STCS Vigyan Vidushi 2025

Cake Cutting

Rohit Vaish

The Model

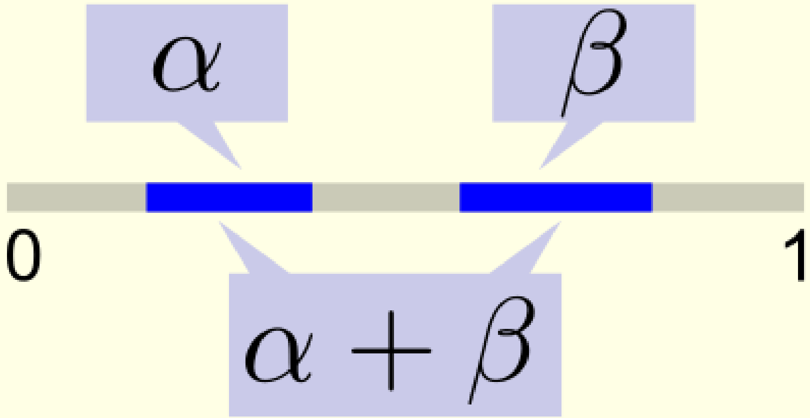
- The resource: Cake [0,1]
- Set of agents {1,2,...,n}
- *Piece of cake*: Finite union of subintervals of [0,1]

Preferences of Agents

- Valuation function v_i: Assigns a non-negative value to any piece of cake

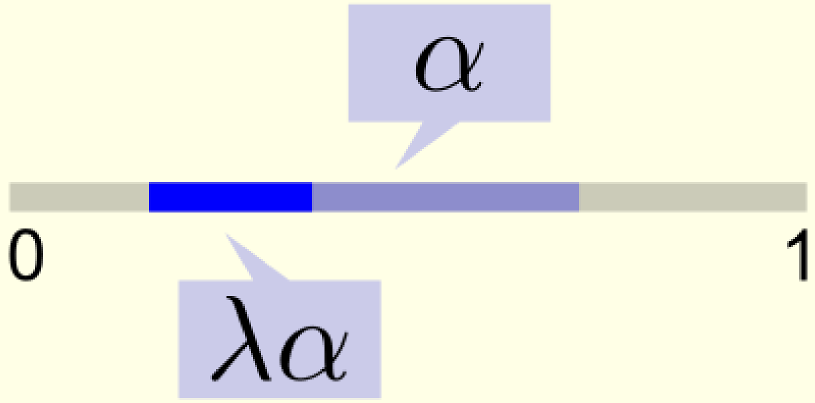
Additivity

for disjoint $X, Y \subseteq [0,1]$,
$v_i(X \cup Y) = v_i(X) + v_i(Y)$



Divisibility

for any $X \subseteq [0,1]$ and any $\lambda \in [0,1]$,
there exists $Y \subseteq X$ s.t. $v_i(Y) = \lambda v_i(X)$



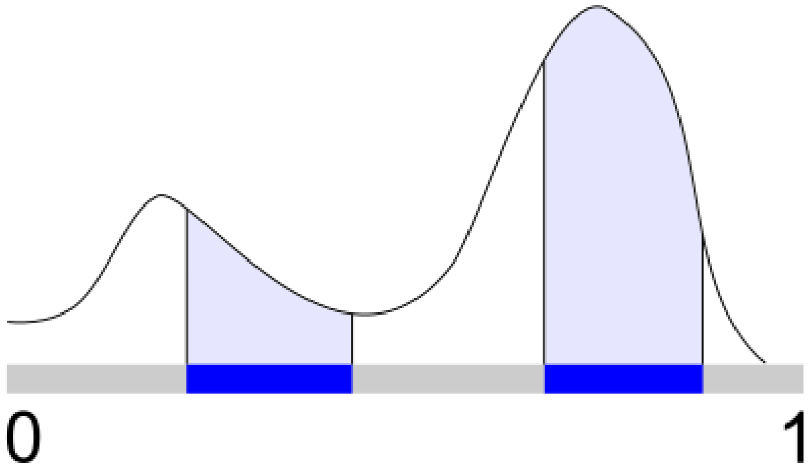
Normalization: for each agent i, $v_i([0,1]) = 1$.

Preferences of Agents

- Valuation function v_i: Assigns a non-negative value to any piece of cake

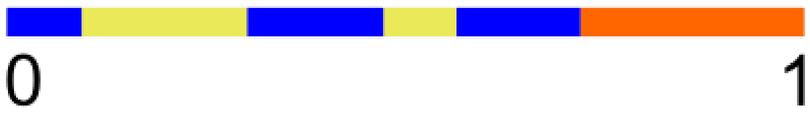
$$v_i(X) = \int_{x \in X} f_i(x) dx$$

value density function



Fairness notions

- *Allocation/Division*: A partition $(A_1, A_2, \ldots, A_n)$ of the cake [0,1] where each A_i is a piece of cake.



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Proportionality

[Steinhaus, 1948]

for each agent i,

$$v_i(A_i) \geq \frac{1}{n}$$

Envy-freeness

[Gamow and Stern, 1958; Foley, 1967]

for every pair of agents i,j,

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For two agents (n=2), is one property stronger than the other?

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What about three or more agents?

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EF implies Prop for *any* number of agents

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EF implies Prop for *any* number of agents

Prop implies EF for *two* agents (but no more)

Robertson-Webb Query Model

Types of queries that can be used to access the valuation functions

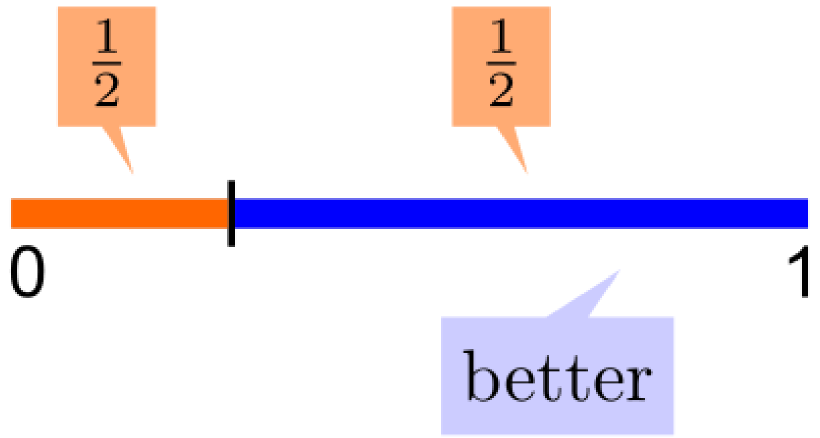
$\texttt{eval}_i(x, y)$: returns $v_i([x, y])$

$\texttt{cut}_i(x, \alpha)$: returns y such that $v_i([x, y]) = \alpha$

Cut and Choose

1. Agent 1 cuts the cake into two equally-valued pieces (as per v_1).

2. Agent 2 chooses its preferred piece (as per v_2), and agent 1 gets the remaining piece.



For two agents, an envy-free/proportional cake division can be computed using two queries.

Dubins-Spanier Procedure

A proportional cake division protocol for any number of agents

Dubins-Spanier Procedure

1. A referee gradually moves a knife from left to right.

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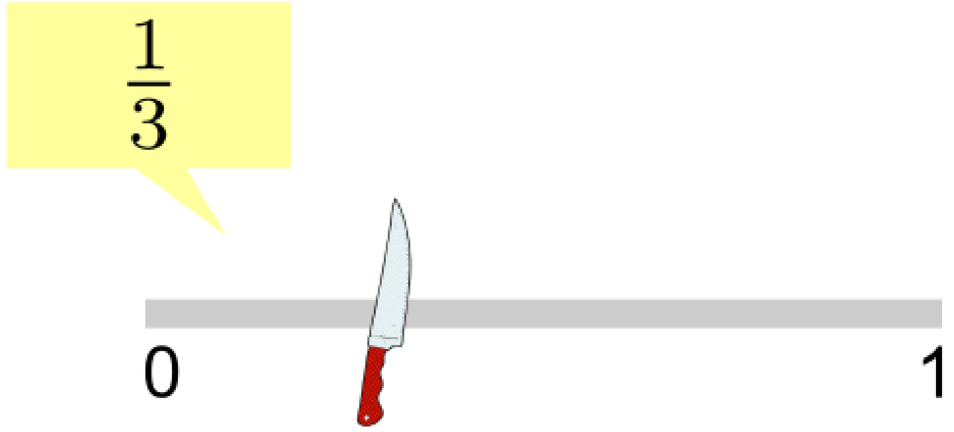

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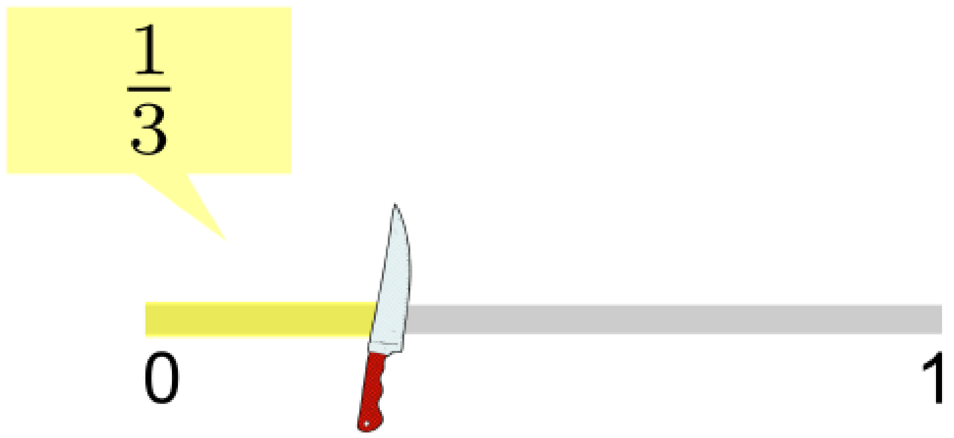

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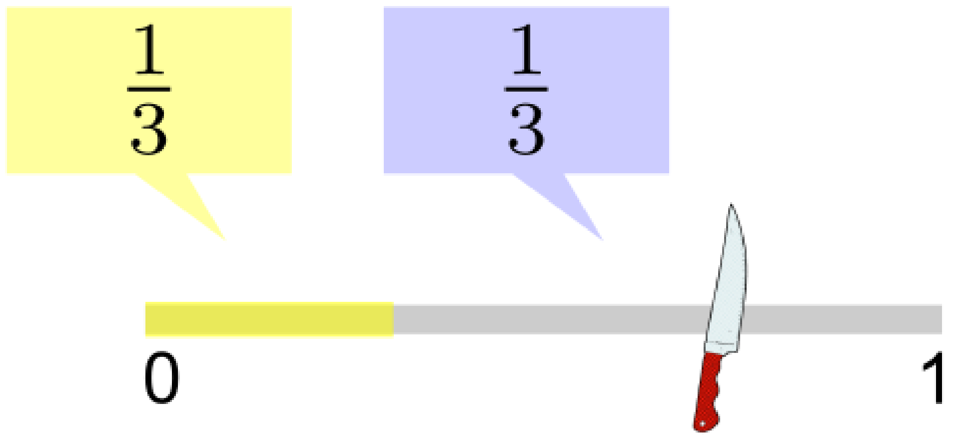

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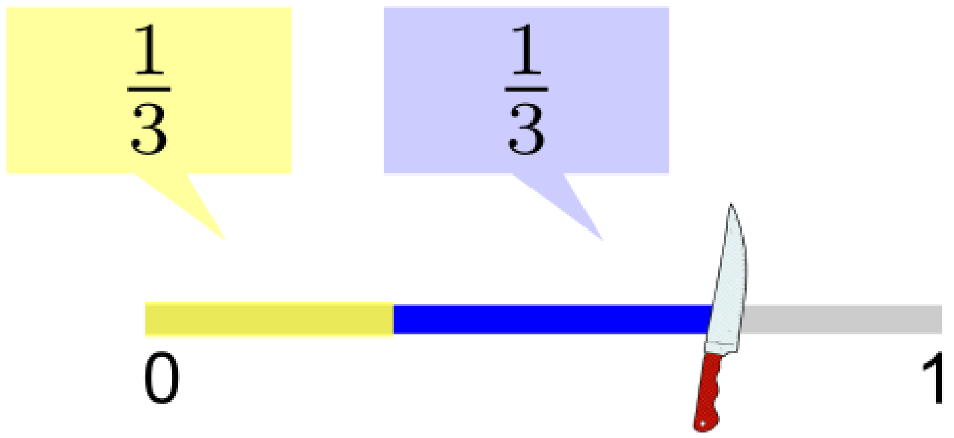

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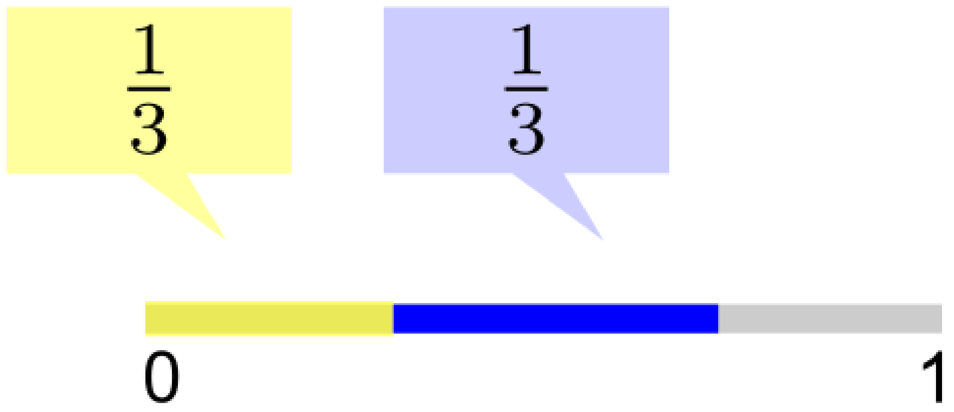

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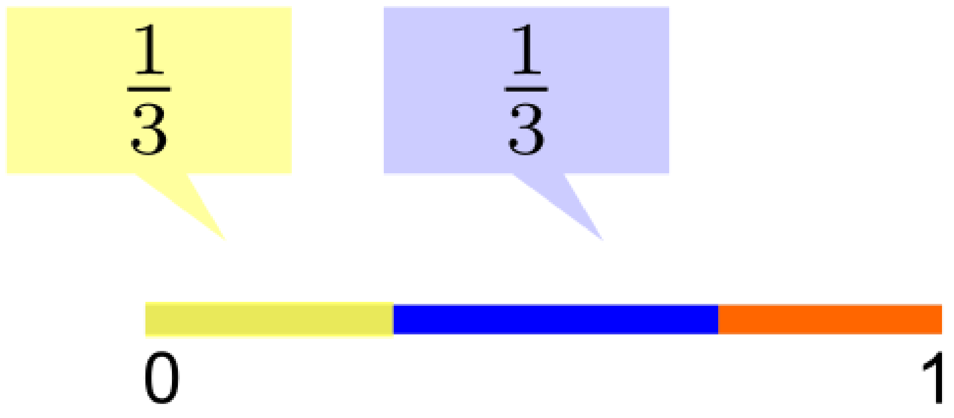

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Why is the resulting allocation proportional?

