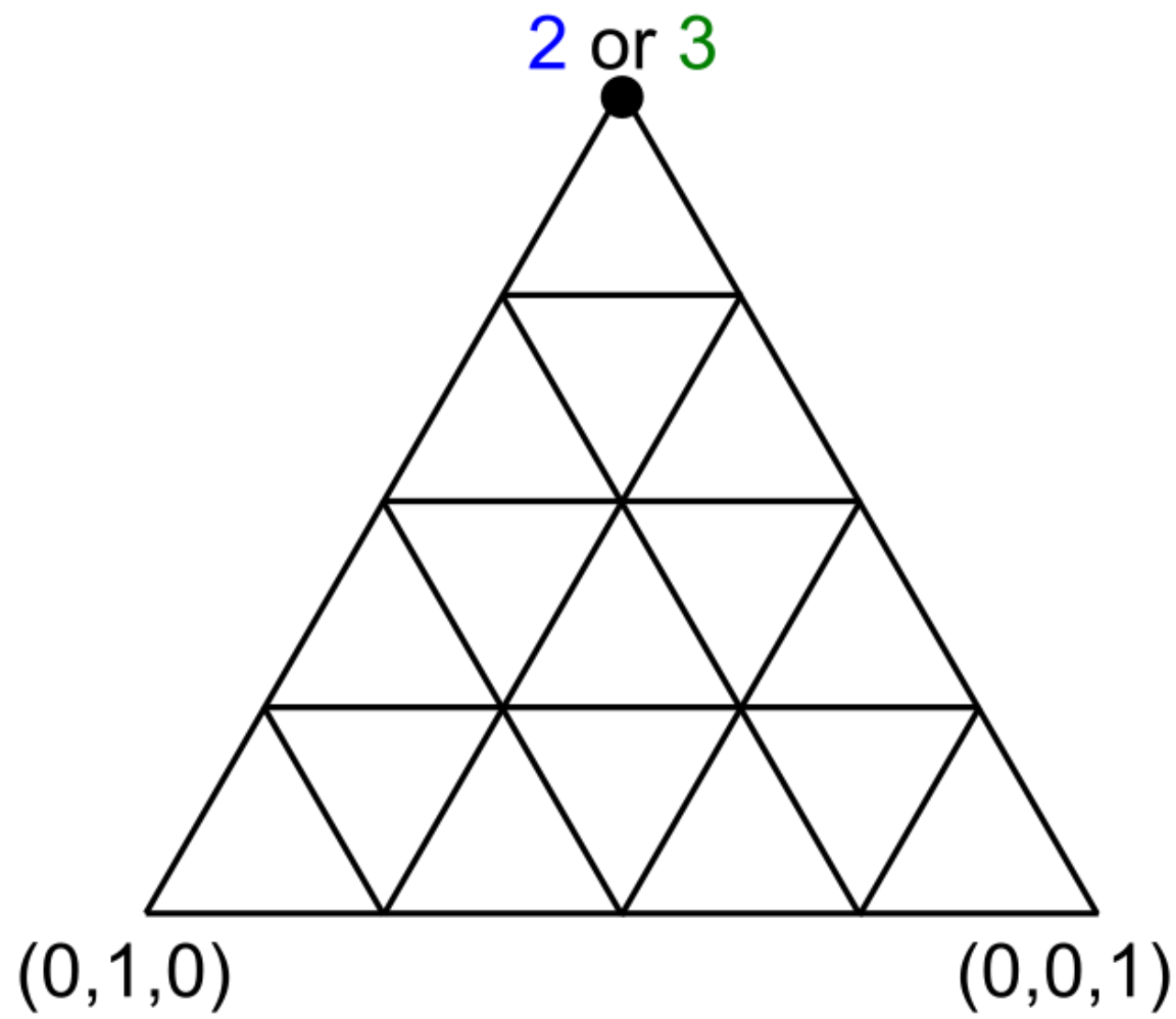
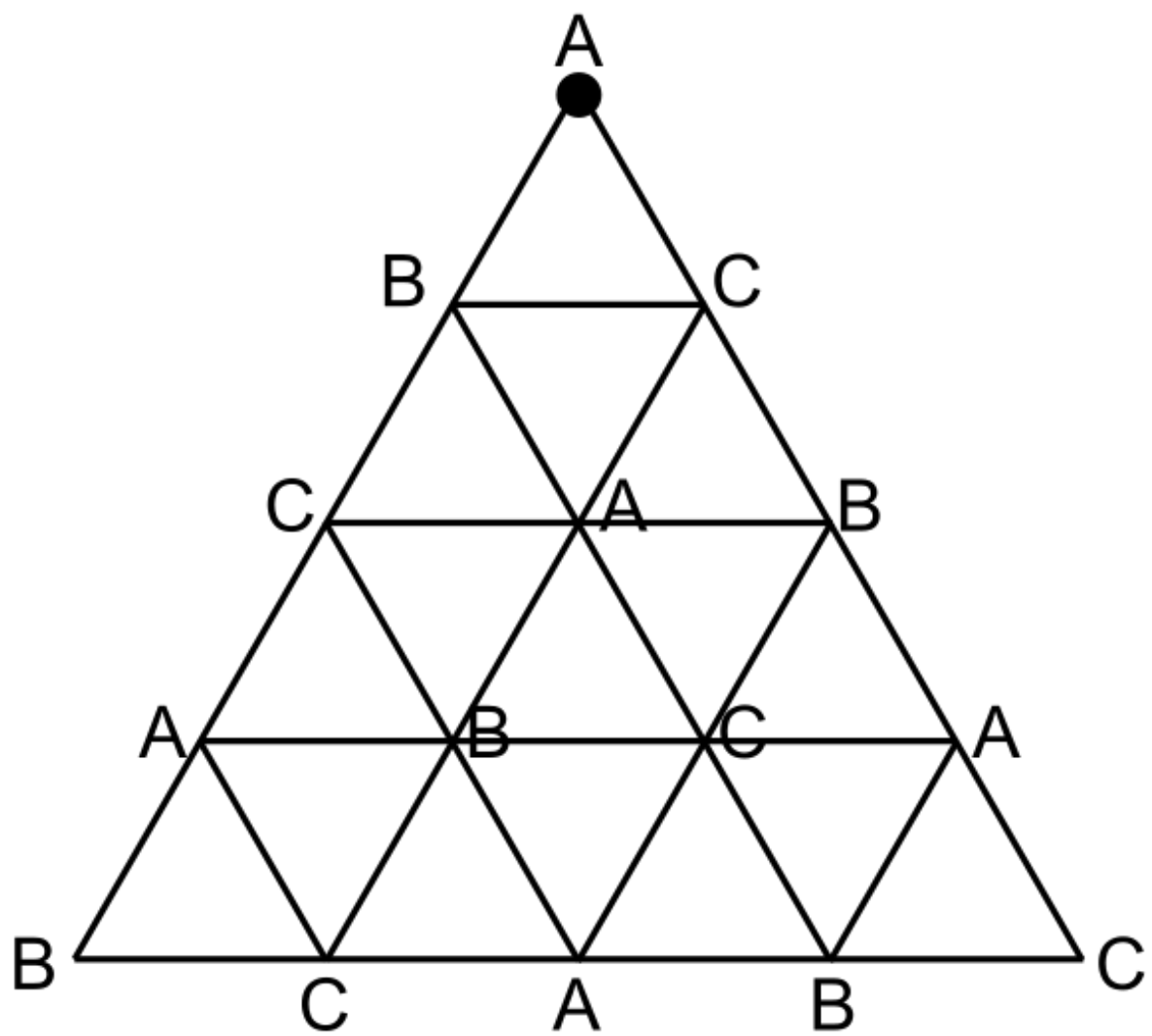


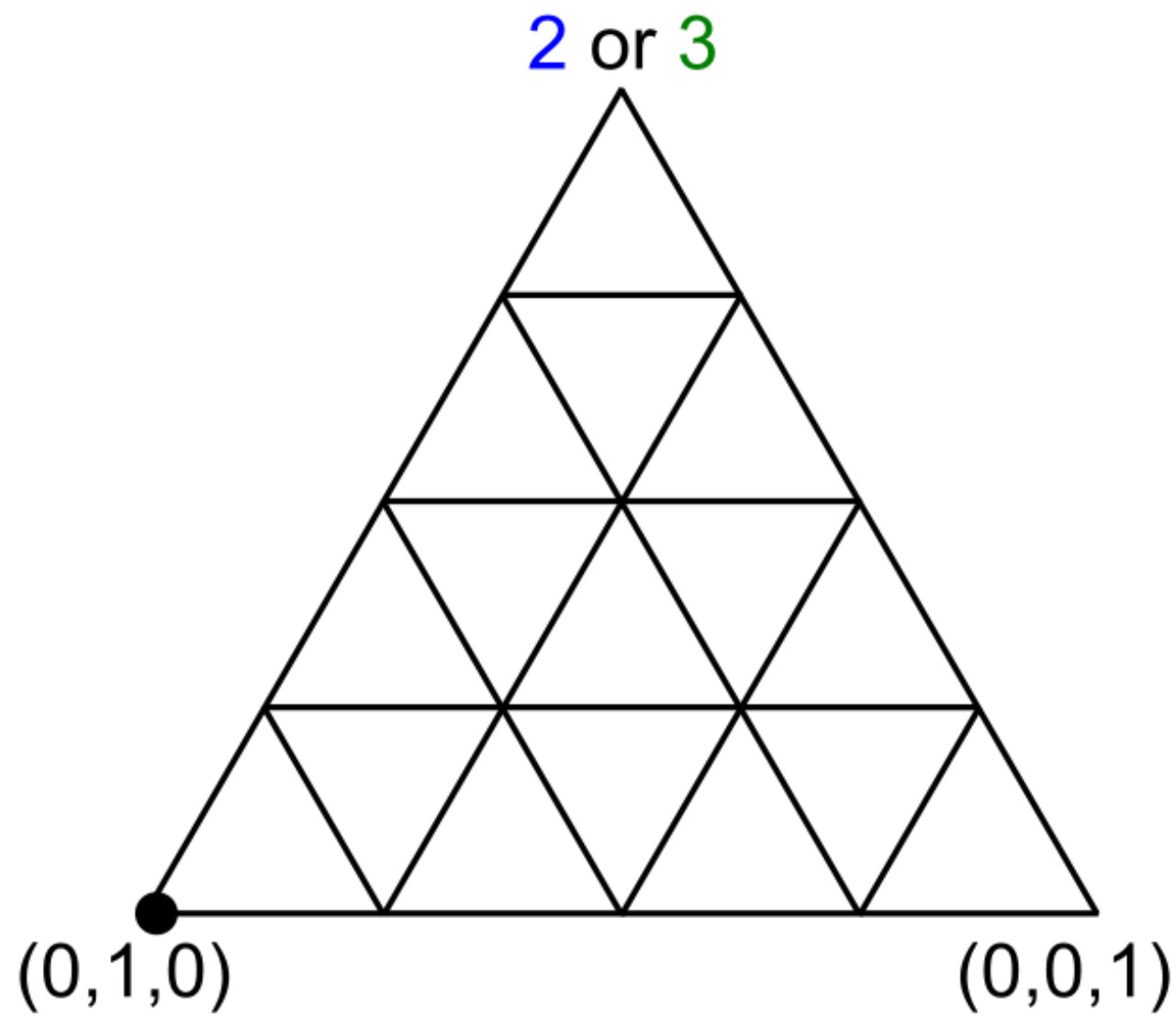
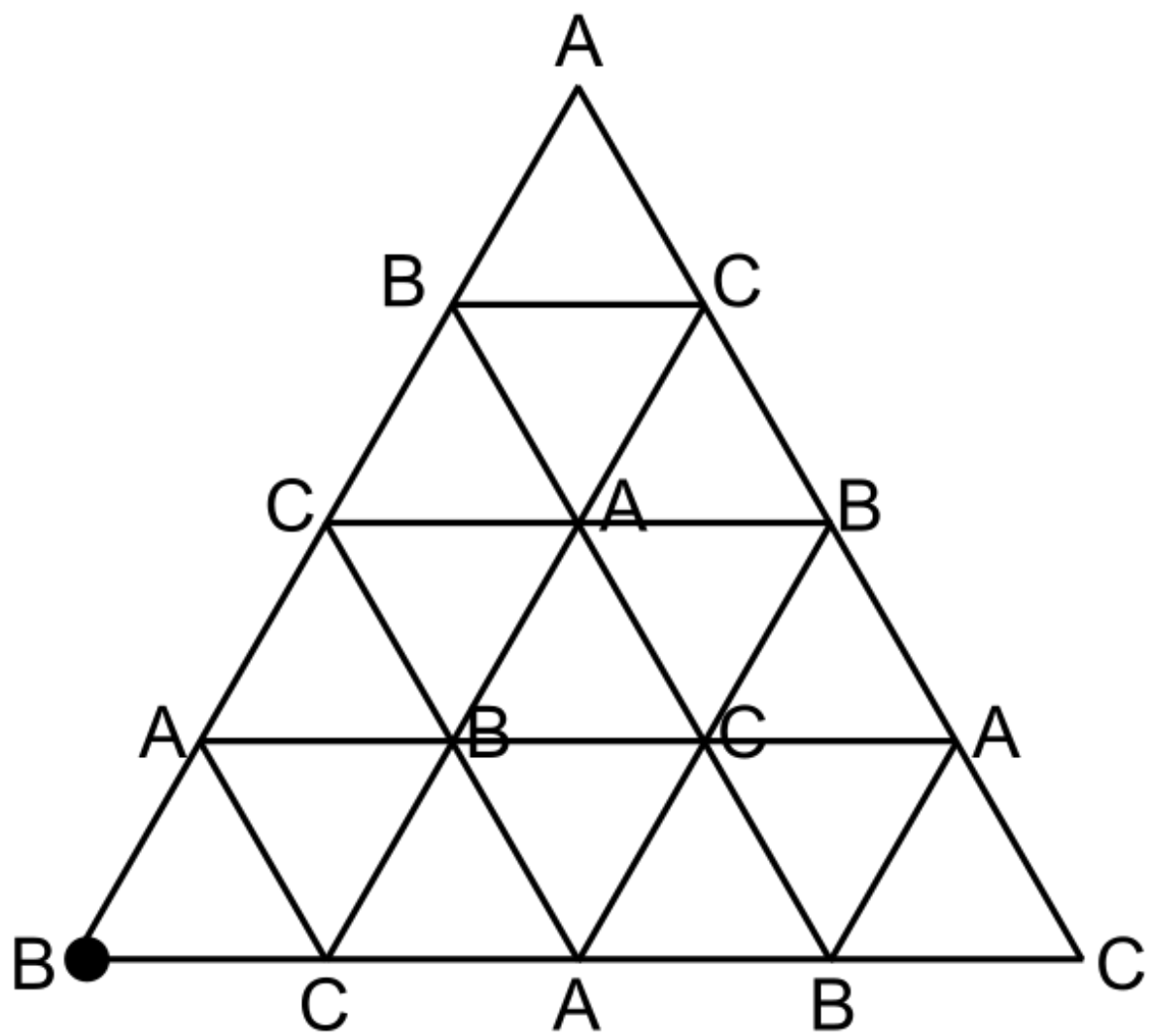


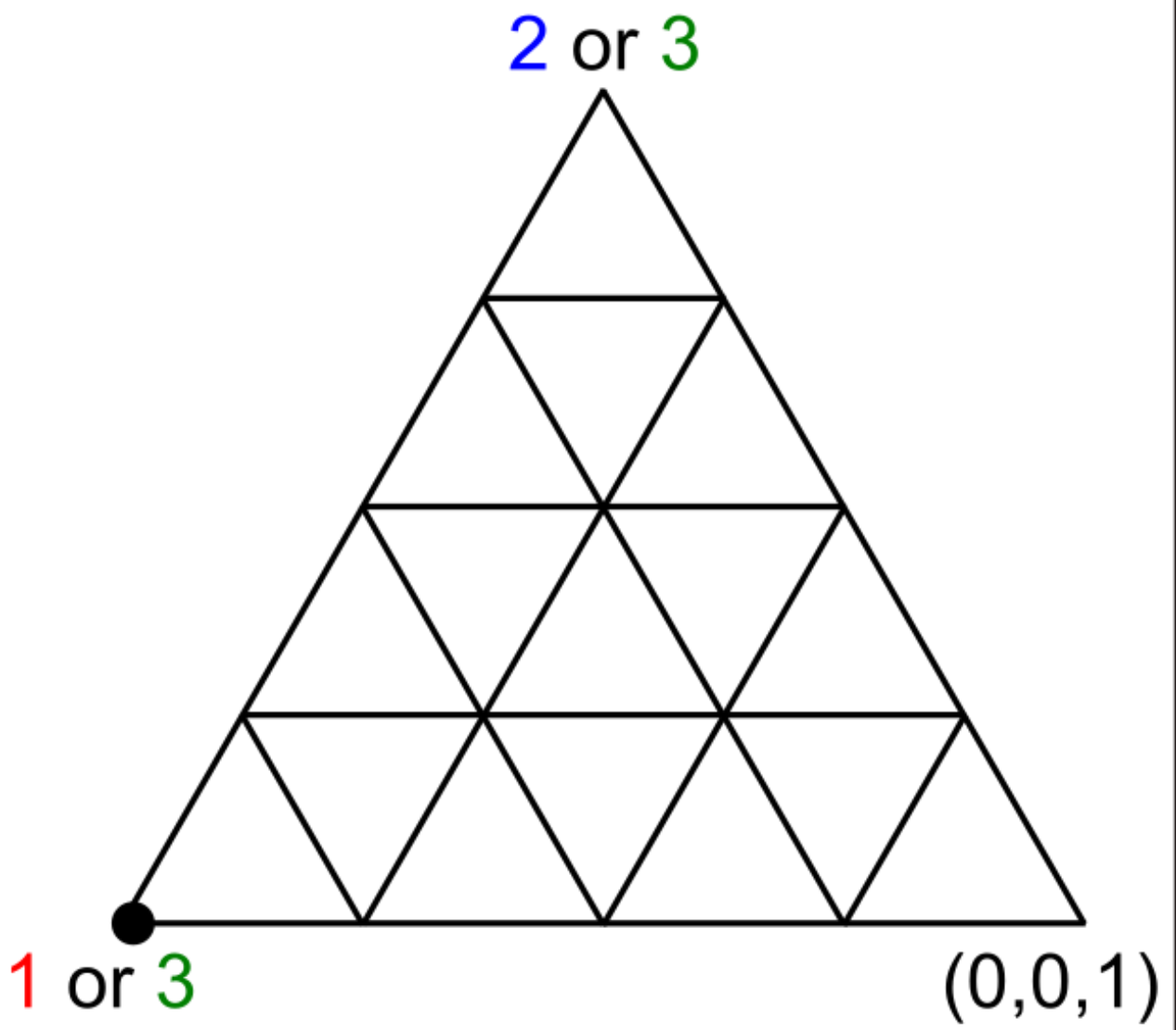
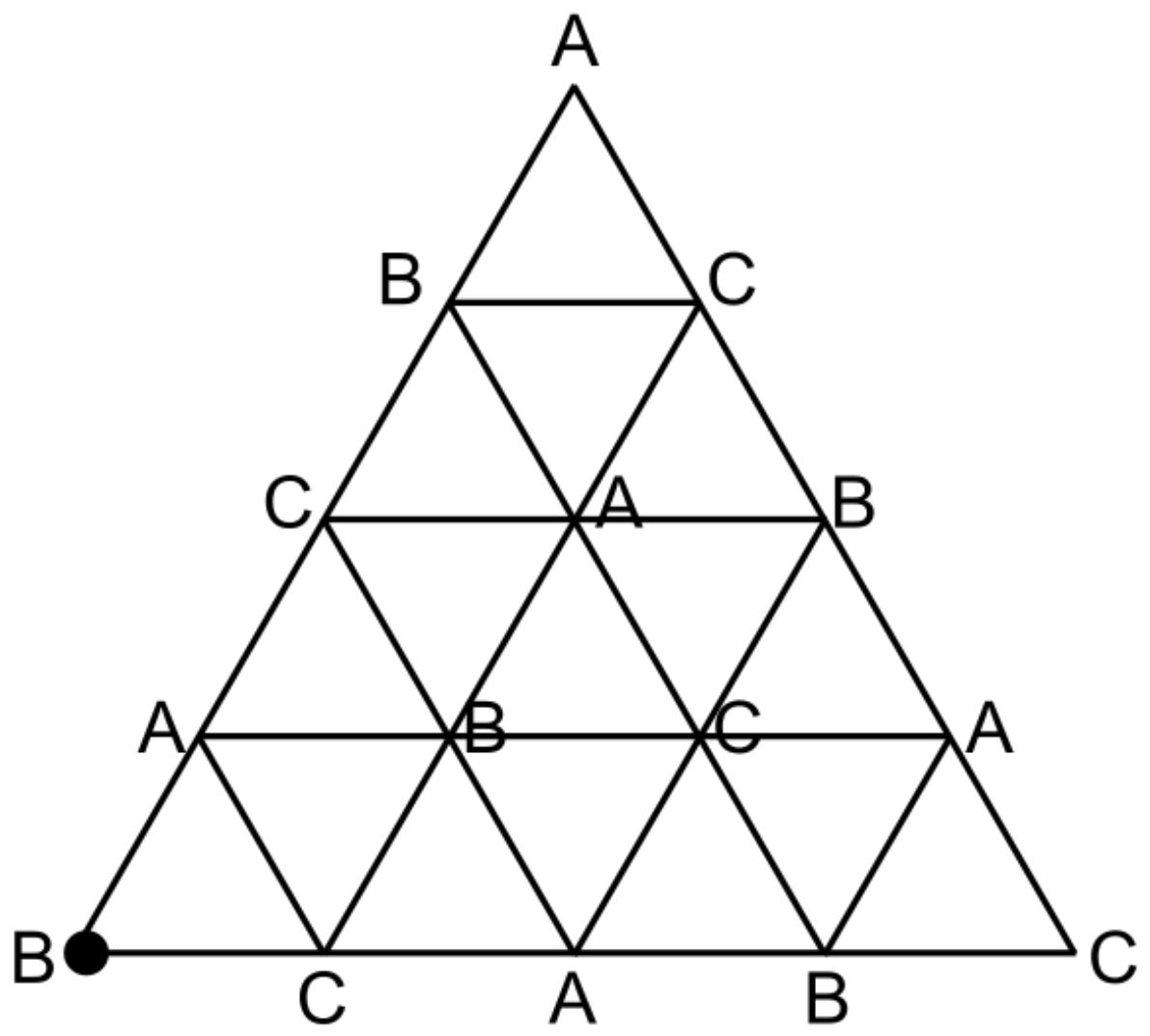
"Miserly agents" assumption
Agents always prefer a *free* room over a room with non-zero rent.

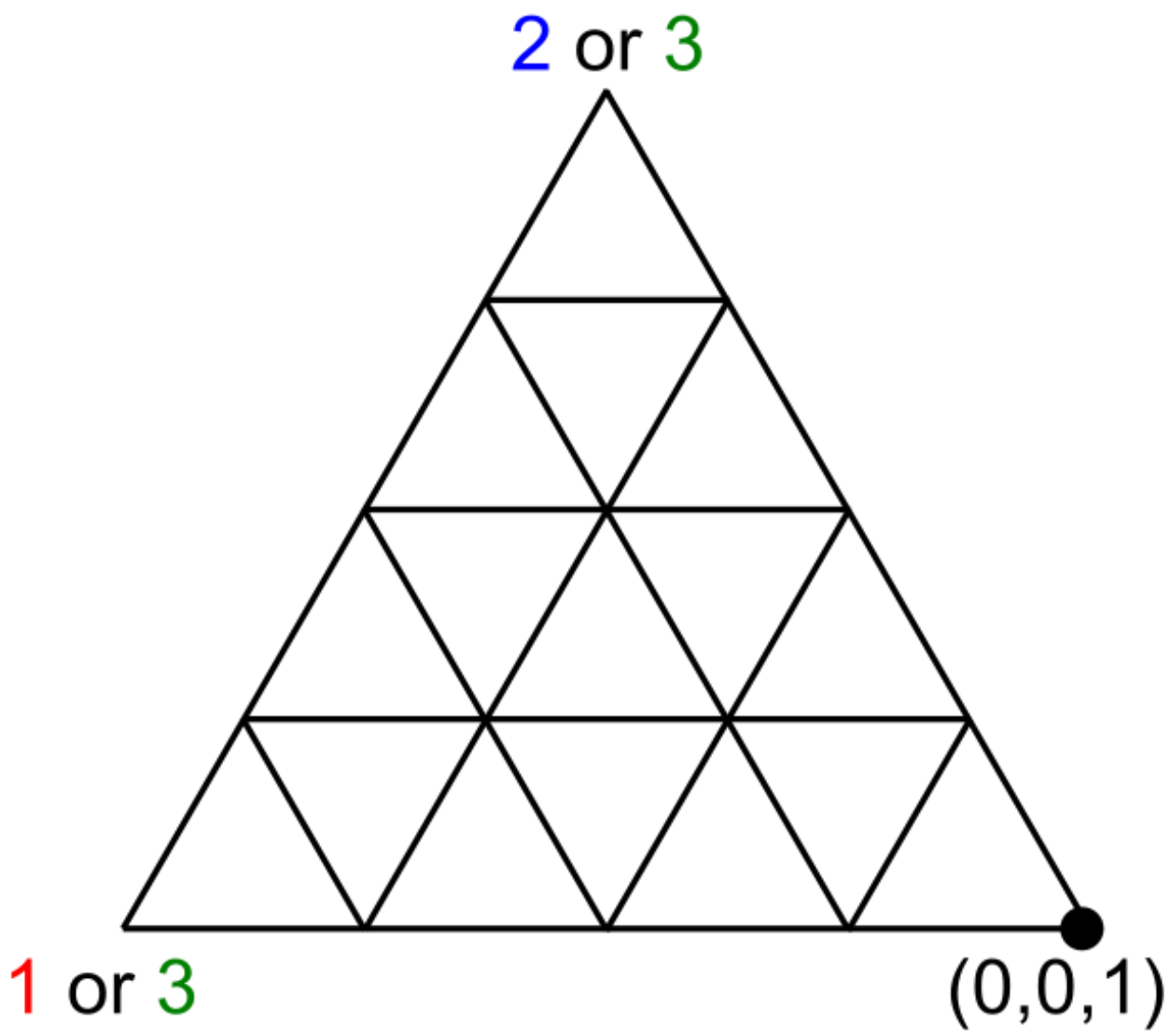
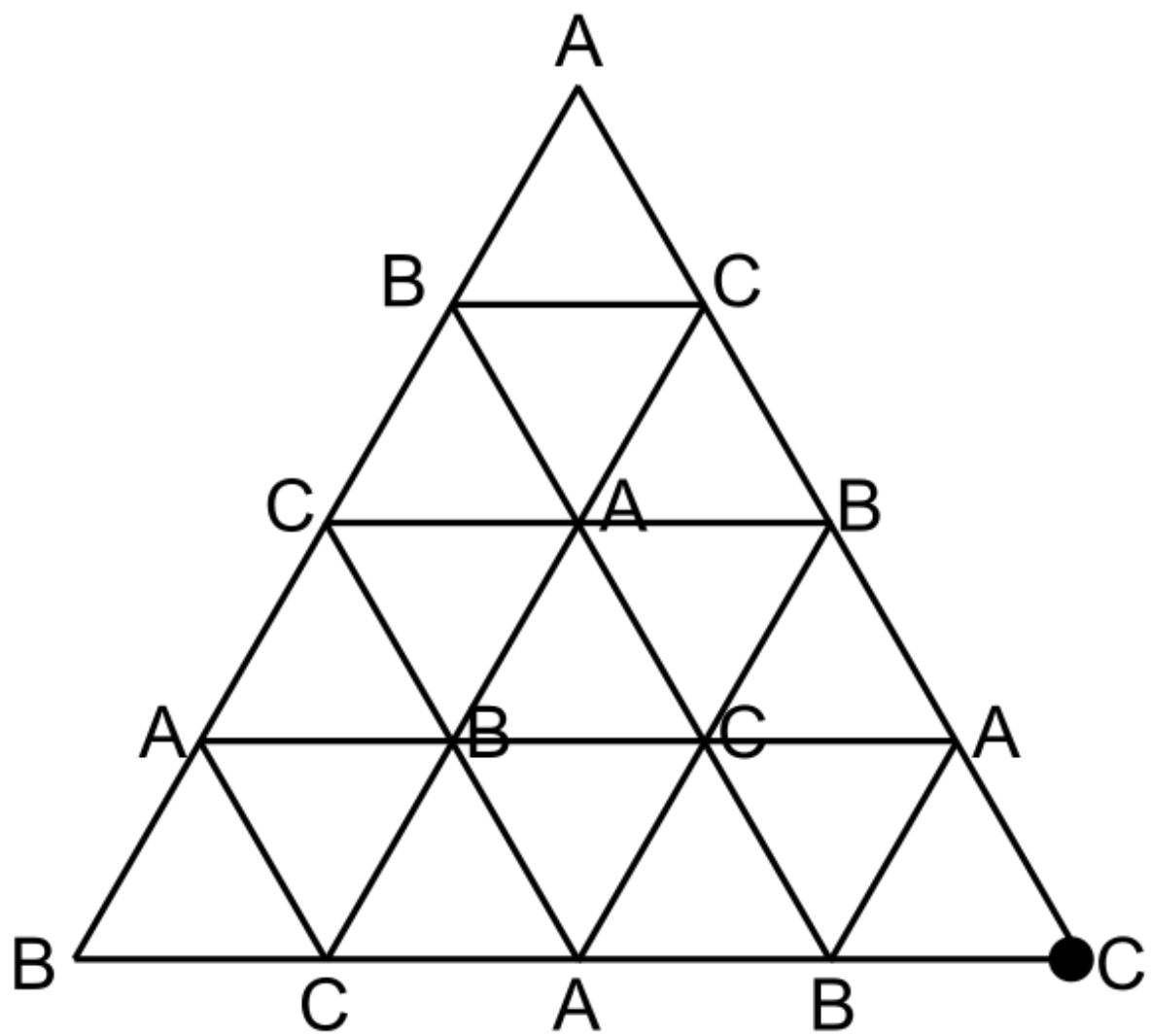


