STCS Vigyan Vidushi 2025

Cake Cutting

Rohit Vaish

Fair Division

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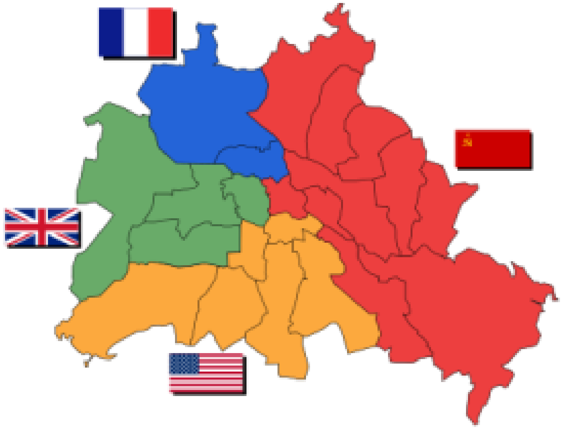

Fair Division

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Cake Cutting

How to fairly divide a cake

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among agents with differing preferences?

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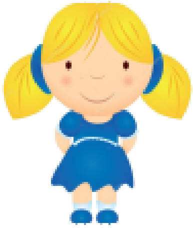

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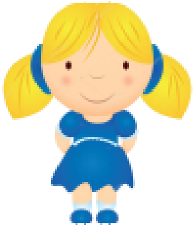

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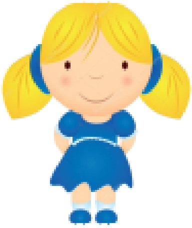

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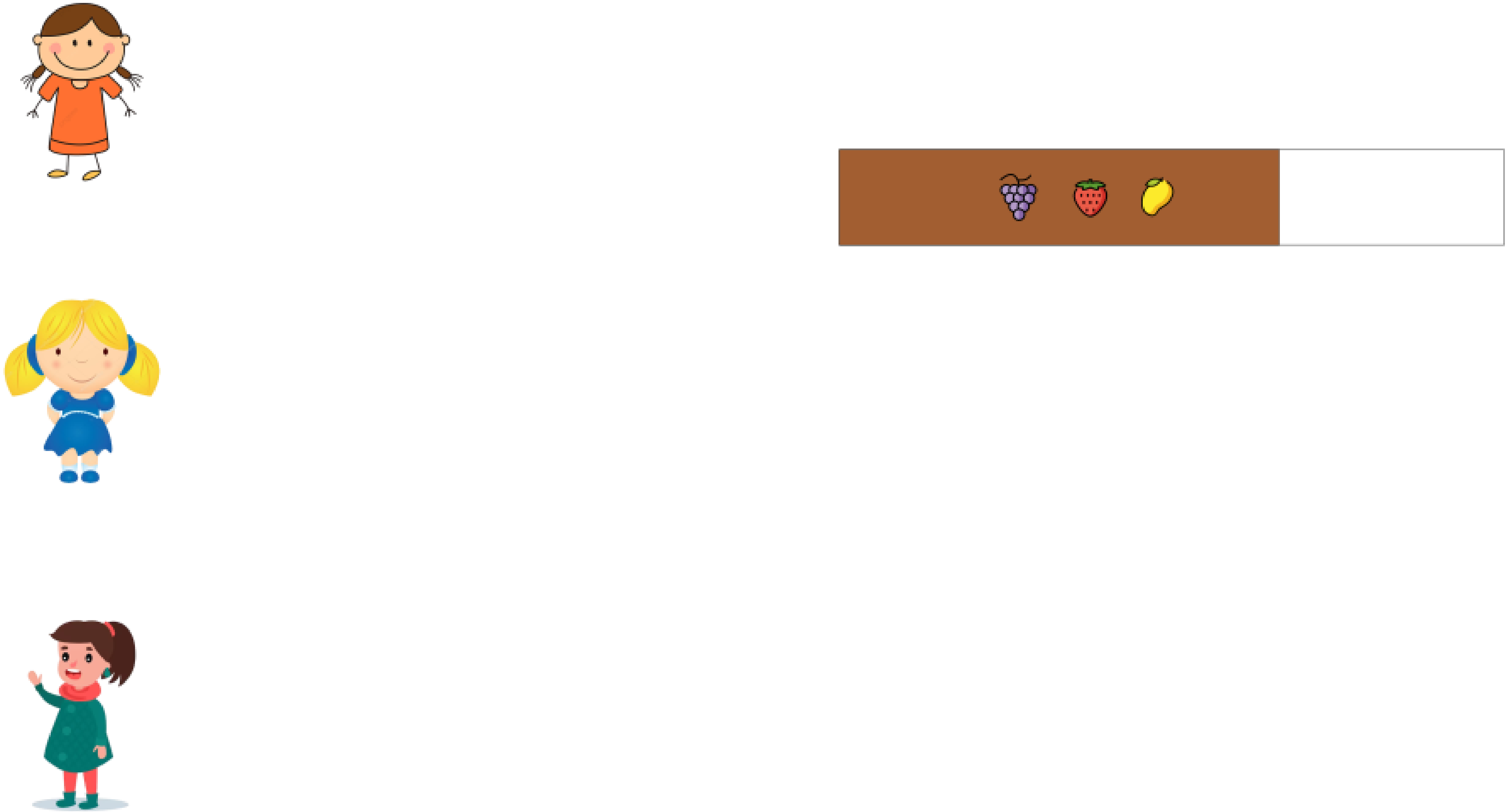