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Why is the resulting allocation proportional?

Every agent except for the last one gets *exactly* 1/n.
The last agent gets *at least* 1/n.

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Can this procedure be implemented in the Robertson-Webb model?

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Yes!

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Query complexity in the Robertson-Webb model?

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Query complexity in the Robertson-Webb model?

$\mathcal{O}(n^2)$ queries (Exercise)

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For n agents, a proportional cake division can be computed using $O(n^2)$ queries.

The Story of Proportionality

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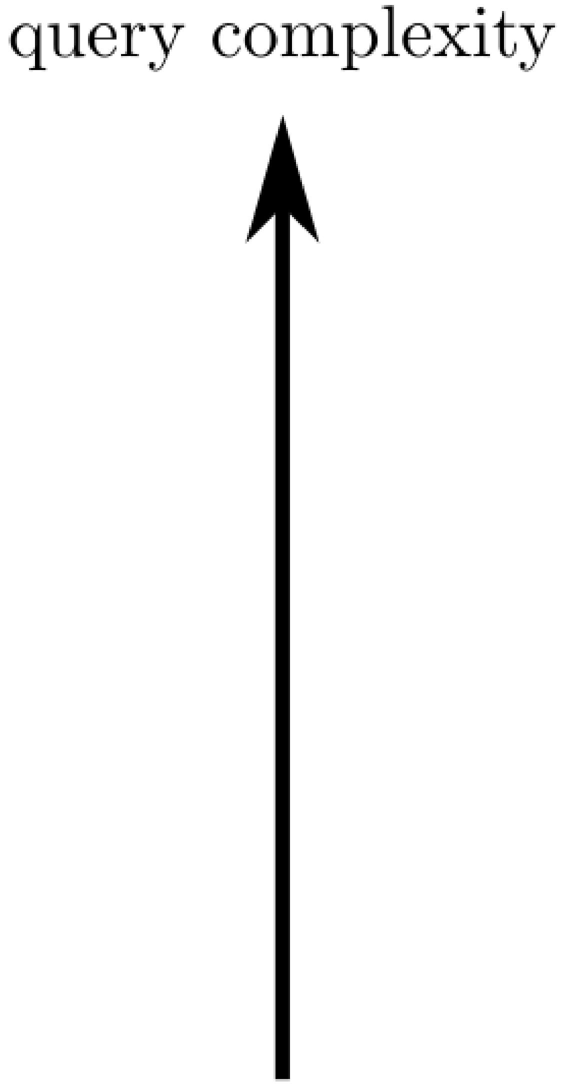

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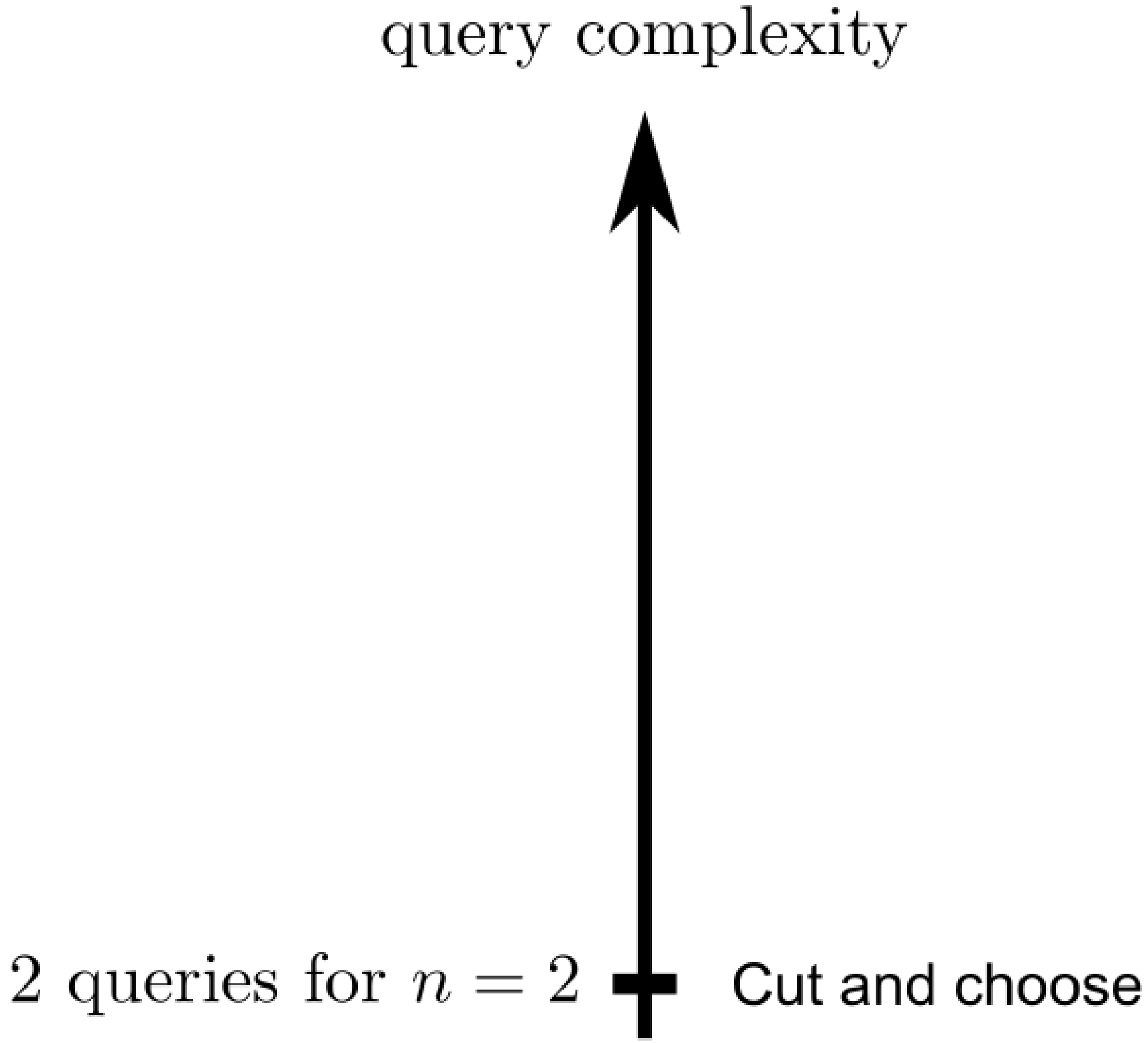

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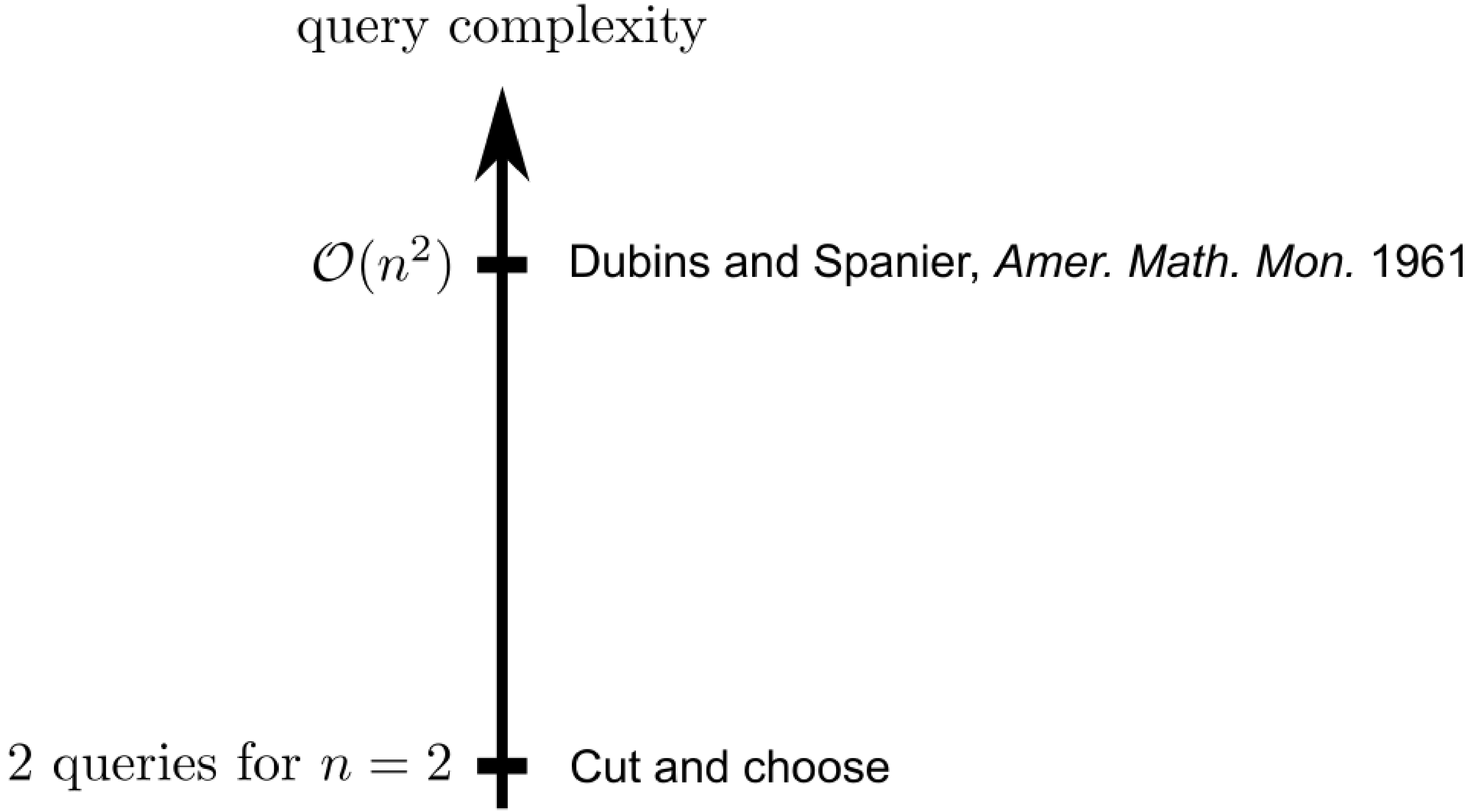

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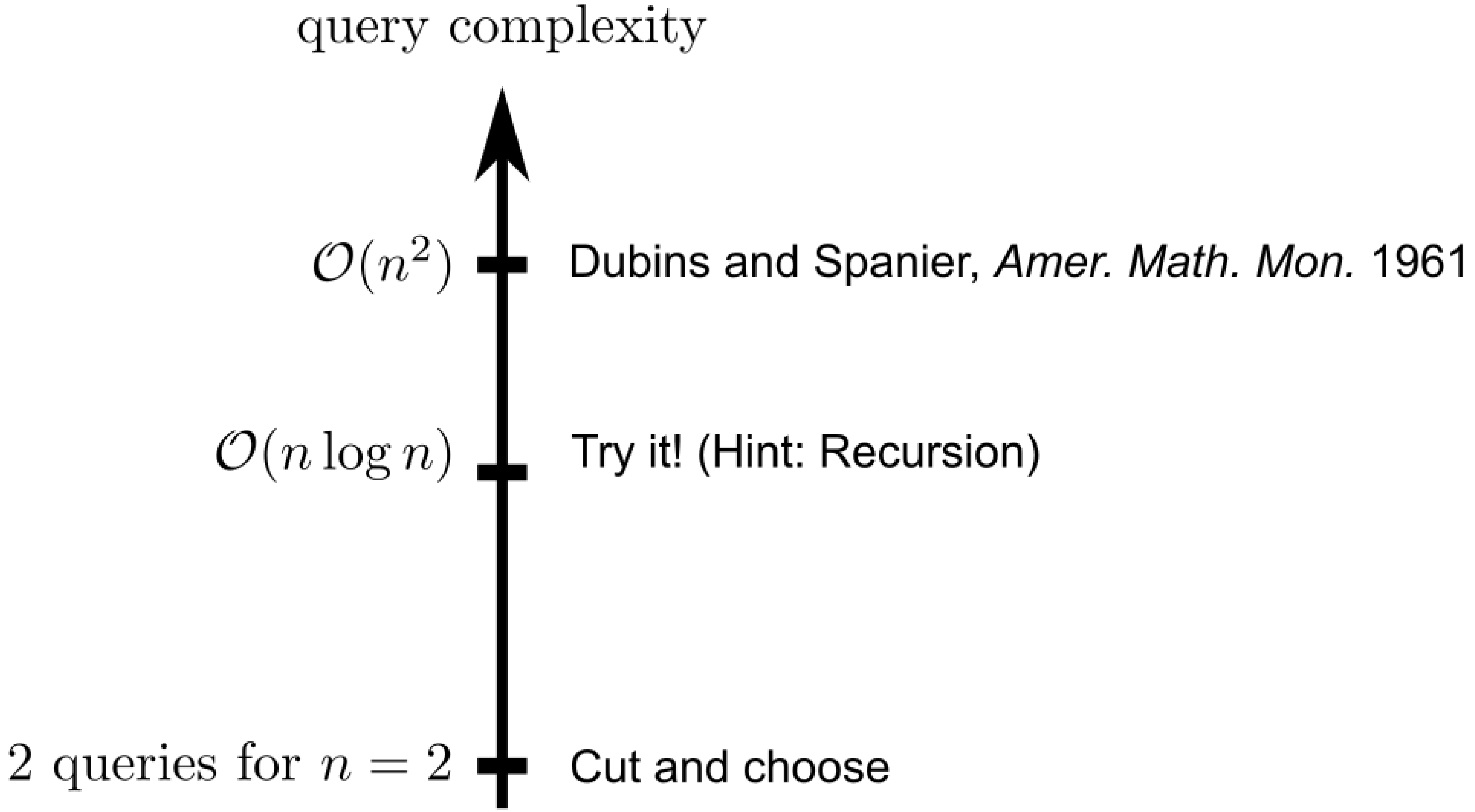

