COL 351 : Analysis & Design of Algorithms

LECTURE 31

NETWORK FLOW III :

MAXIMUM FLOWS AND MINIMUM CUTS (CONTD.)

OCT 23, 2026 | ROHIT VAISH

FORD - FULKERSON ALGORITHM

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① initialize $f_e = 0 \quad \forall e$

② repeat:

* search for an $s-t$ path P in G_f
  such that every edge in P has positive residual capacity

// possible in $O(|E|)$ time via BFS / DFS

* if no such path, return current flow $\{f_e\}_{e \in E}$
* else, let $\Delta := \min_{e \in P}$ e's residual capacity in G_f

  for all $e \in G$ with forward edges in P, add Δ to f_e.
  for all $e \in G$ with reverse edges in P, subtract Δ from f_e.

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Ford-Fulkerson terminates in finite time with a valid flow.

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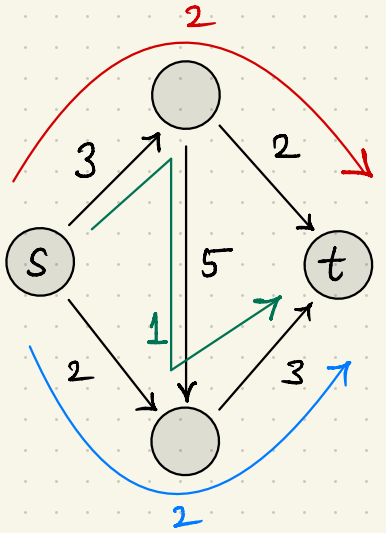
🤔 HOW DO WE KNOW WHEN WE ARE DONE?

OPTIMALITY CONDITIONS

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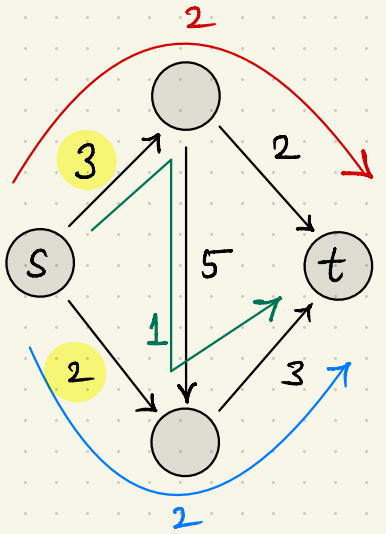
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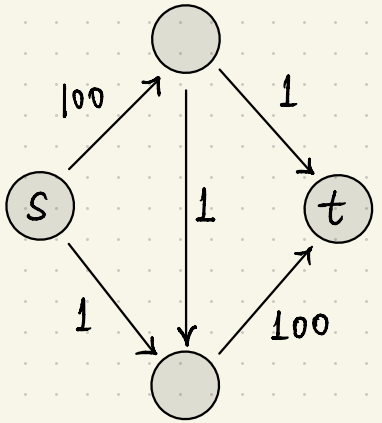
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easy!

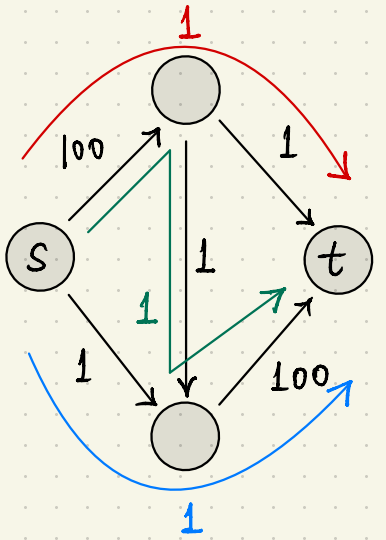
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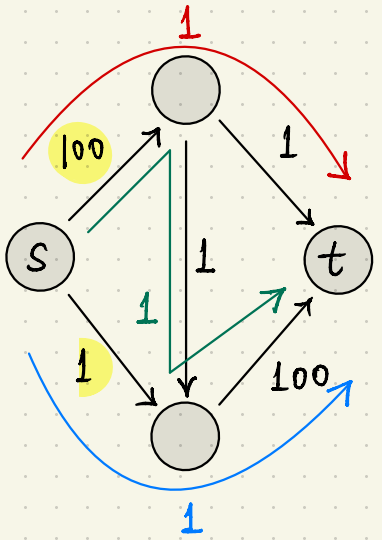
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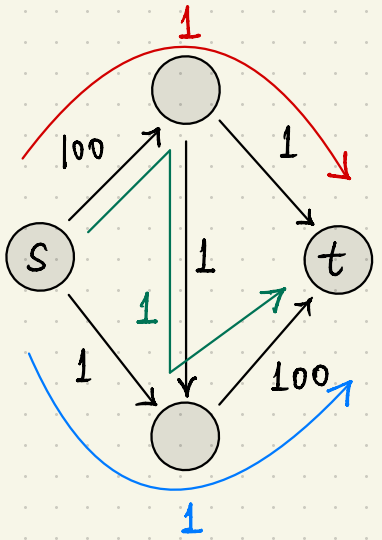


101 >> 3

Could this be a max flow?

OPTIMALITY CONDITIONS

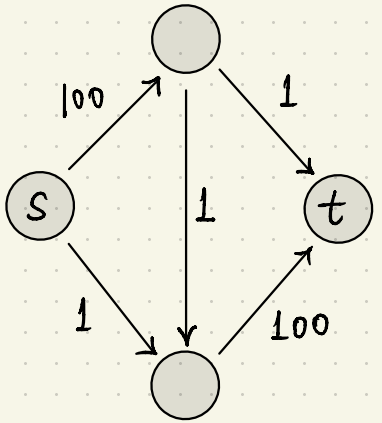
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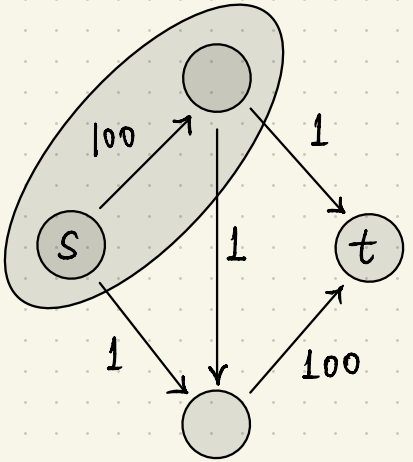
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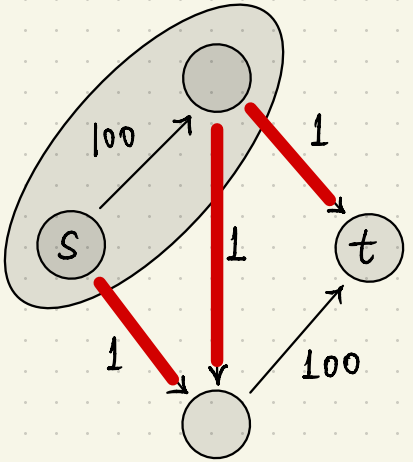
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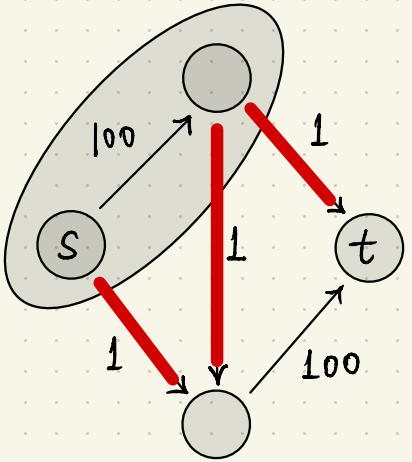
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All flow going into the sink has to cross this set.