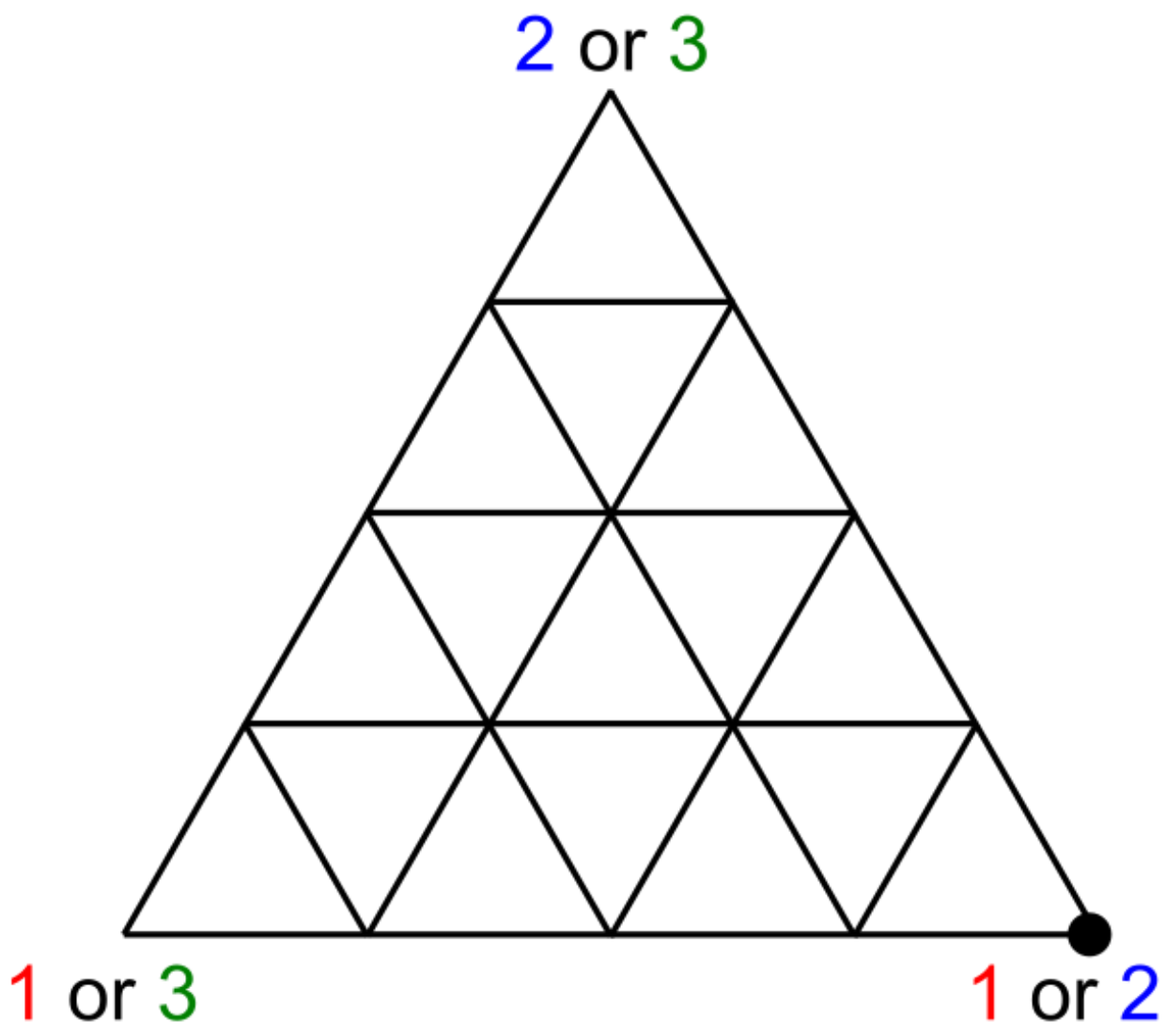
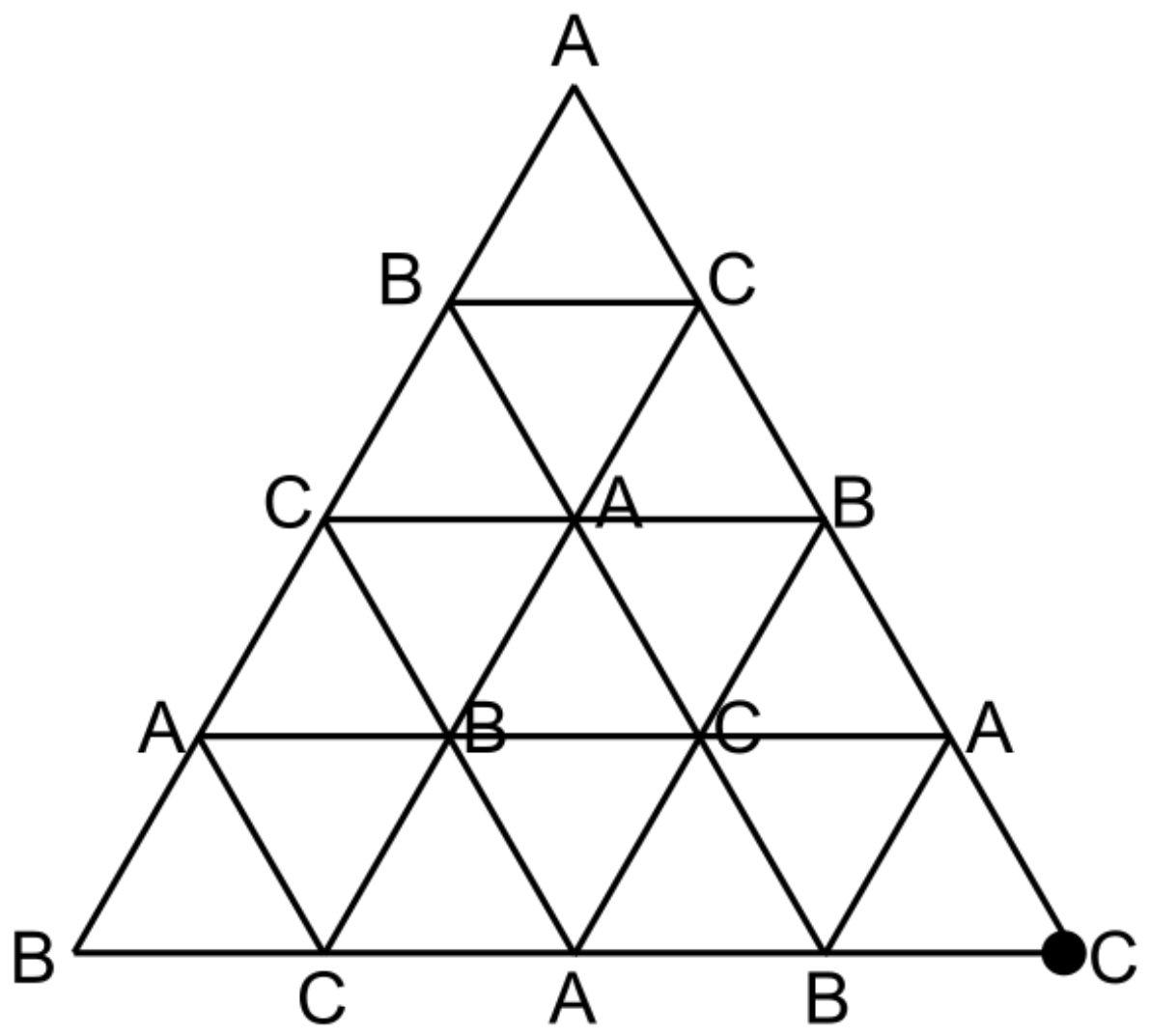


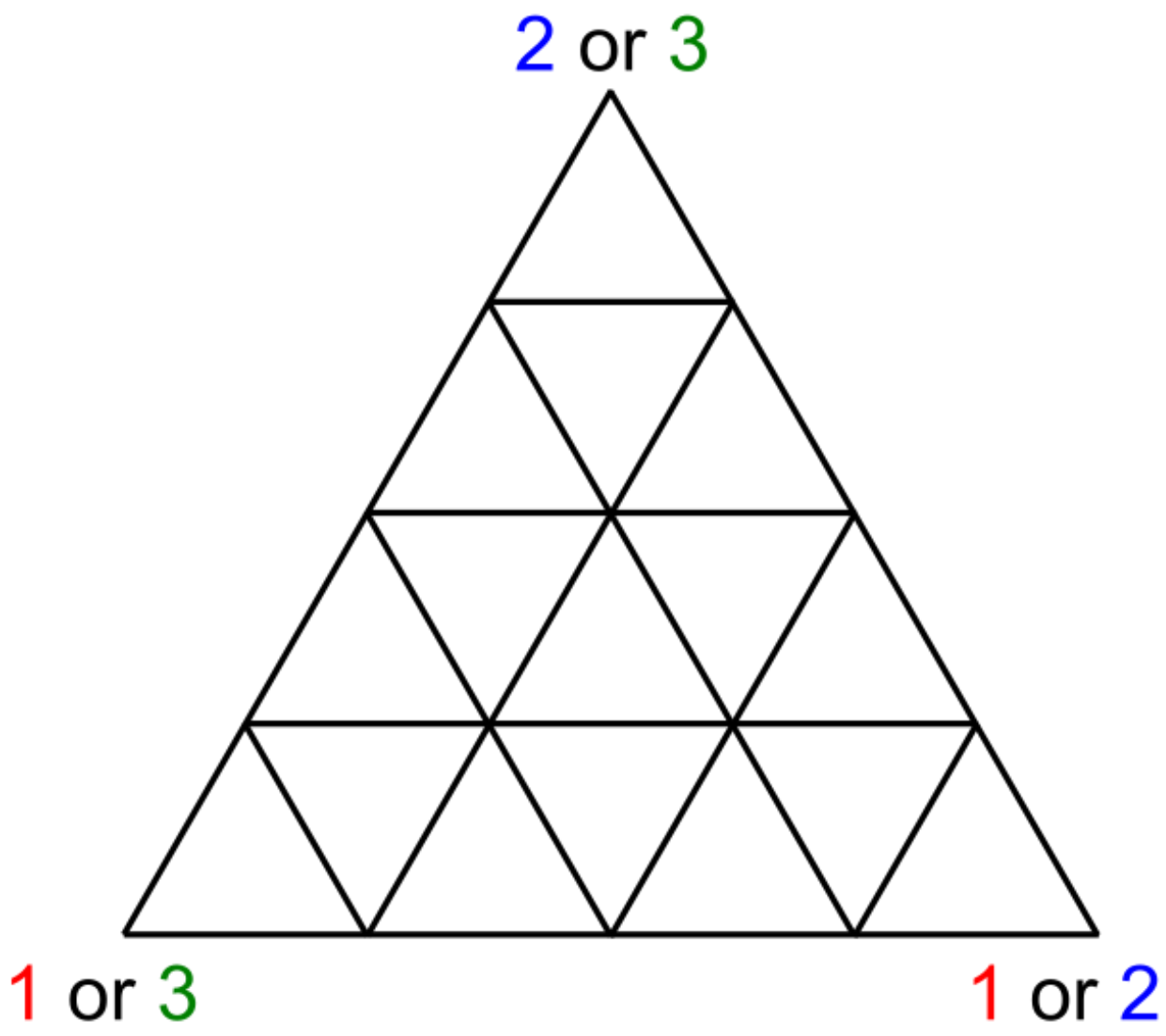
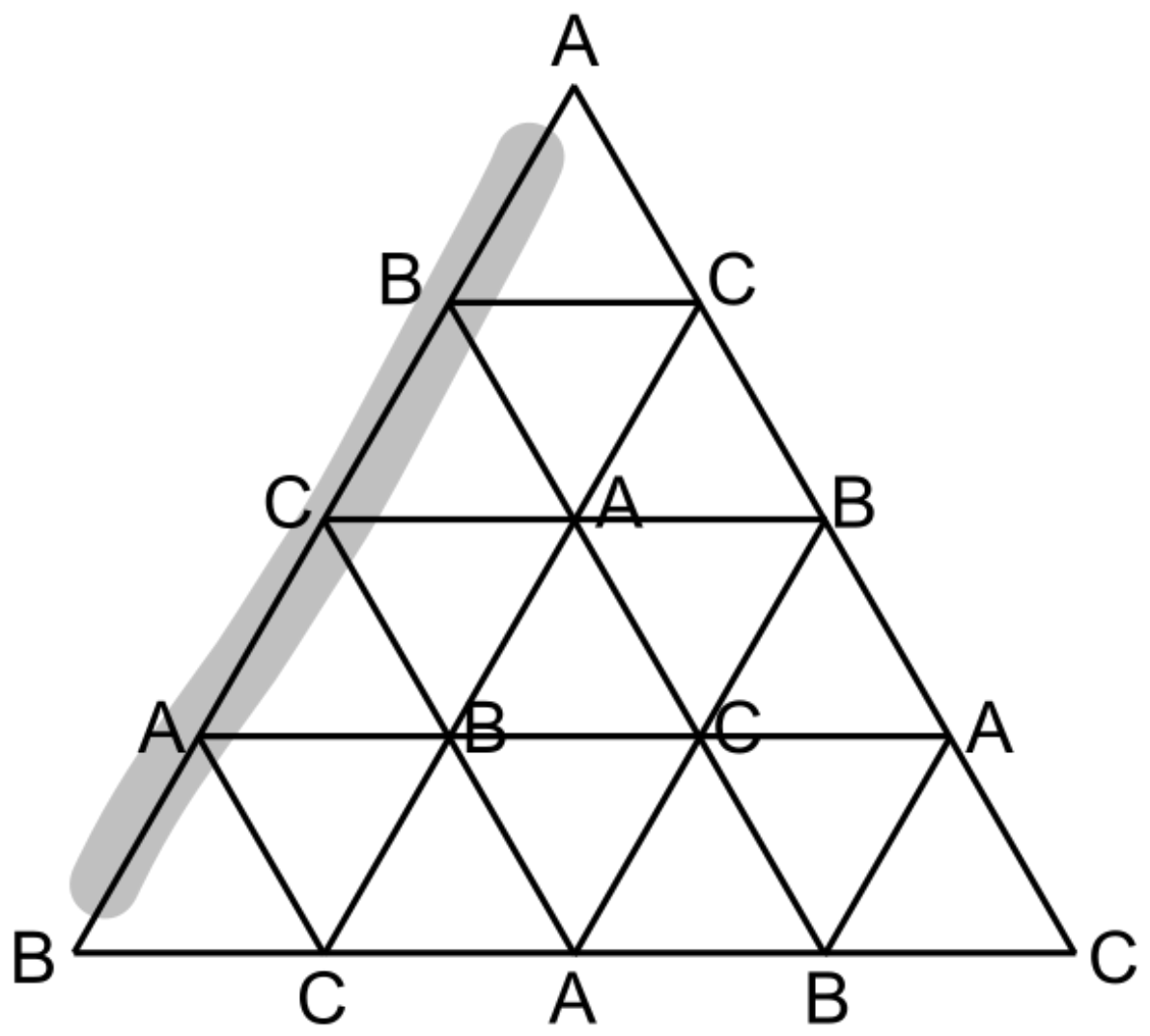