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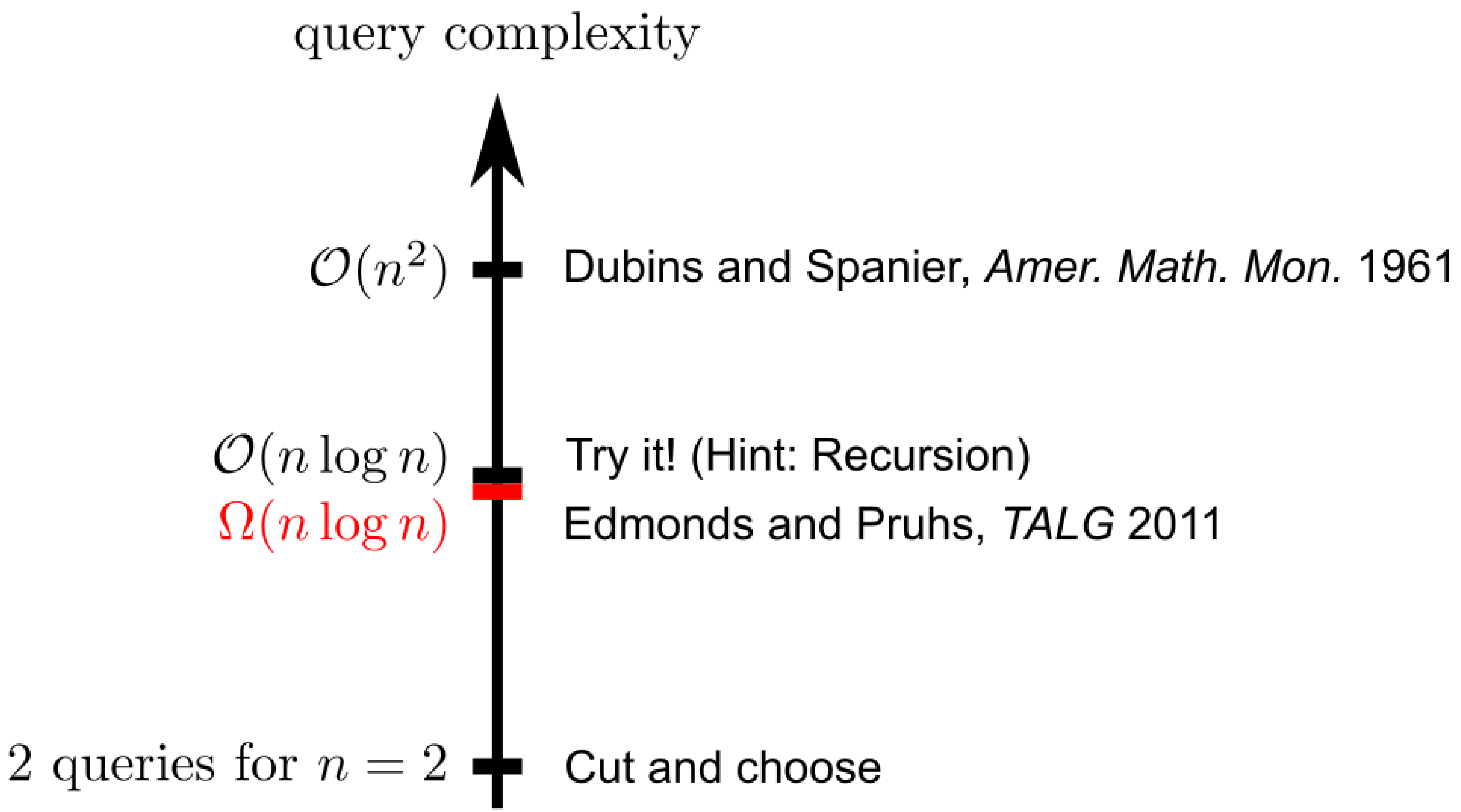


The Story of Envy-freeness

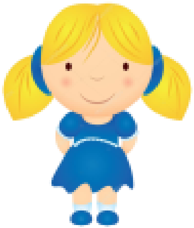
Selfridge-Conway Procedure

An envy-free cake division protocol for three agents

Envy-free for three

Envy-free for three

A 

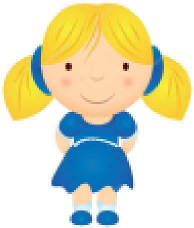
B 

C 

Envy-free for three

A

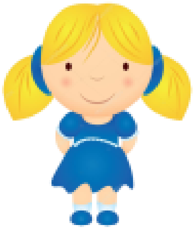
B



C

Envy-free for three

A 

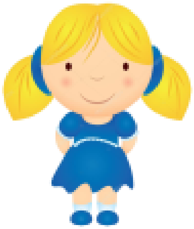
B 

C 

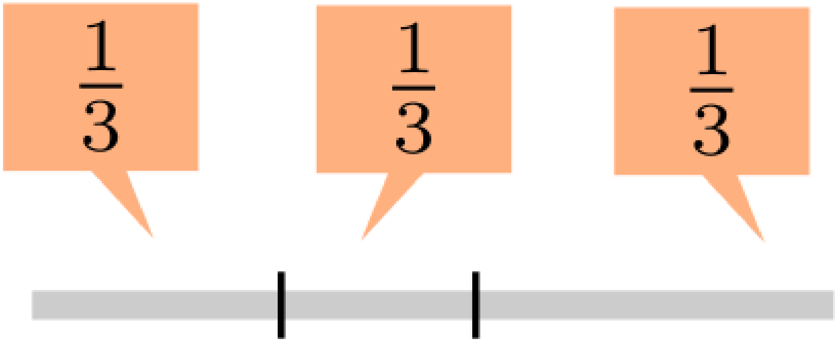
Envy-free for three

A

B



C



Envy-free for three

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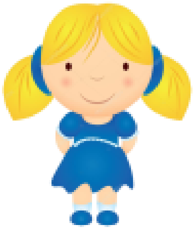
B

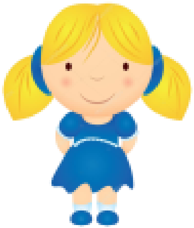


C

Envy-free for three

A

B

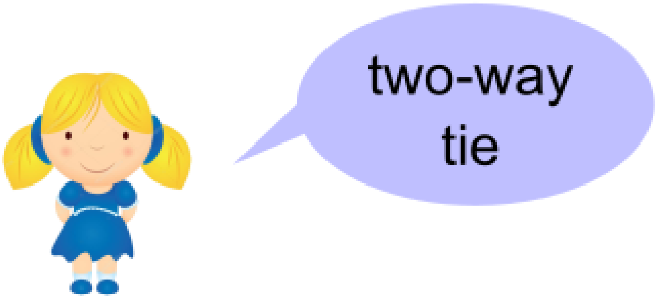


C

Envy-free for three

A

B



C

Envy-free for three

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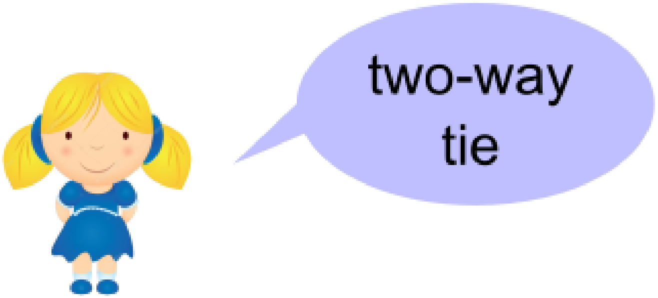

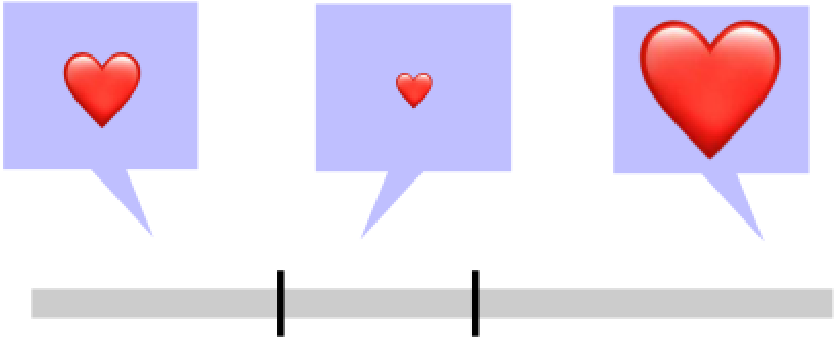

C

Envy-free for three

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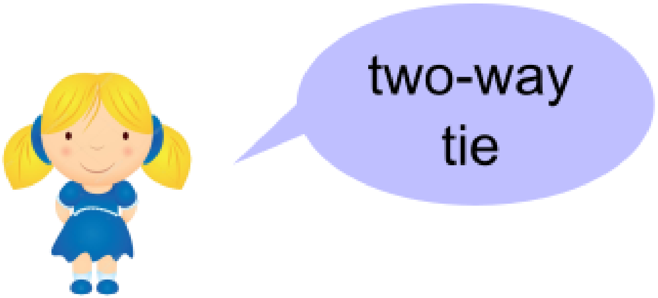

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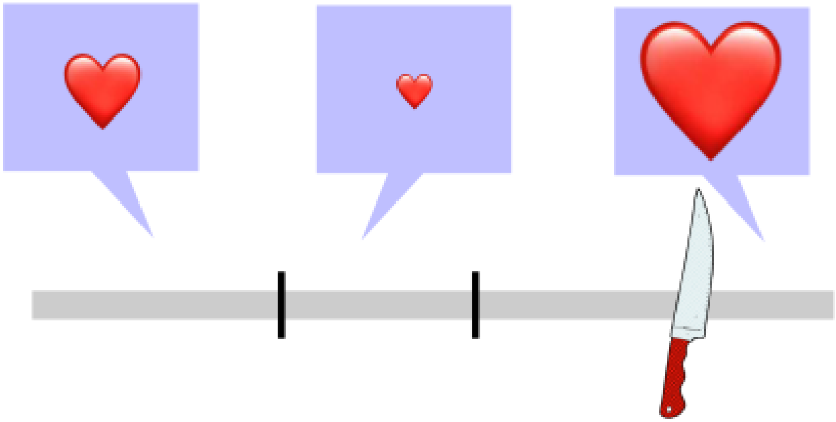

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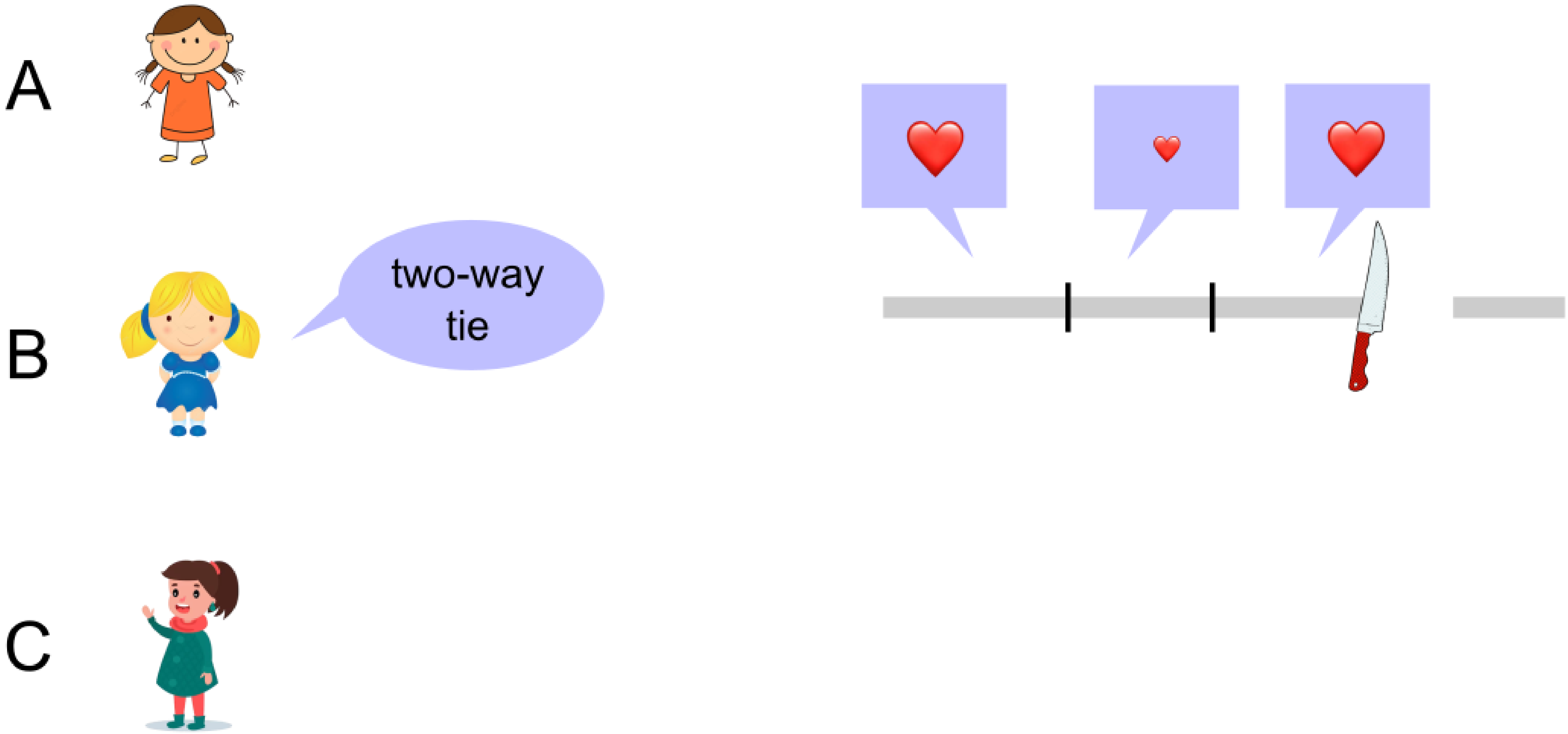