Identify a bottleneck subset of vertices

OPTIMALITY CONDITIONS

A general recipe :

(1) (Structure) Identify "optimality conditions".

(2) (Algorithm) Design an algorithm that terminates with the optimality conditions satisfied.

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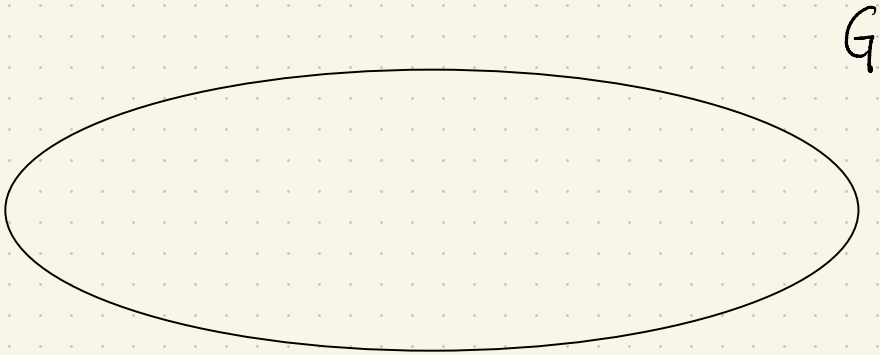
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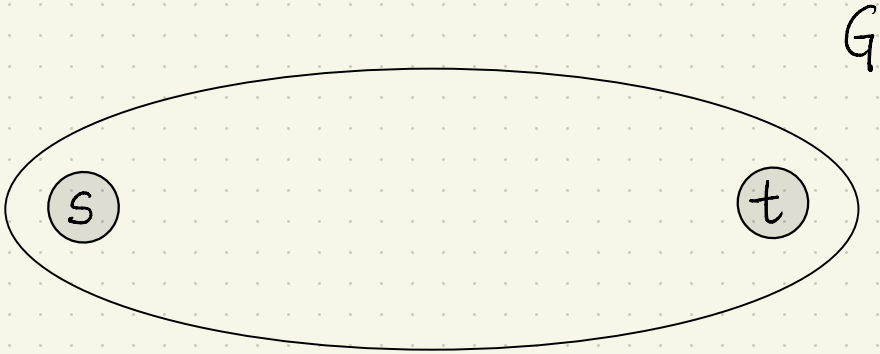
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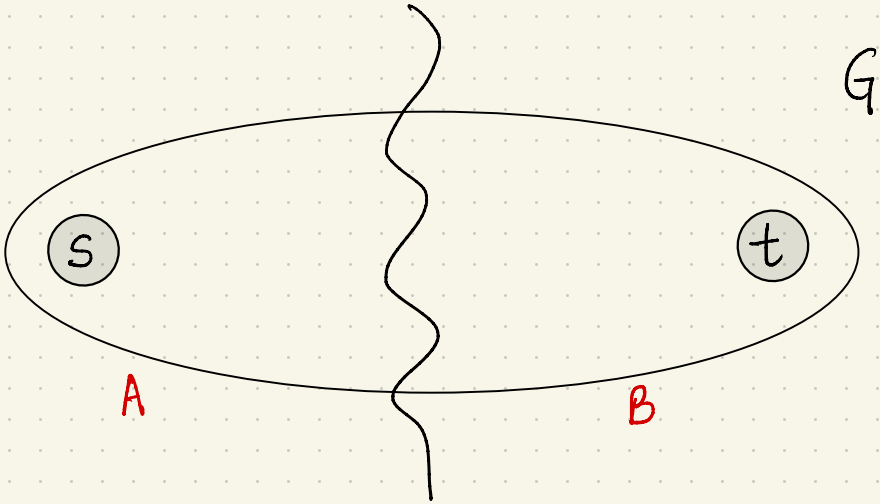
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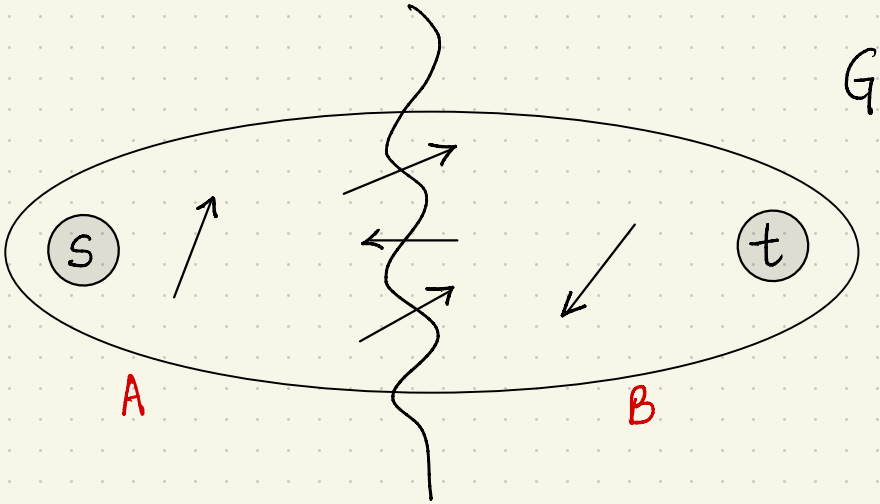
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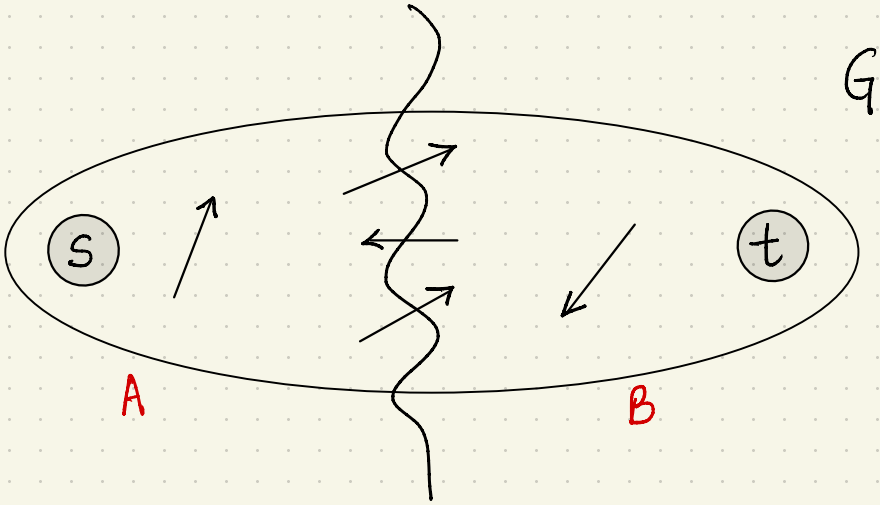
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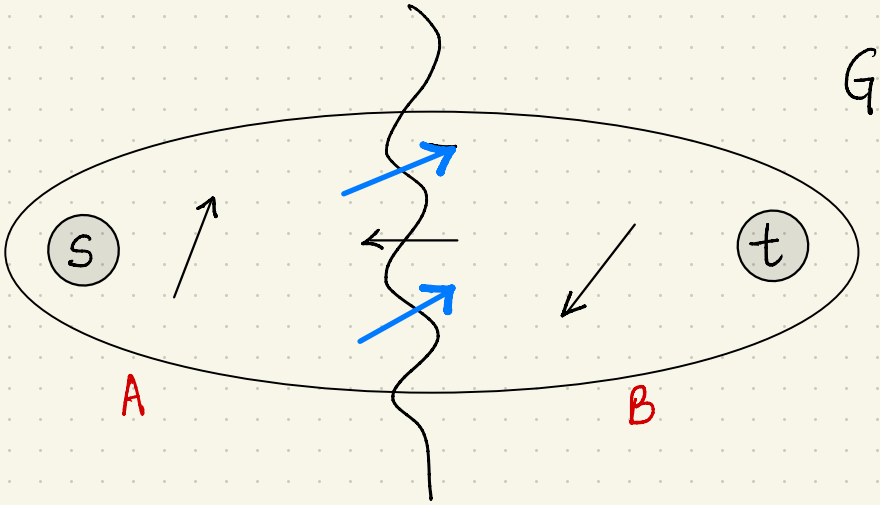
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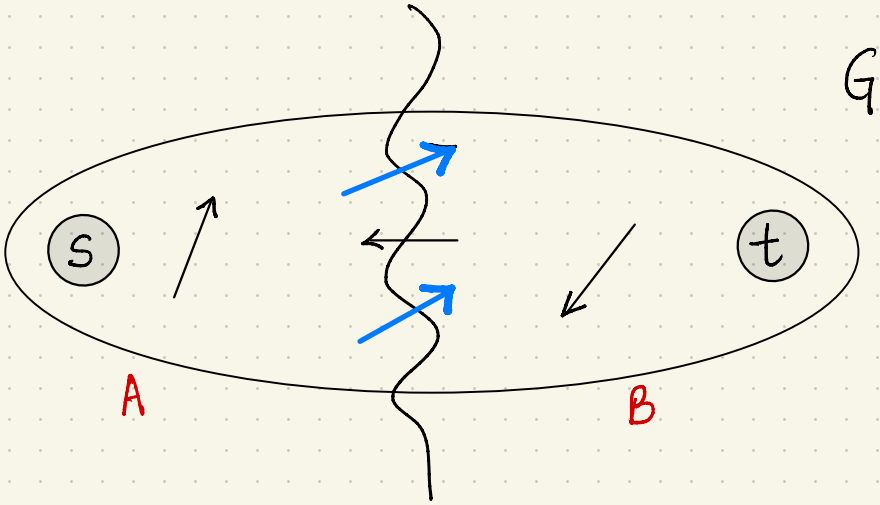
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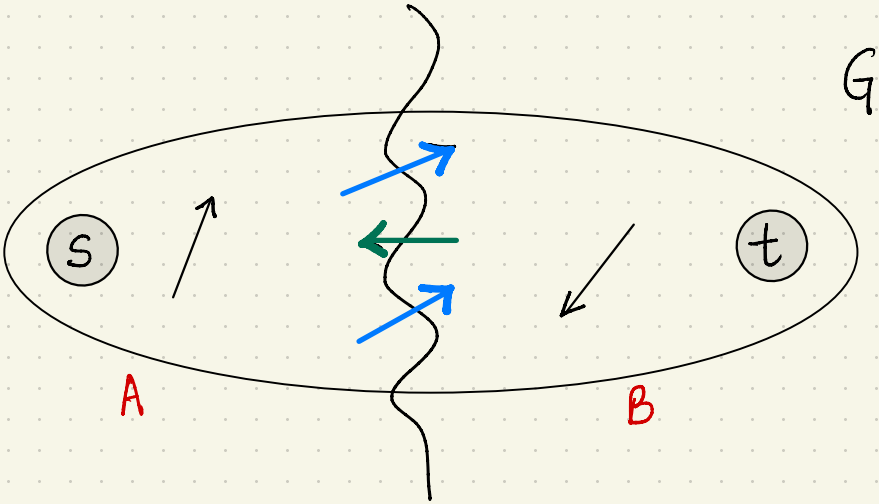