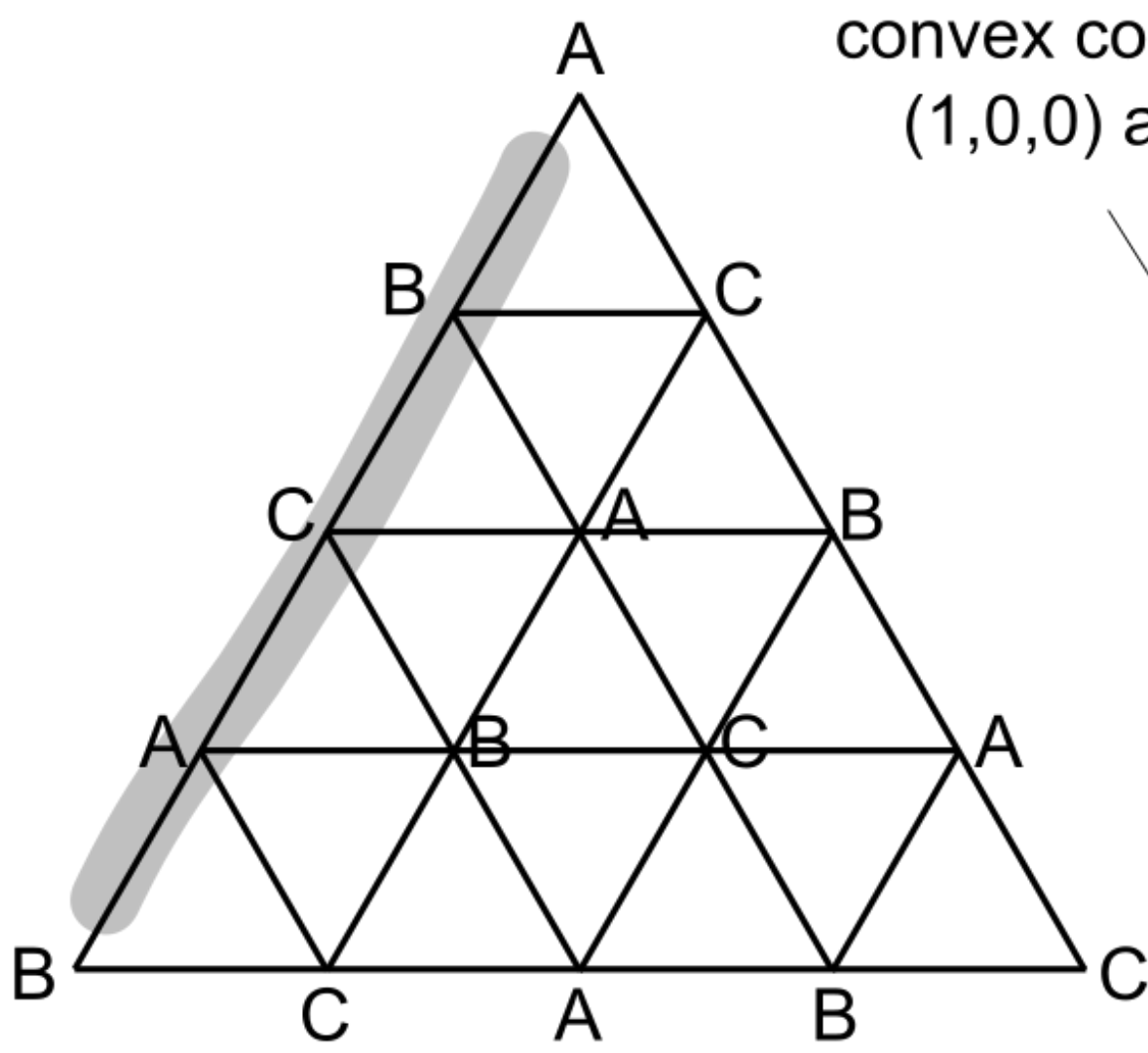




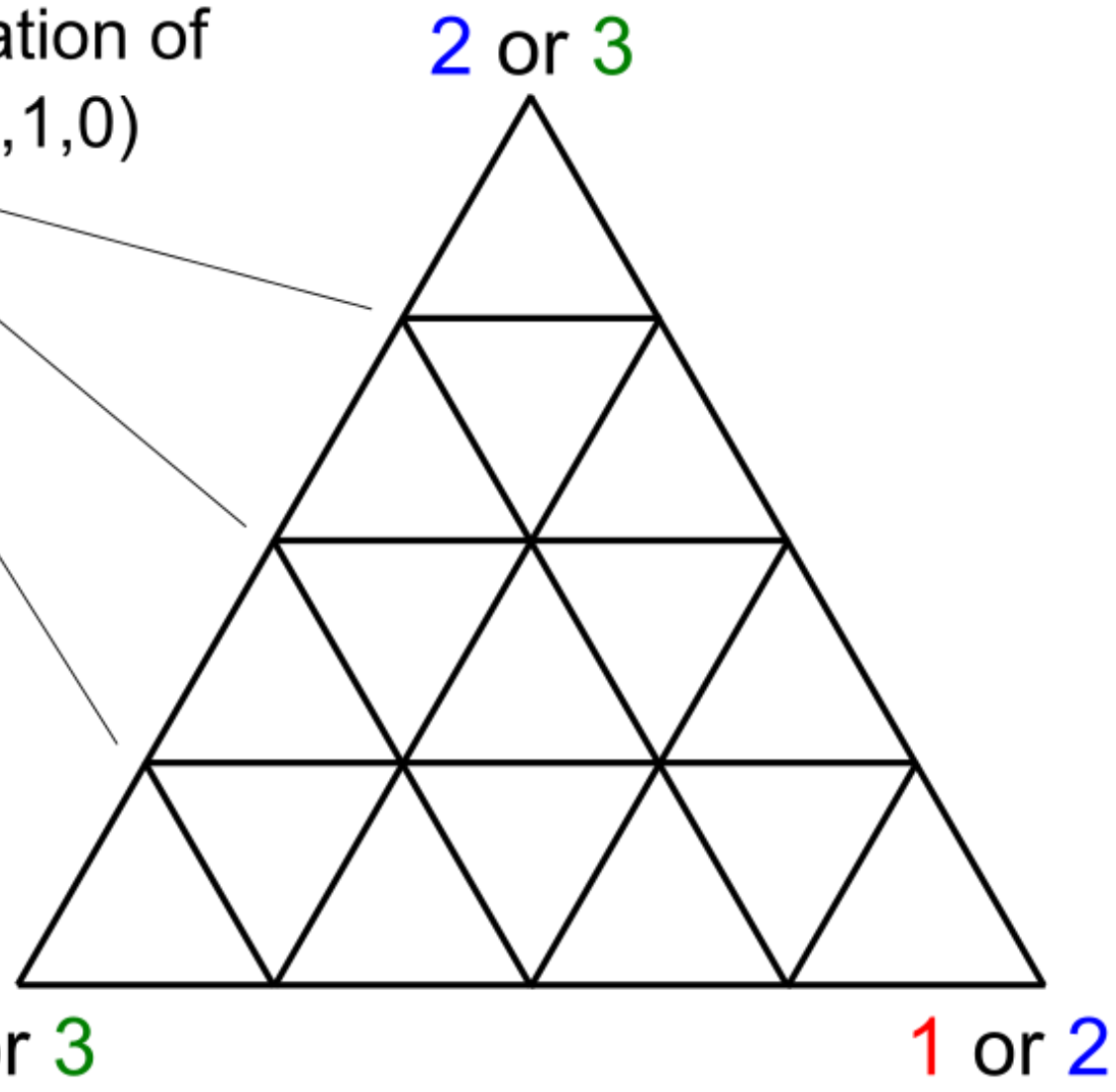


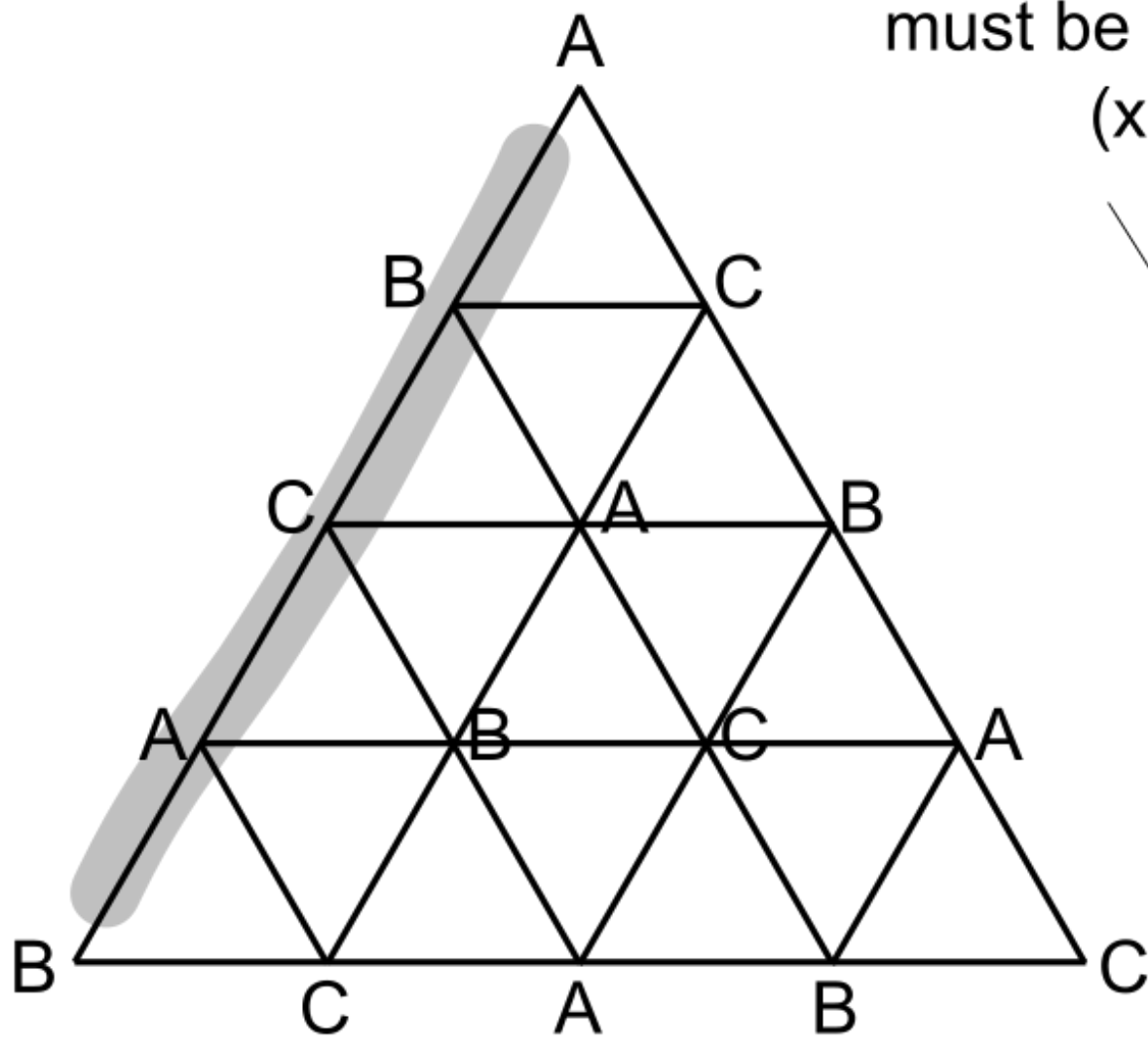




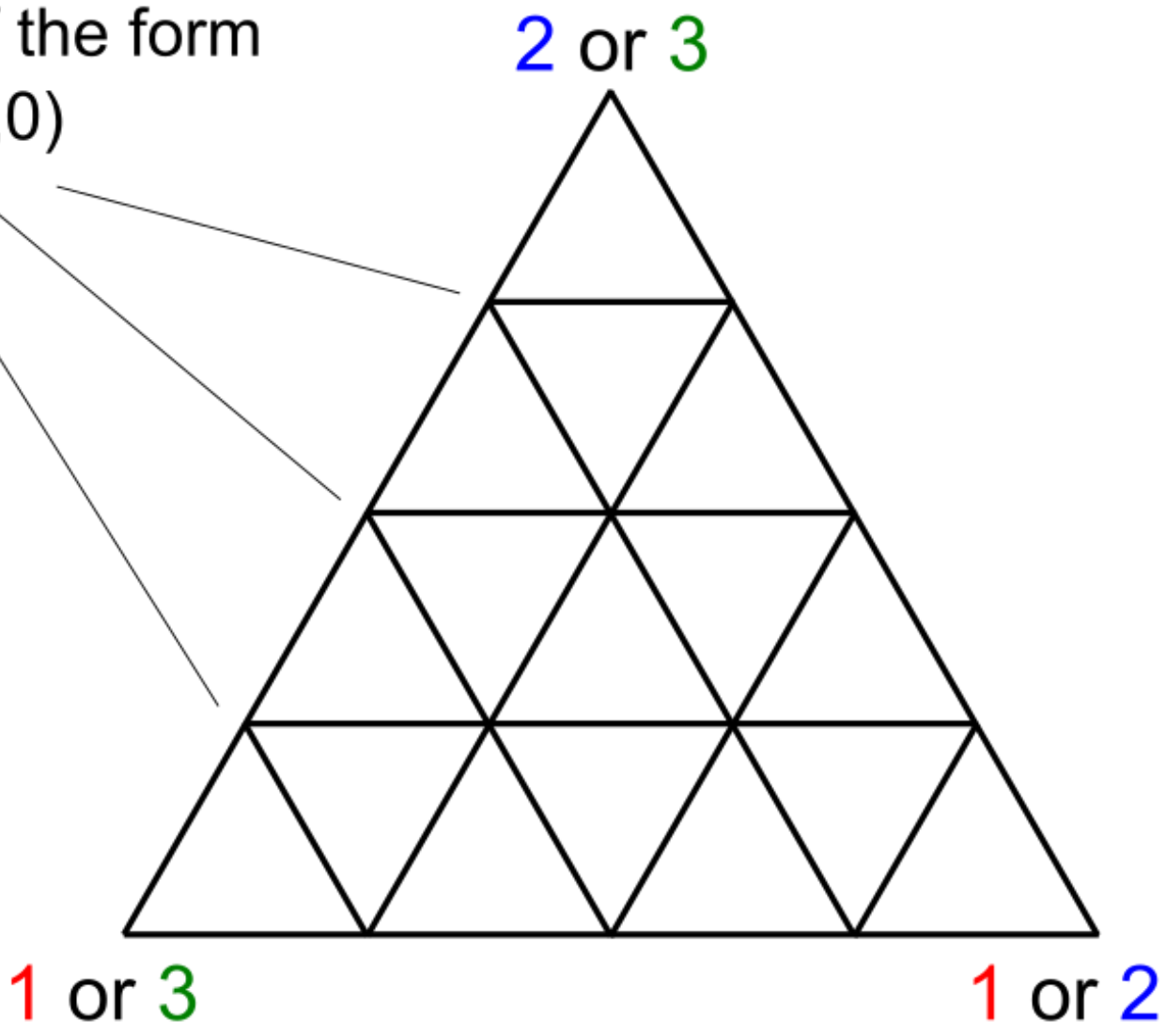


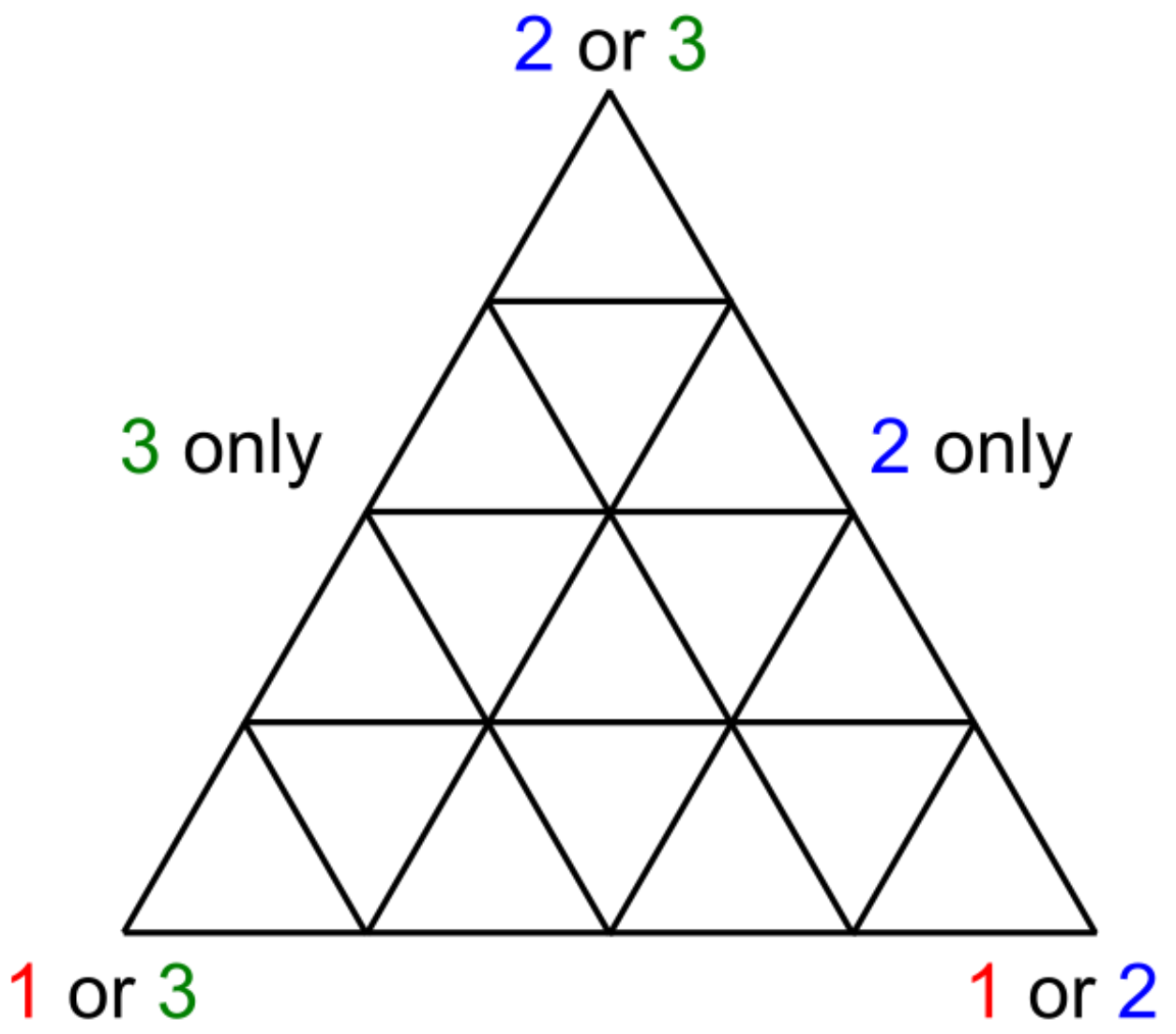
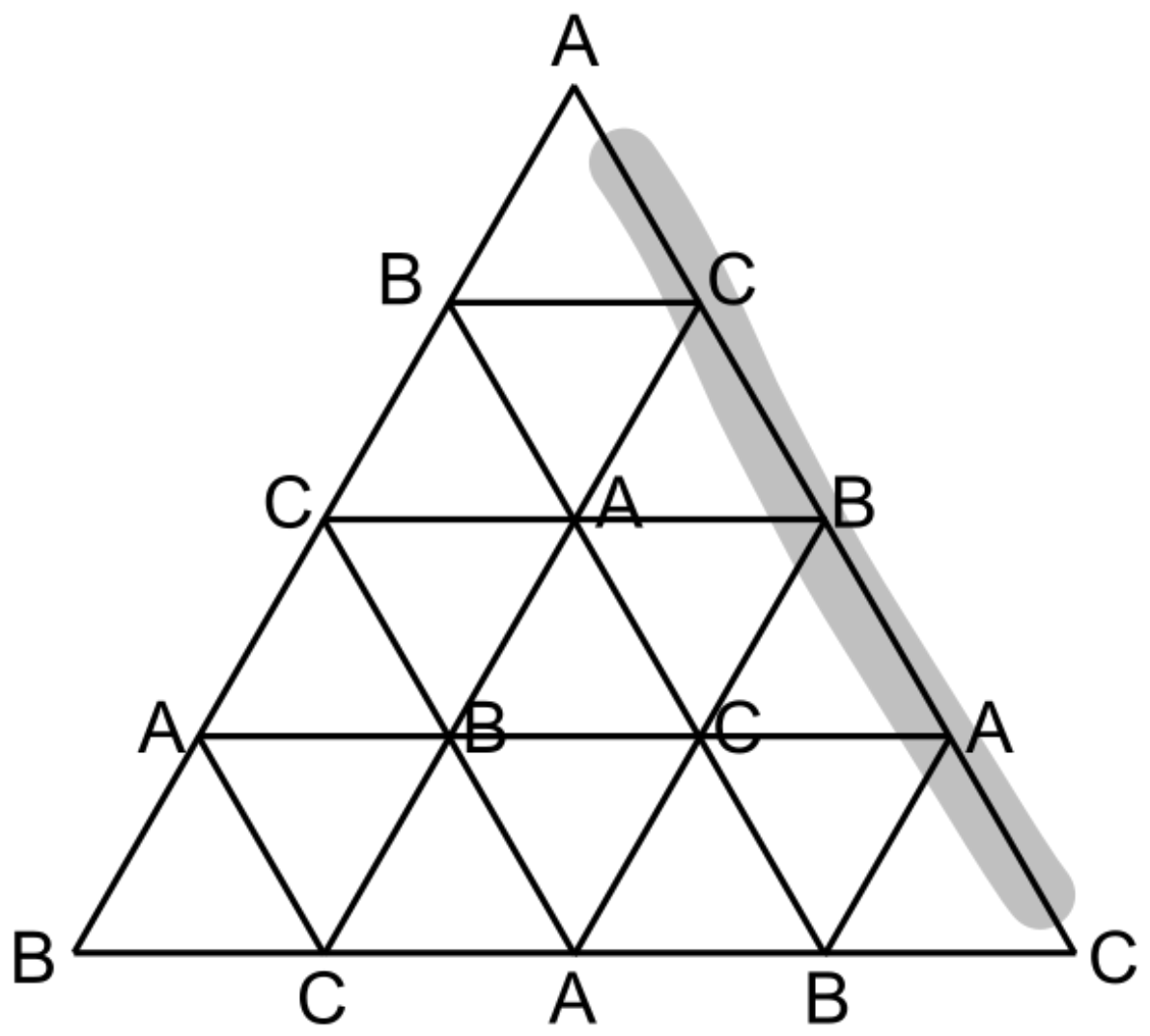
convex combination of
 $(1,0,0)$ and $(0,1,0)$

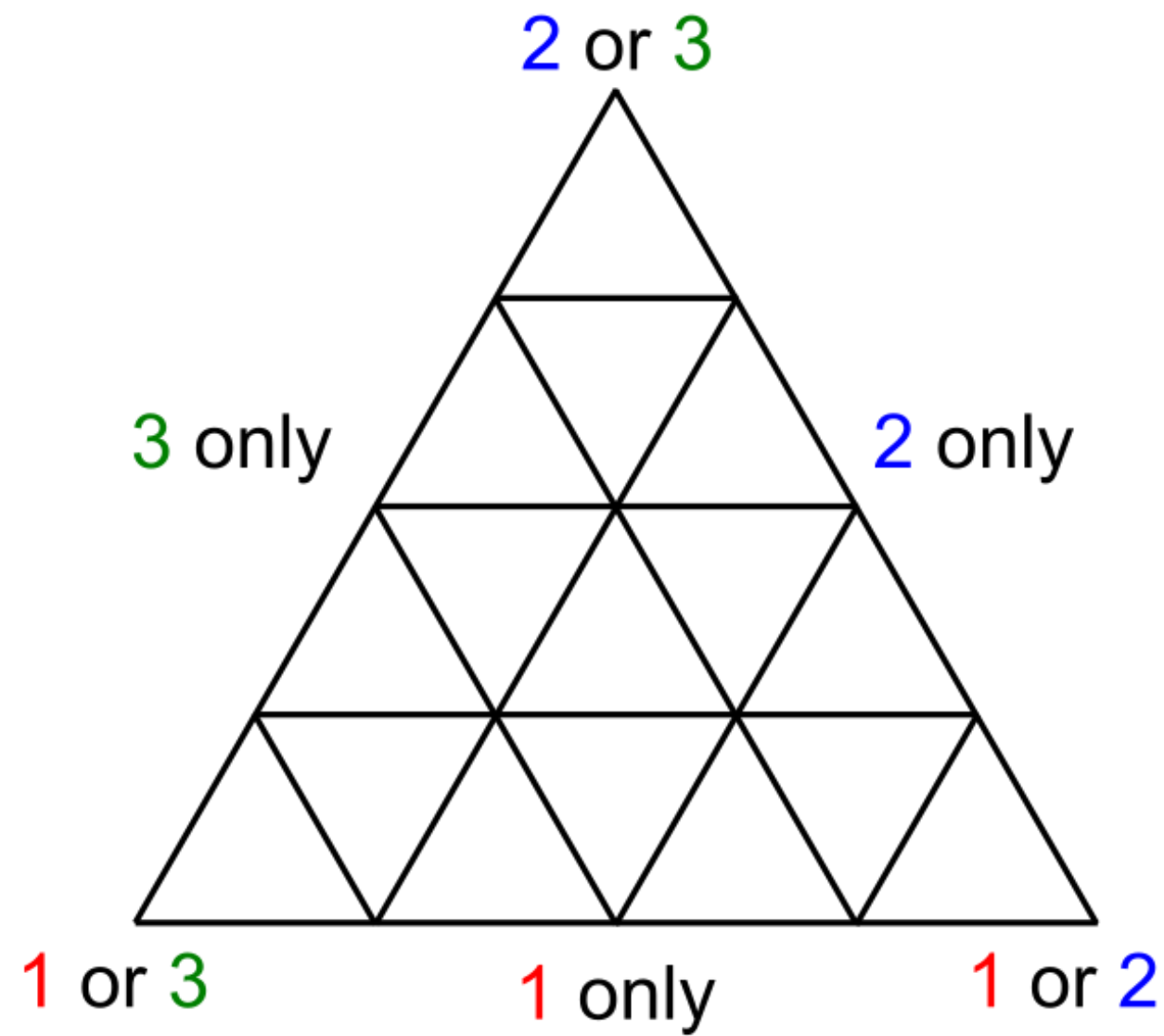
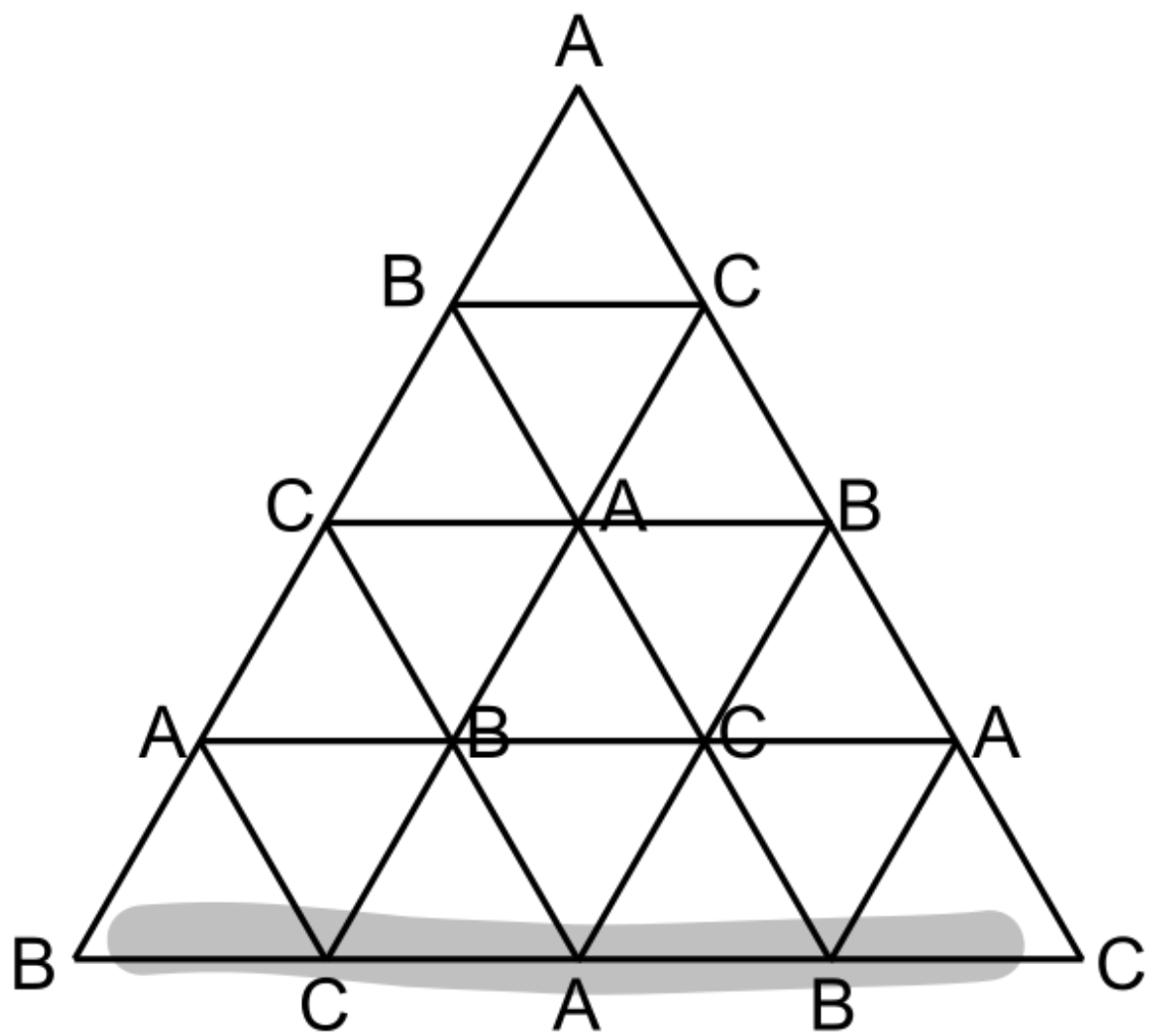


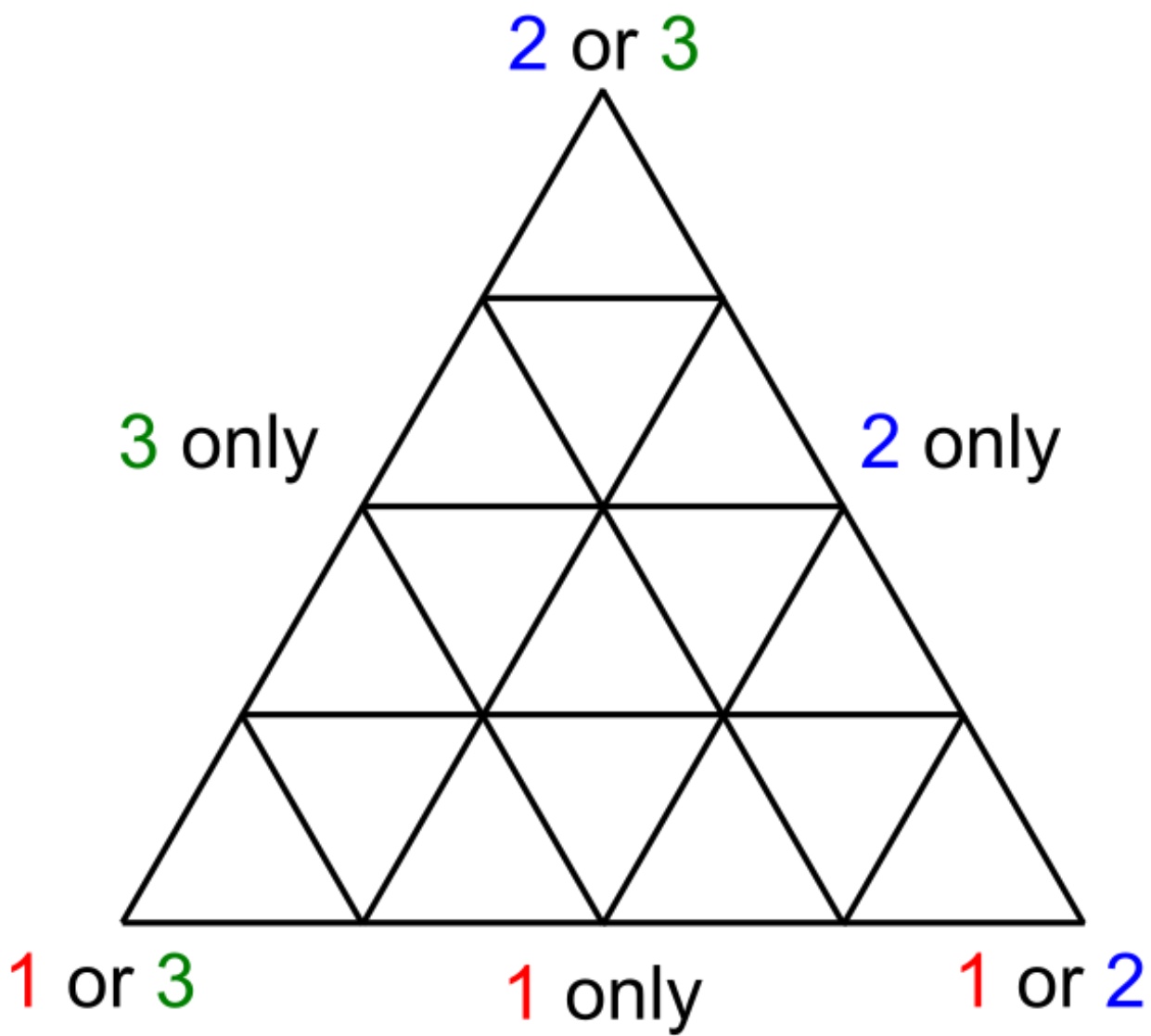
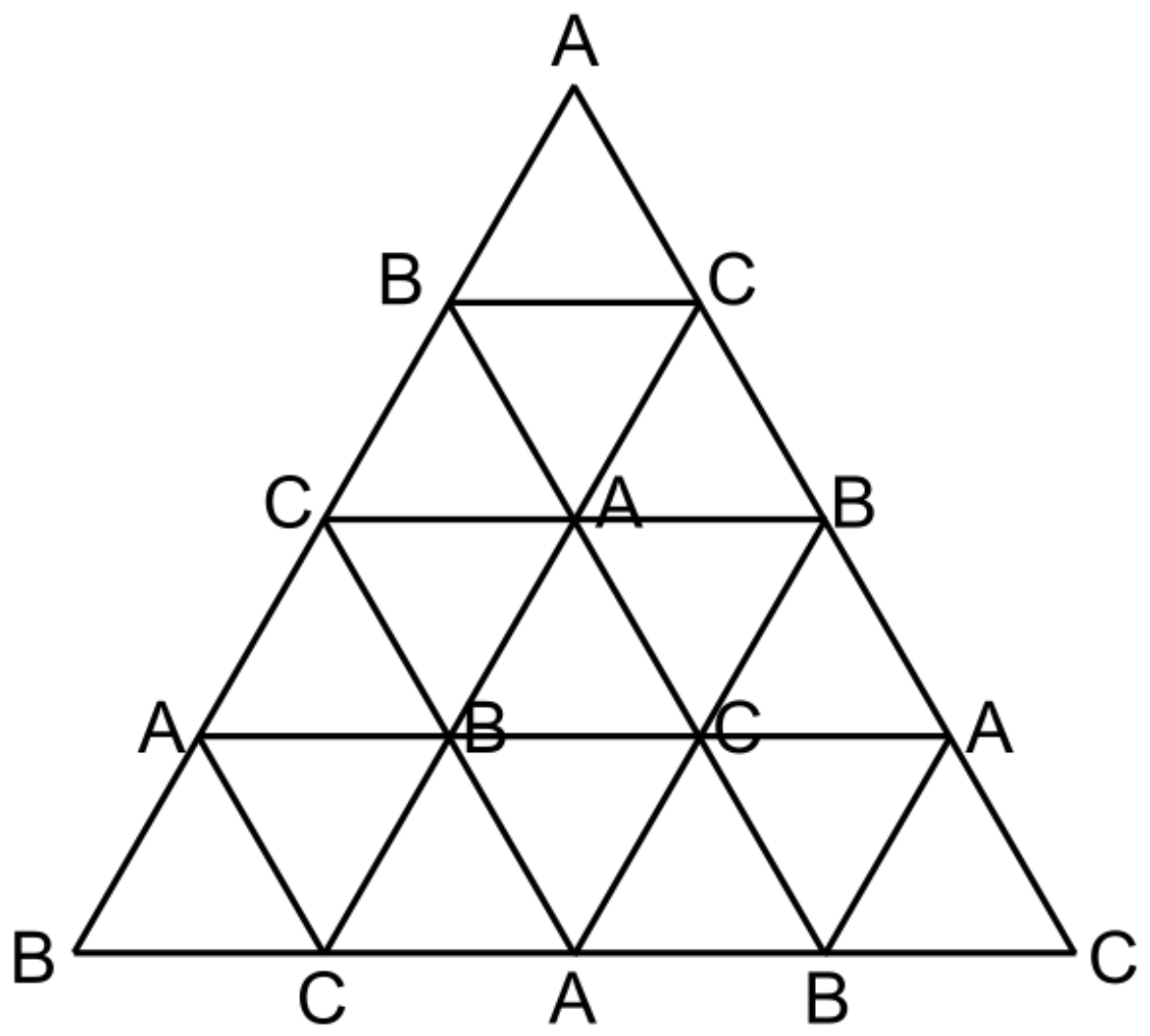