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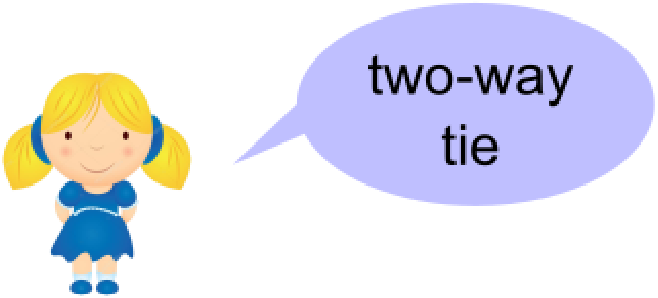

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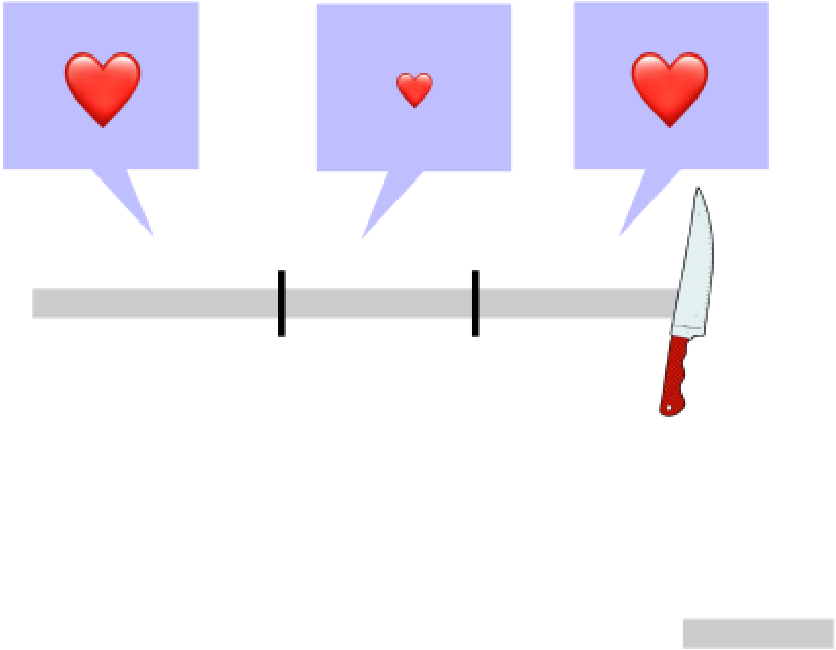

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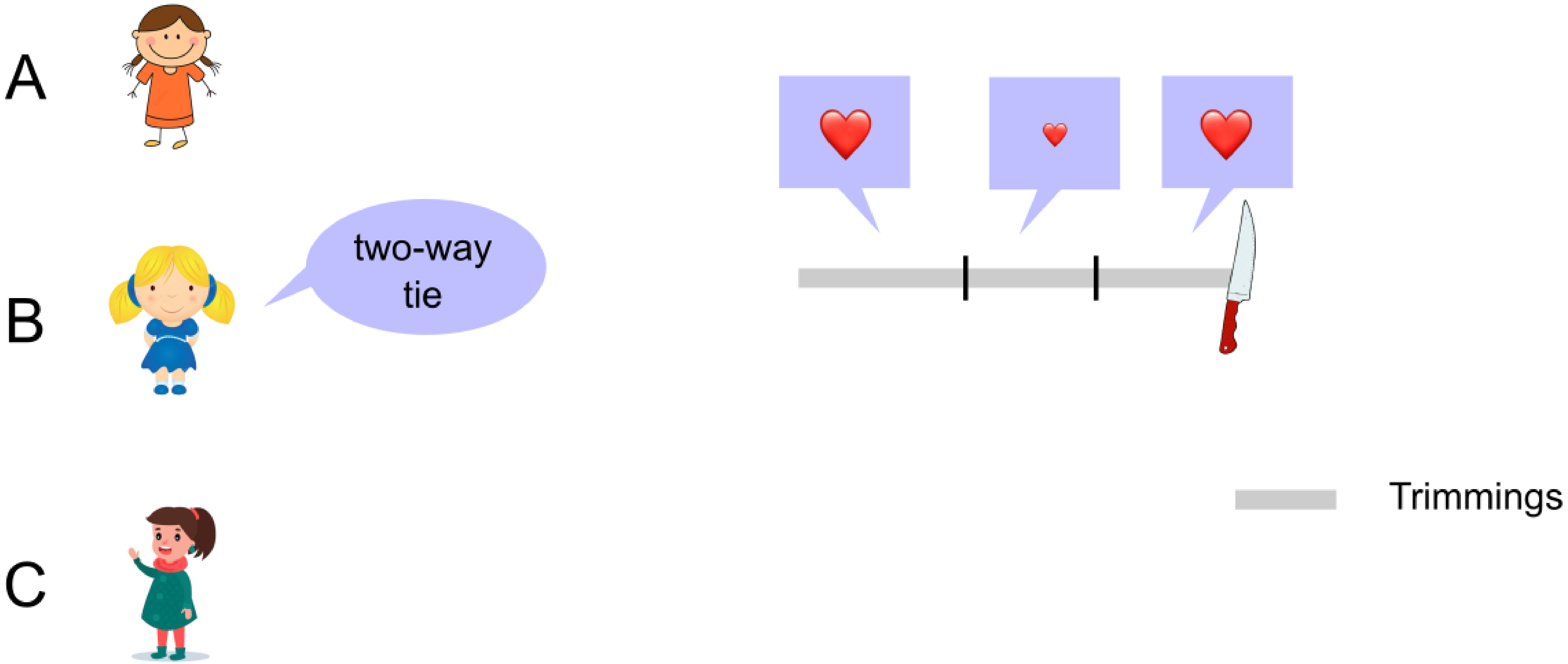

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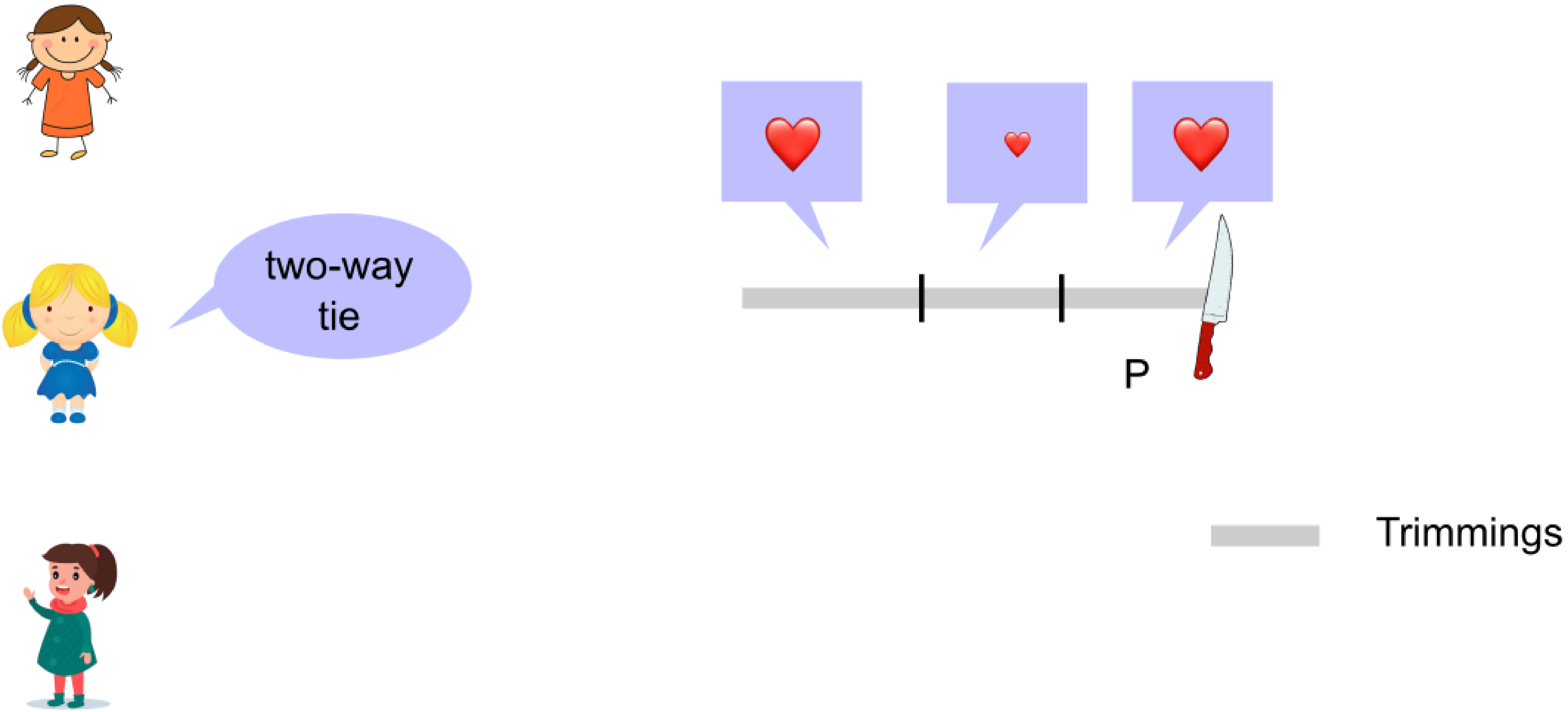

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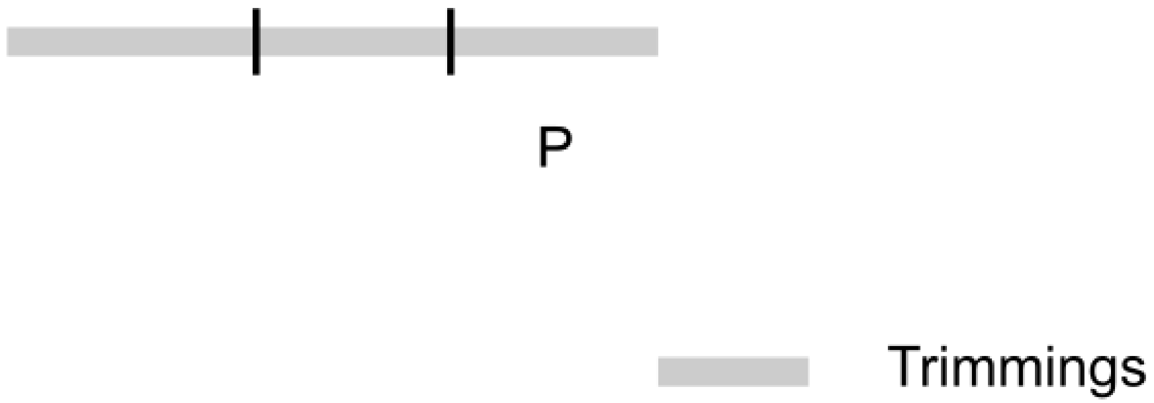
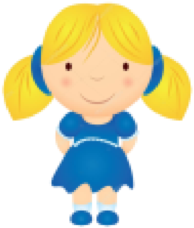


Envy-free for three

A

B



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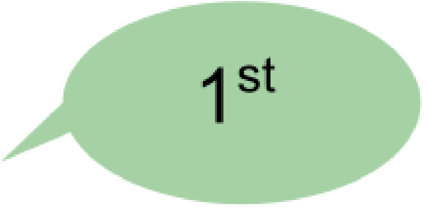
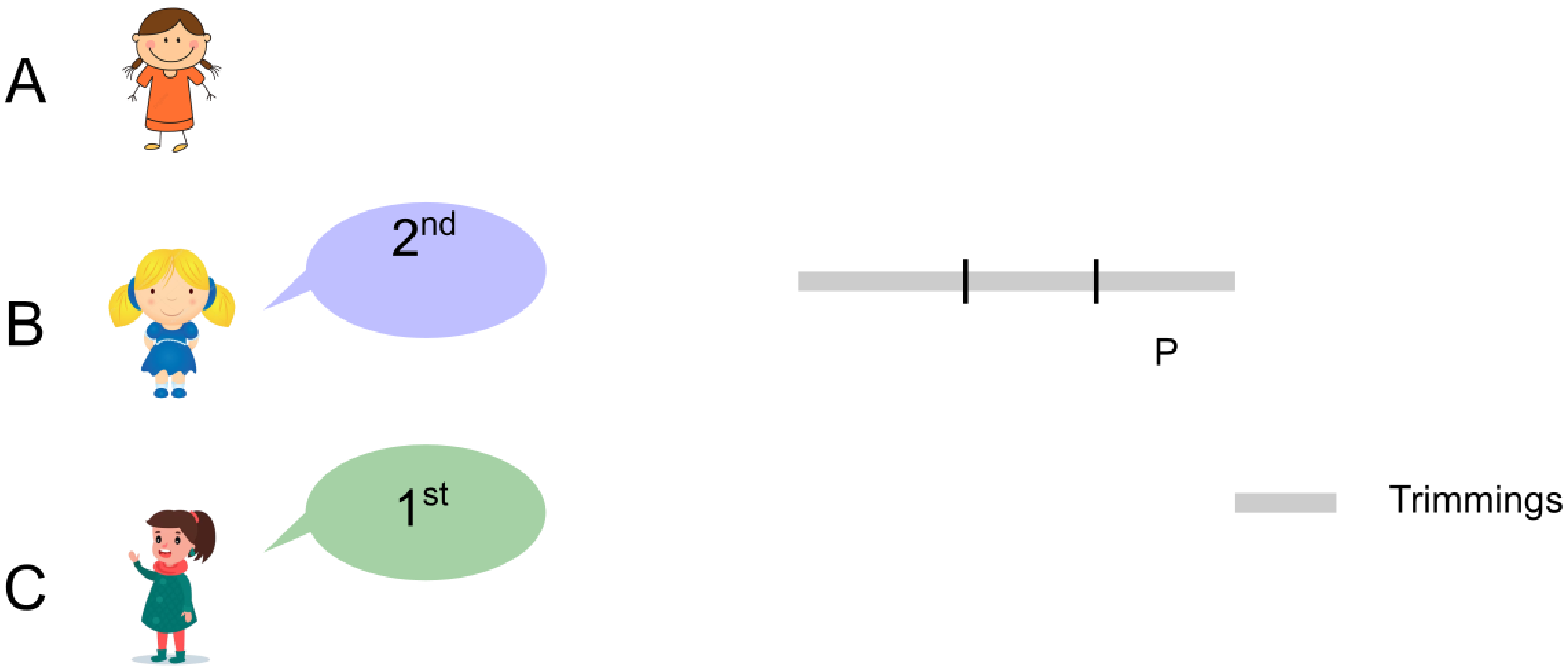