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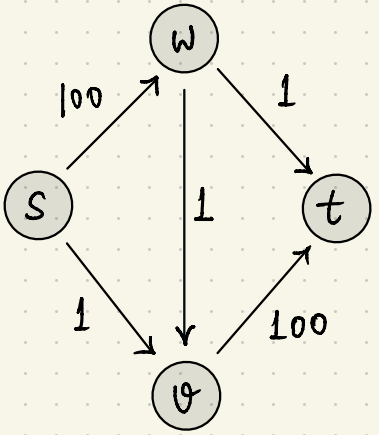
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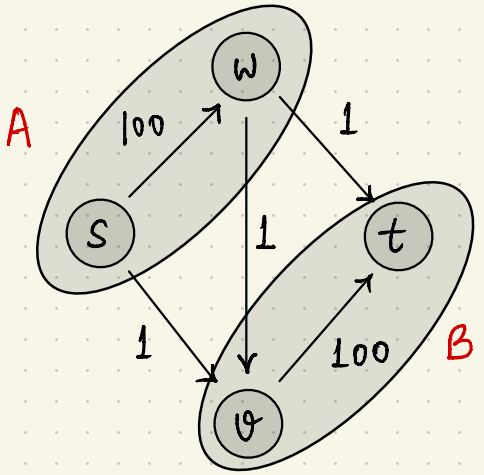
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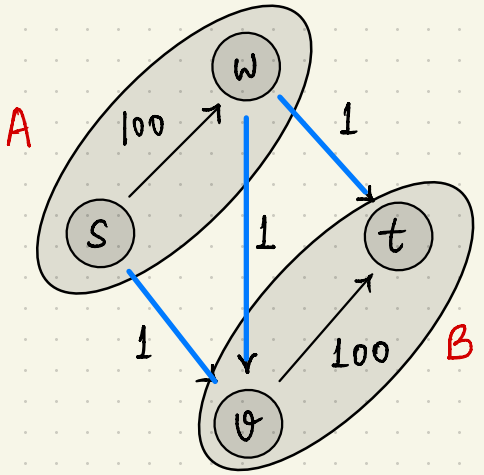
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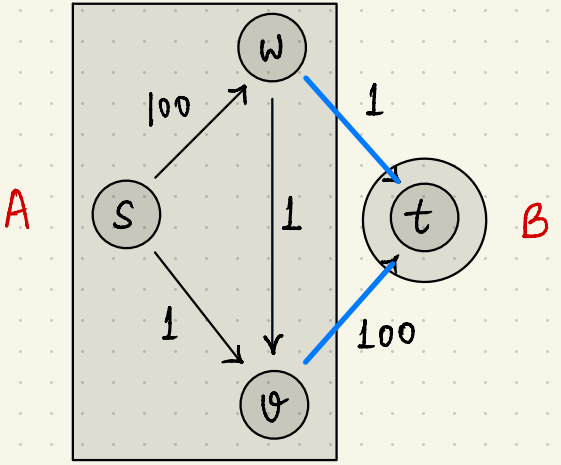
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Amounts to showing that (c) $\Rightarrow$ (a).