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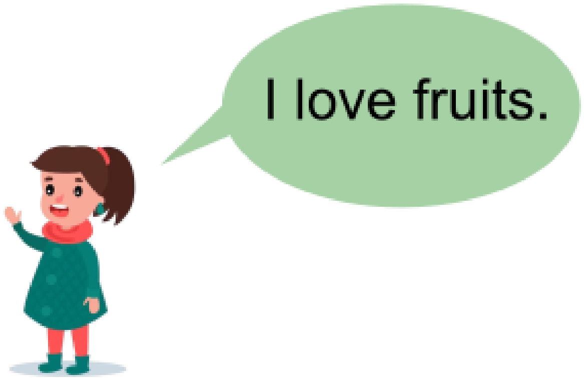

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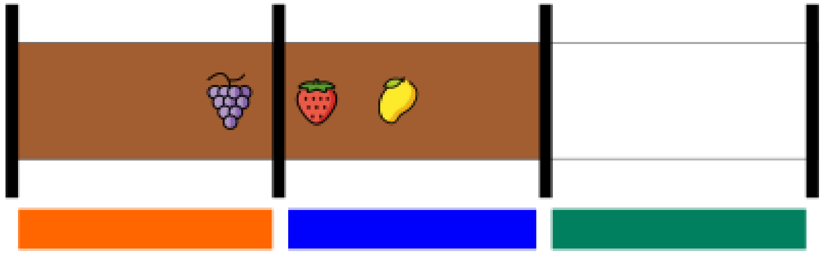

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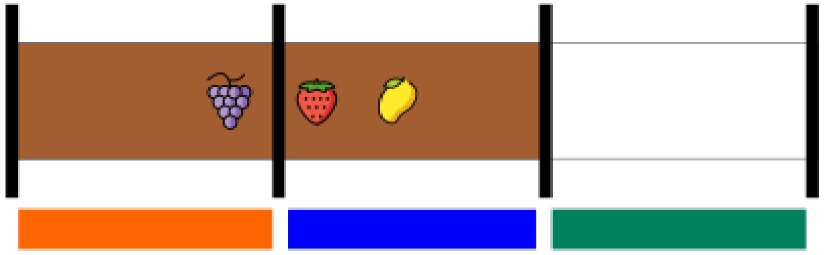

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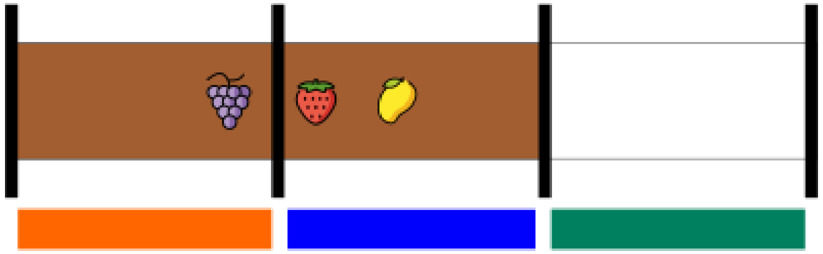

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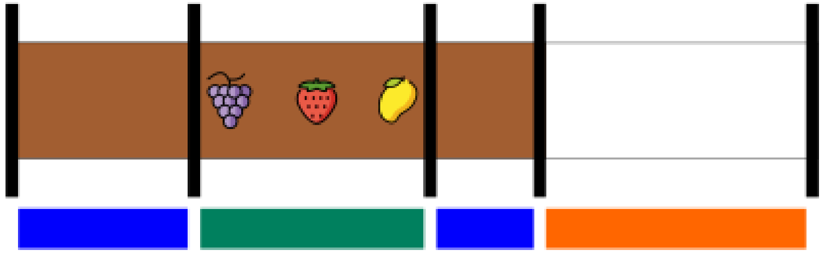

A fairer division 

Why is this problem interesting?