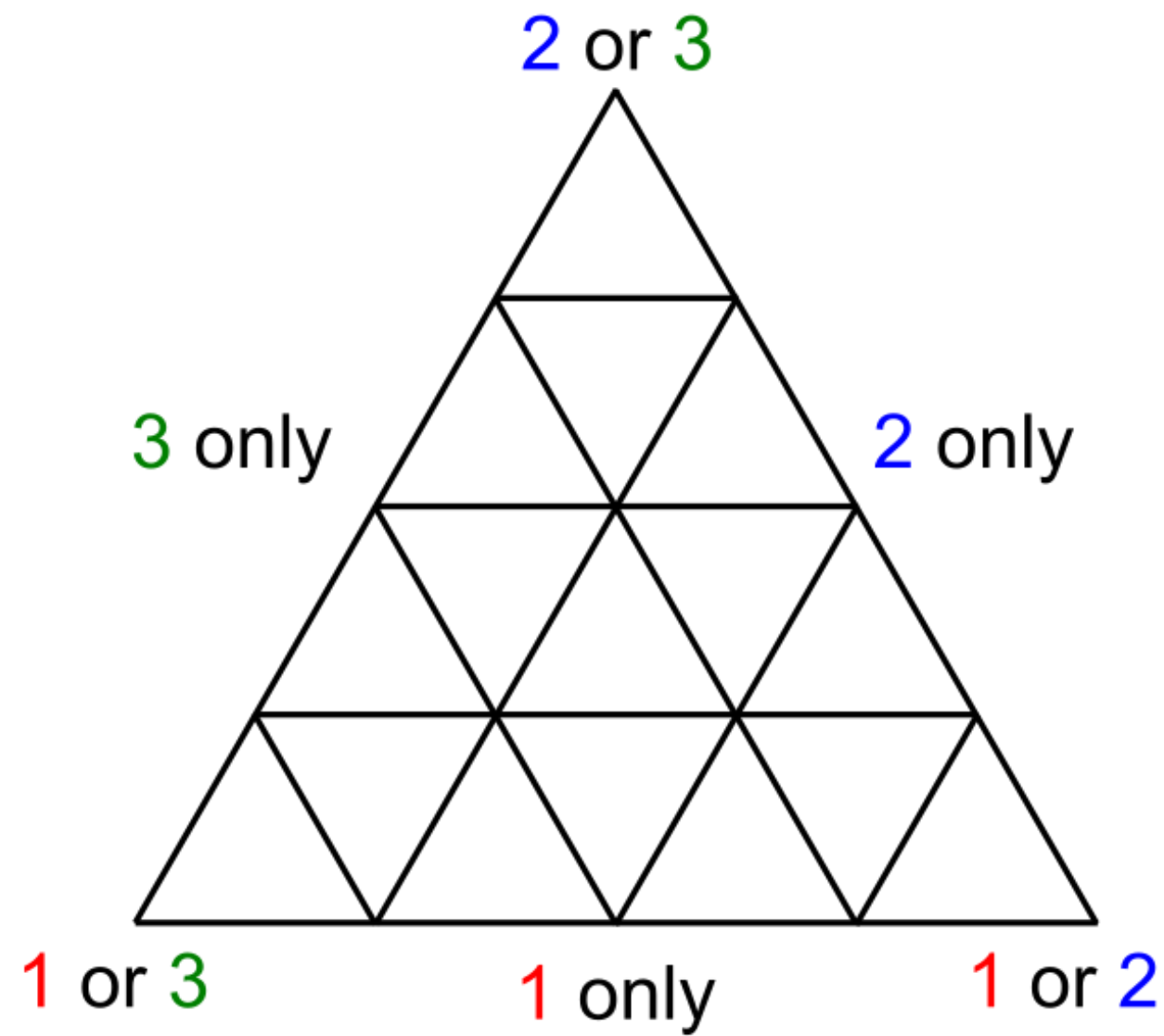
must be of the form
 $(x,y,0)$



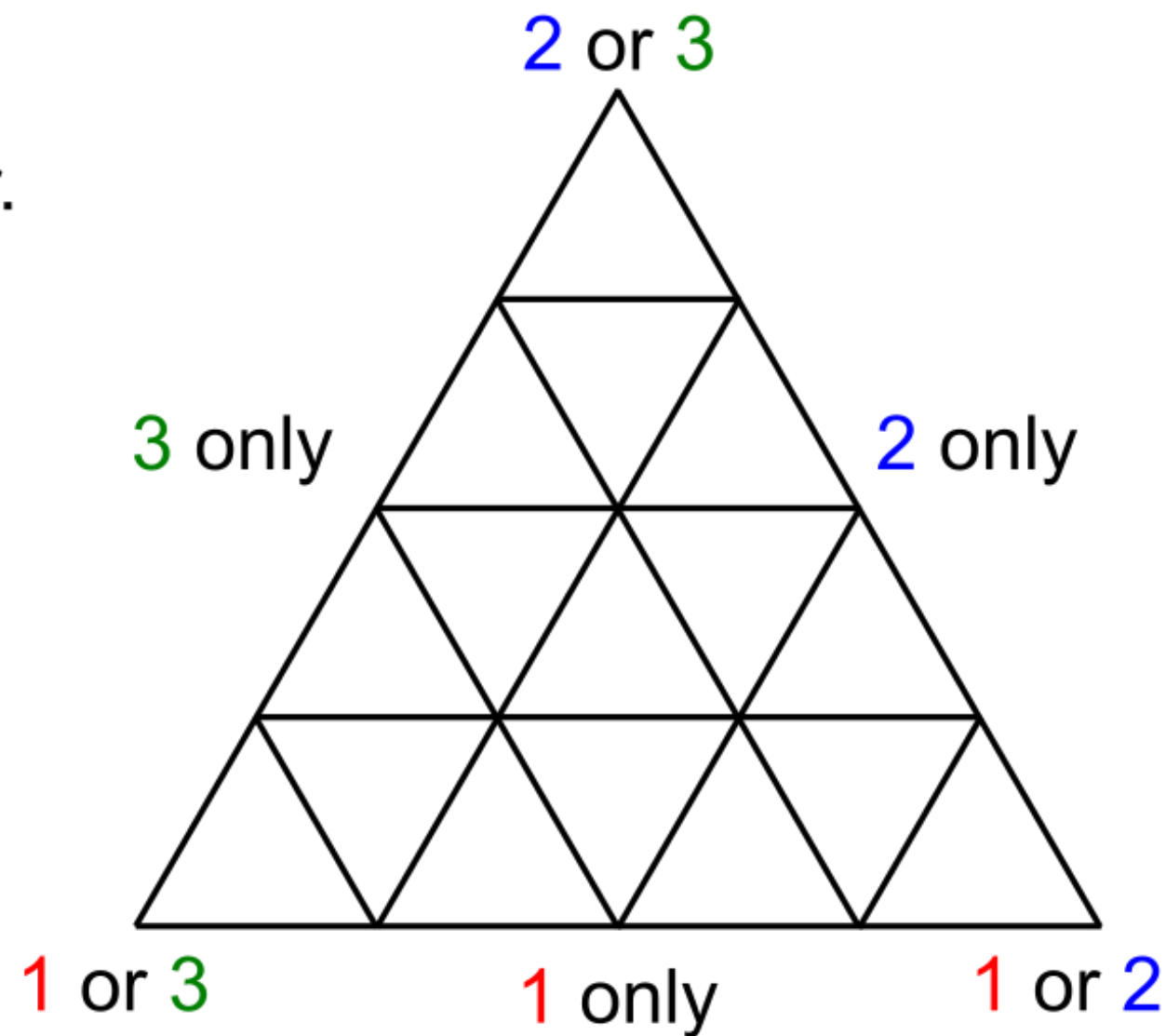






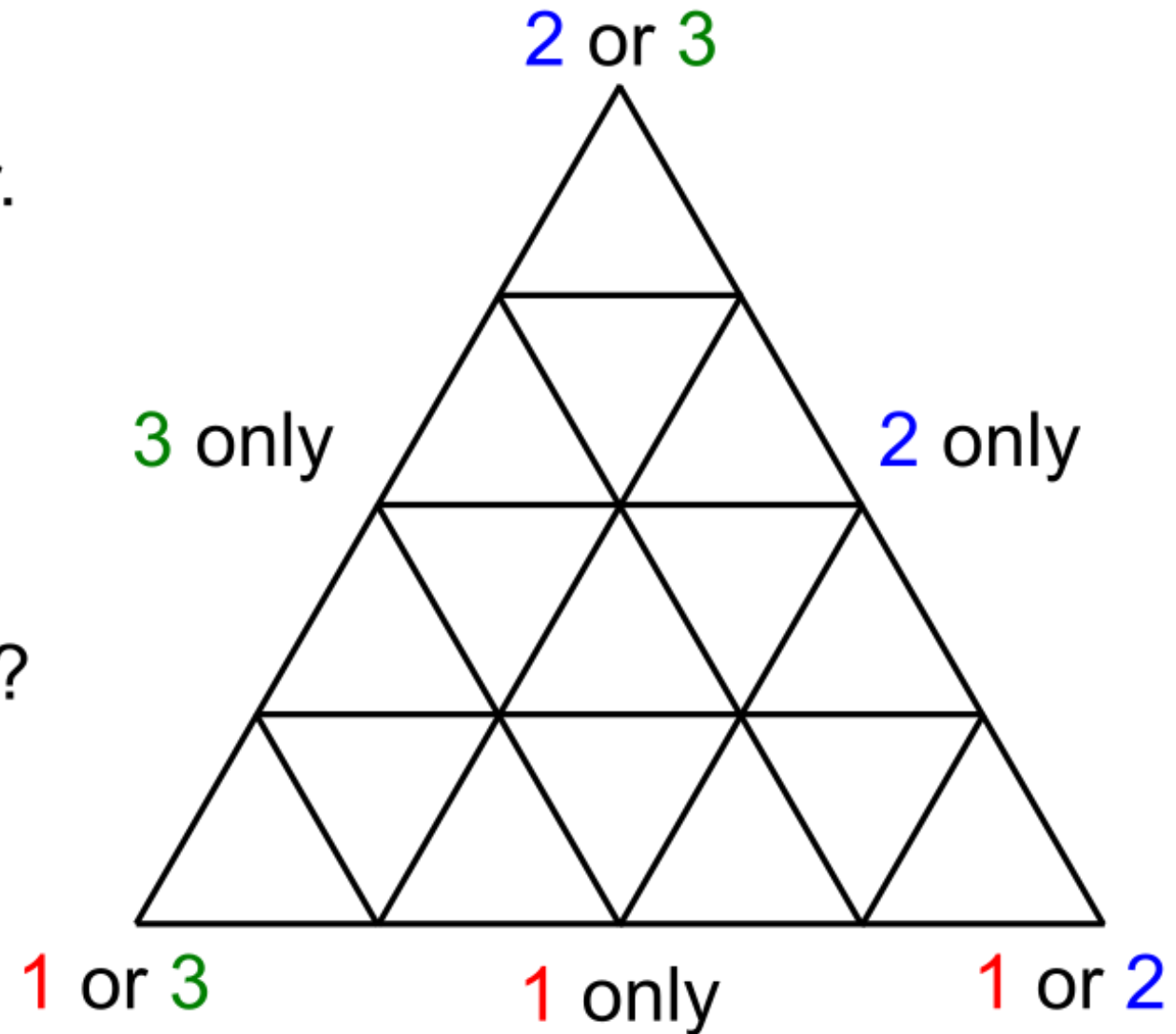


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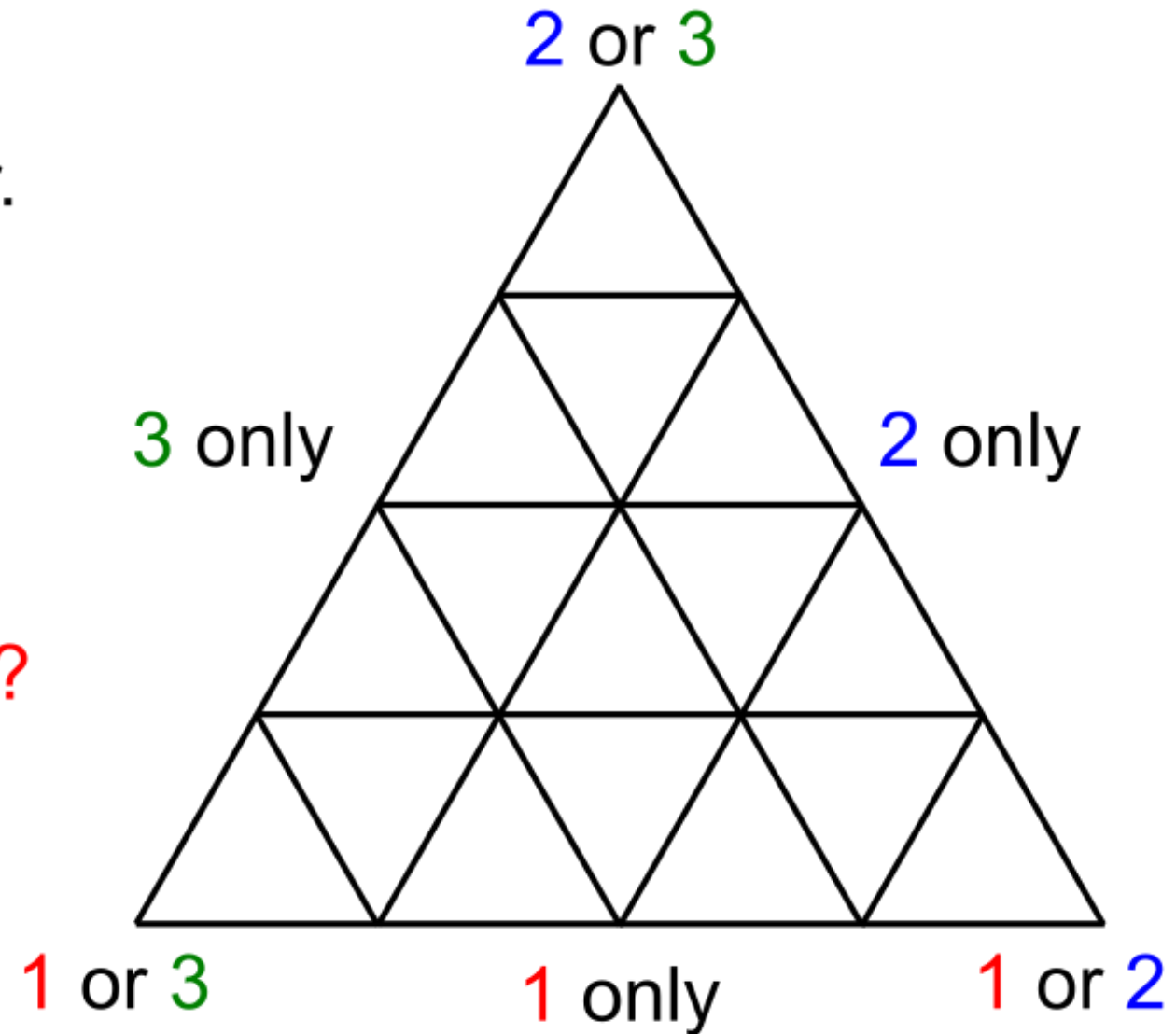
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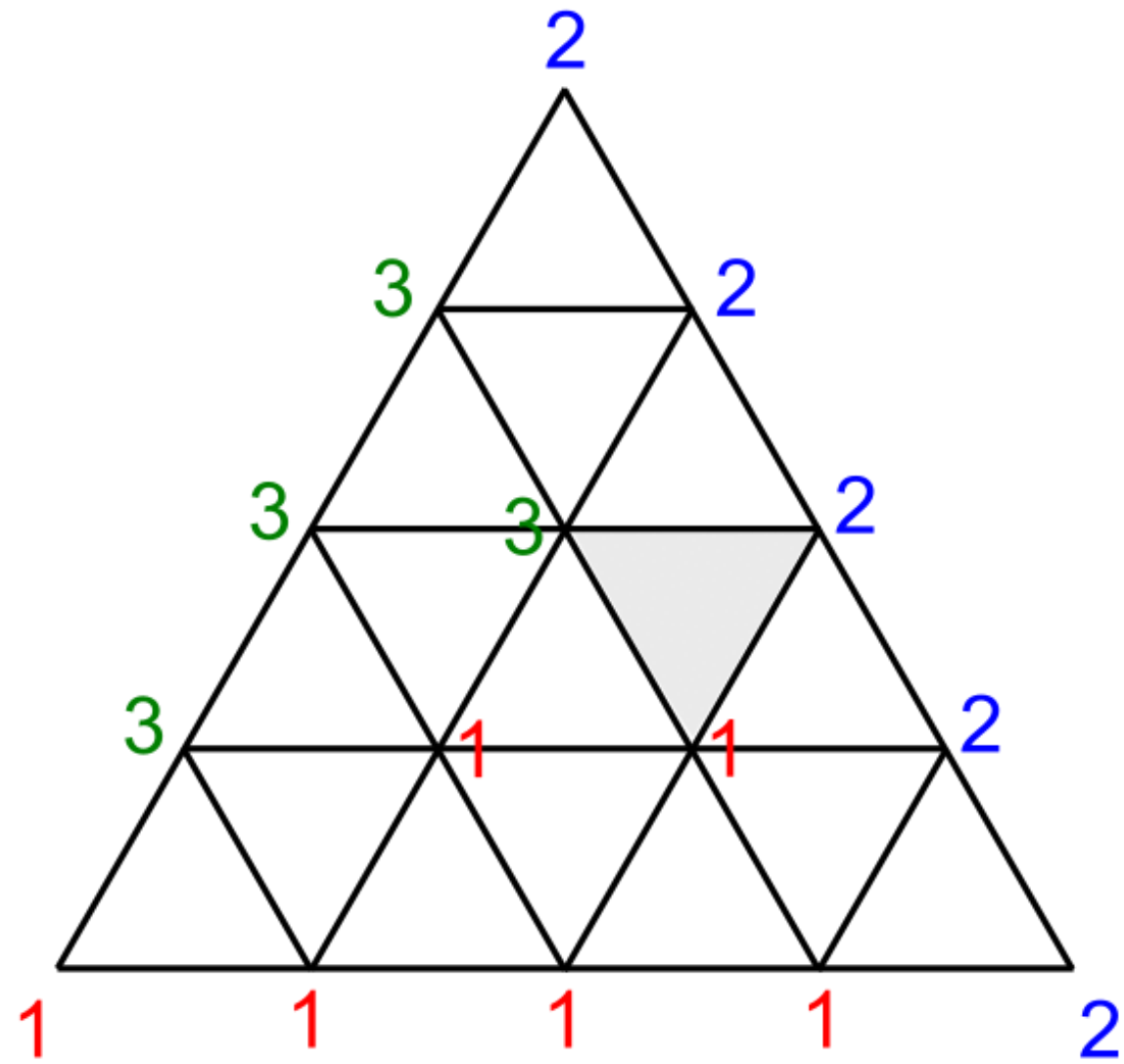
Why is a fully labeled triangle useful for envy-free rent division?



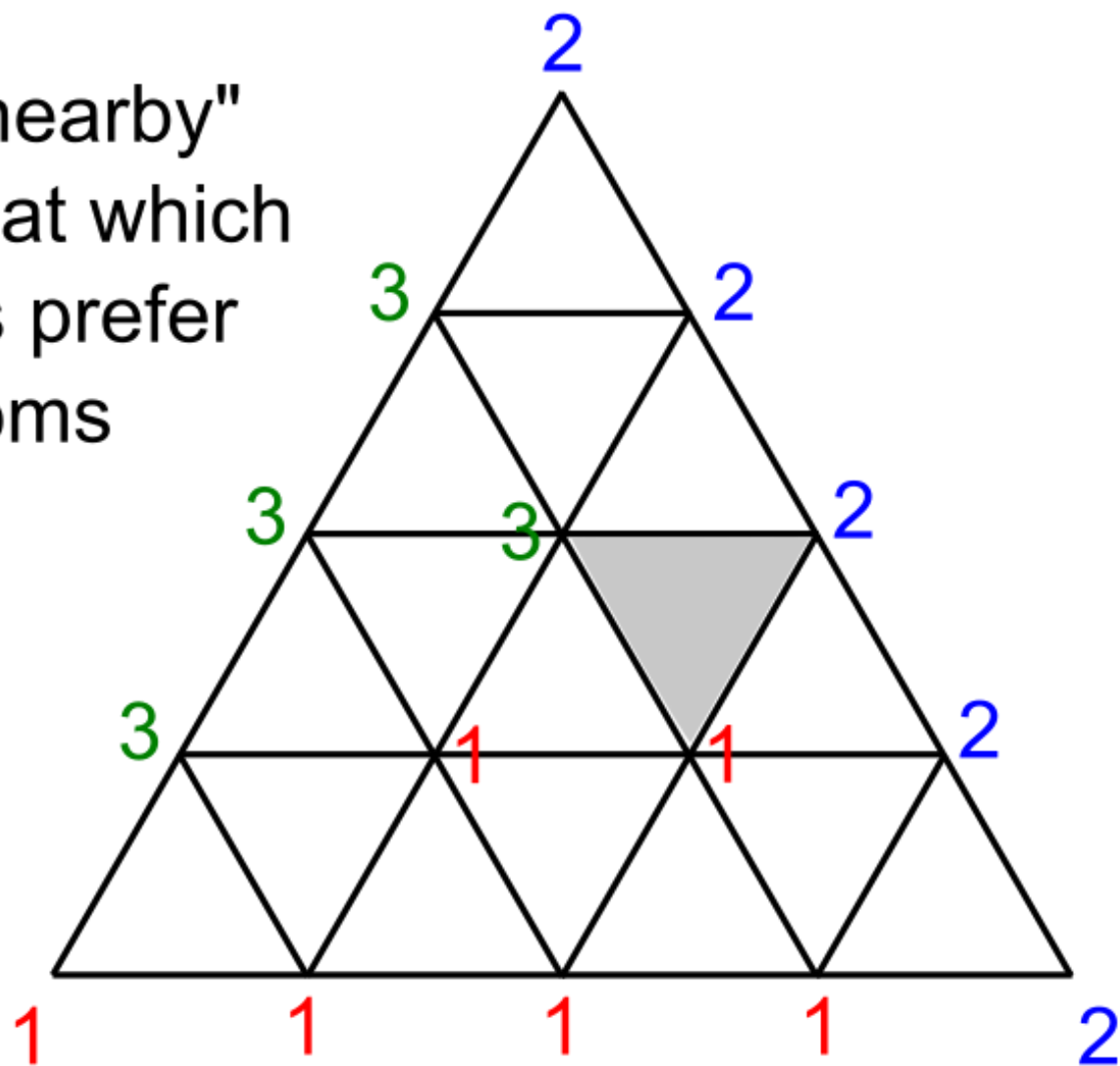
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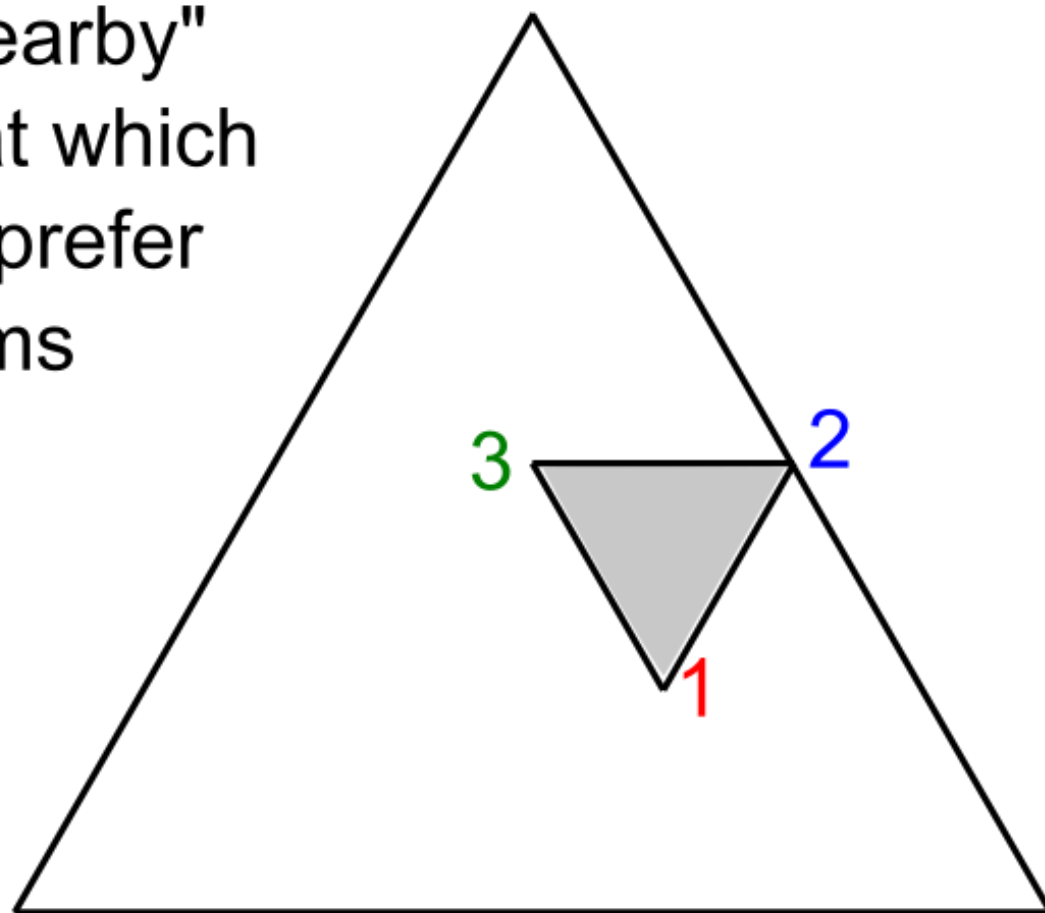




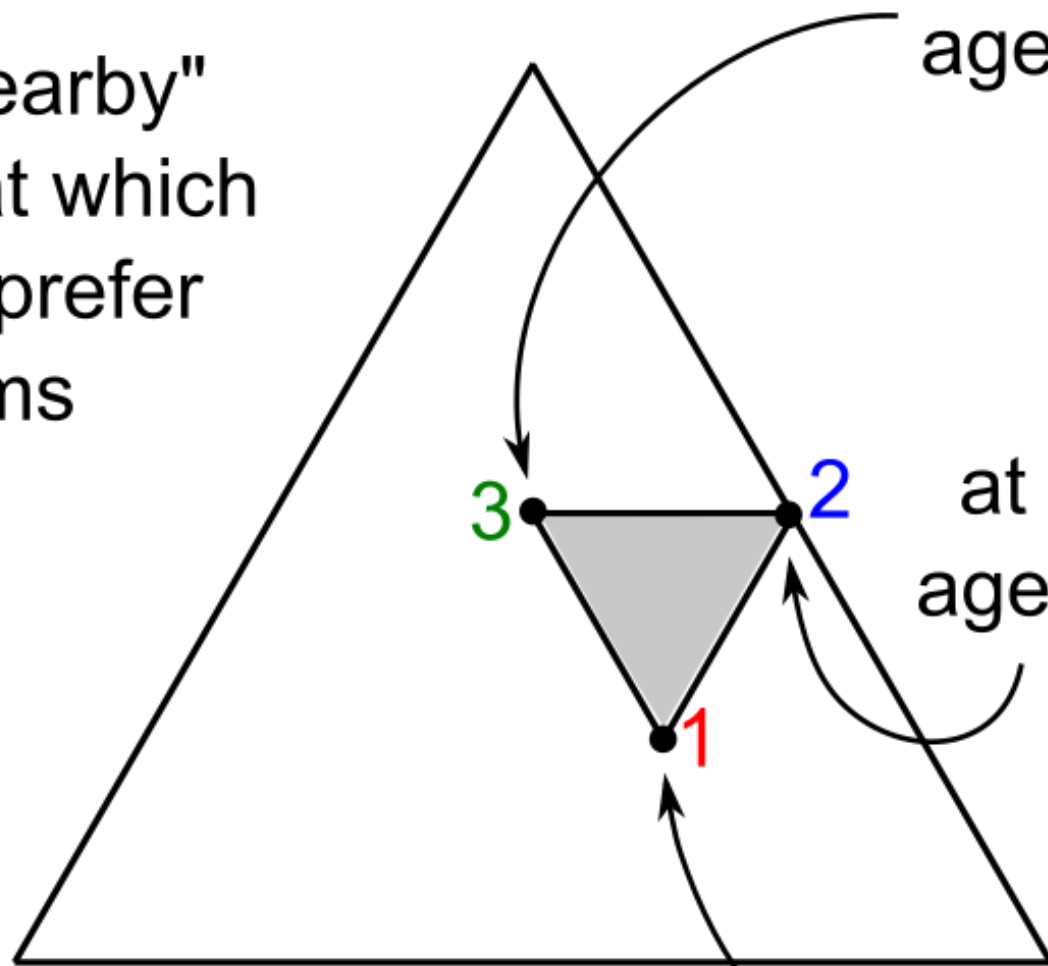
a set of three "nearby"
pricing schemes at which
different agents prefer
different rooms



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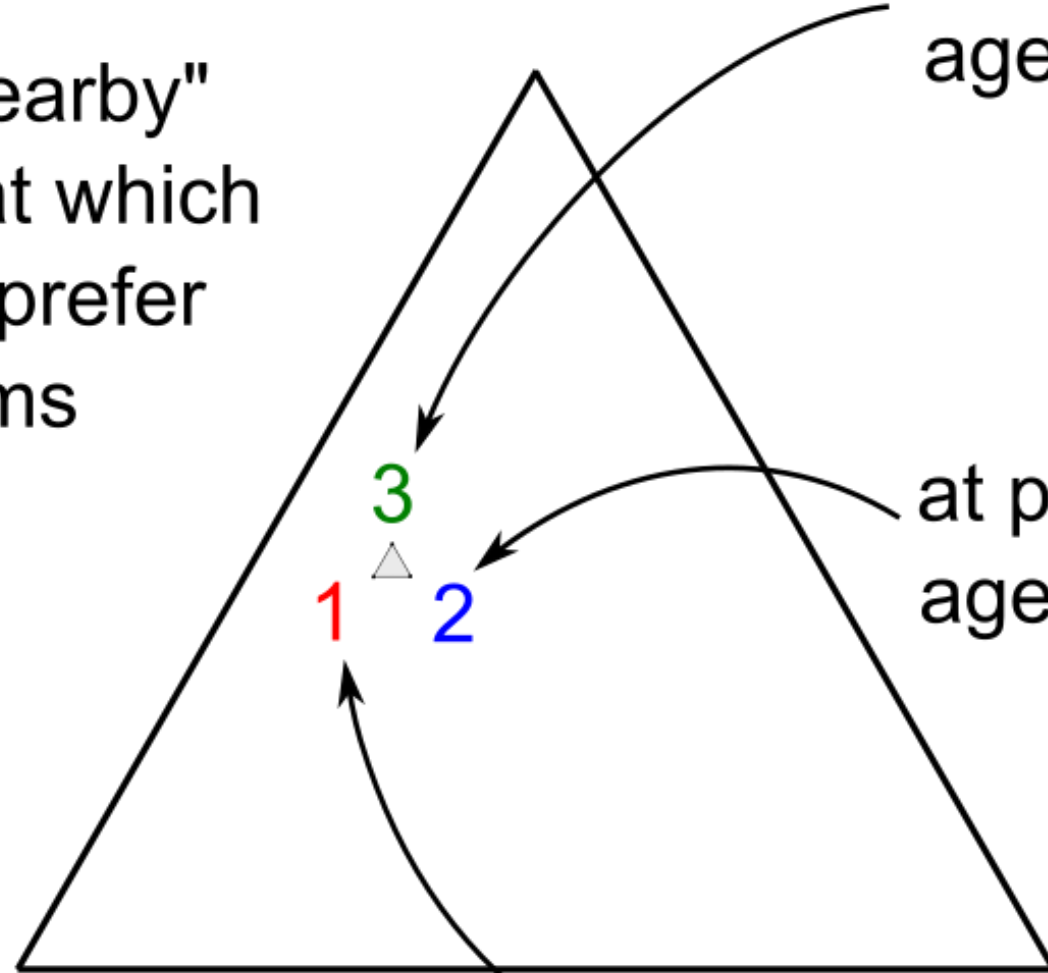


at prices (500,250,250)
agent A prefers room 3

at prices (500,0,500)
agent B prefers room 2

at prices (250,250,500)
agent C prefers room 1

a set of three "nearby"
pricing schemes at which
different agents prefer
different rooms

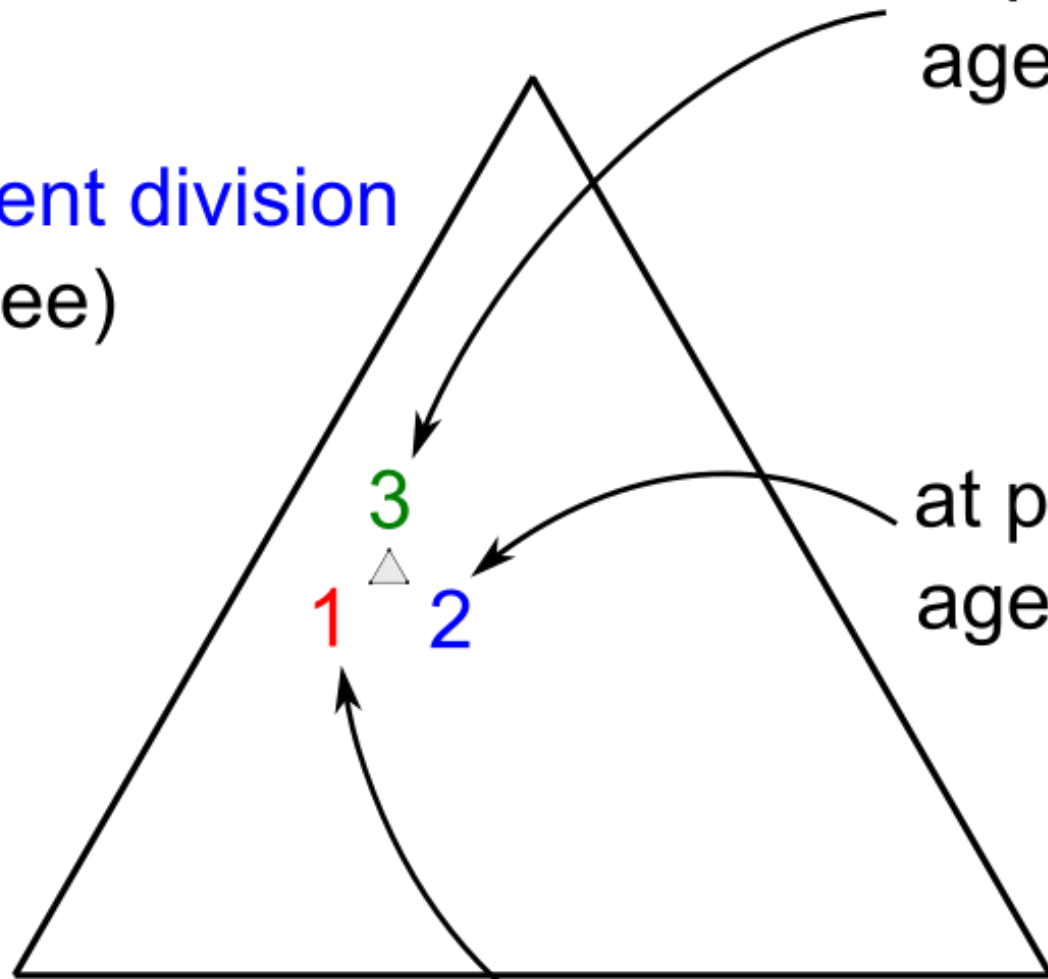


at prices (426, 424, 150)
agent A prefers room 3

at prices (425, 424, 151)
agent B prefers room 2

at prices (425, 425, 150)
agent C prefers room 1

Approx. envy-free rent division
(up to a rupee)