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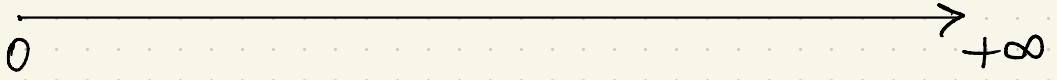
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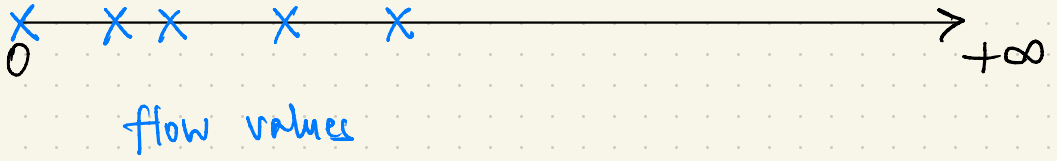


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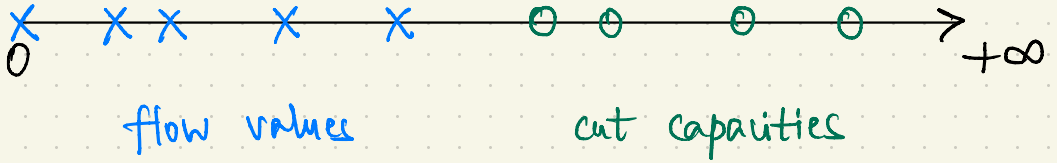


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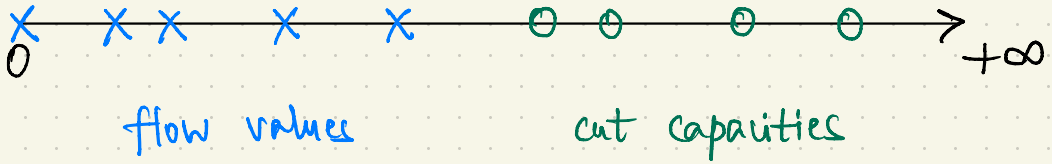


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"tight" (s,t) cut $\Rightarrow$ X coincides with O $\Rightarrow$ maximum possible value

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Proof: Fix a flow f and a cut (A,B).

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vacuous term (no edges into s)

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Conservation constraint

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Adding (2) for all $v \in A \setminus \{s\}$ and (1):

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$$\text{value of } f = \sum_{e \in \delta^+(s)} f_e = \sum_{e \in \delta^+(s)} f_e - \sum_{e \in \delta^-(s)} f_e \qquad \text{——— (1)}$$

For any $v \notin \{s,t\}$:

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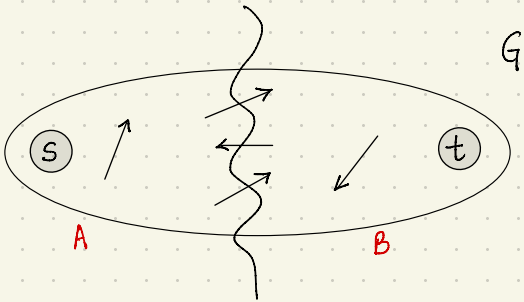
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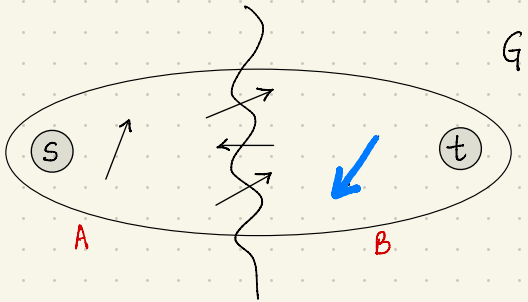
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No contribution

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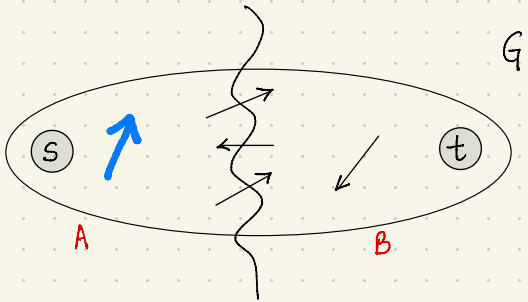
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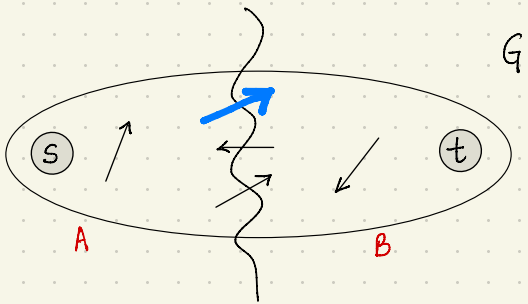
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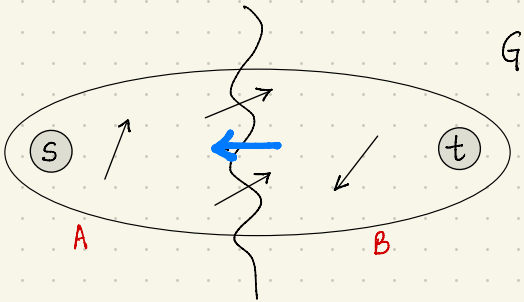
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OPTIMALITY CONDITIONS FOR MAX FLOW

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(a) f is a maximum flow of G

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Will prove (b) $\Rightarrow$ (a) $\Rightarrow$ (c) $\Rightarrow$ (b).