Preferences matter!

The Model

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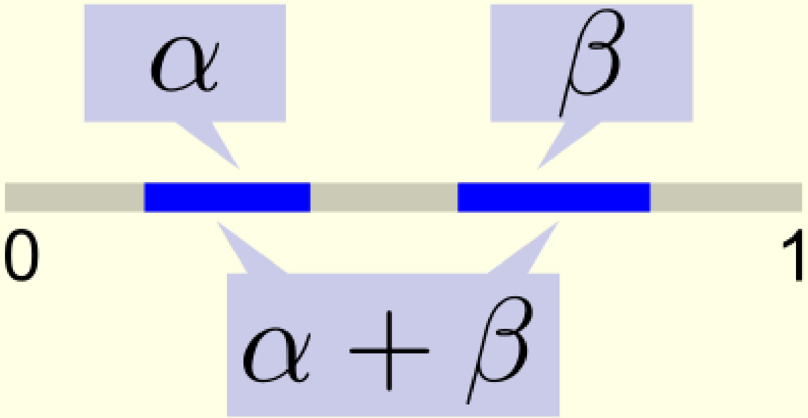

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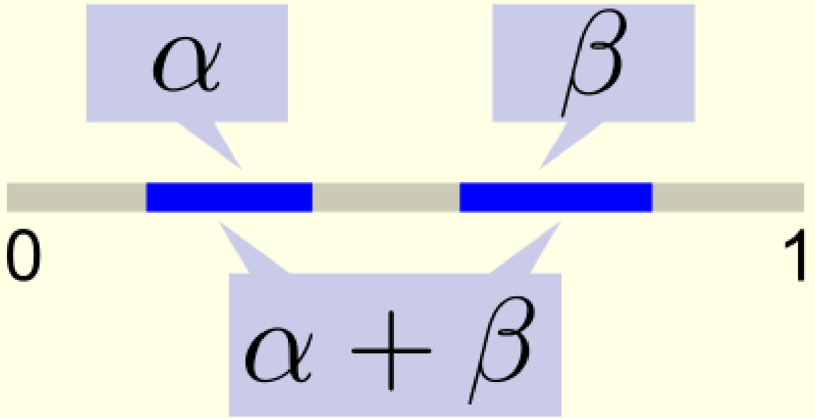

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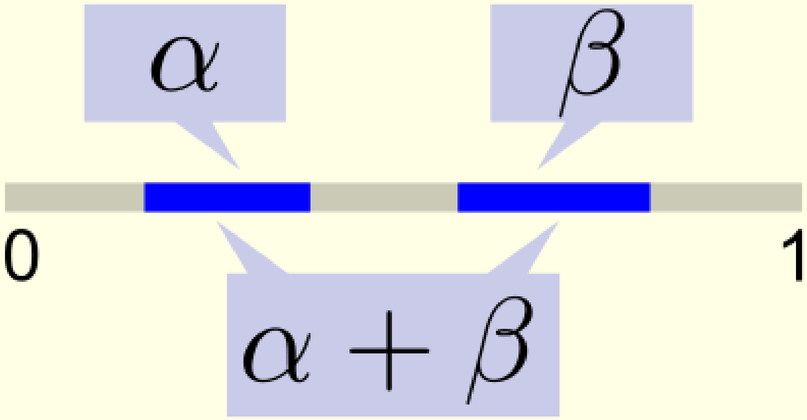

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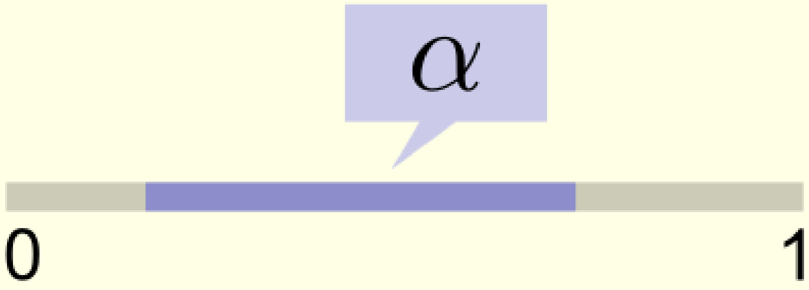