Trimmings

Envy-free for three



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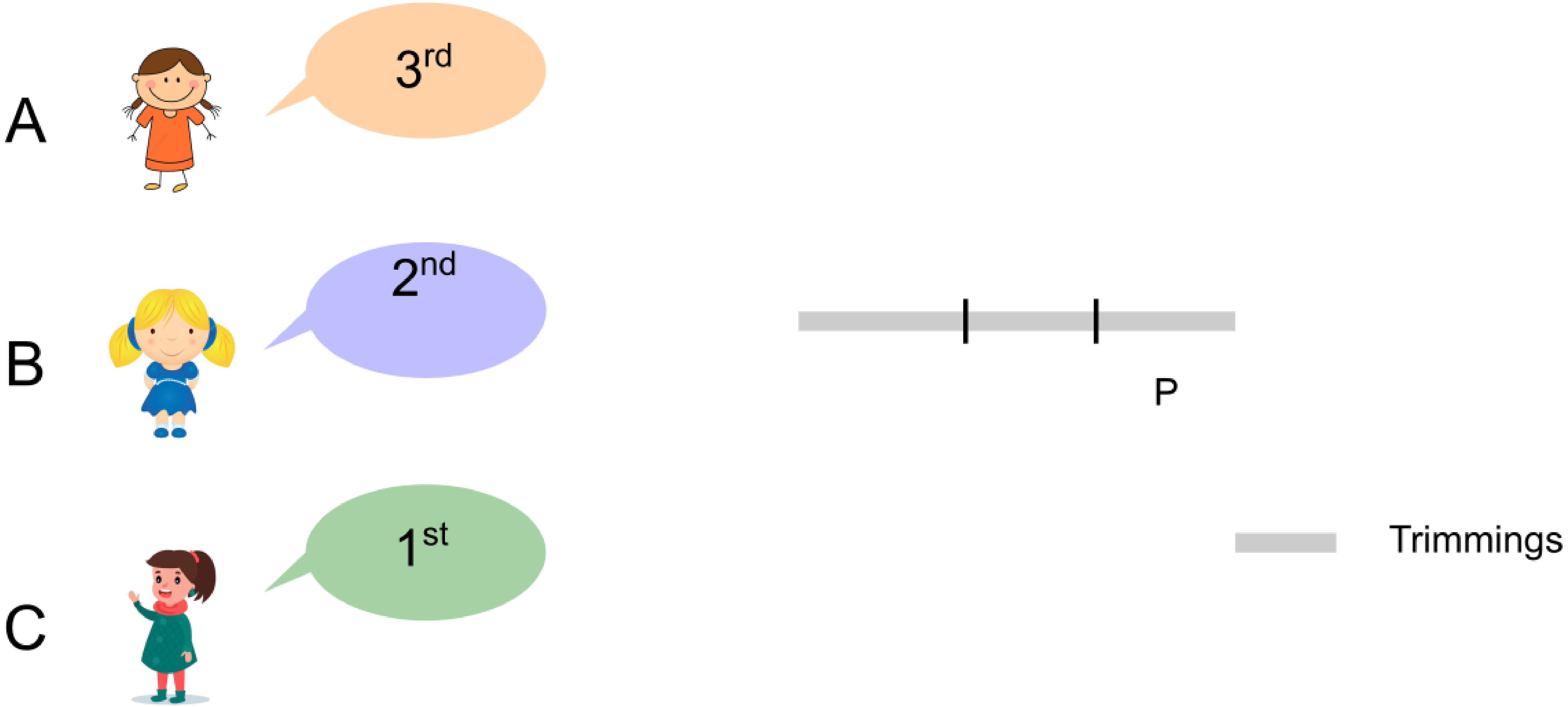

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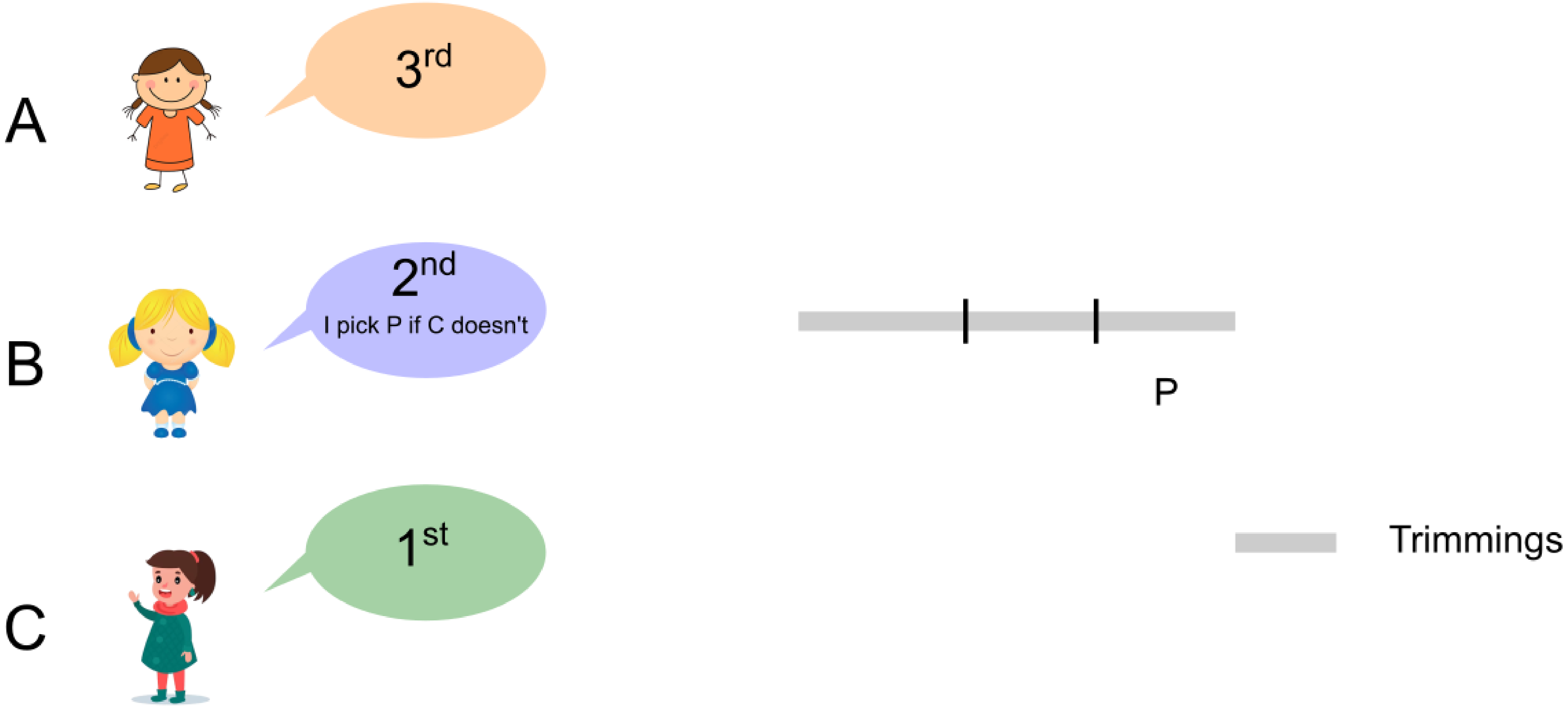

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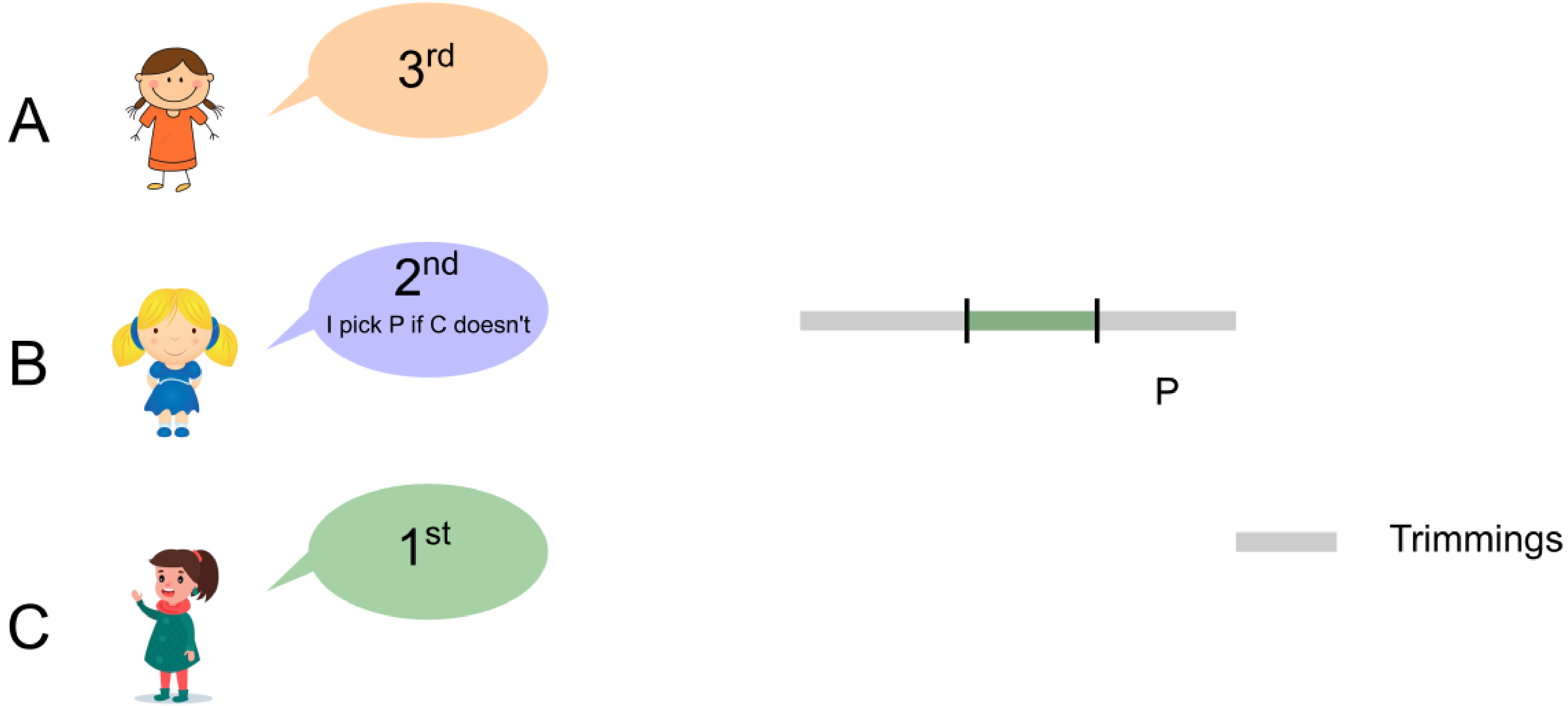
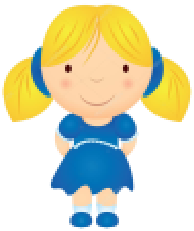


Envy-free for three

A

3^{rd}

B

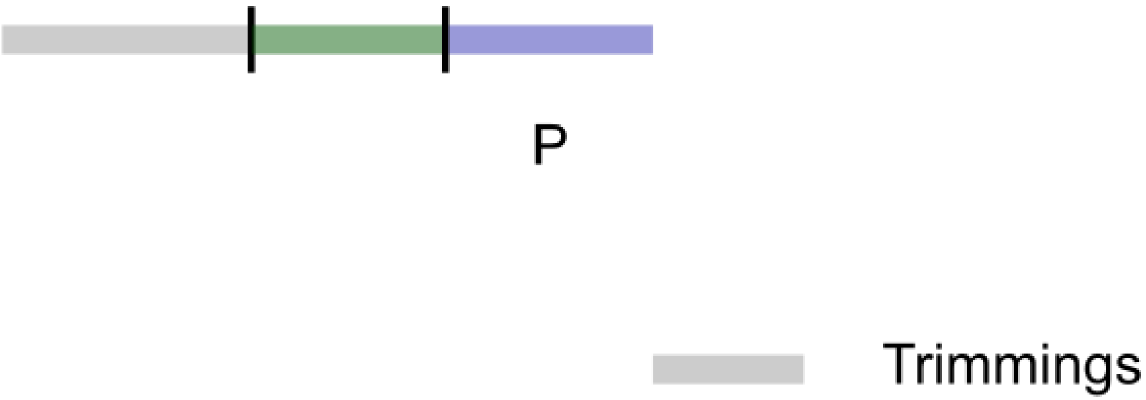


2^{nd}

I pick P if C doesn't

C

1^{st}



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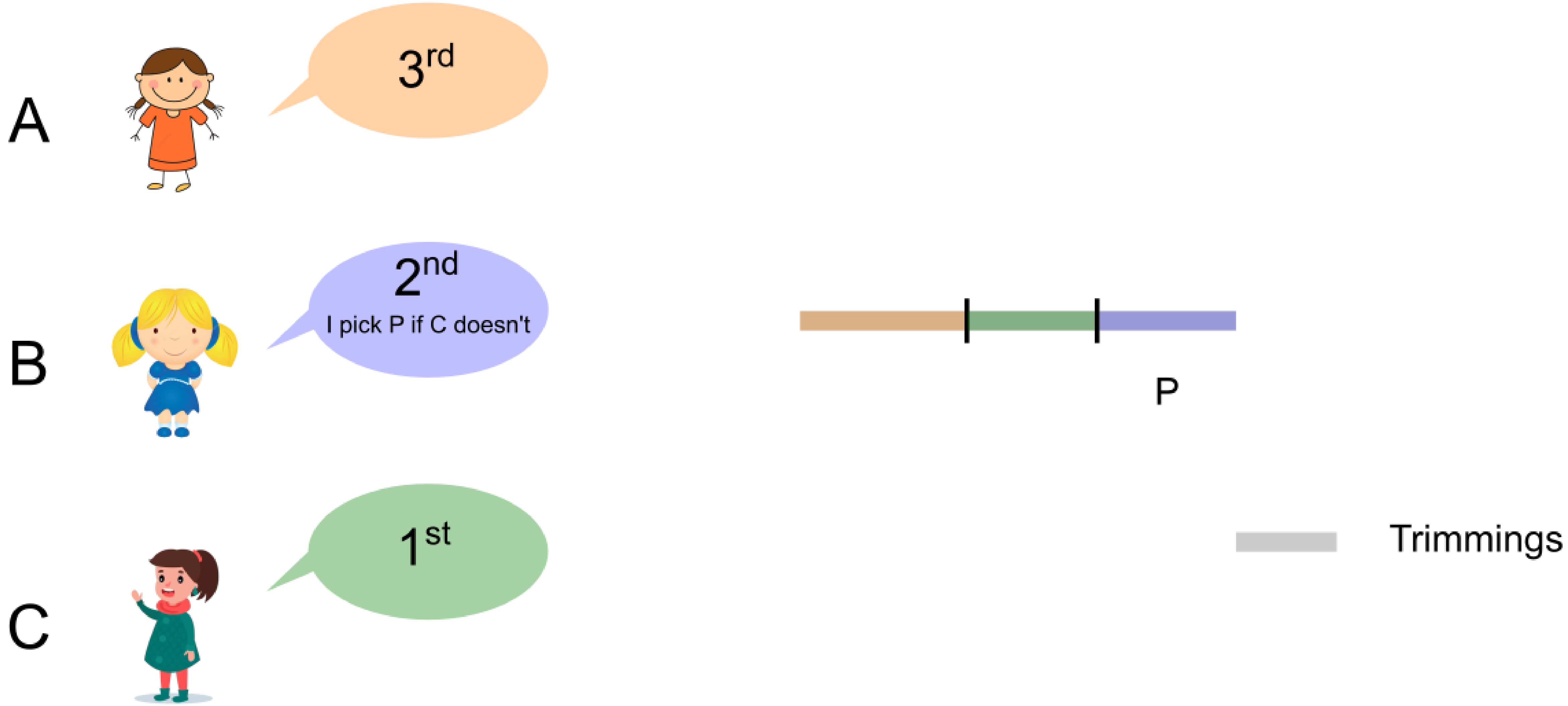
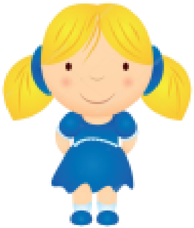