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Easy! (Exercise)

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No (s,t) PATH IN $G_f \Rightarrow$ "TIGHT" CUT

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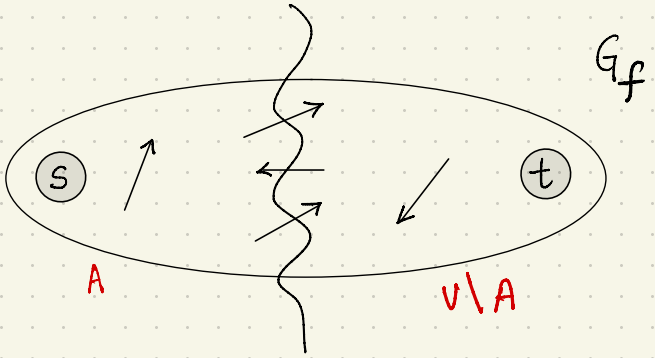
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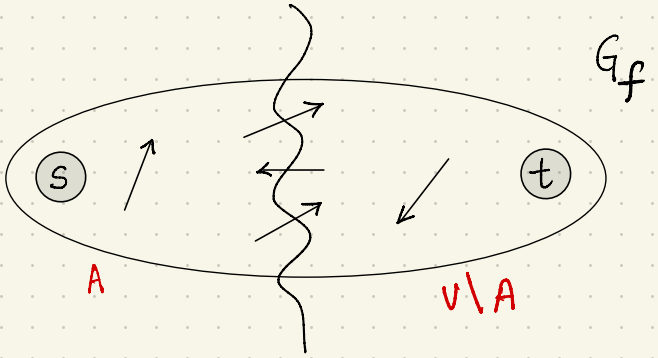


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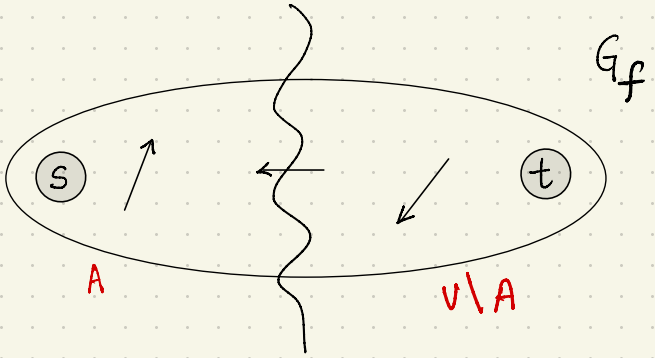
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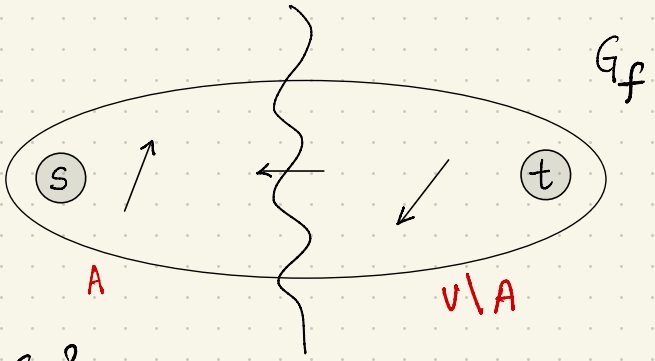
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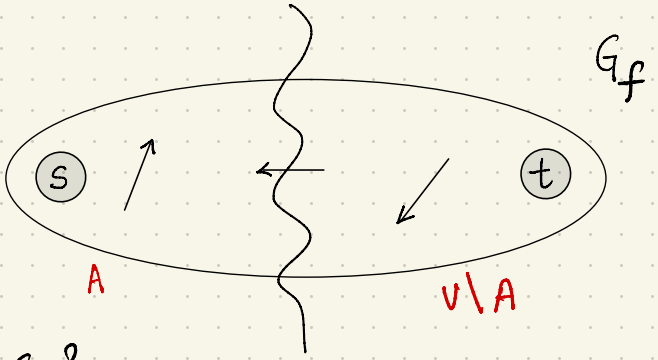
What does it mean for the underlying graph G?