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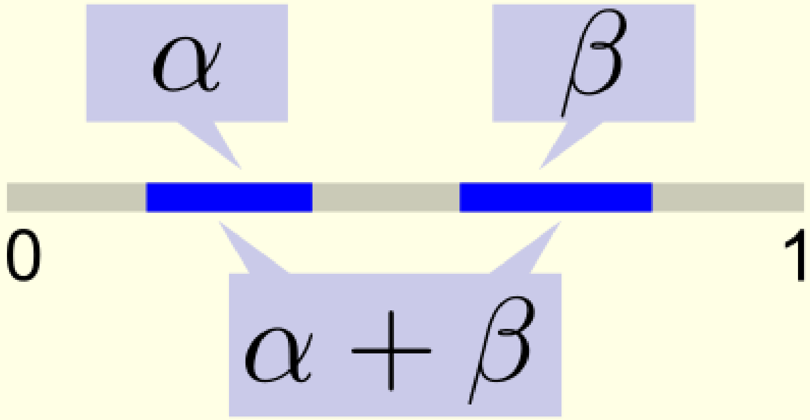

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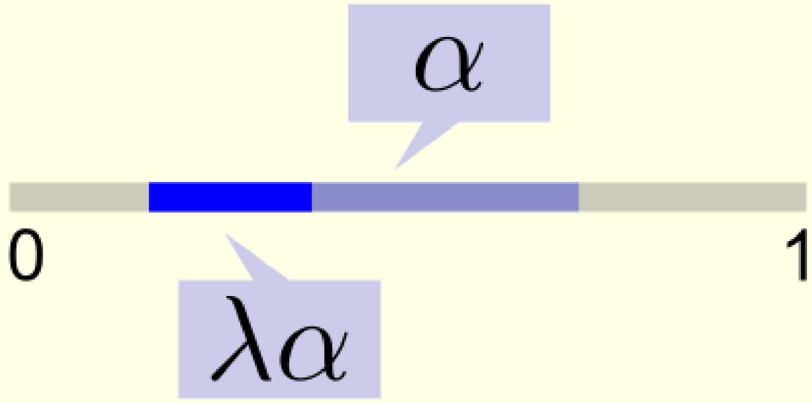

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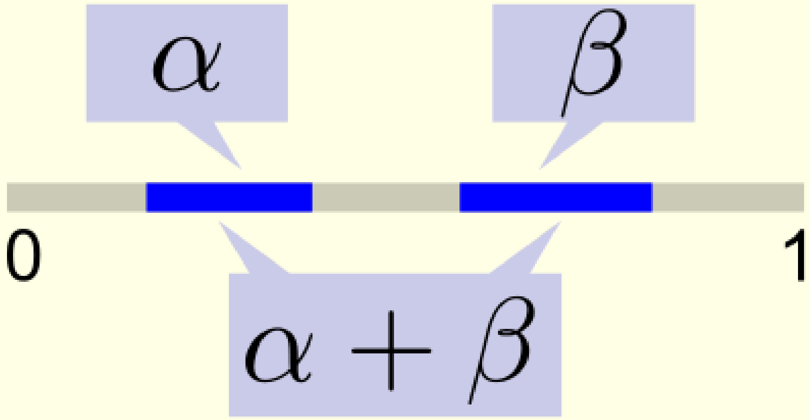

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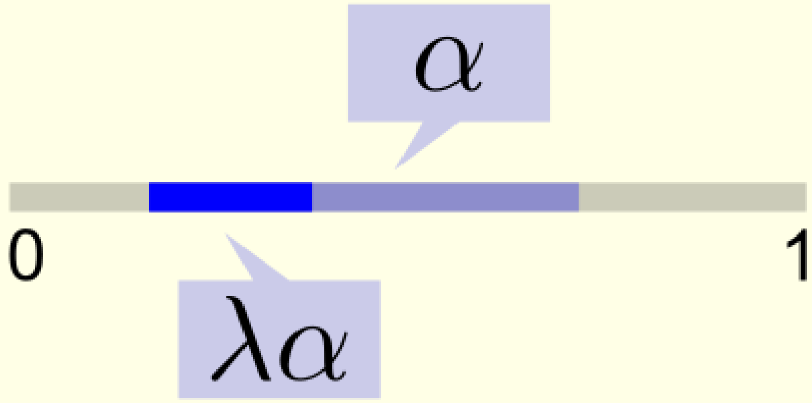

Normalization: for each agent i, $v_i([0,1]) = 1$.