Envy-free for three

A

EF because of equal cuts

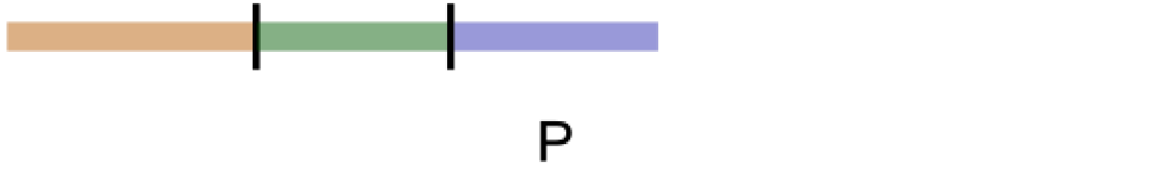
B



EF because of two-way tie

C

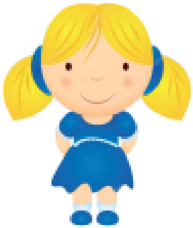
EF because I picked first



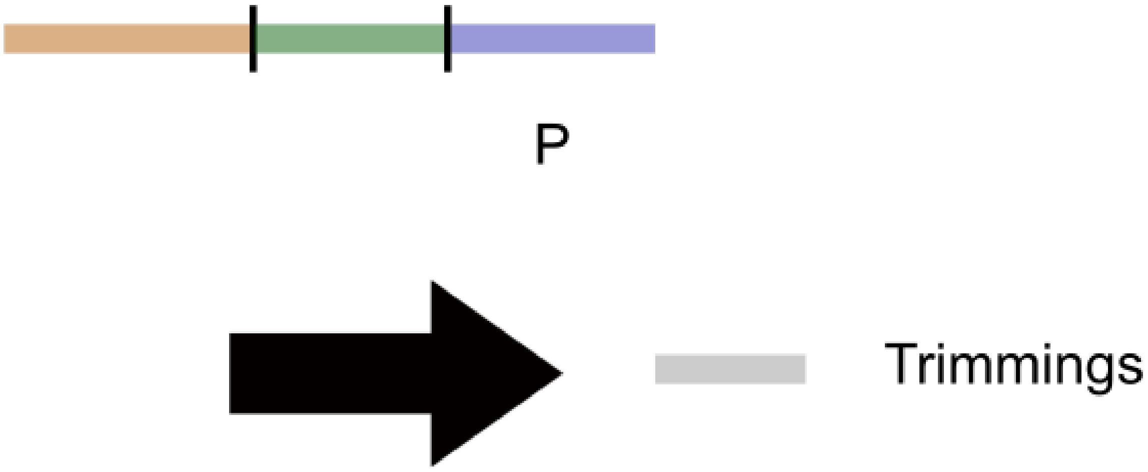
Envy-free for three

A

B



C

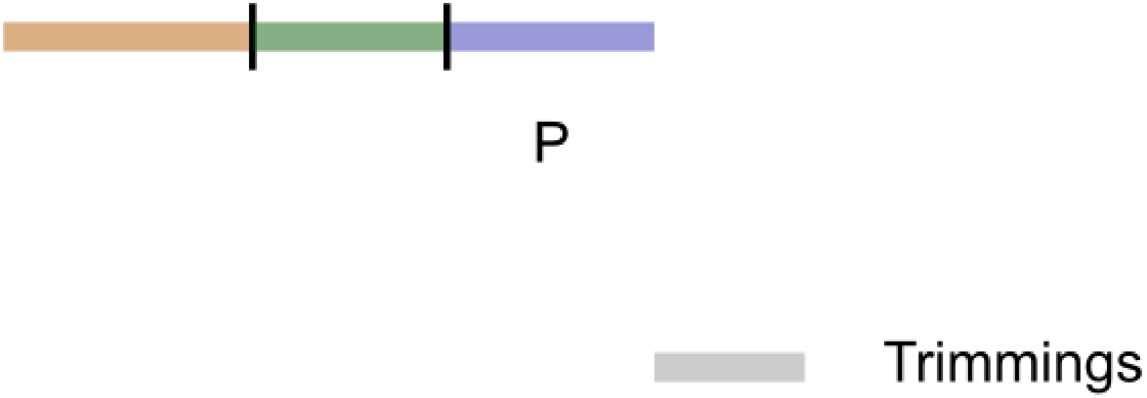


Envy-free for three

A

B

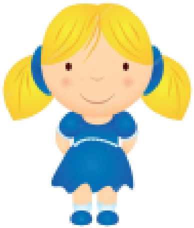
C



Envy-free for three

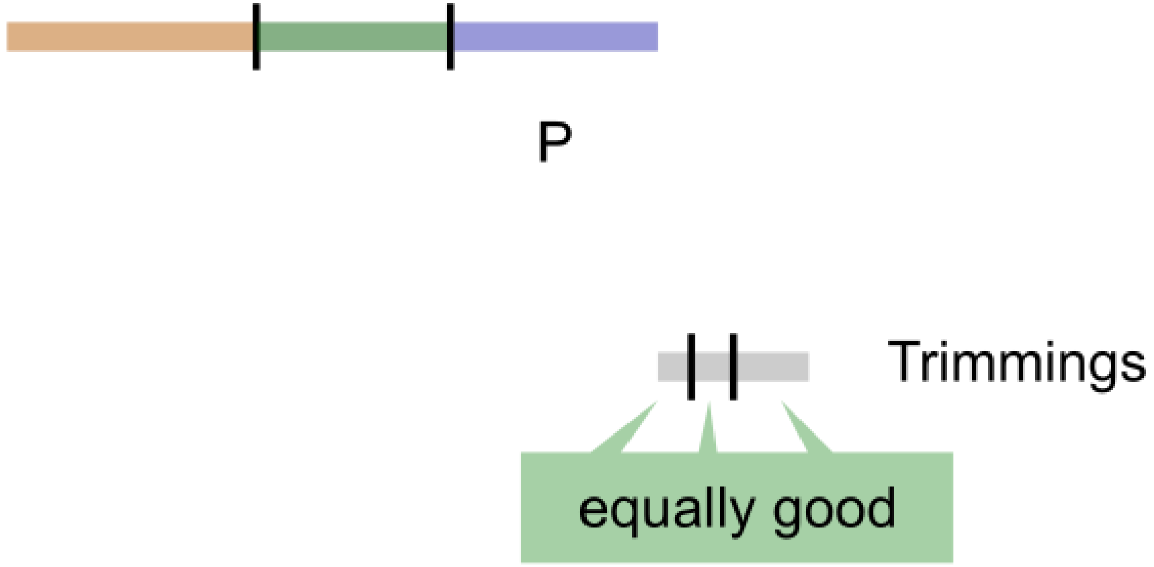
A

B



C

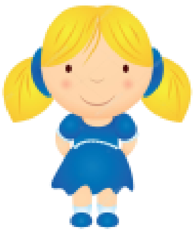
I equidivide



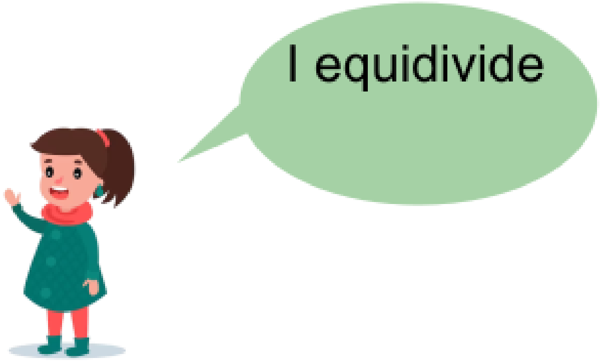
Envy-free for three

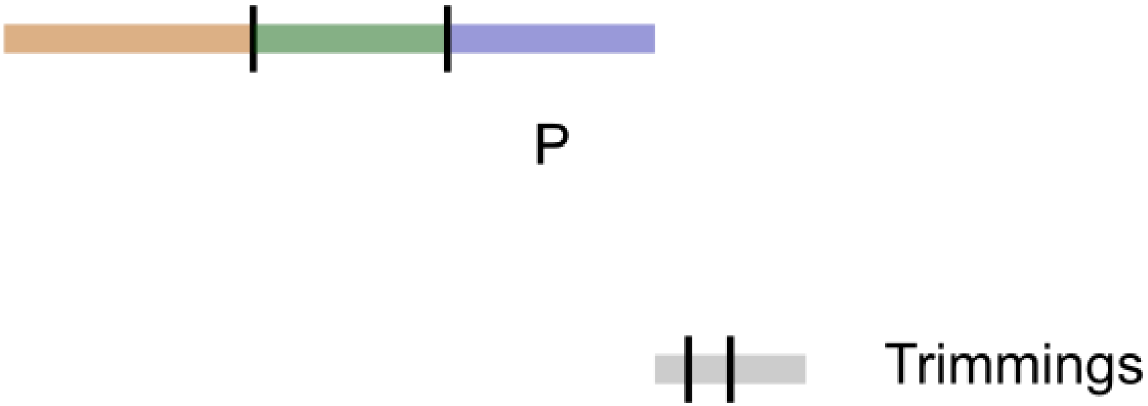
A

B



C

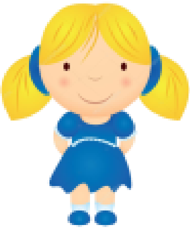




Envy-free for three

A

B

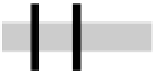


I pick first yay!

C

I equidivide

P



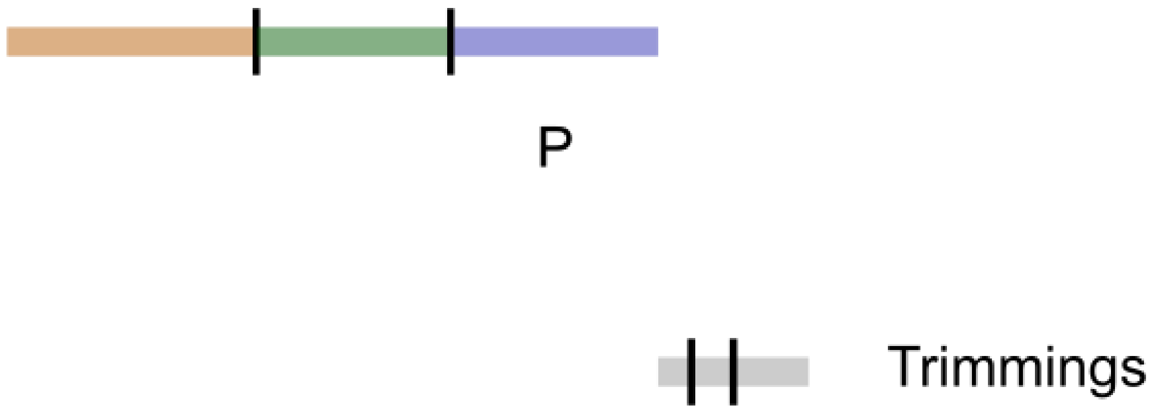
Trimmings

Envy-free for three

A

B

C

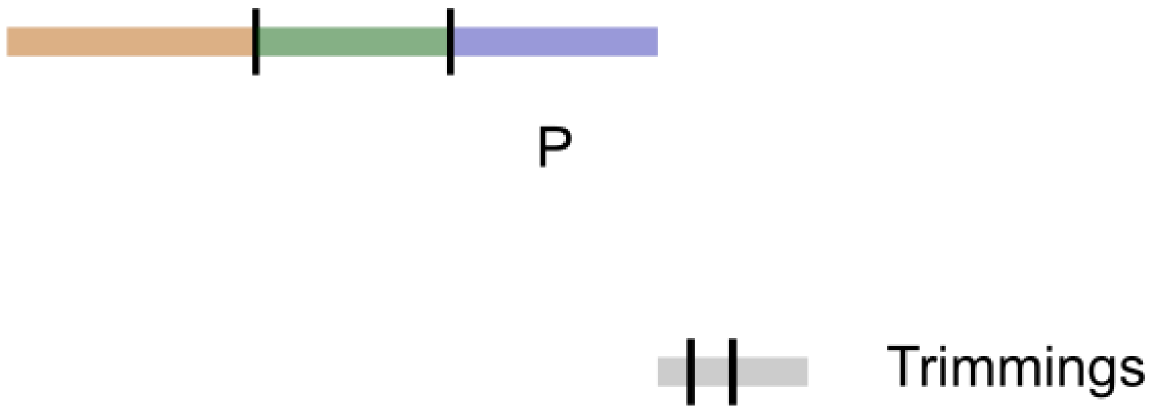


Envy-free for three

A

B

C

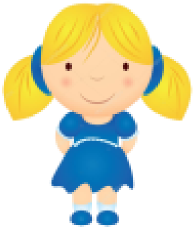


Envy-free for three

A

I can pick after B does

B

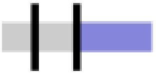


I pick first yay!