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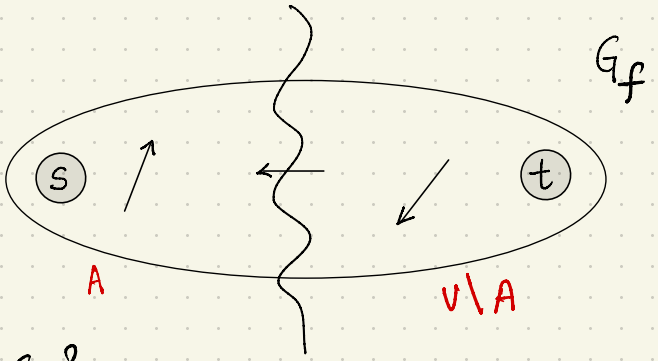
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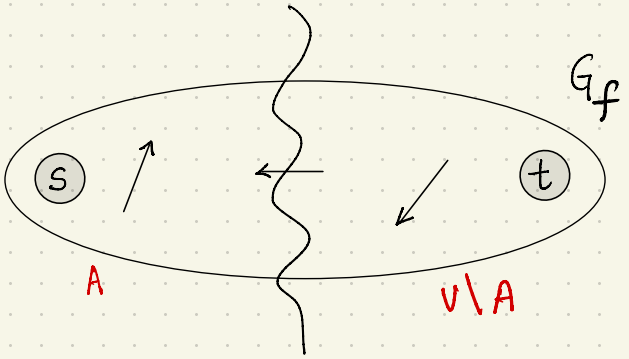
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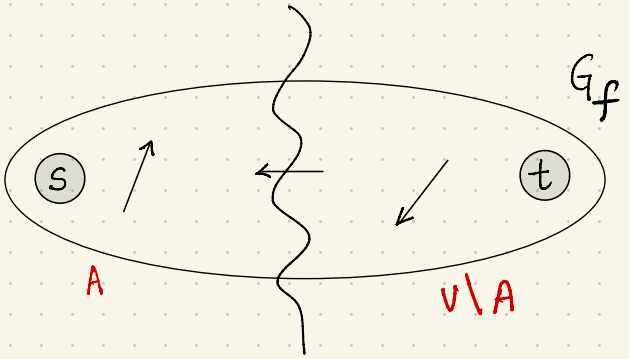
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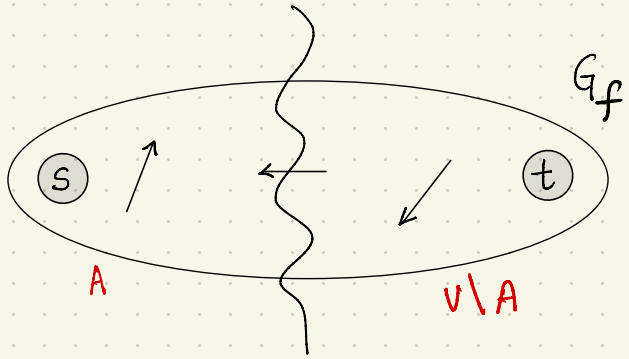
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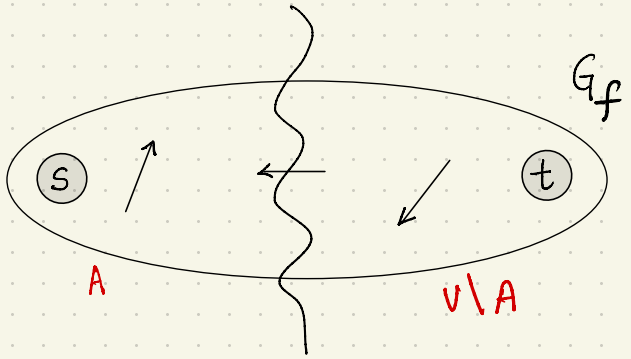
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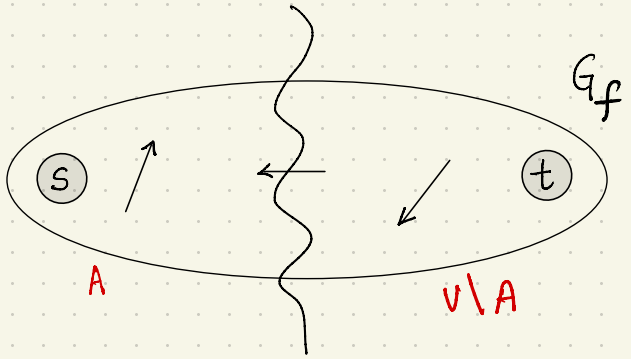
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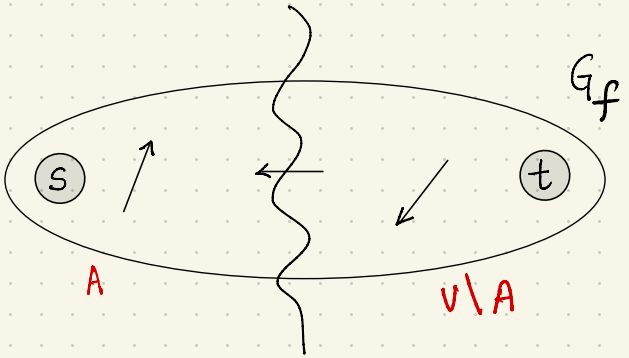
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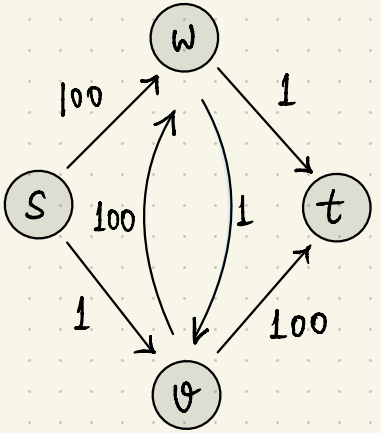
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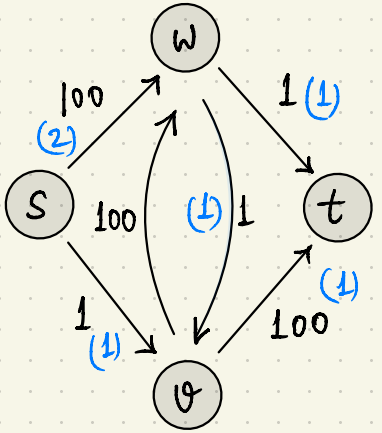
(c) there is no (s,t) path in the residual graph G_f.

Will prove (b) $\Rightarrow$ (a) $\Rightarrow$ (c) $\Rightarrow$ (b).

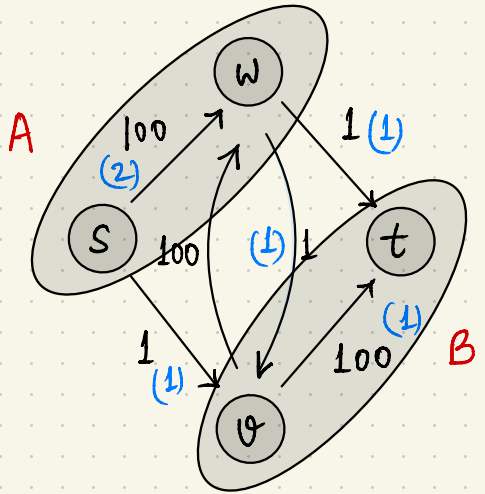
OPTIMALITY CONDITIONS FOR MAX FLOW



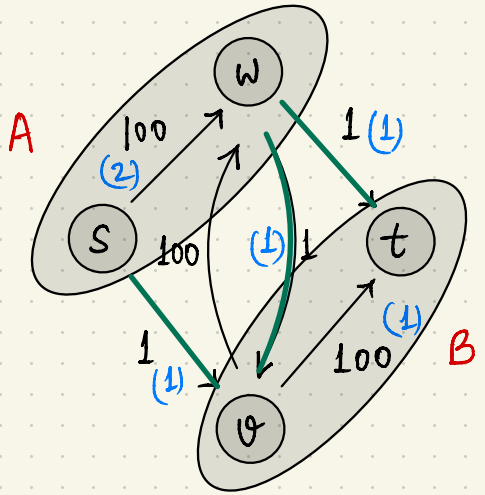
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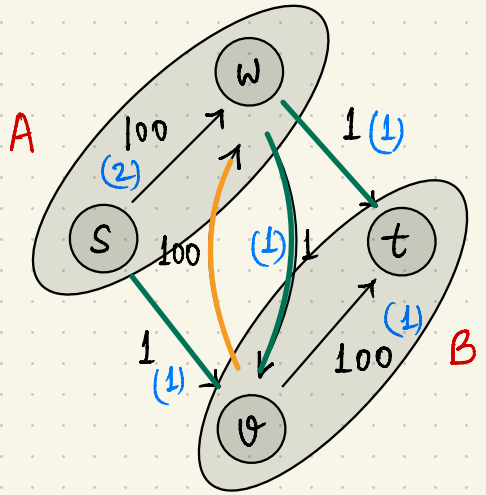