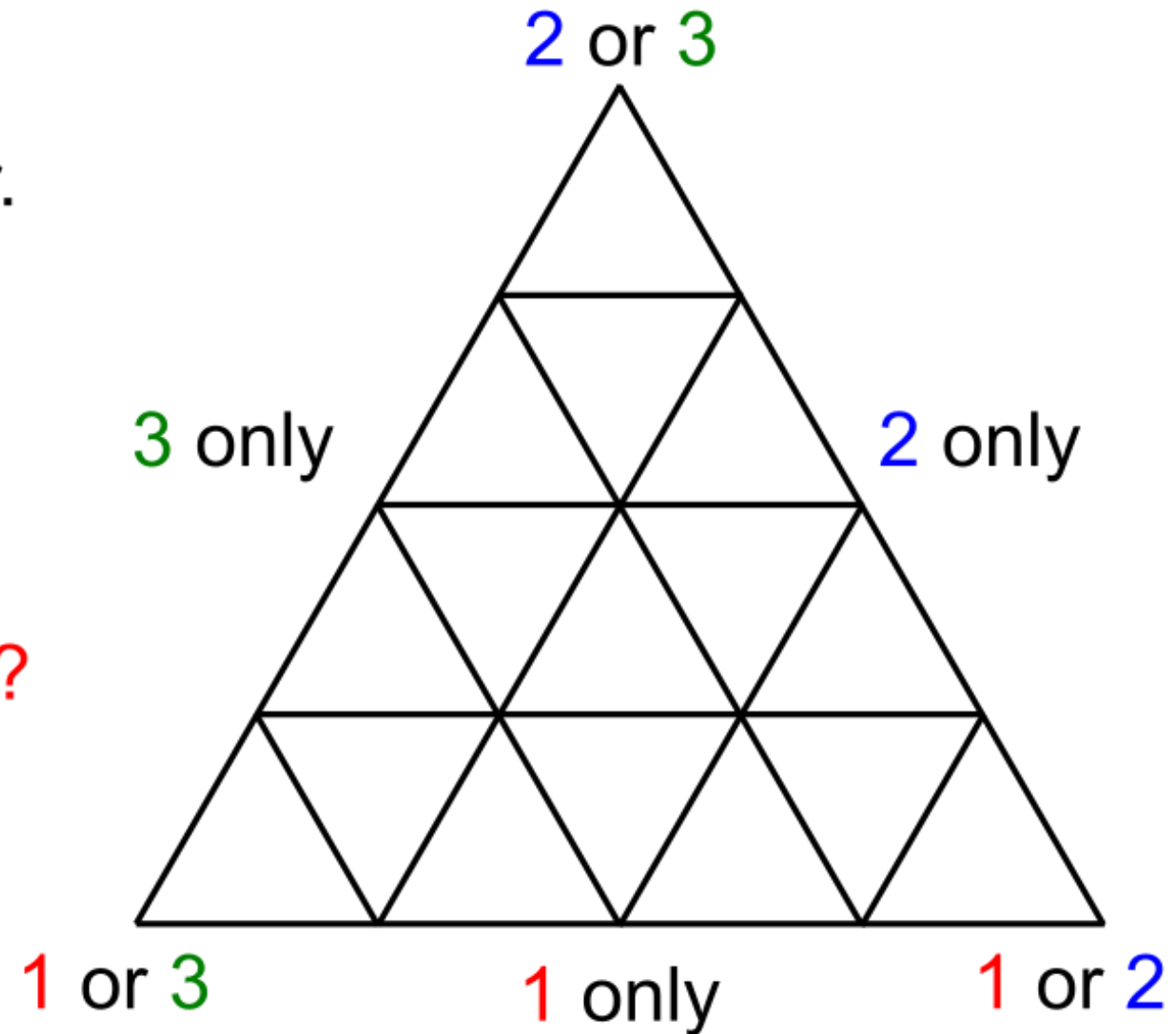
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agent C prefers room 1

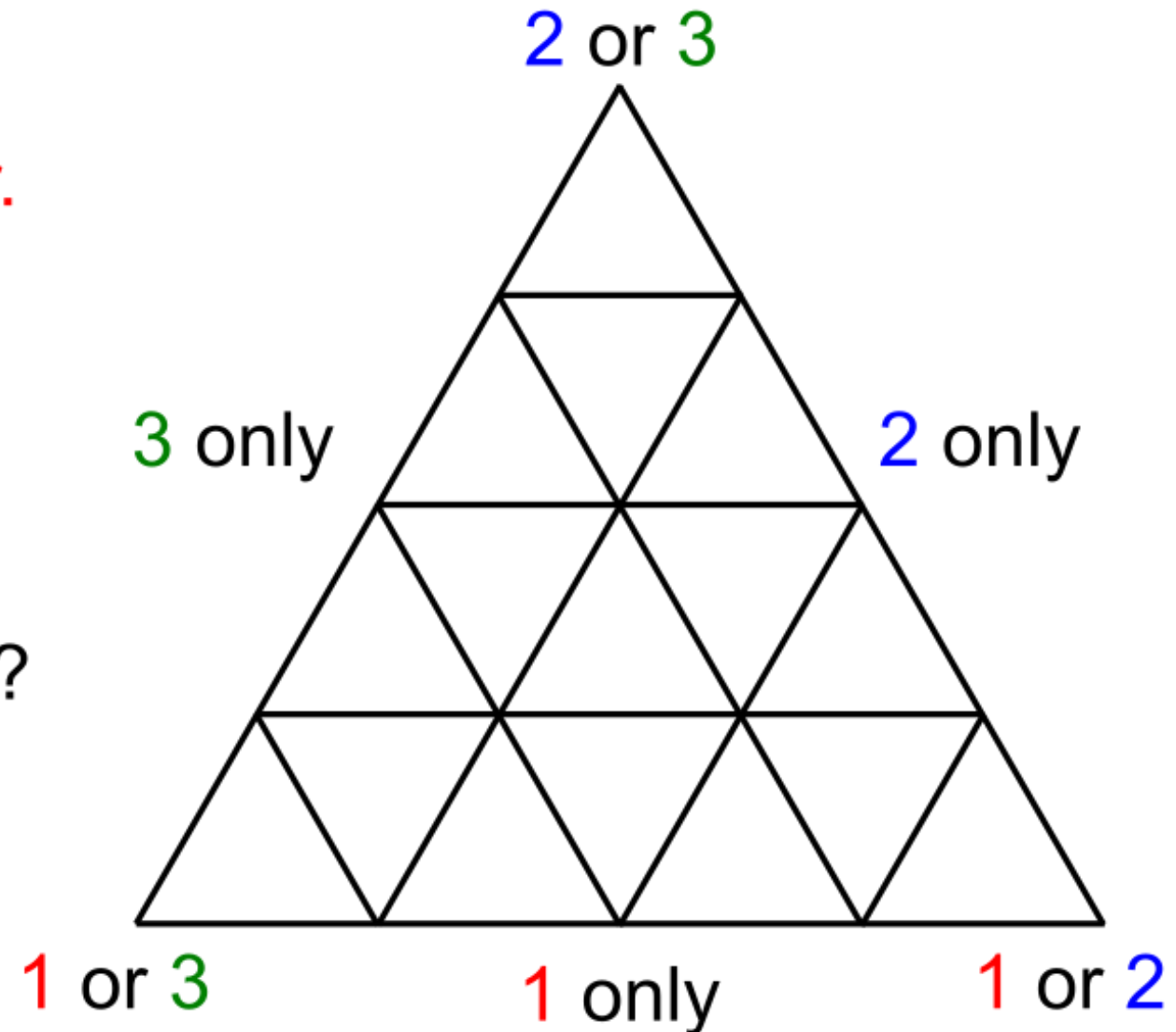
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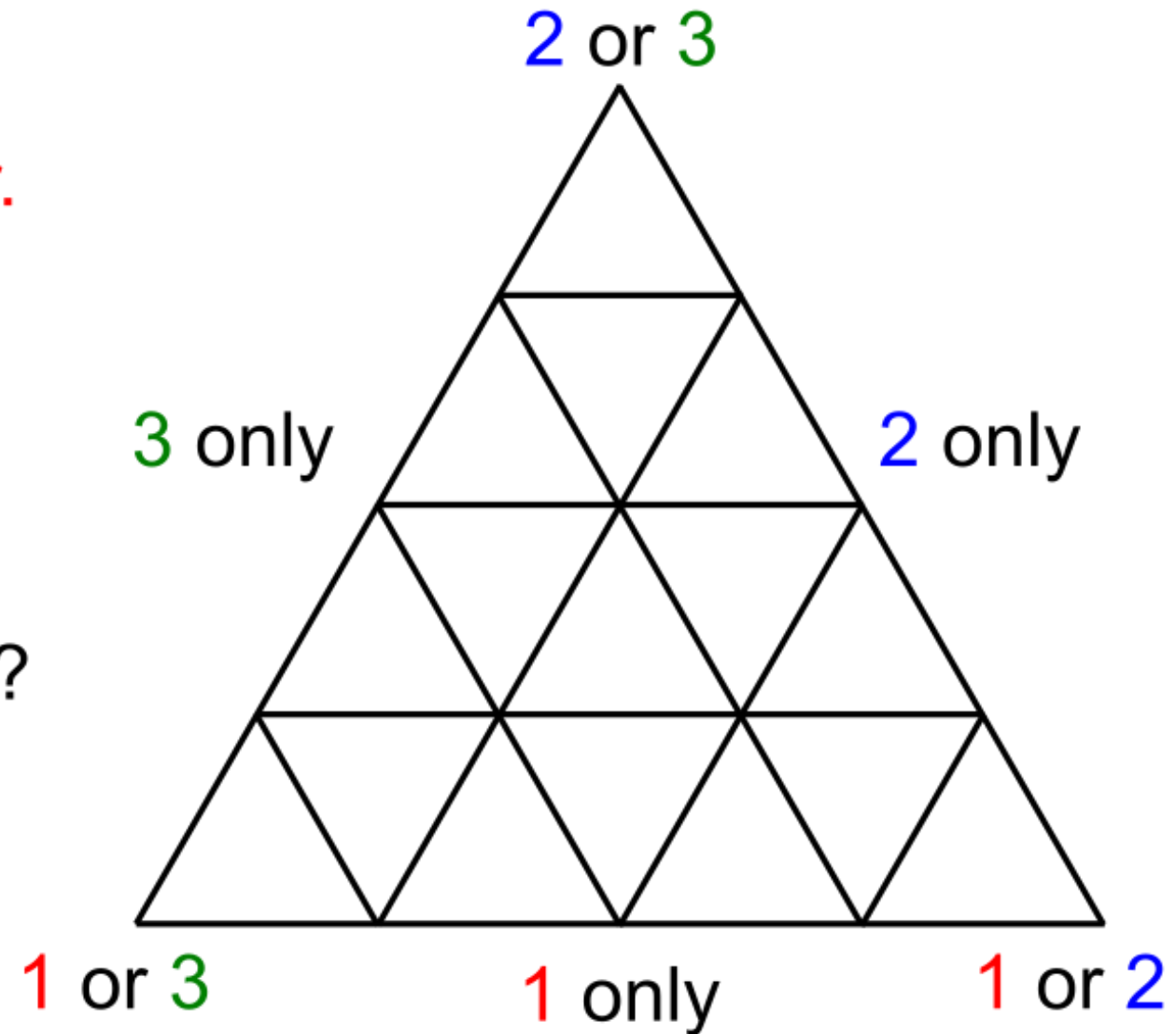
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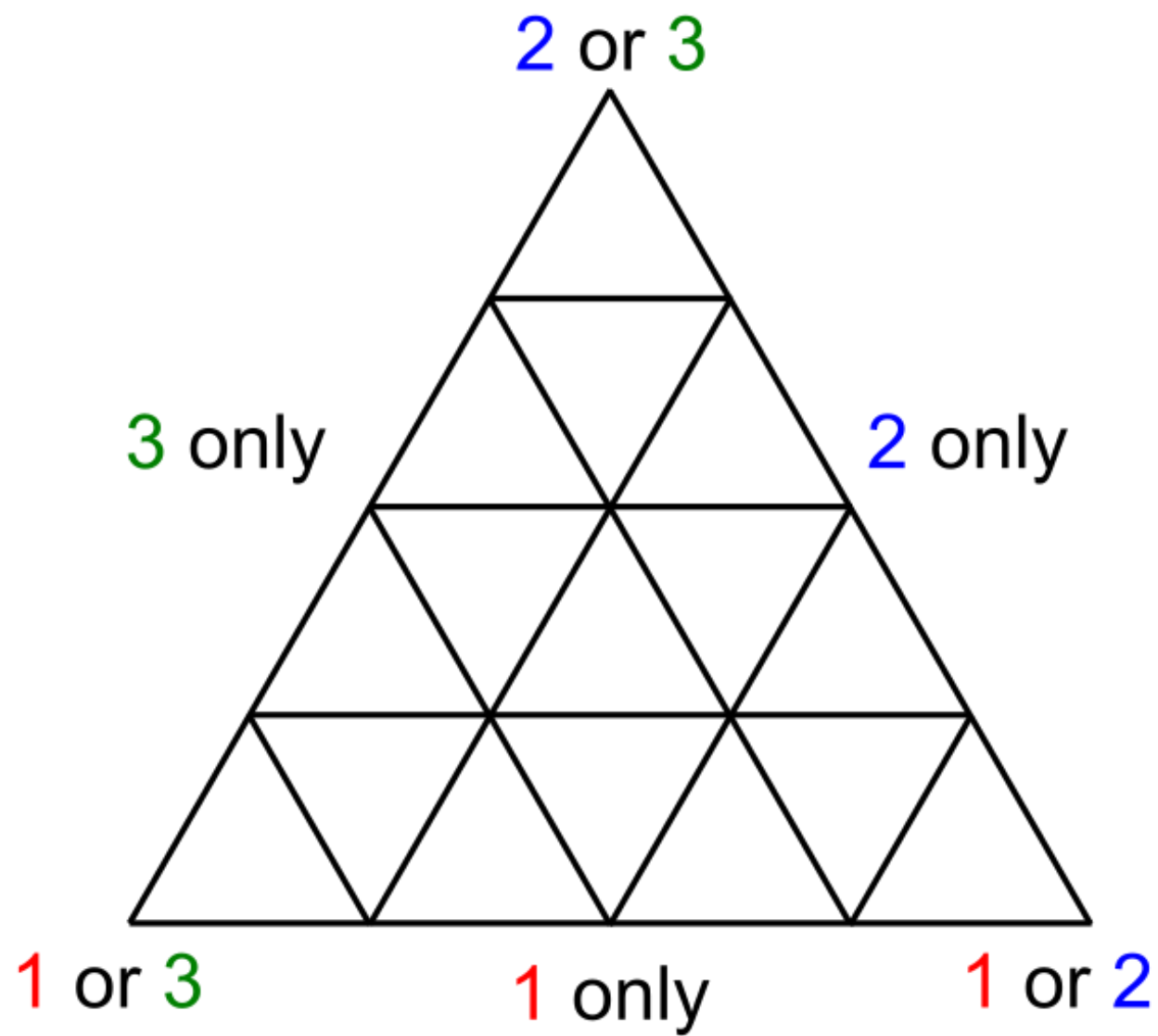
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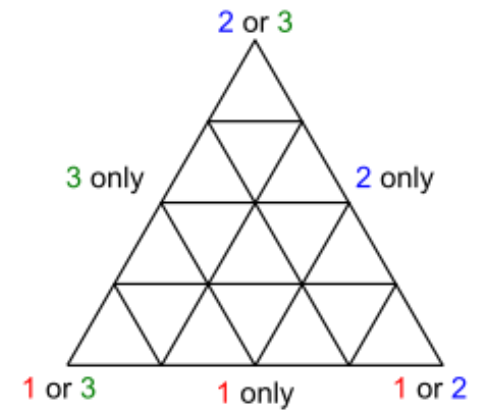


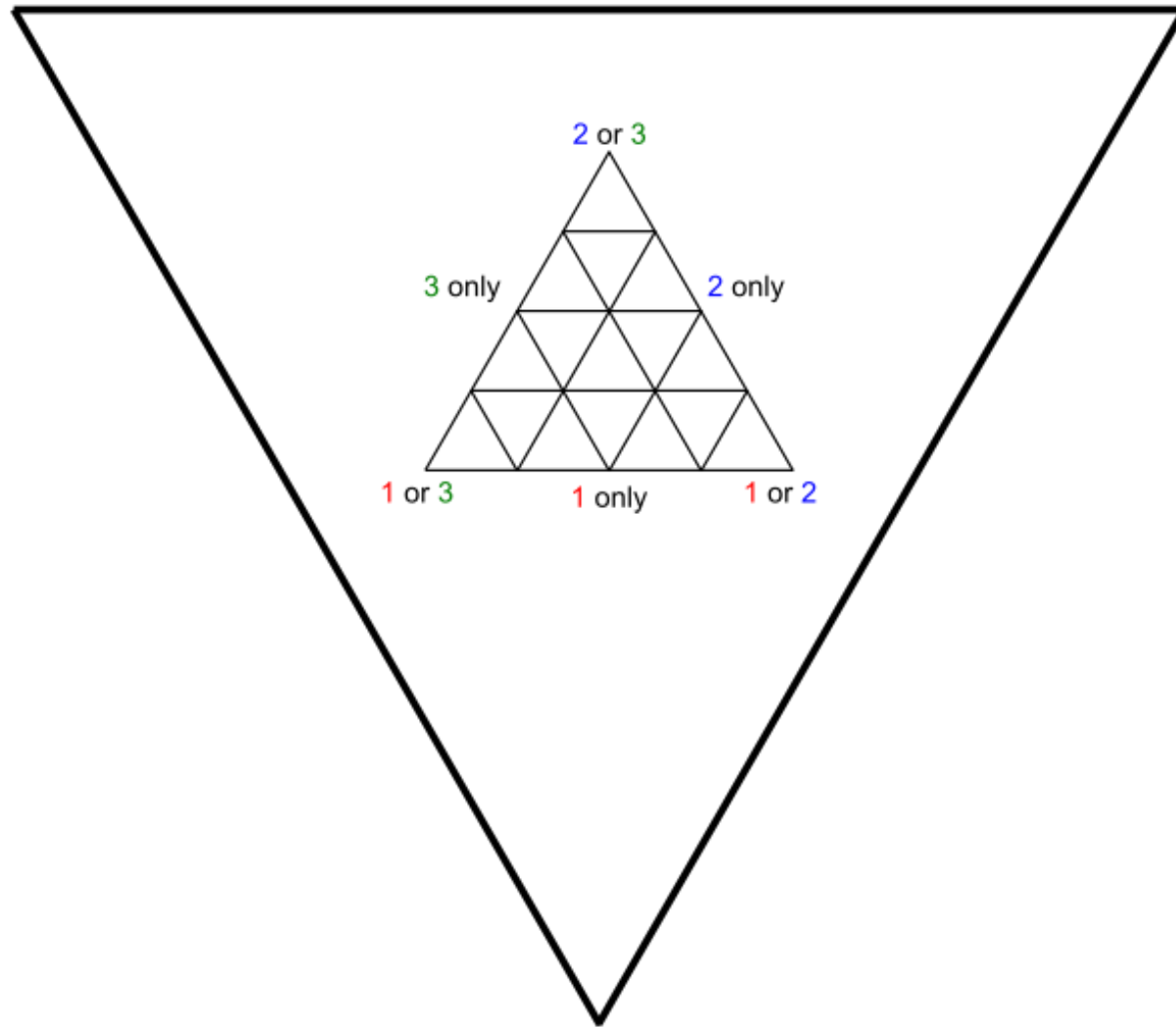
This looks nothing like Sperner.
Think outside the box!

Why is a fully labeled triangle
useful for envy-free rent division?