C

I equidivide and pick last

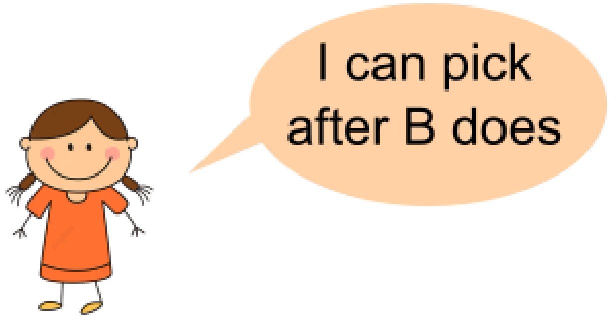
P



Trimmings

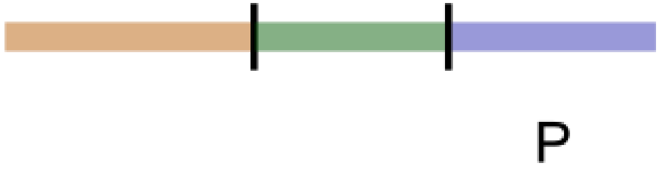
Envy-free for three

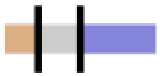
A



B

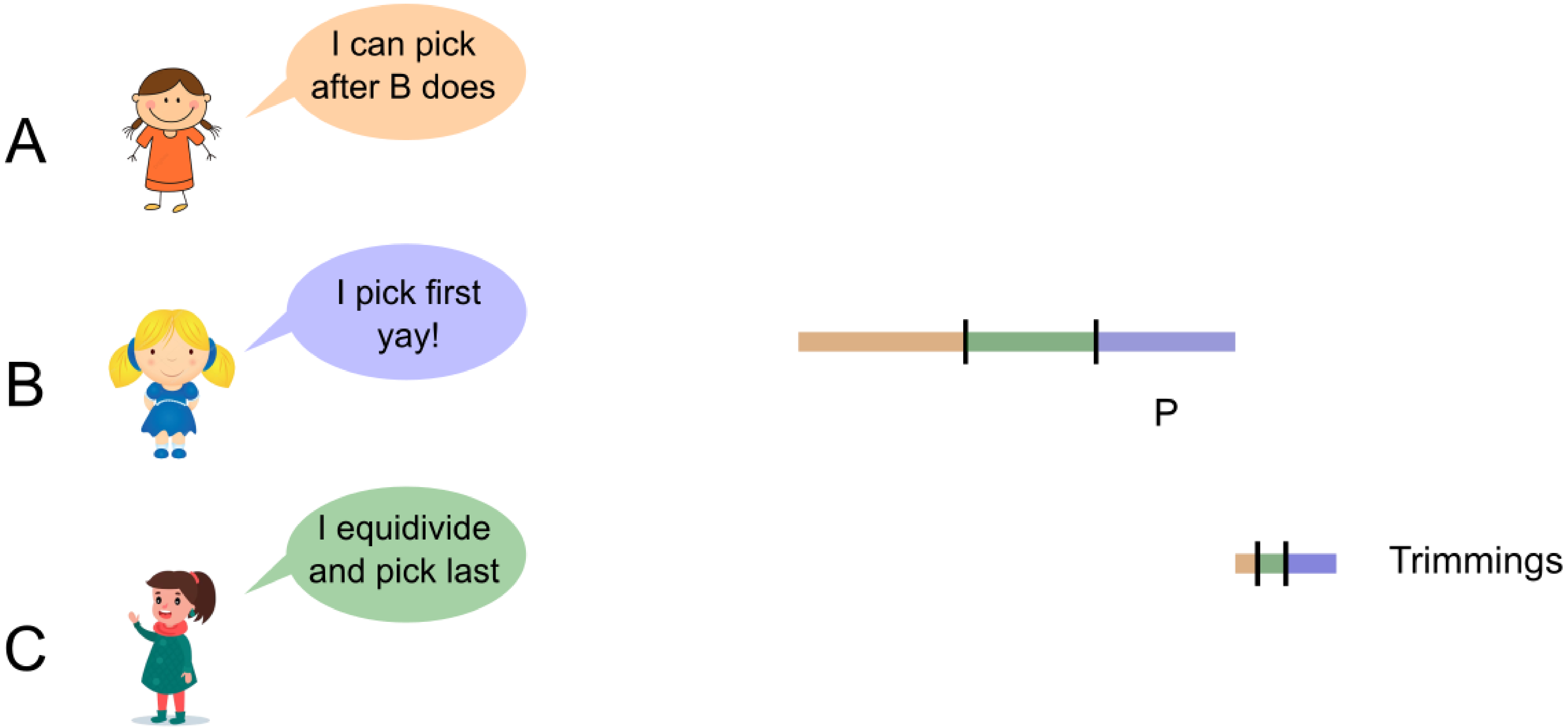
C





Trimmings

Envy-free for three



Envy-free for three

A

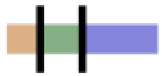
Irrevocable advantage

B

EF because I picked first

C

EF because I equidivided

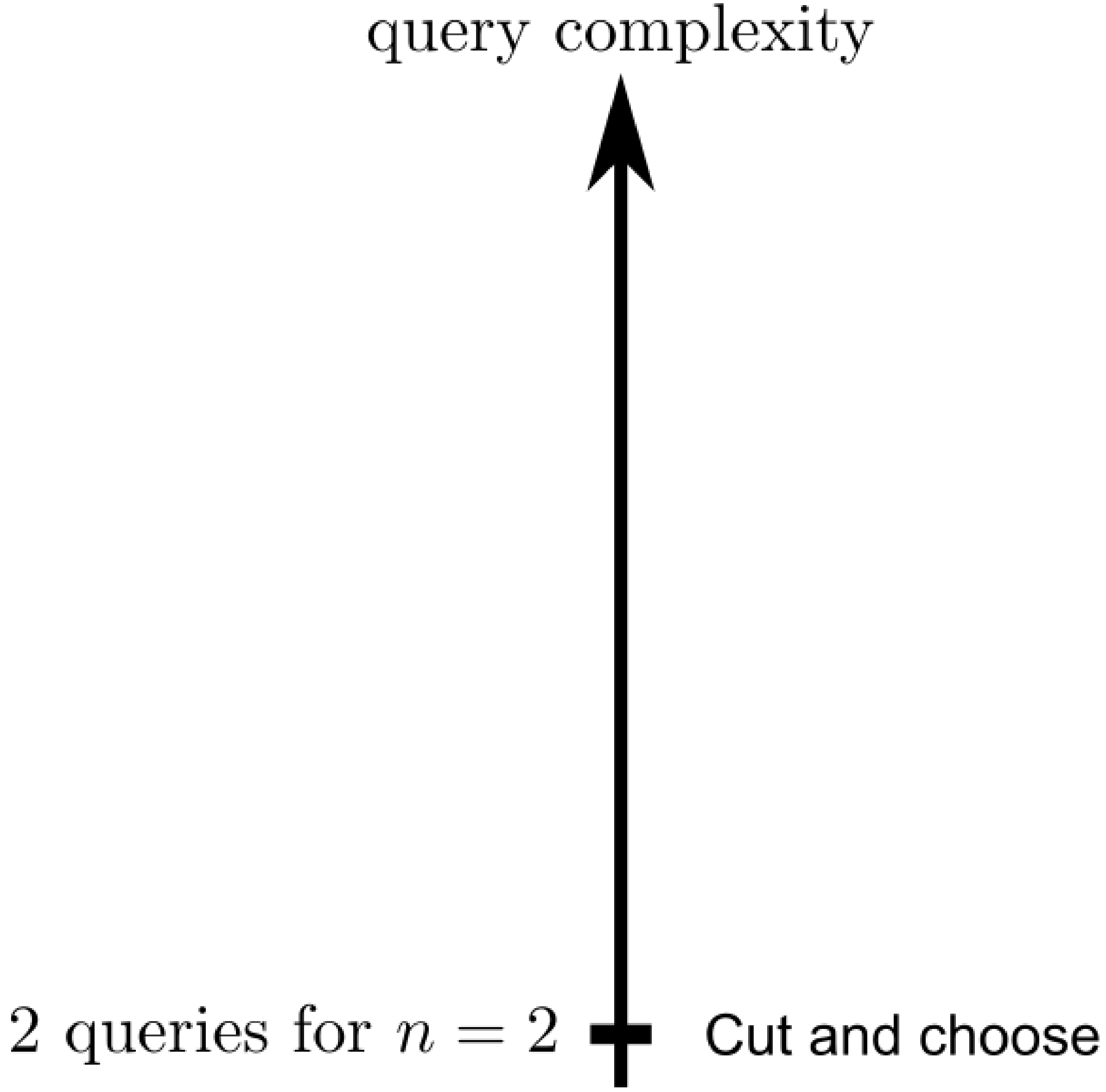


Exercise

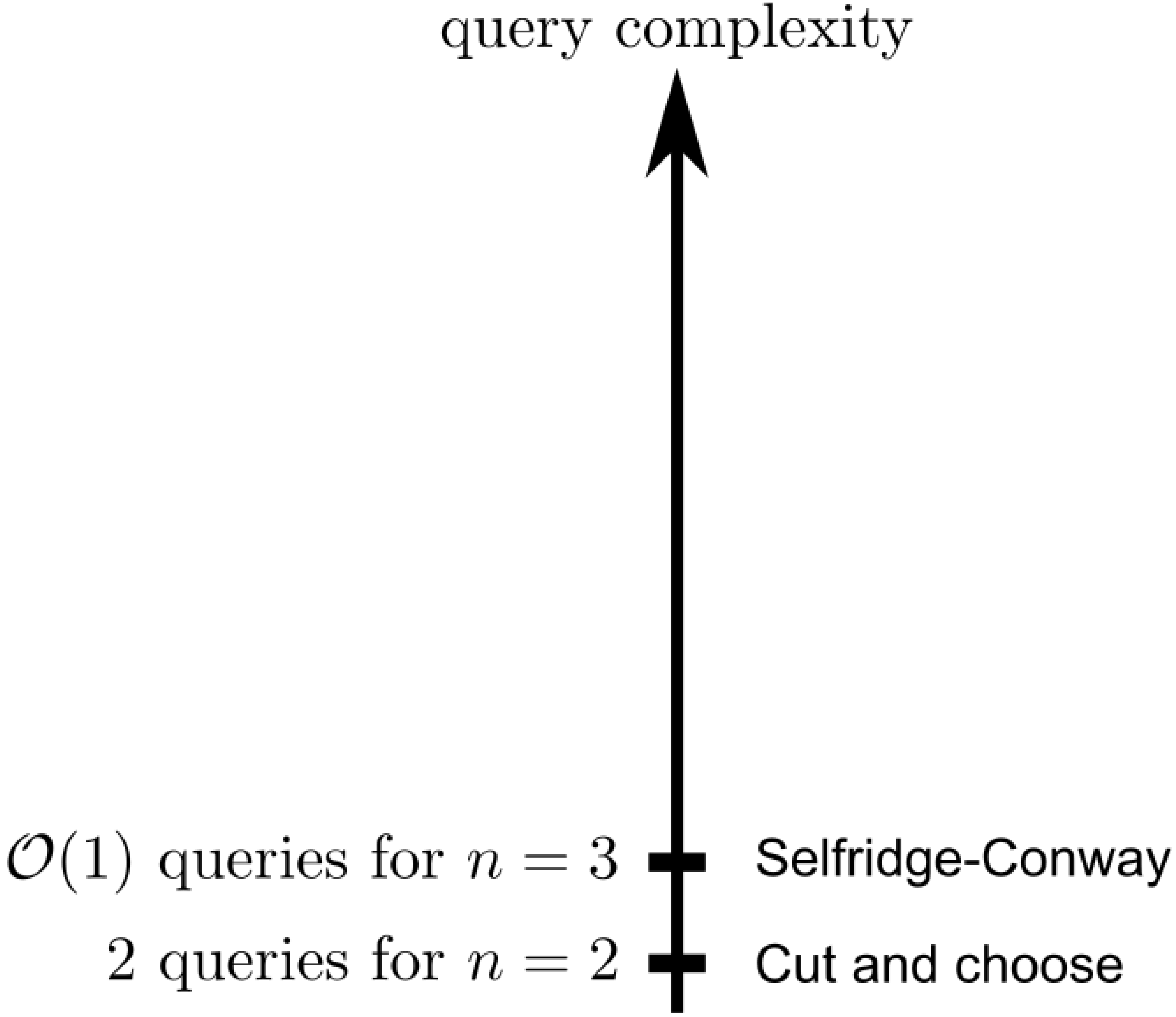
How many queries does the three-person EF protocol require?