OPTIMALITY CONDITIONS FOR MAX FLOW



$\delta^+(A)$: saturated

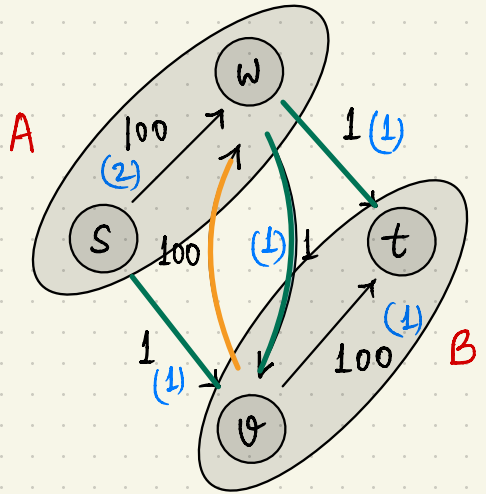
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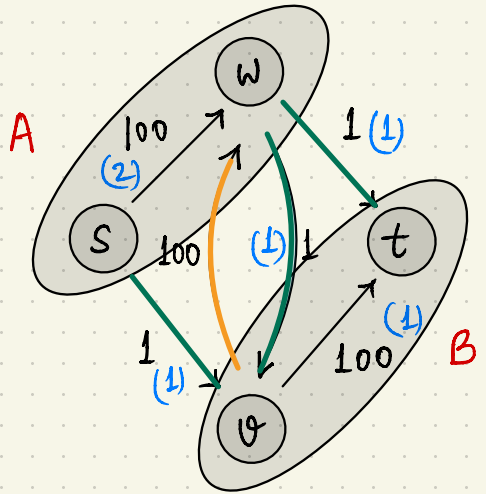


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Max-flow Min-cut Theorem

MAX-FLOW MIN-CUT THEOREM

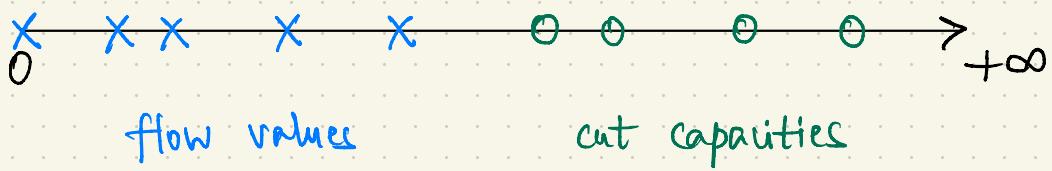
Corollary: For every network,

maximum value of a flow = minimum capacity of an (s,t)-cut

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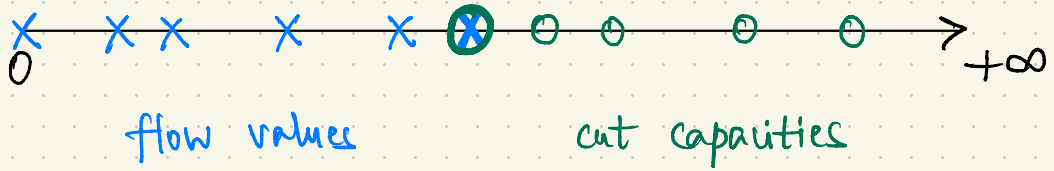
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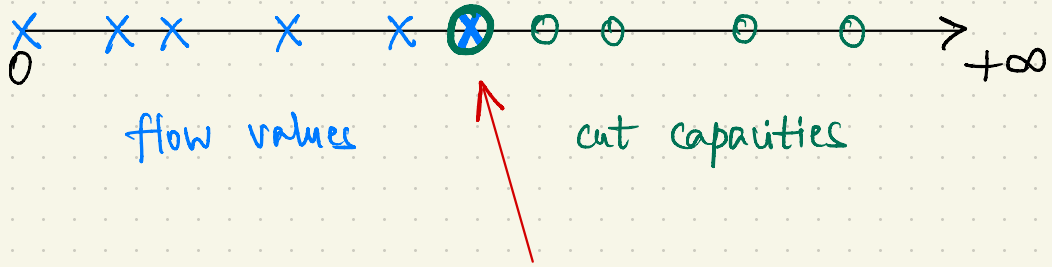
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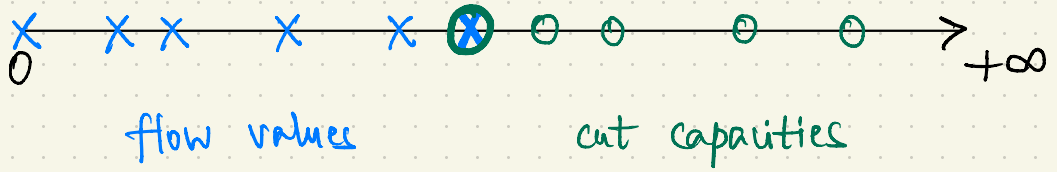
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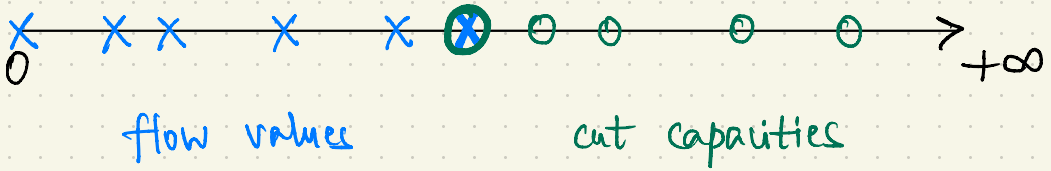


Corollary (of Corollary): Given a maximum flow, a minimum cut can be recovered in $O(|E|)$ time.