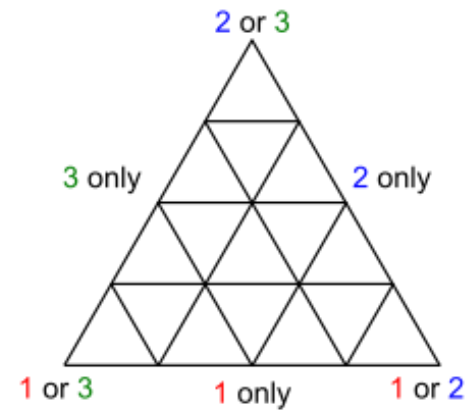






3

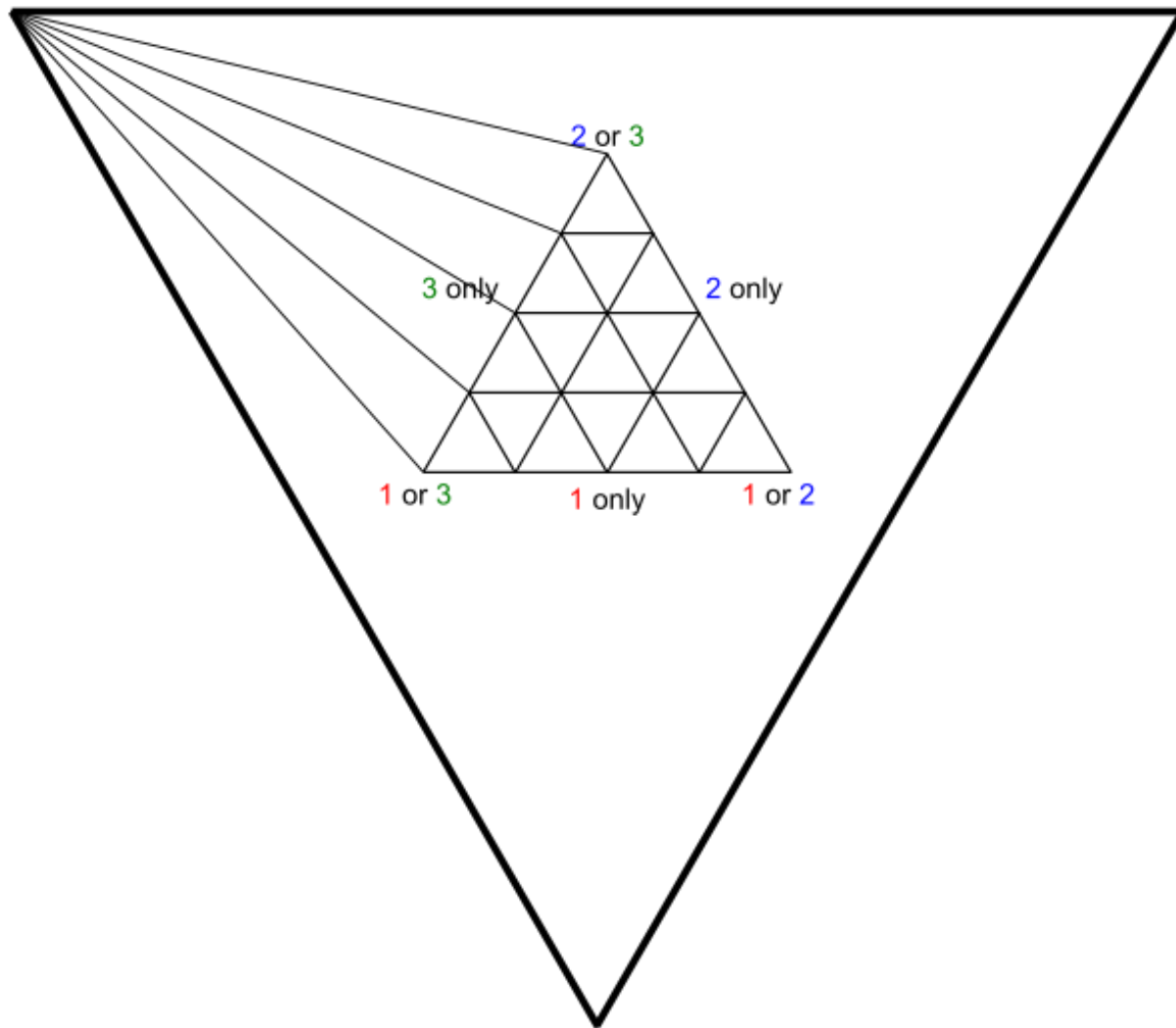
2



1

3

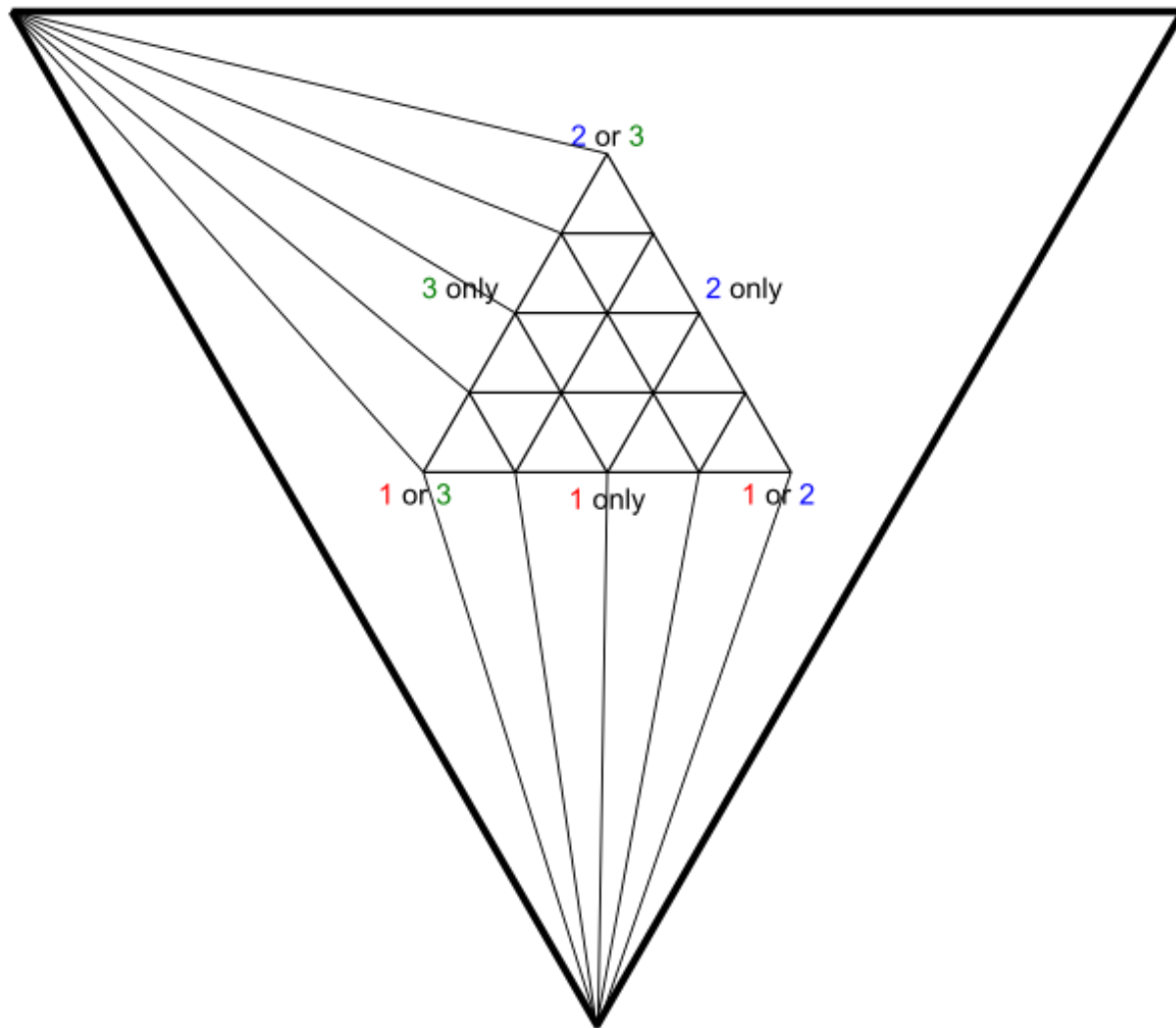
2



1

3

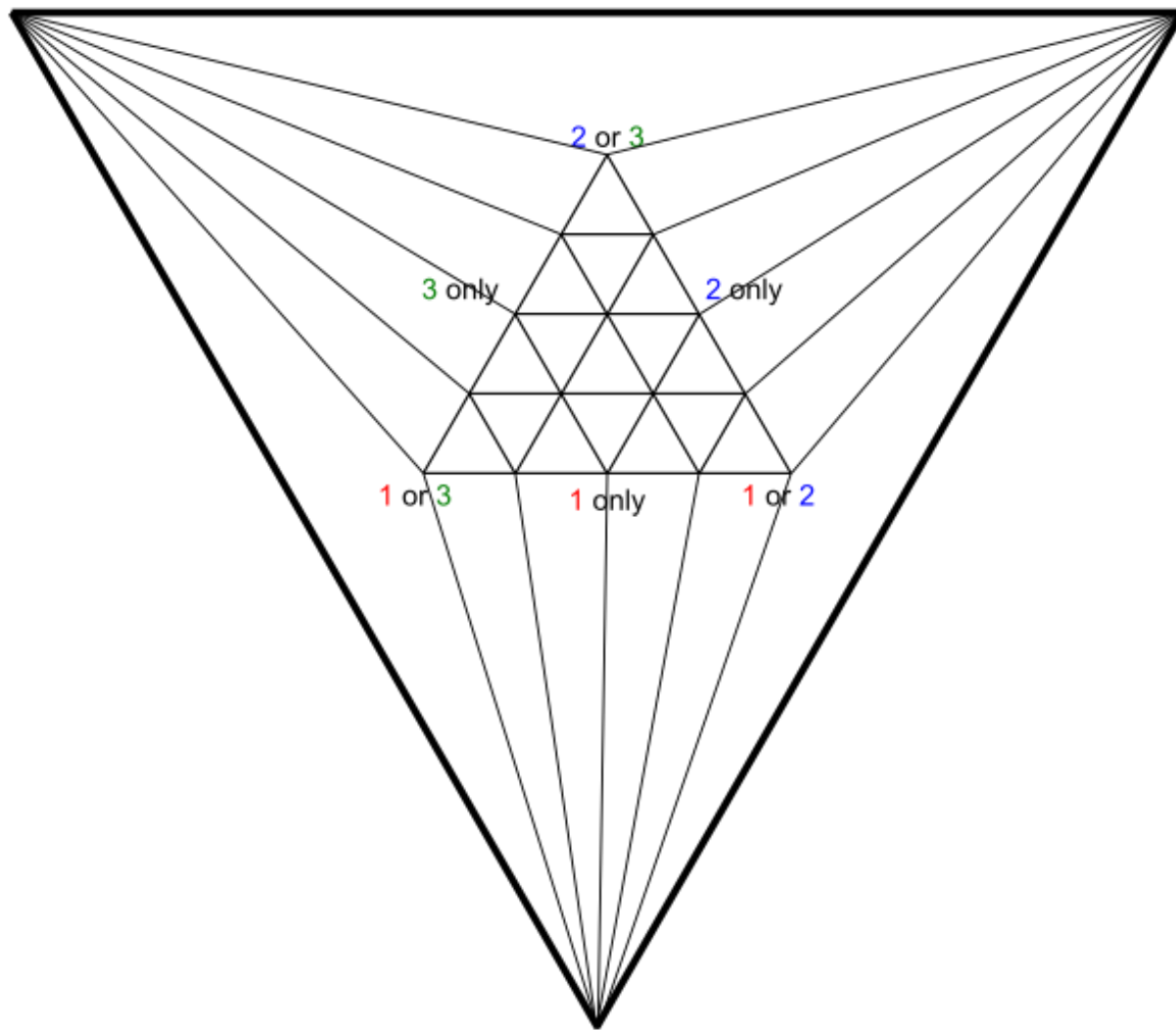
2



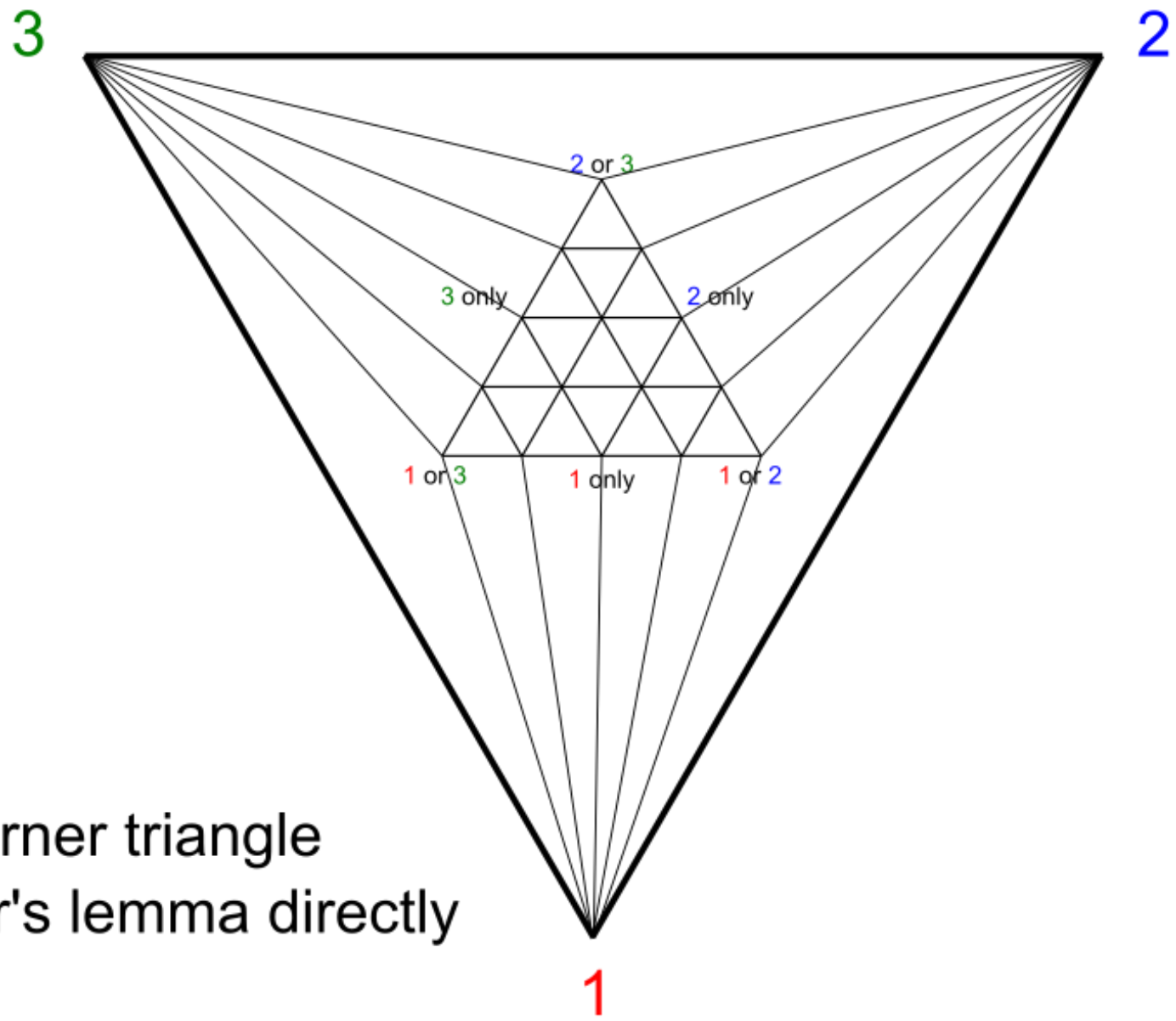
1

3

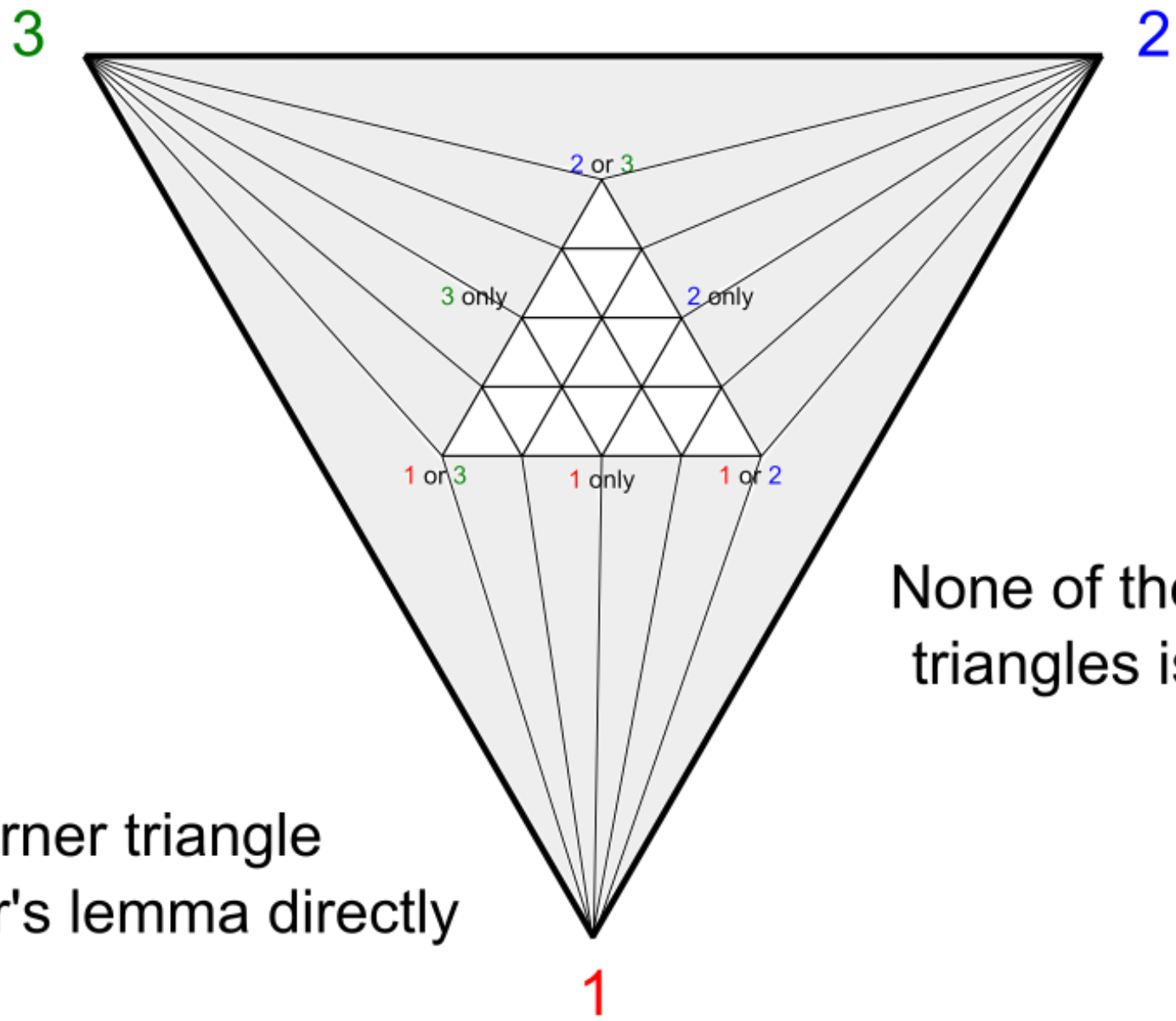
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1

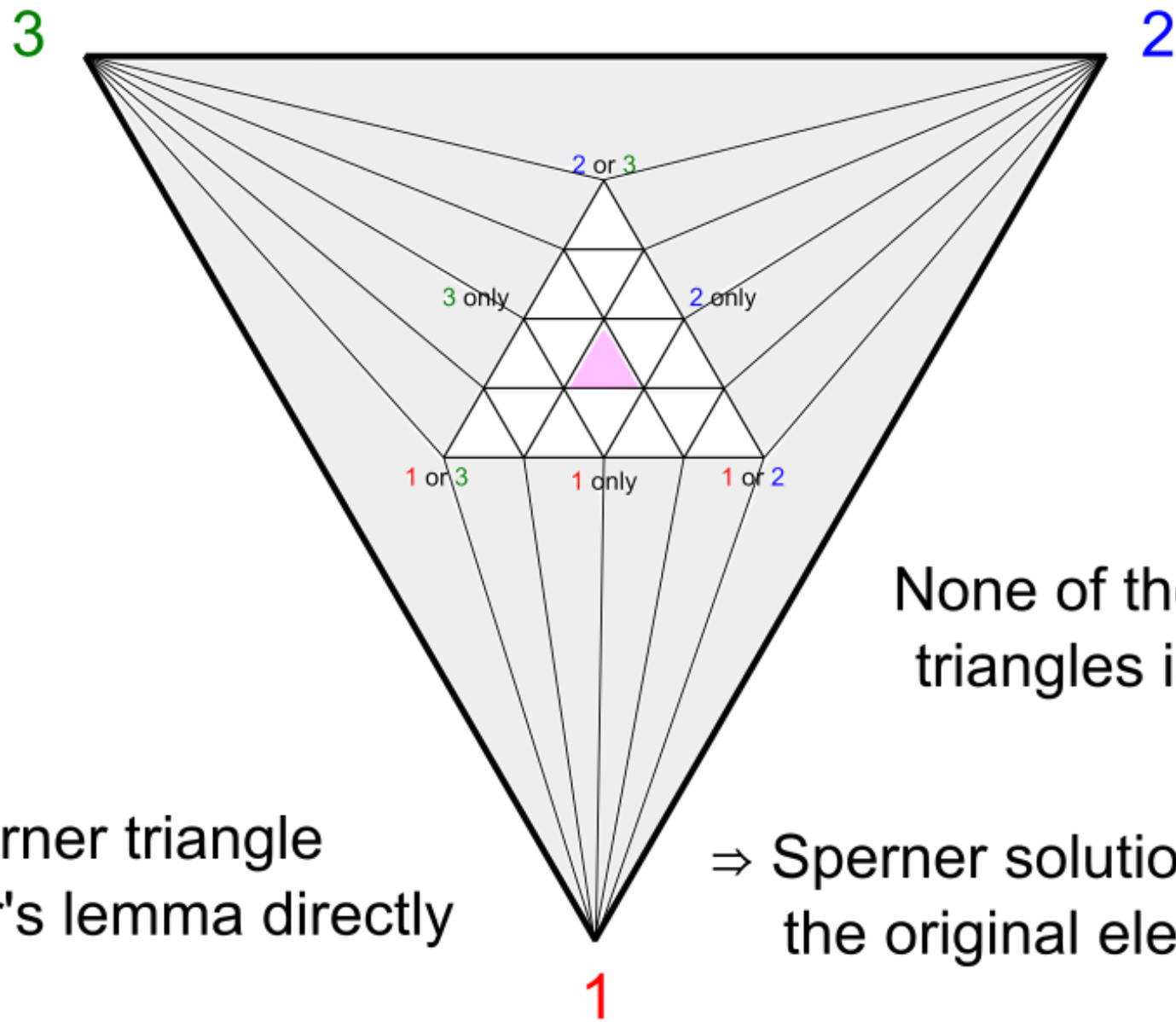


A valid Sperner triangle
⇒ apply Sperner's lemma directly



None of the newly added triangles is fully labeled

A valid Sperner triangle
⇒ apply Sperner's lemma directly



None of the newly added triangles is fully labeled

A valid Sperner triangle
 ⇒ apply Sperner's lemma directly

⇒ Sperner solution must be one of the original elementary triangles

1

Cake Cutting

Rental harmony
via
Sperner's lemma



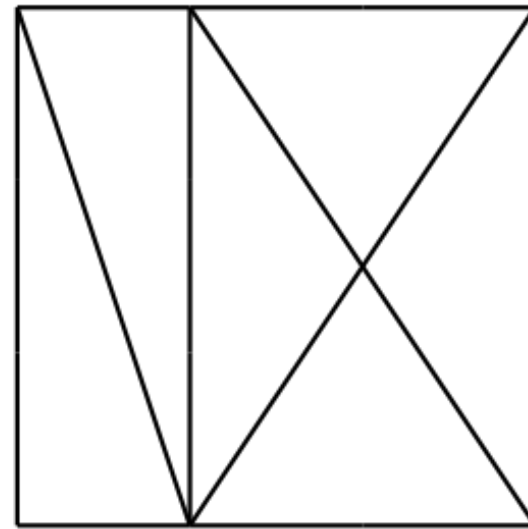
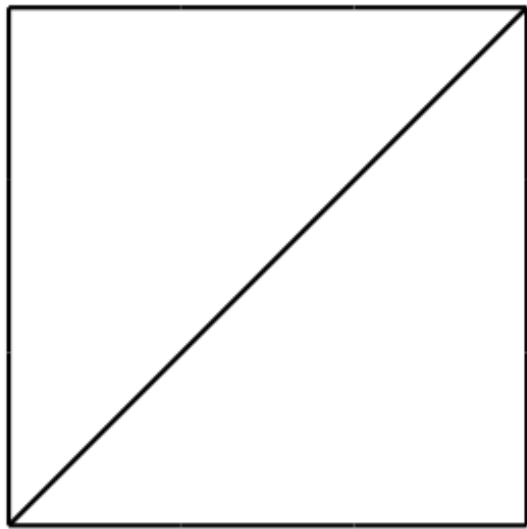
Fair cake cutting
via
Sperner's lemma
(next lecture)

Rental harmony
via
Sperner's lemma

Other Applications of Sperner's Lemma

Monsky's theorem


A square cannot be dissected into an odd number of triangles with equal area.



Brouwer's fixed point theorem

Every continuous function from a compact convex set to itself has a fixed point.

Brouwer's fixed point theorem

$$f(x) = x$$


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References

- Fair cake cutting and rent division via Sperner's Lemma

Francis Su

“Rental Harmony: Sperner's Lemma in Fair Division”

American Mathematical Monthly, 106, 1999 pg 930-942

- Fun videos on the topic:

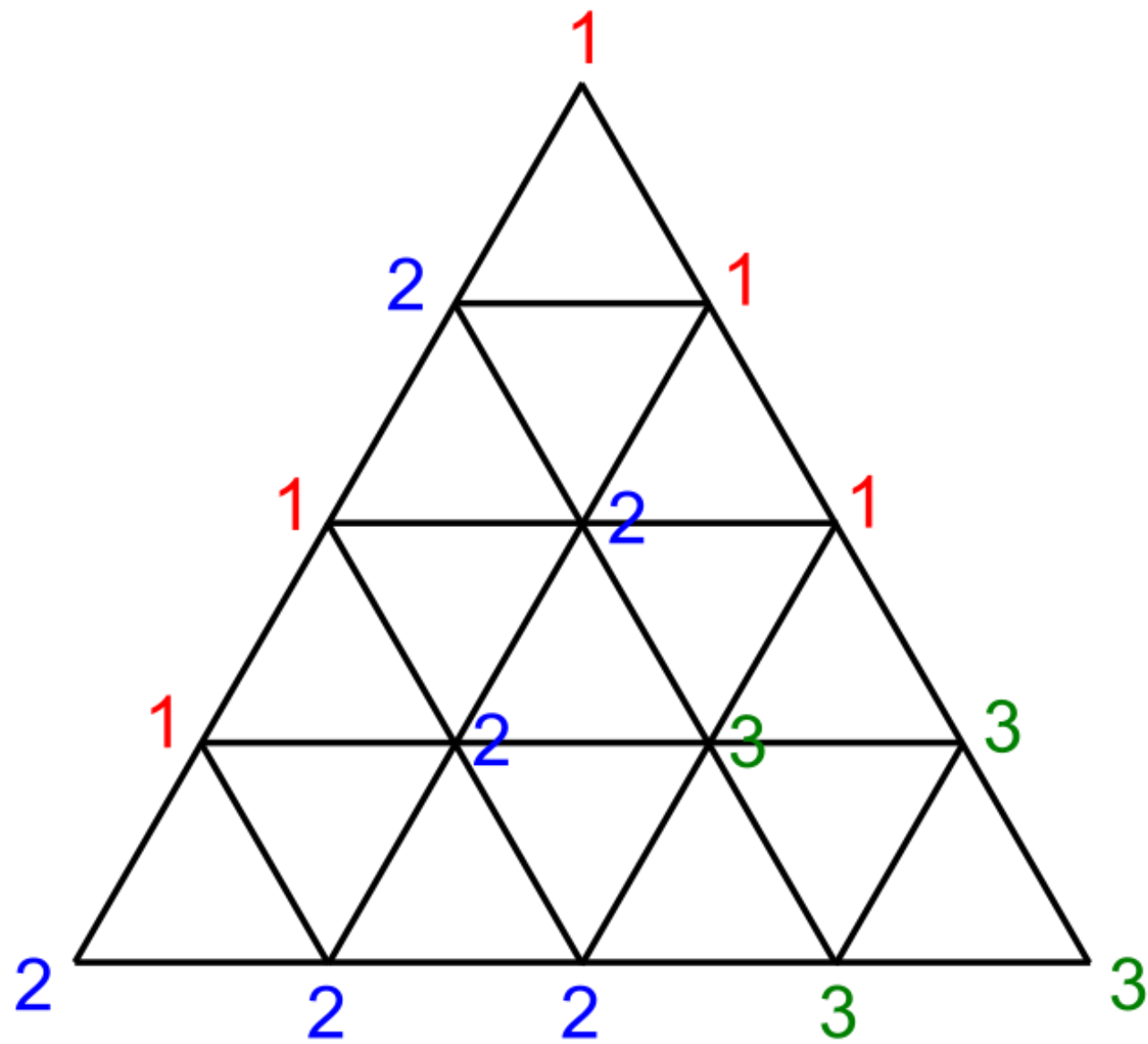
Mathologer: <https://www.youtube.com/watch?v=7s-YM-kcKME>

MoMath: <https://www.youtube.com/watch?v=CBVg8x4LWZE>

PBS Infinite Series: <https://www.youtube.com/watch?v=48oBEvpdYSE>

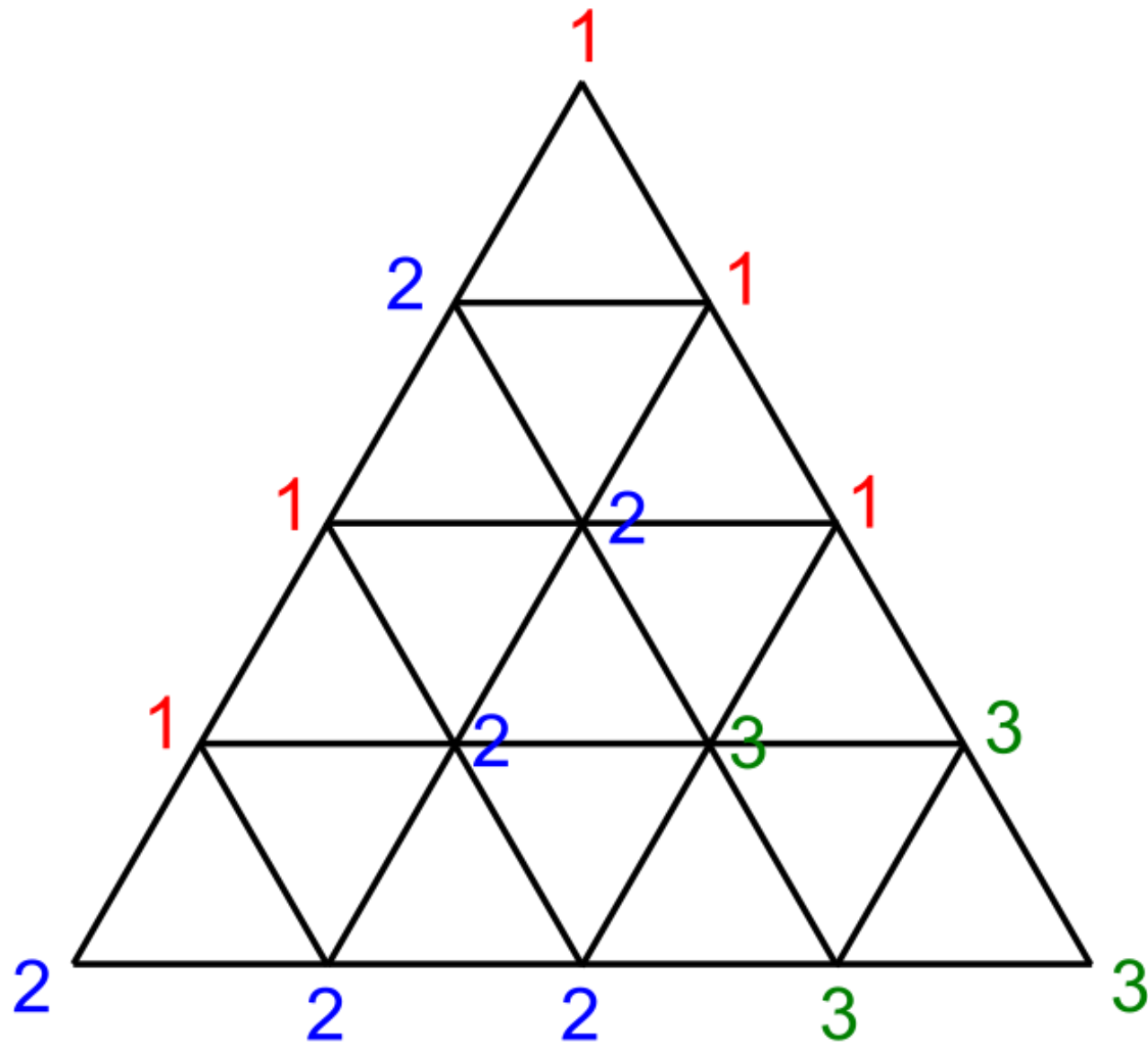
Proof Sketch of Sperner's Lemma

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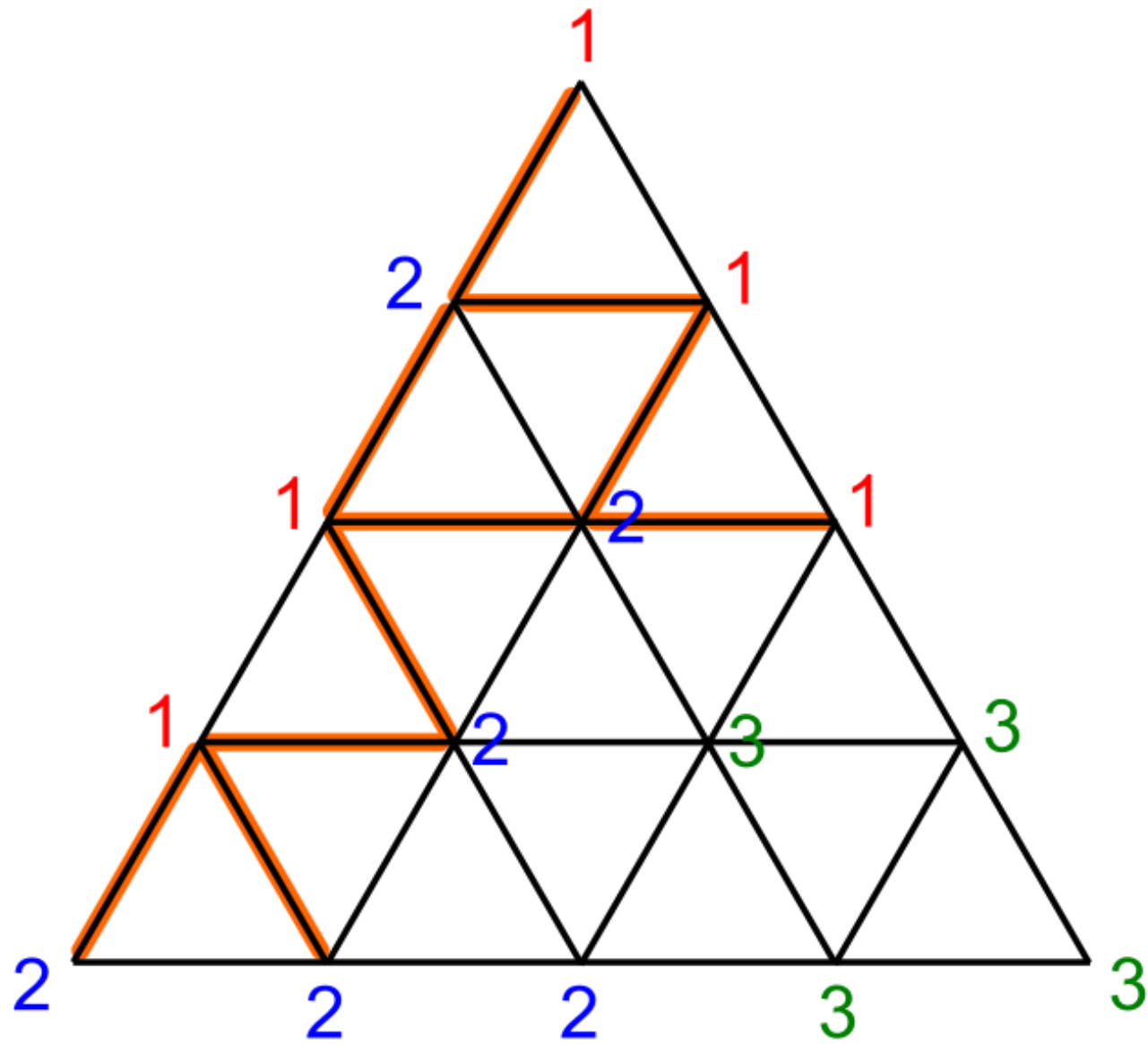
Proof Sketch of Sperner's Lemma

Call any 1—2 edge a "door".

