

COL 351 : ANALYSIS & DESIGN OF ALGORITHMS

LECTURE 26

DYNAMIC PROGRAMMING II : BELLMAN FORD ALGORITHM (CONTD.)

OCT 10, 2026

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ROHIT VAISH

SINGLE - SOURCE SHORTEST PATH PROBLEM

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input: A directed graph $G = (V, E)$, starting vertex s ,
a possibly negative length l_e for each edge $e \in E$

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SINGLE - SOURCE SHORTEST PATH PROBLEM

input: A directed graph $G = (V, E)$, starting vertex s ,
a possibly negative length l_e for each edge $e \in E$

output: ① for all destinations $v \in V$, compute the length of the
shortest $s-v$ path, or
② output a negative cycle.

OPTIMAL SUBSTRUCTURE

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Lemma: Let $G = (V, E)$ be a directed graph with edge lengths l_e and starting vertex s .

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For every destination $v \in V$ and every $i \in \{1, 2, \dots\}$, let

P be the shortest $s-v$ path with at most i edges.



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P be the shortest $s-v$ path with at most i edges.

Case I: P has $(i-1)$ or fewer edges

$\Rightarrow P$ is the shortest $s-v$ path with at most $(i-1)$ edges.

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Case II: P has i edges

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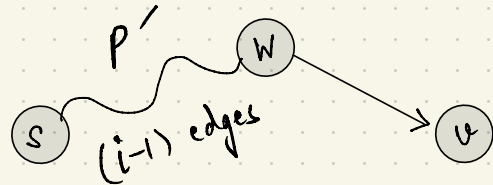
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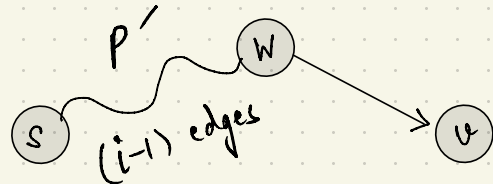
P be the shortest $s-v$ path with at most i edges.

Case I: P has $(i-1)$ or fewer edges

$\Rightarrow P$ is the shortest $s-v$ path with at most $(i-1)$ edges.

Case II: P has i edges

$\Rightarrow P'$ is the shortest $s-w$ path with at most $(i-1)$ edges



THE RECURRENCE

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Define $L_{i,v} :=$ length of shortest $s-v$ path with $\leq i$ edges

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Define $L_{i,v} :=$ length of shortest $s-v$ path with $\leq i$ edges

with cycles allowed

defined as $+\infty$ if no $s-v$ path with $\leq i$ edges

THE RECURRENCE

Define $L_{i,v} :=$ length of shortest $s-v$ path with $\leq i$ edges

For every $v \in V$ and every $i \in \{0, 1, 2, \dots\}$

$$L_{i,v} = \min \begin{cases} L_{(i-1),v} & // \text{Case I} \\ \min_{(w,v) \in E} L_{(i-1),w} + l_{w,v} & // \text{Case II} \end{cases}$$

BELLMAN-FORD ALGORITHM

BELLMAN-FORD ALGORITHM

$A := n \times n$ dimensional array

// budget $\times$ destinations

// base cases

$A[0, s] := 0$, $A[0, v] := +\infty$ for all $v \neq s$

// recurrence

for $i \in \{1, 2, \dots, n-1\}$

for each $v \in V$

$$A[i, v] = \min \begin{cases} A[i-1, v] \\ \min_{(w, v) \in E} A[i-1, w] + l_{w, v} \end{cases}$$