MAX-FLOW MIN-CUT THEOREM

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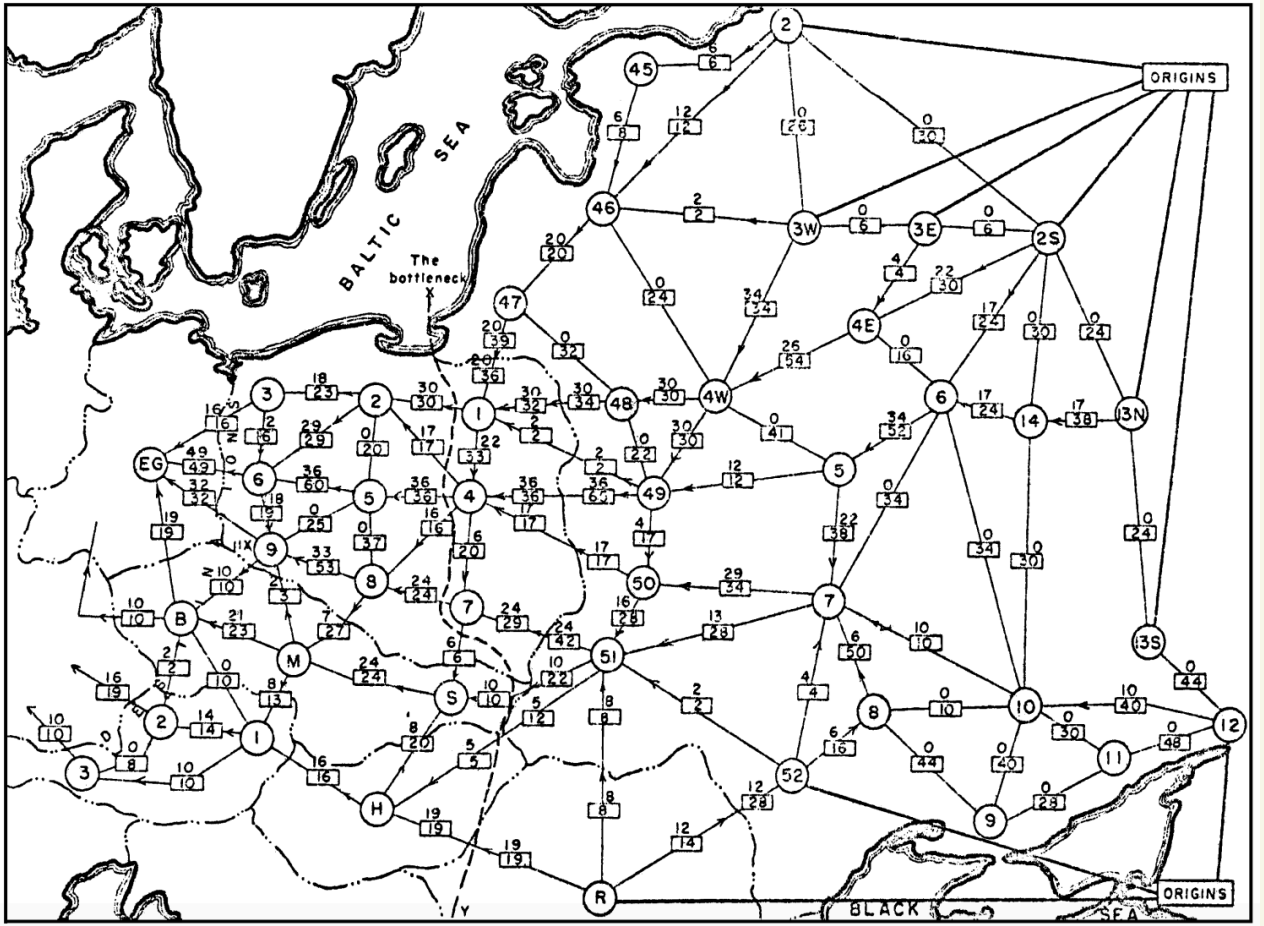
Proof: Do BFS/DFS from s, construct the set A of reachable vertices. Then, $(A, V \setminus A)$ is a min cut. ∎

MAX-FLOW MIN-CUT THEOREM

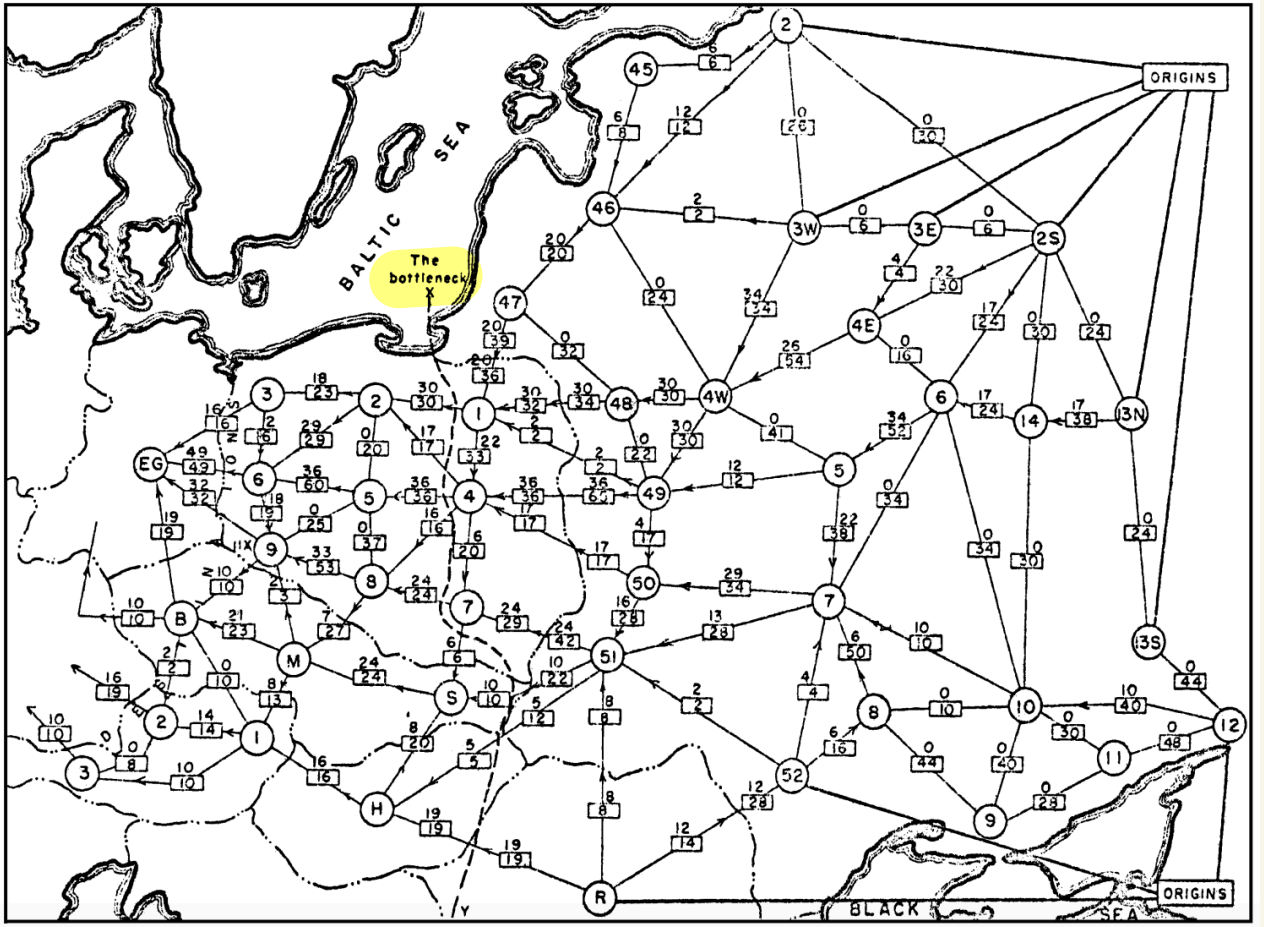
On the history of combinatorial optimization (till 1960)

Alexander Schrijver[1]

MAX-FLOW MIN-CUT THEOREM



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