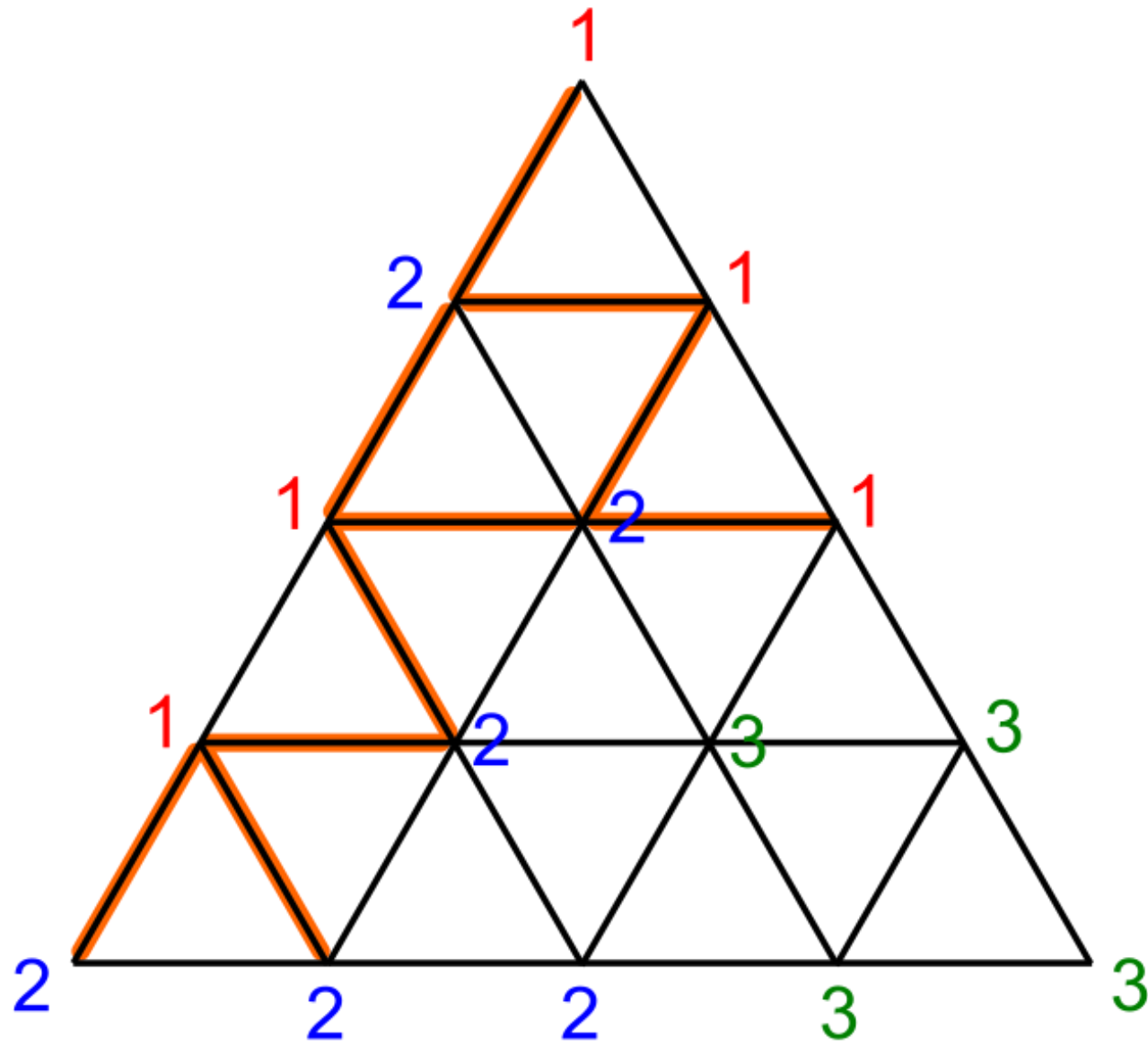


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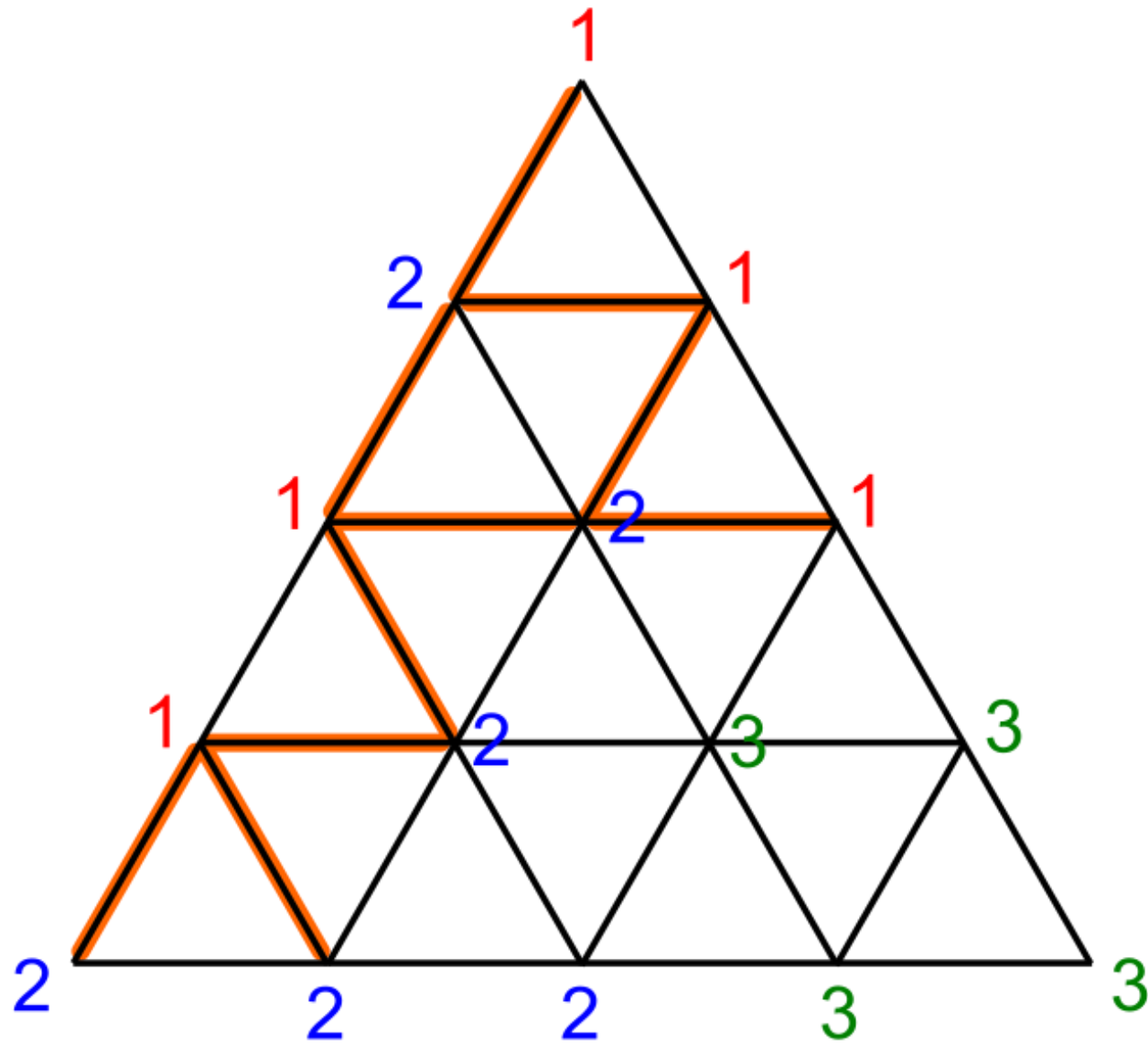
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Call any elementary triangle a "room".

Proof Sketch of Sperner's Lemma



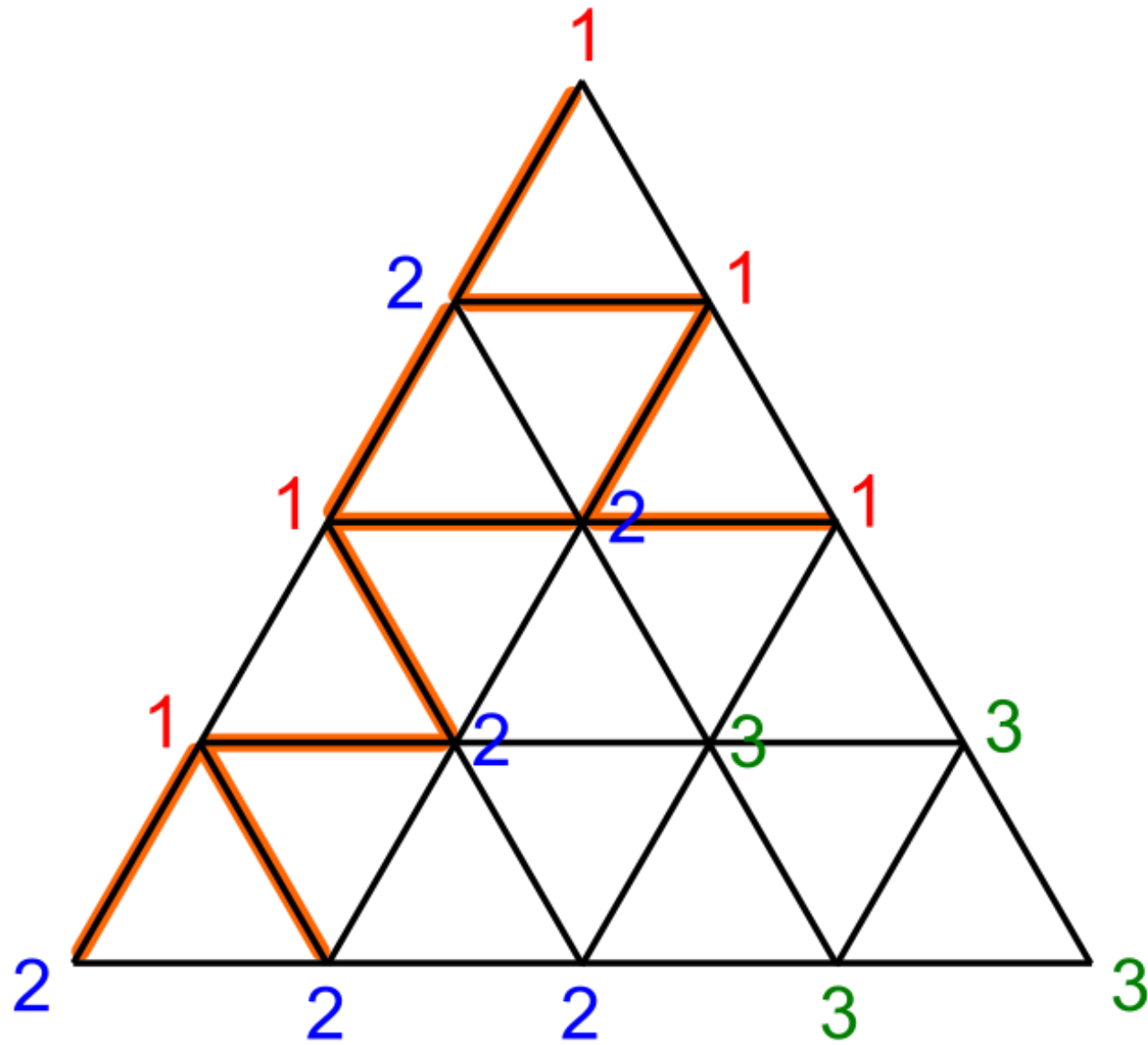
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Observe:

- No. of doors on the boundary is odd.

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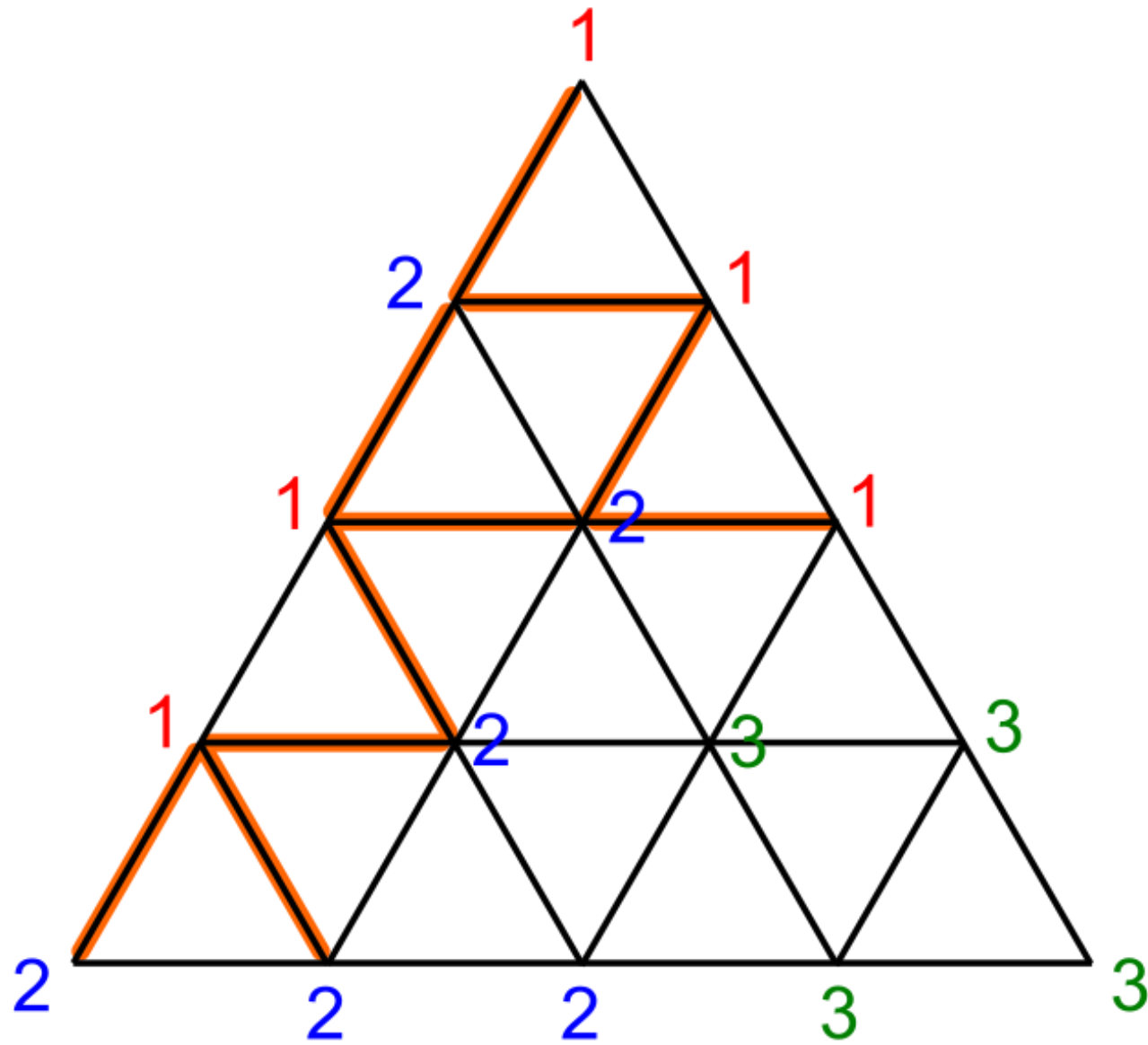
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Proof Sketch of Sperner's Lemma

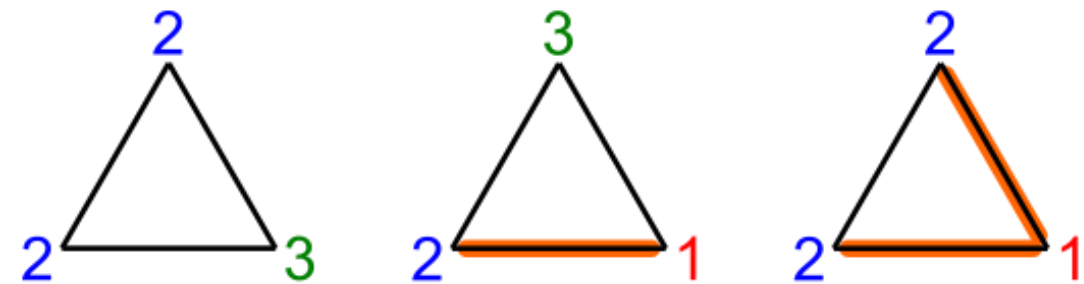


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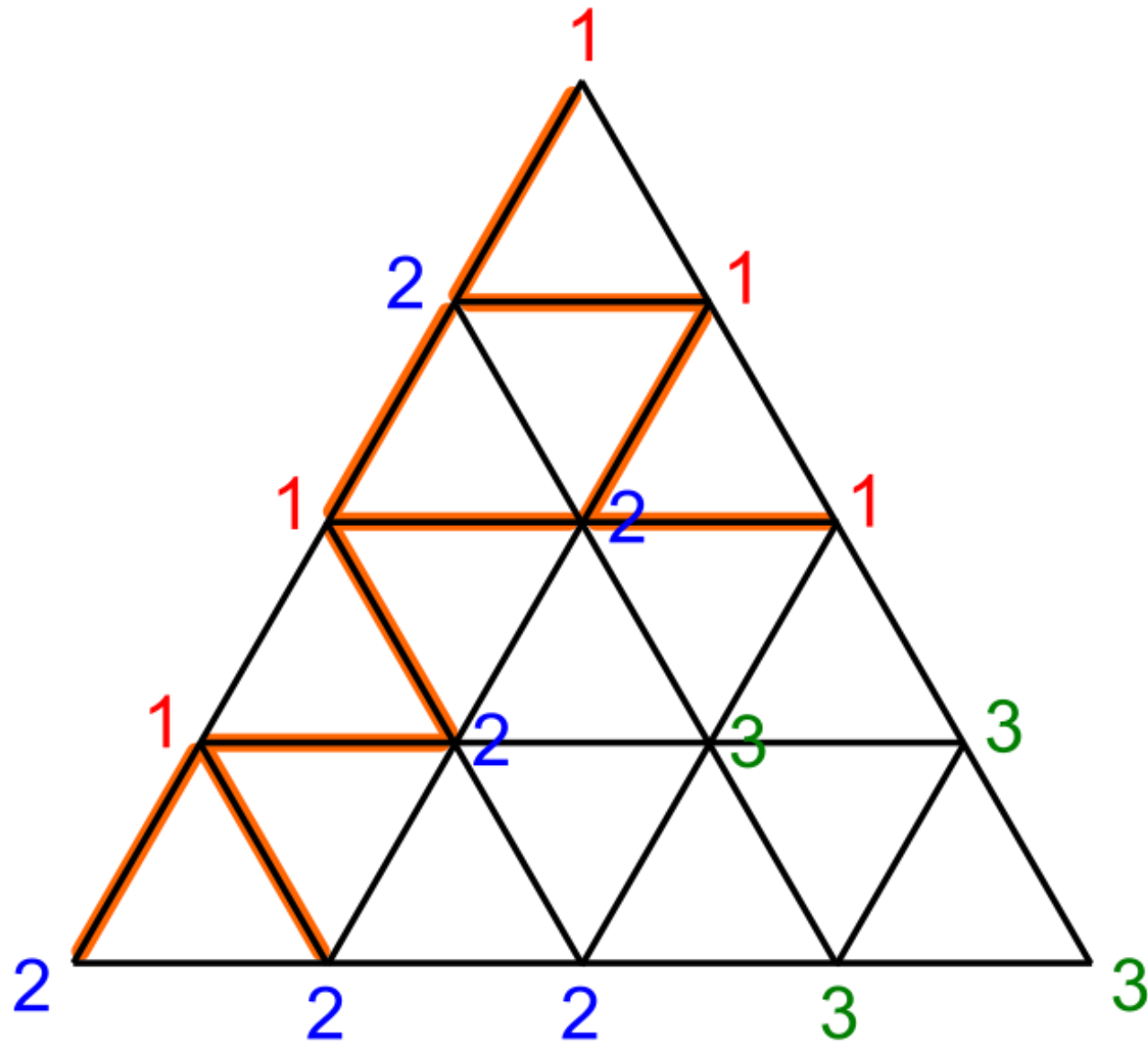
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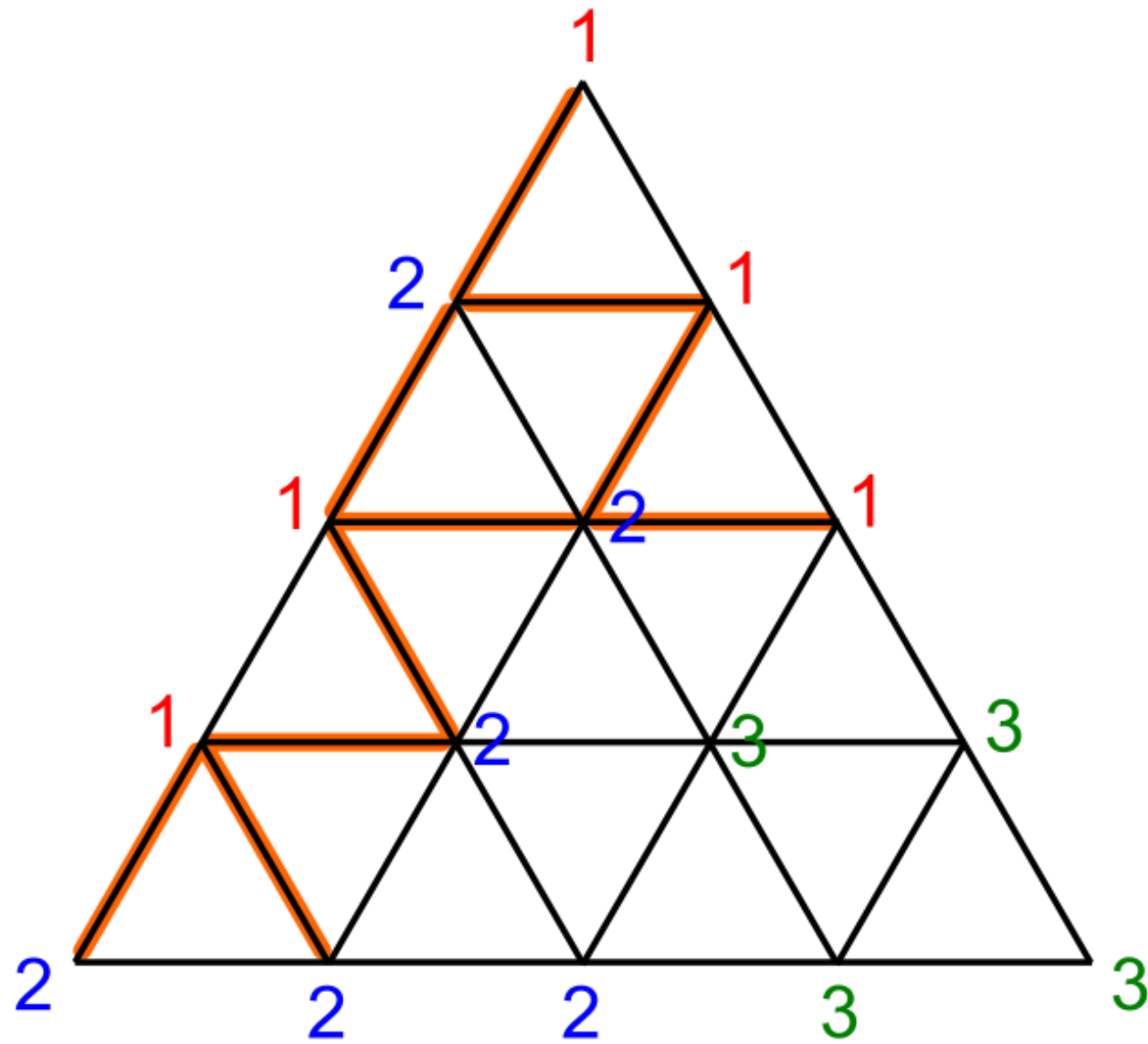
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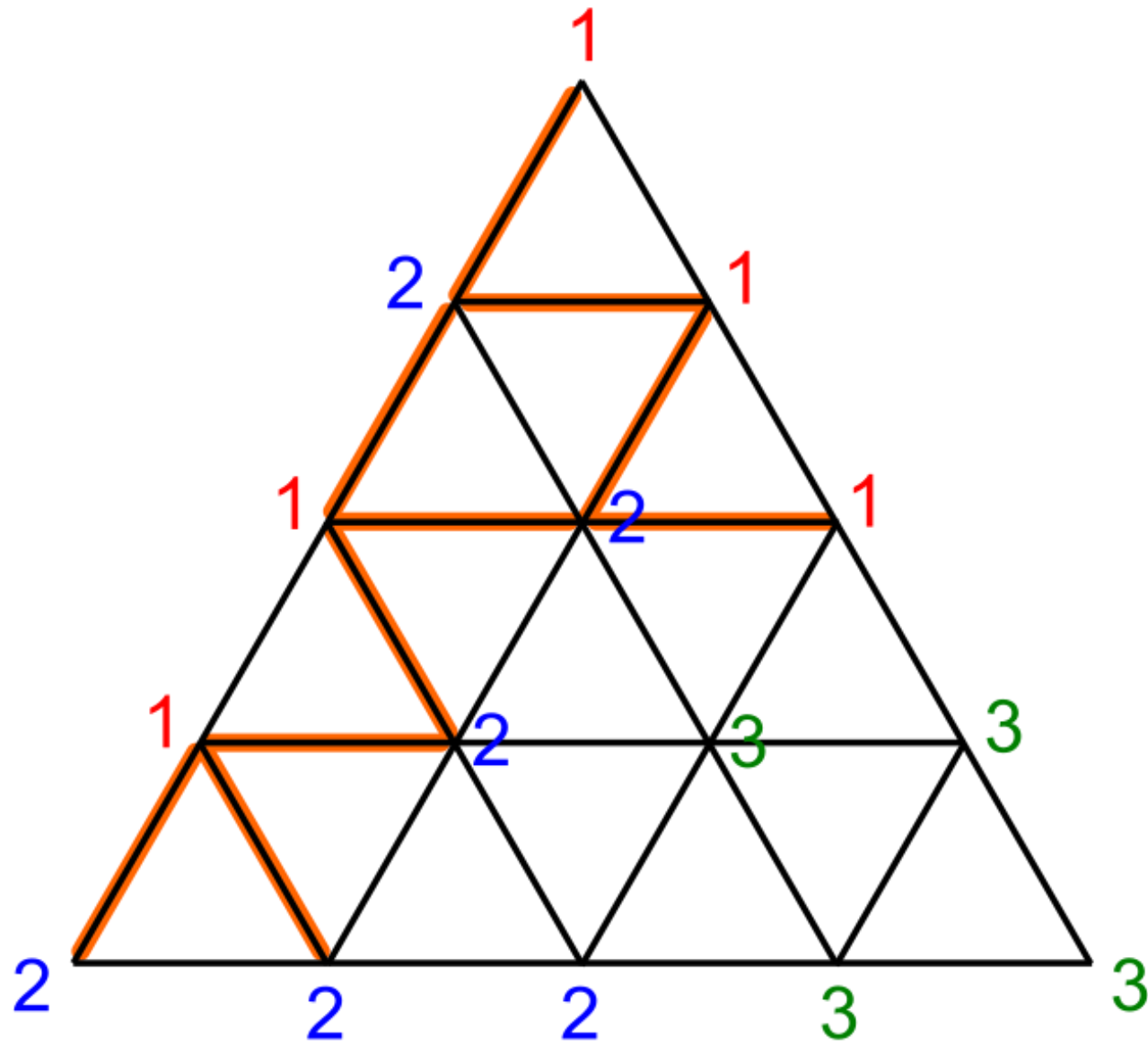
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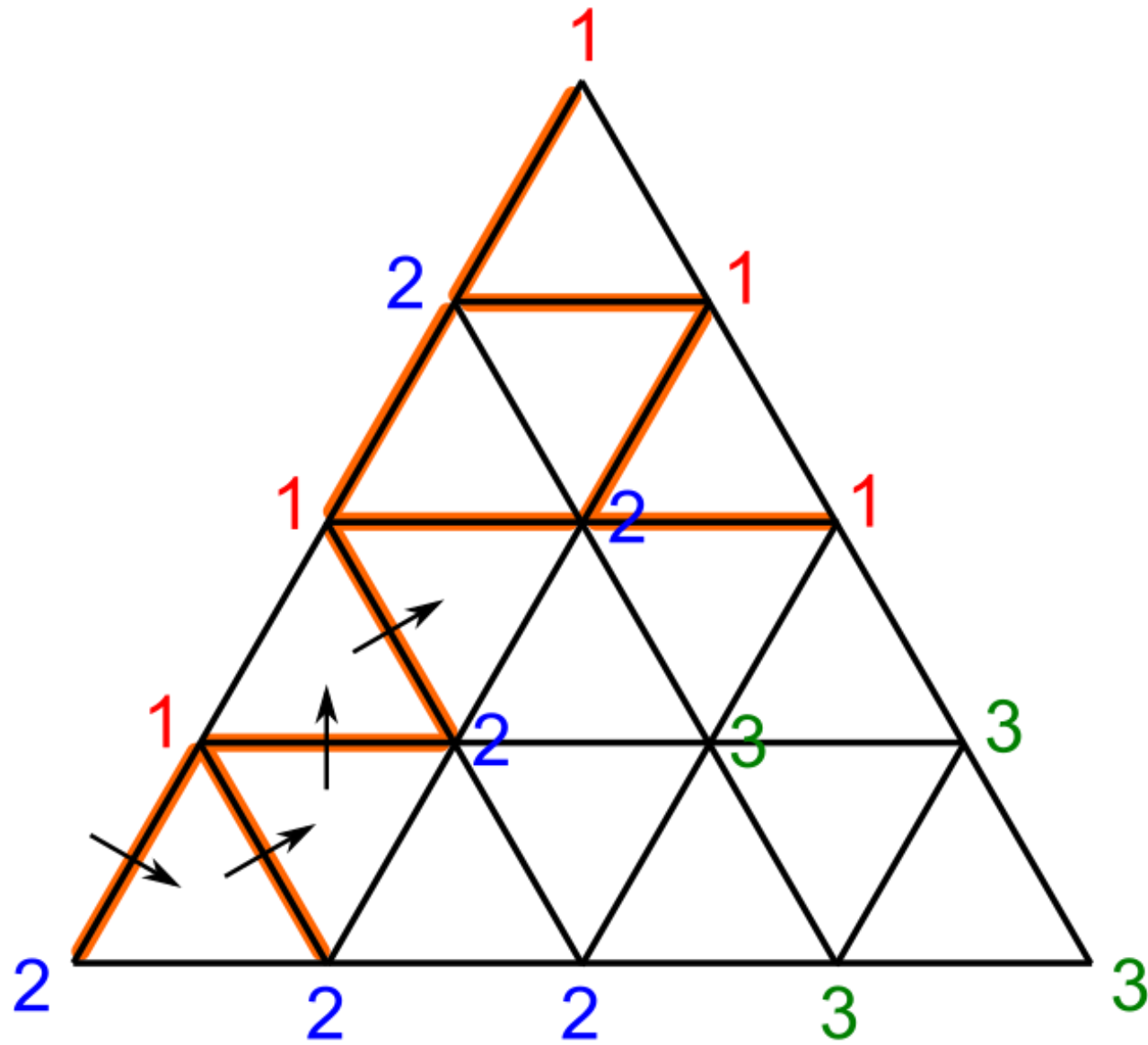


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Start walking!