The Story of Envy-freeness

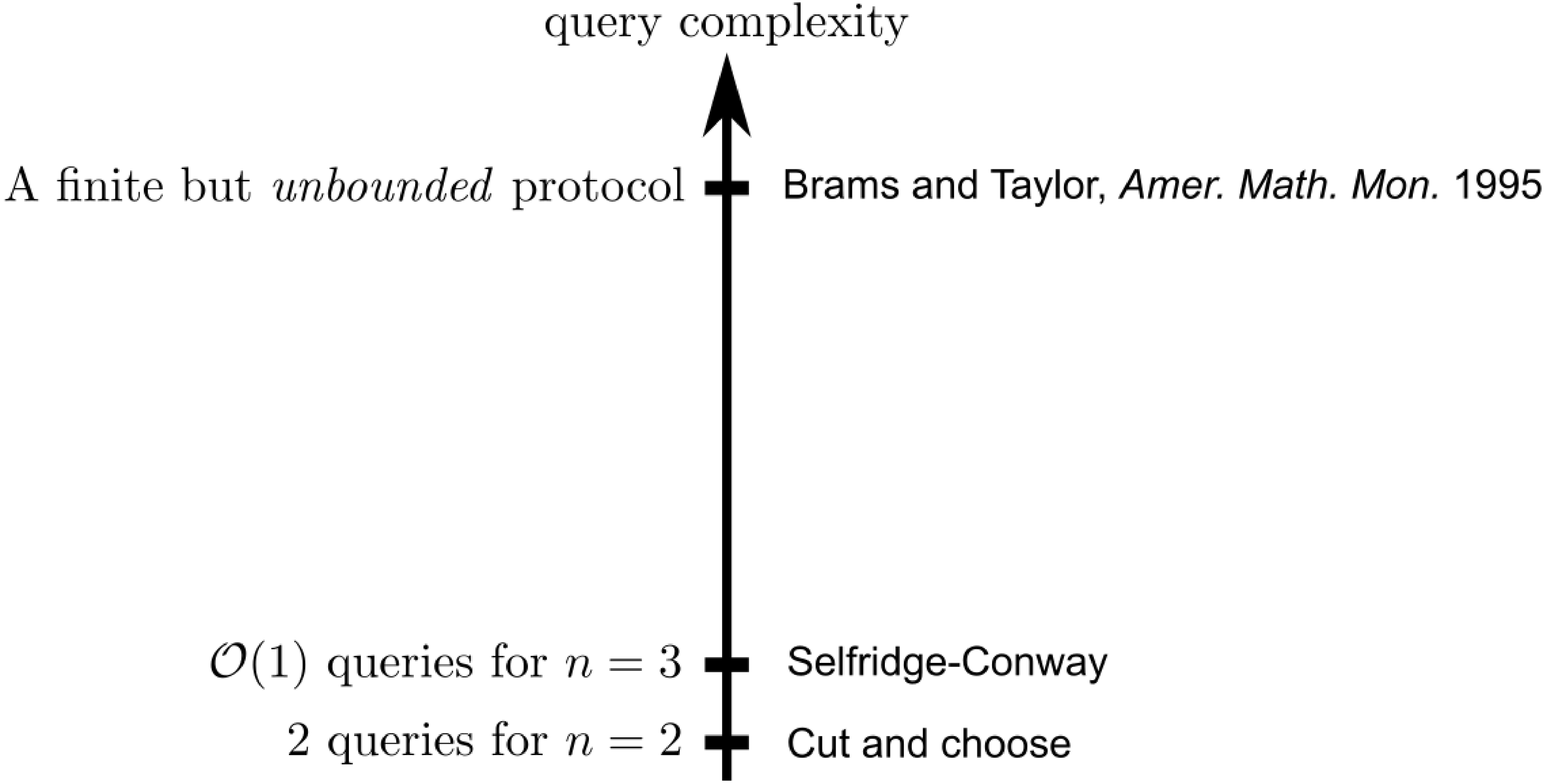
The Story of Envy-freeness



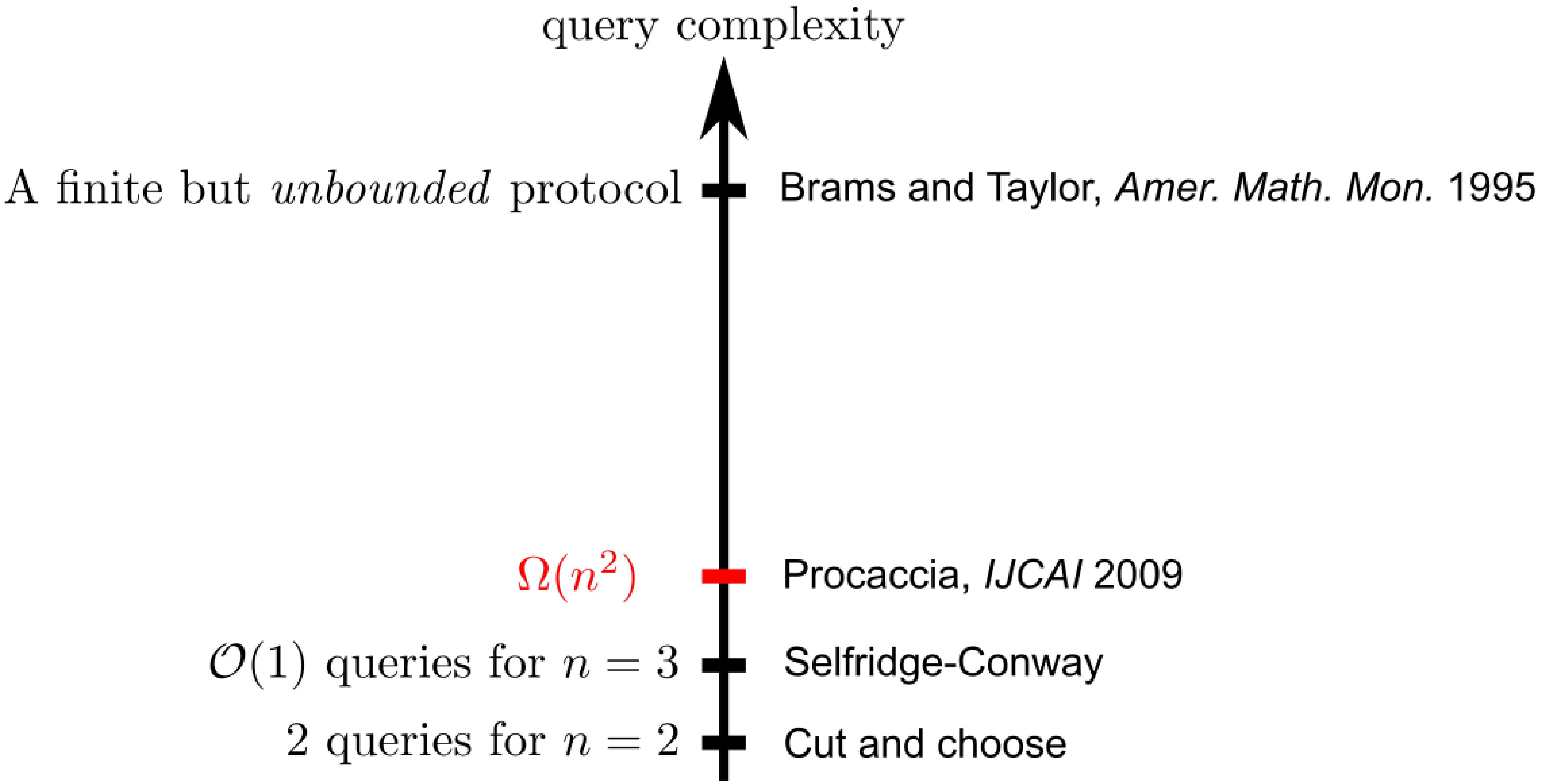
The Story of Envy-freeness



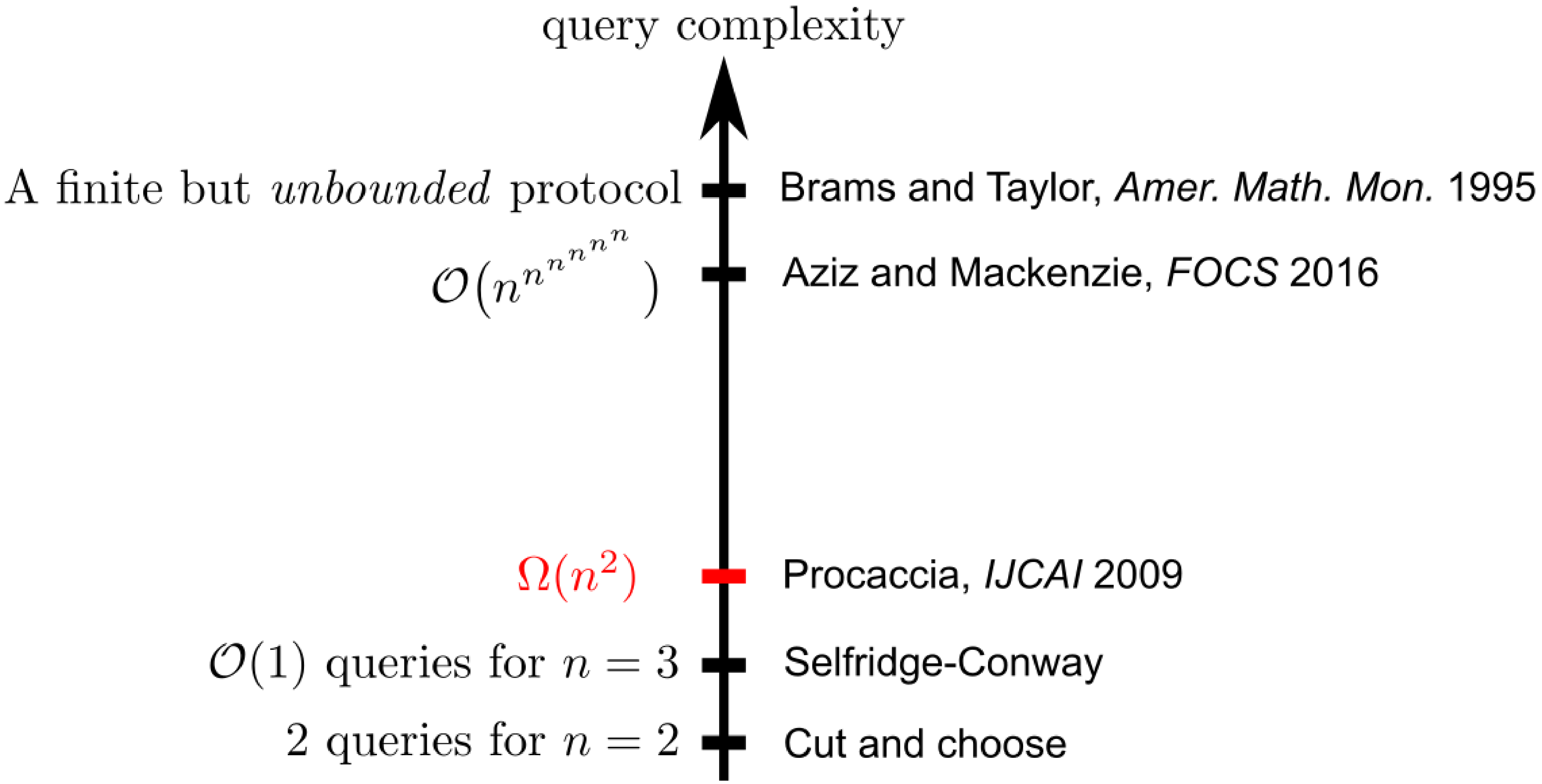
The Story of Envy-freeness



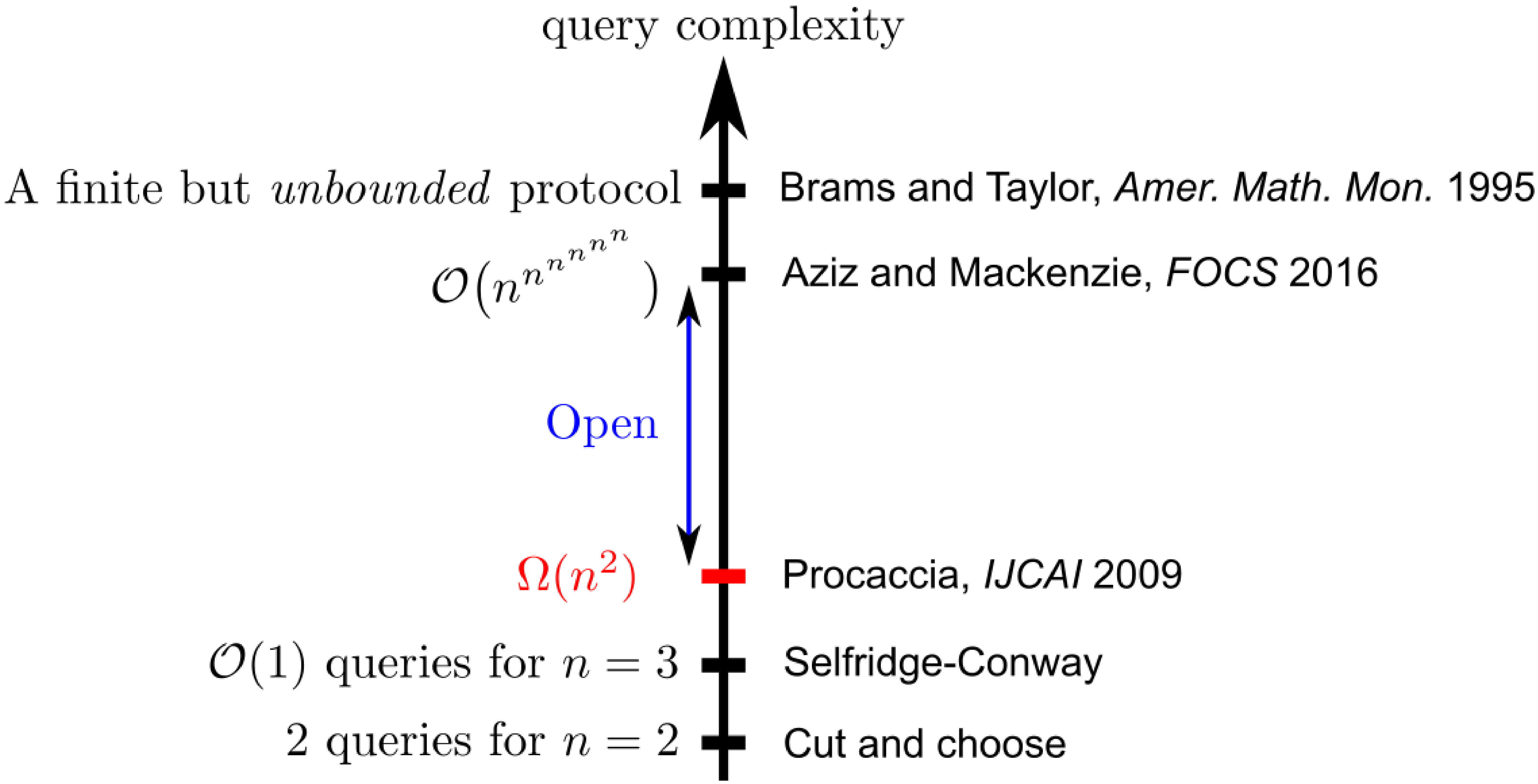
The Story of Envy-freeness



The Story of Envy-freeness



The Story of Envy-freeness



References

- Introduction to cake-cutting algorithms.

  Ariel Procaccia
  "*Cake Cutting Algorithms*"
  Chapter 13 in Handbook of Computational Social Choice

- Lecture by Ariel Procaccia on "Cake cutting" in the *Optimized Democracy* course.
  https://sites.google.com/view/optdemocracy/schedule