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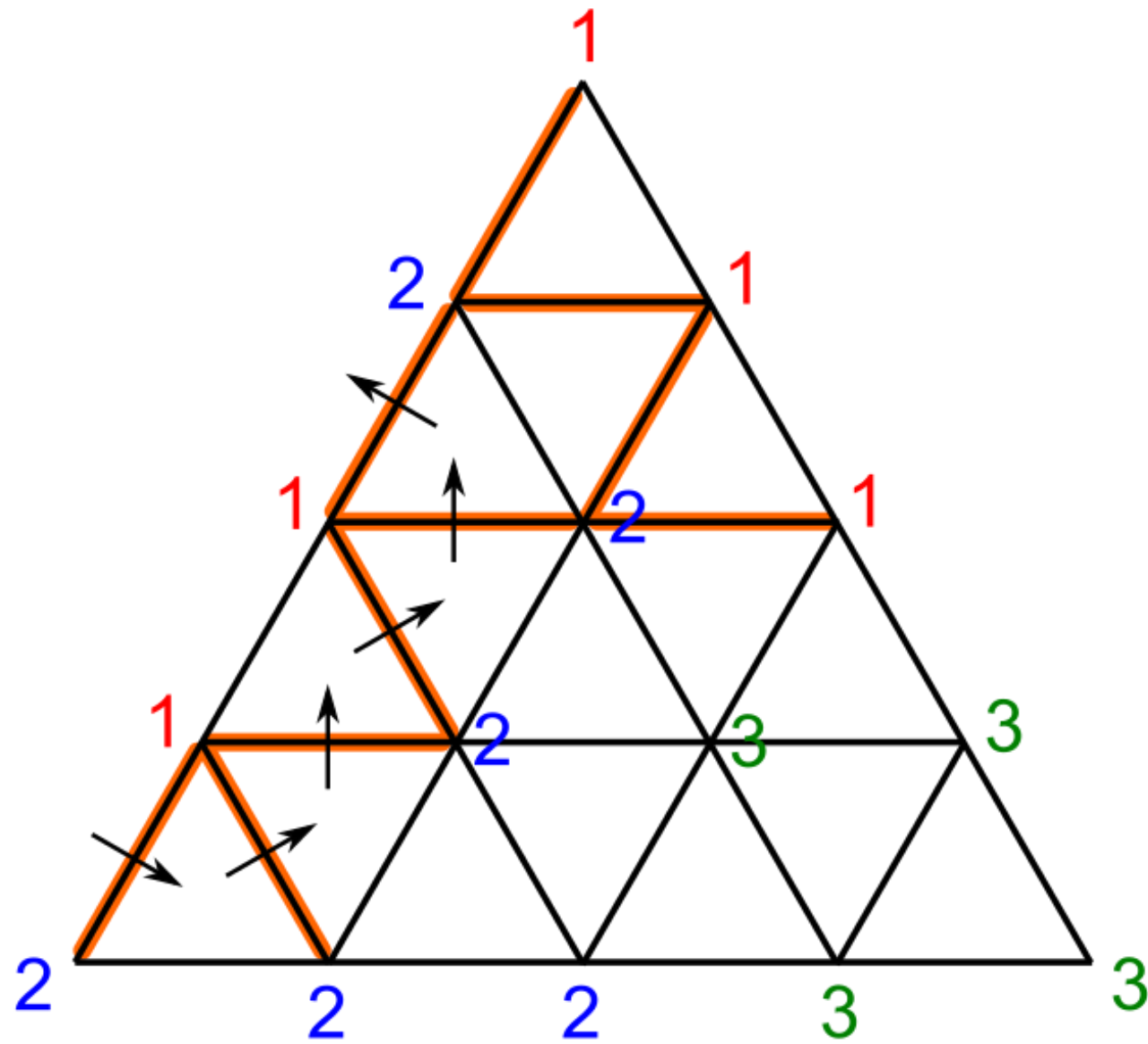


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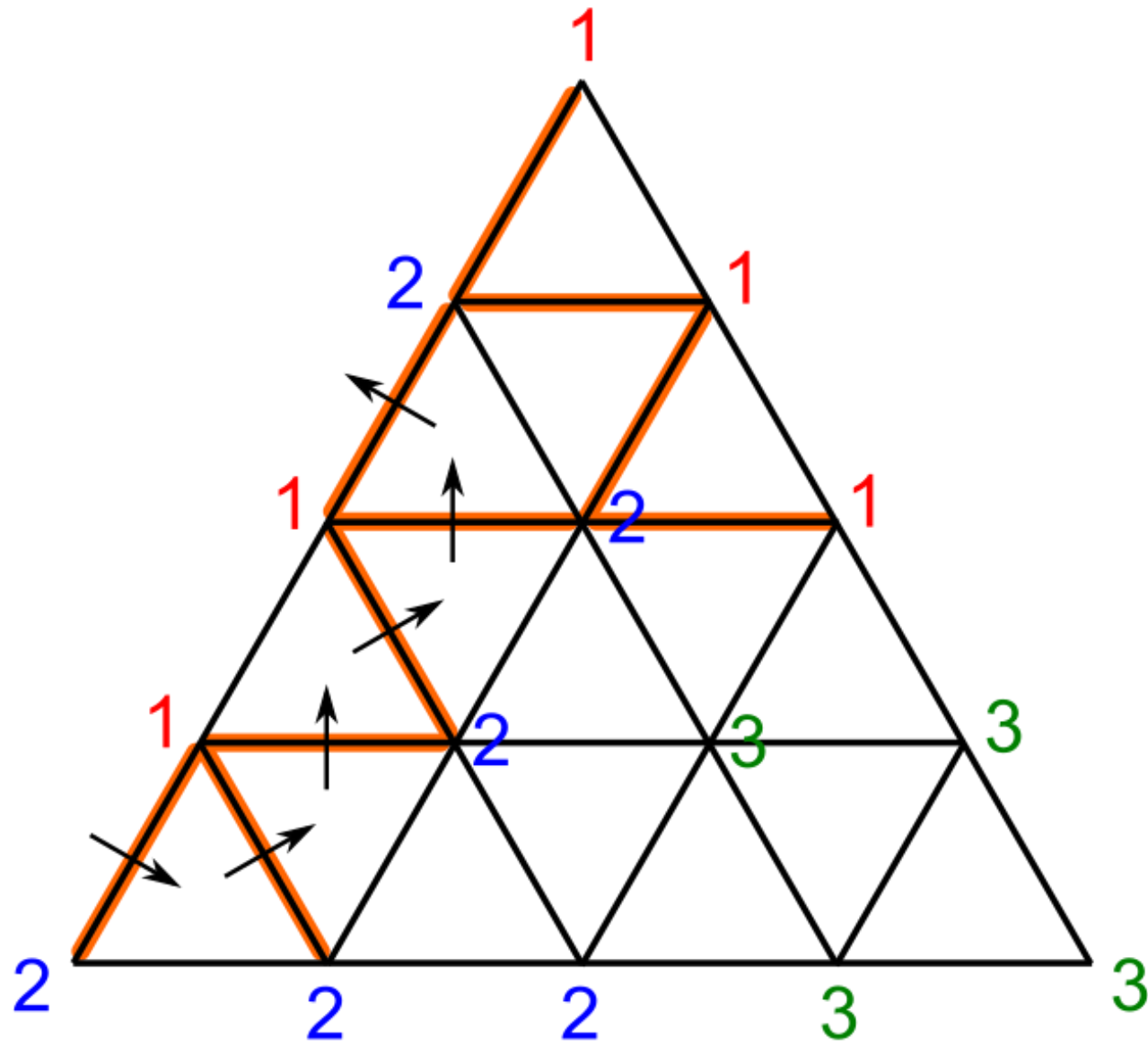


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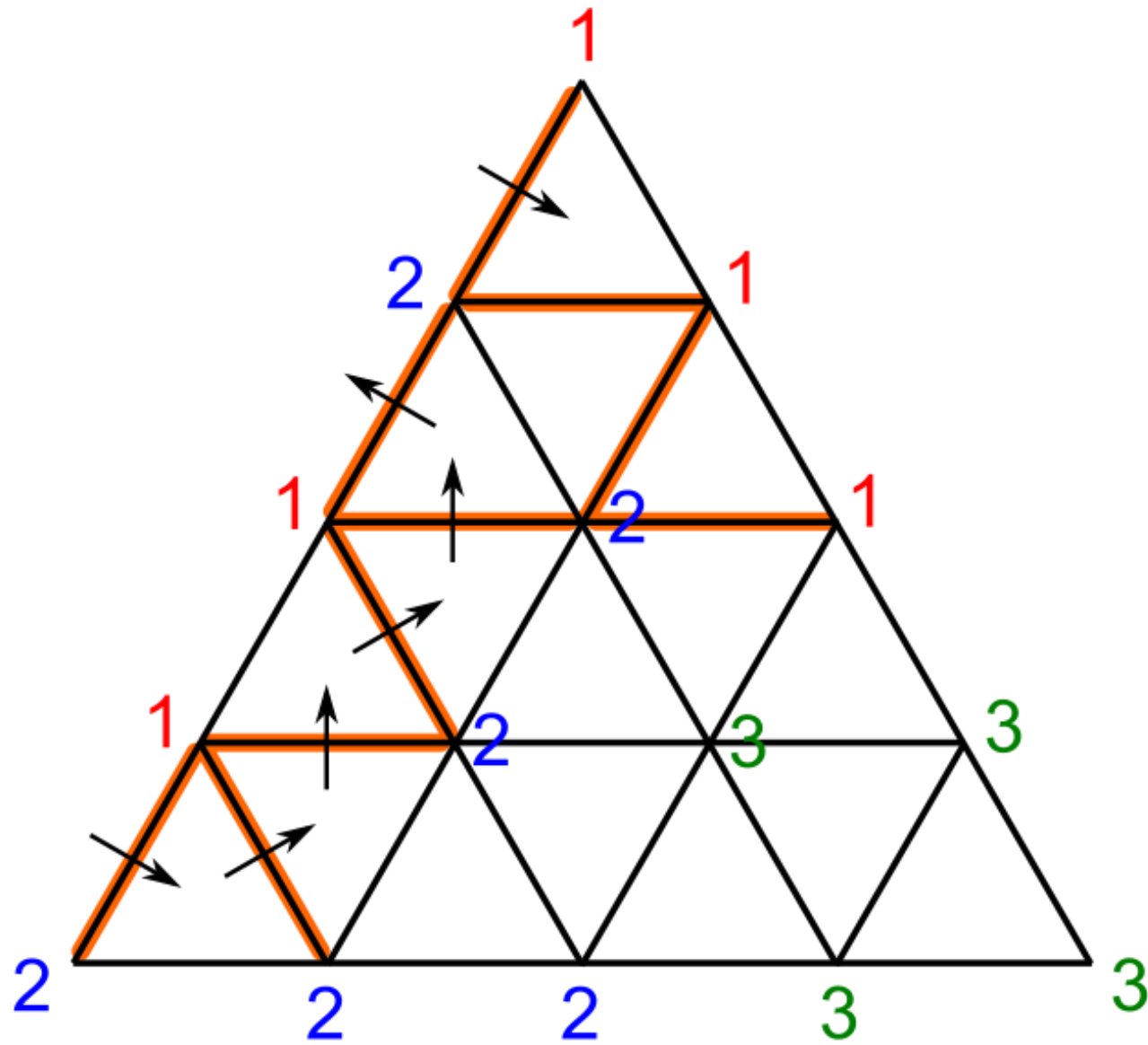
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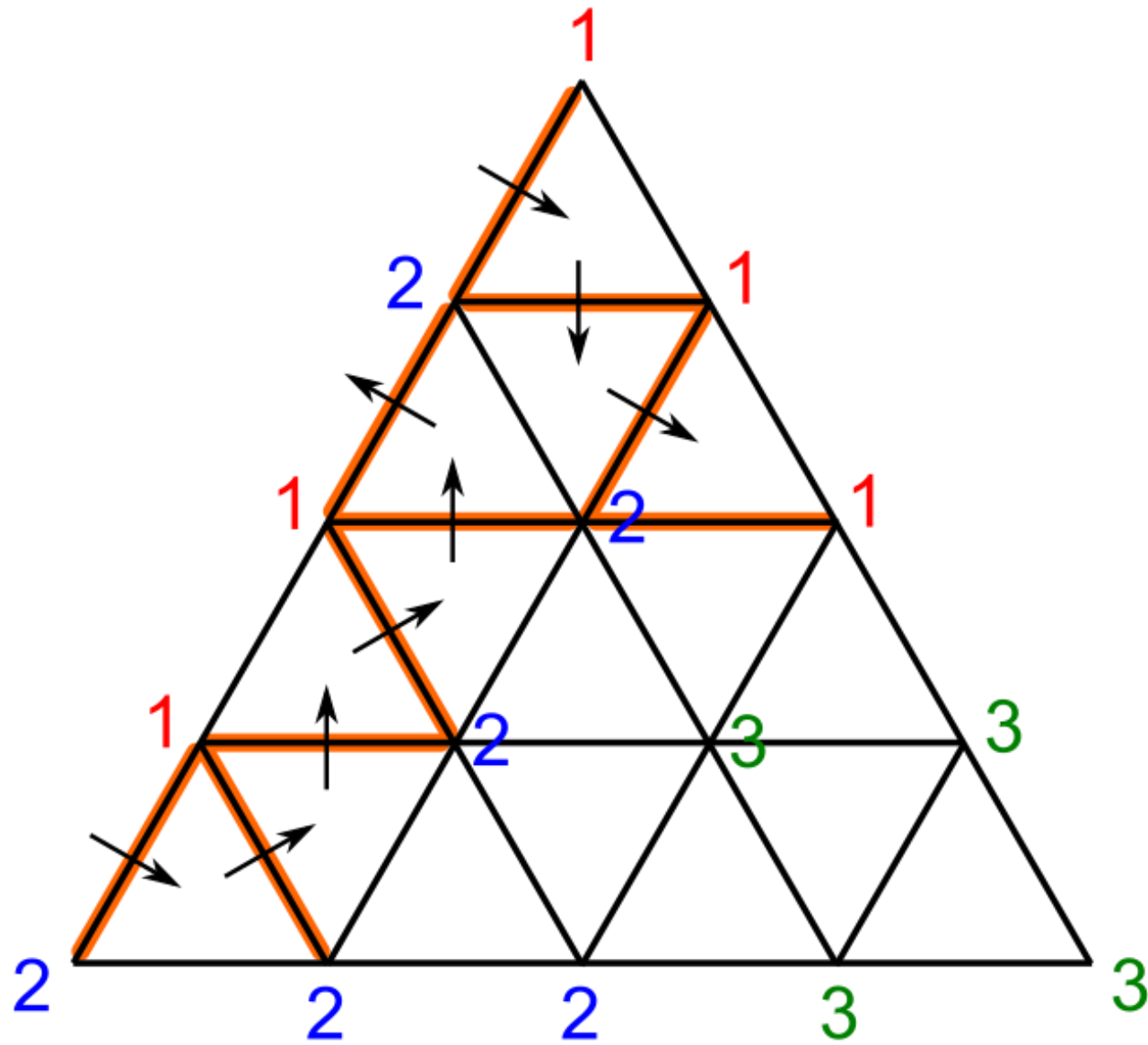
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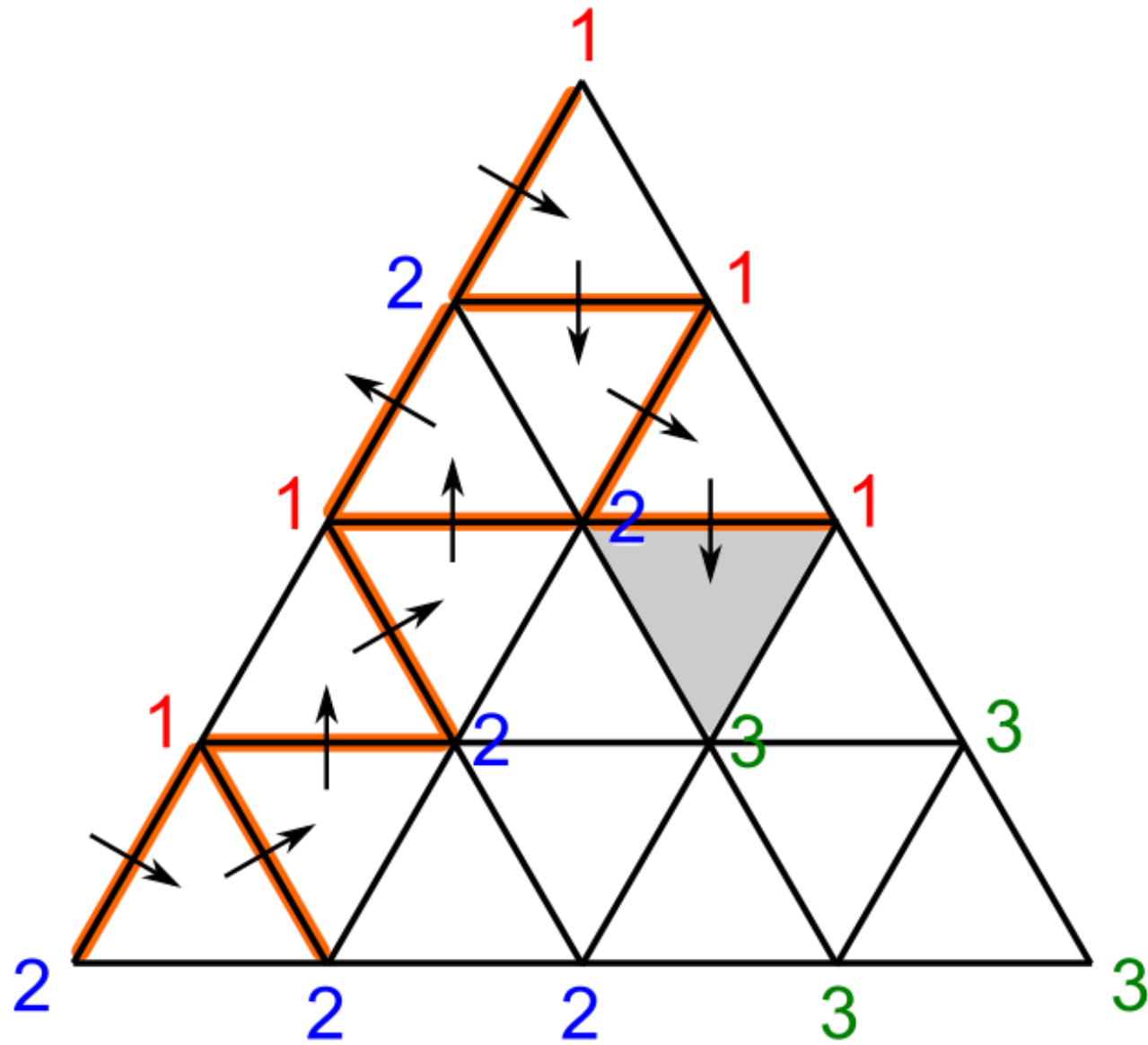
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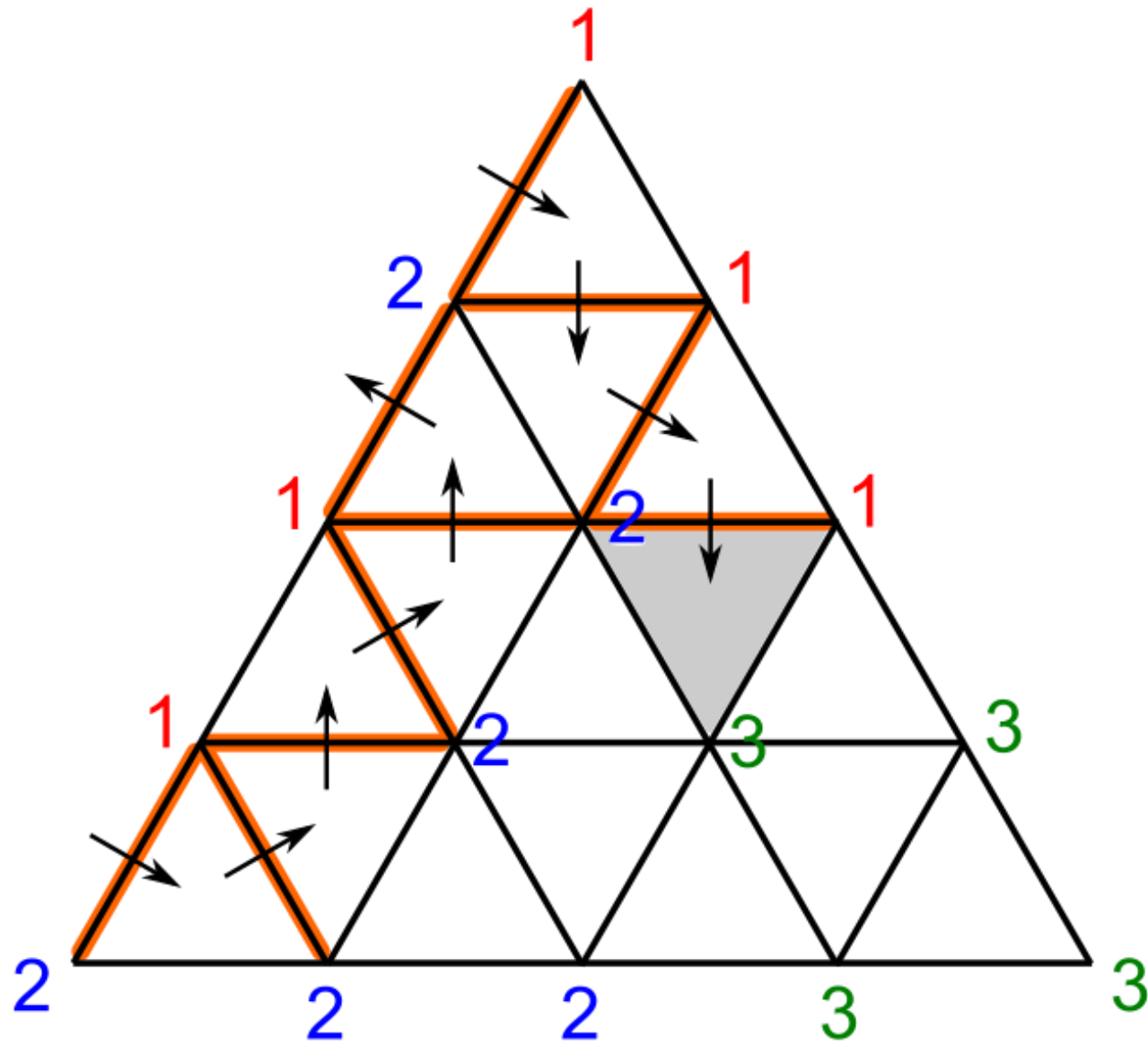
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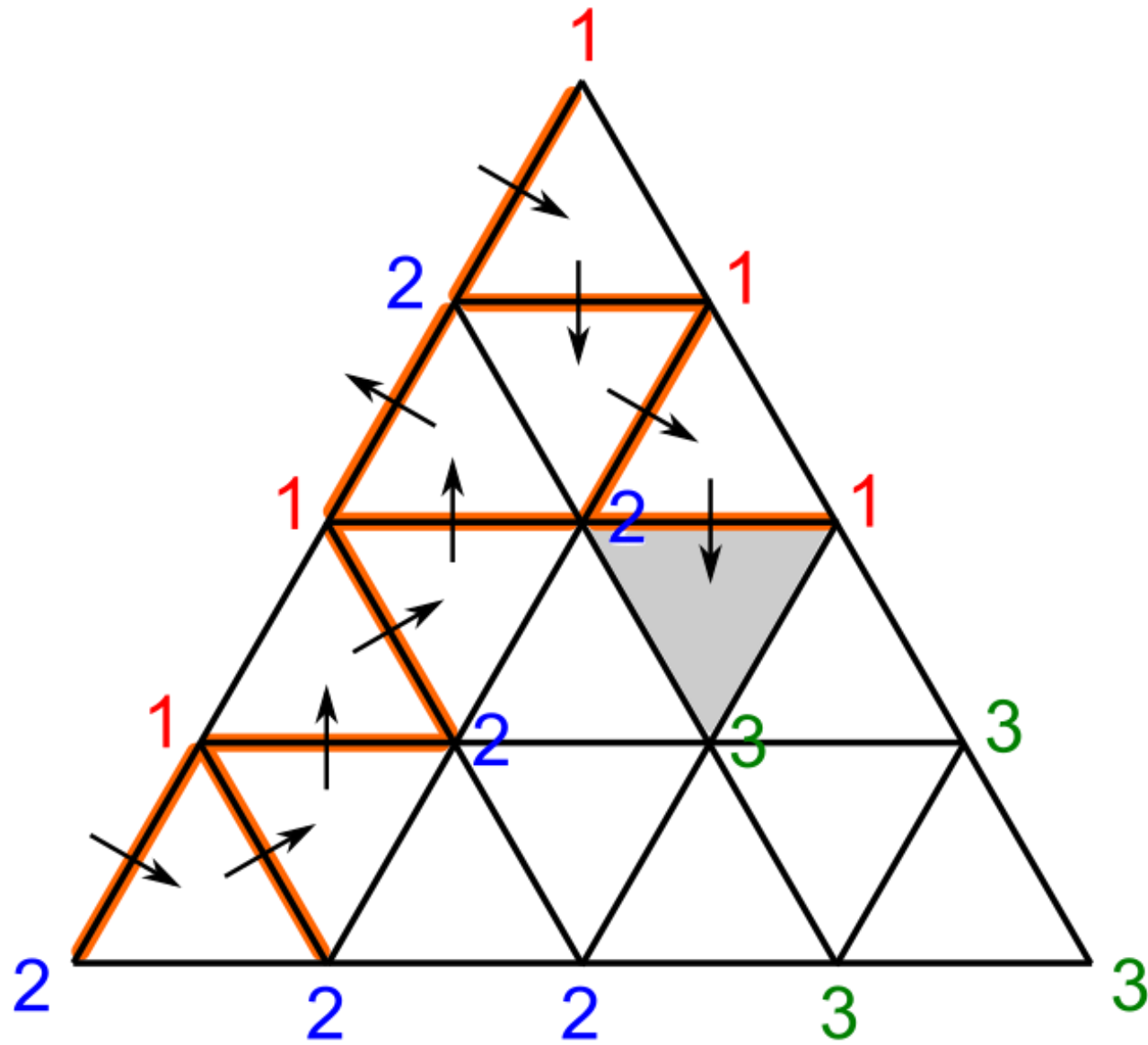
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Why can't such a walk cycle?

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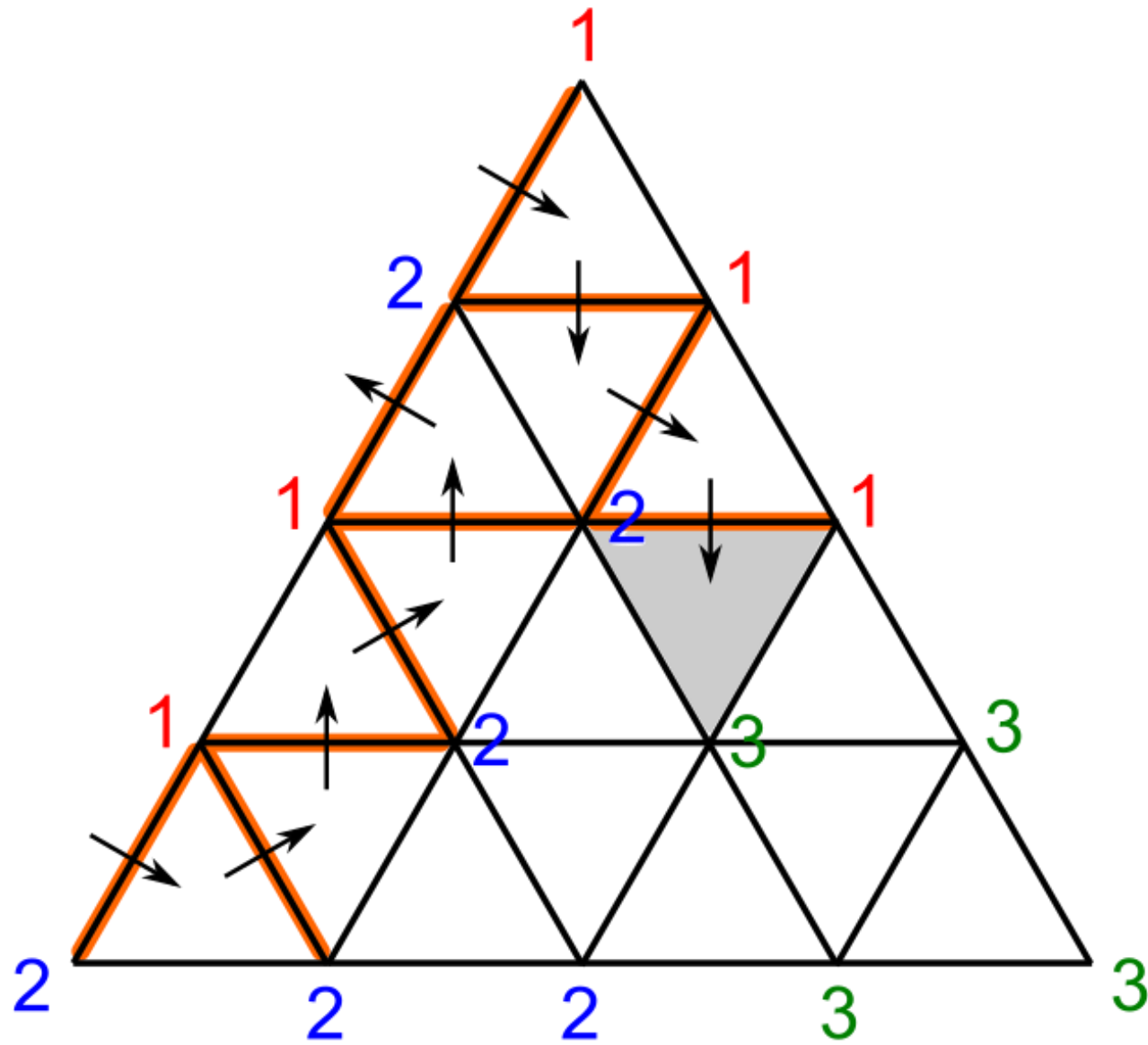
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