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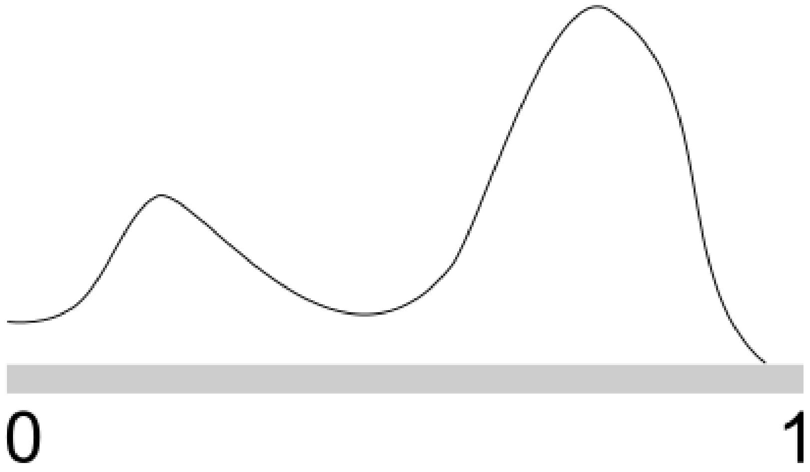
$$v_i(X) = \int_{x \in X} f_i(x)dx$$



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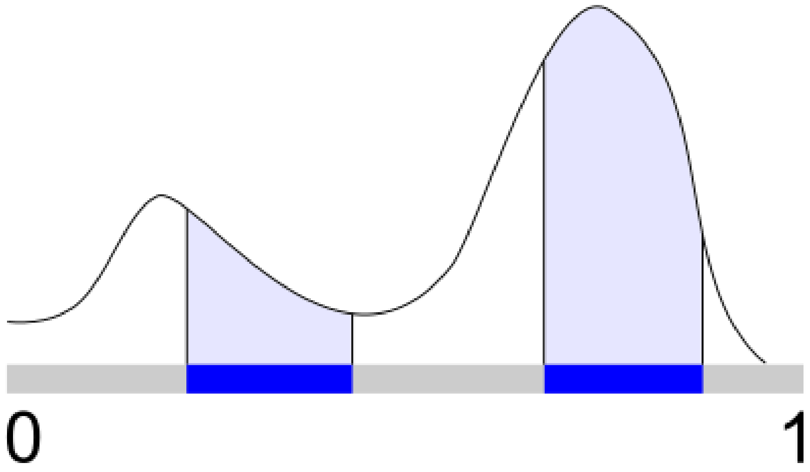
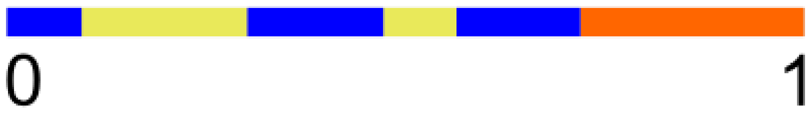
value density function



Fairness notions

- *Allocation/Division*: A partition $(A_1, A_2, \ldots, A_n)$ of the cake [0,1] where each A_i is a piece of cake.



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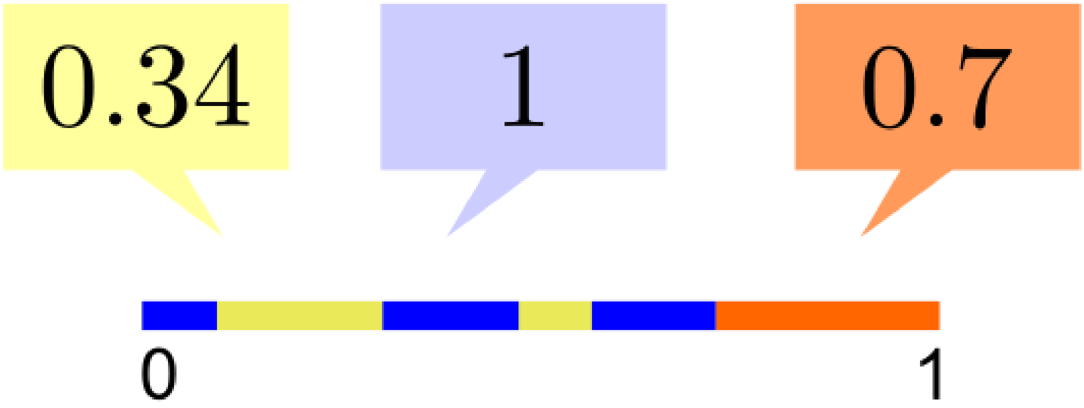

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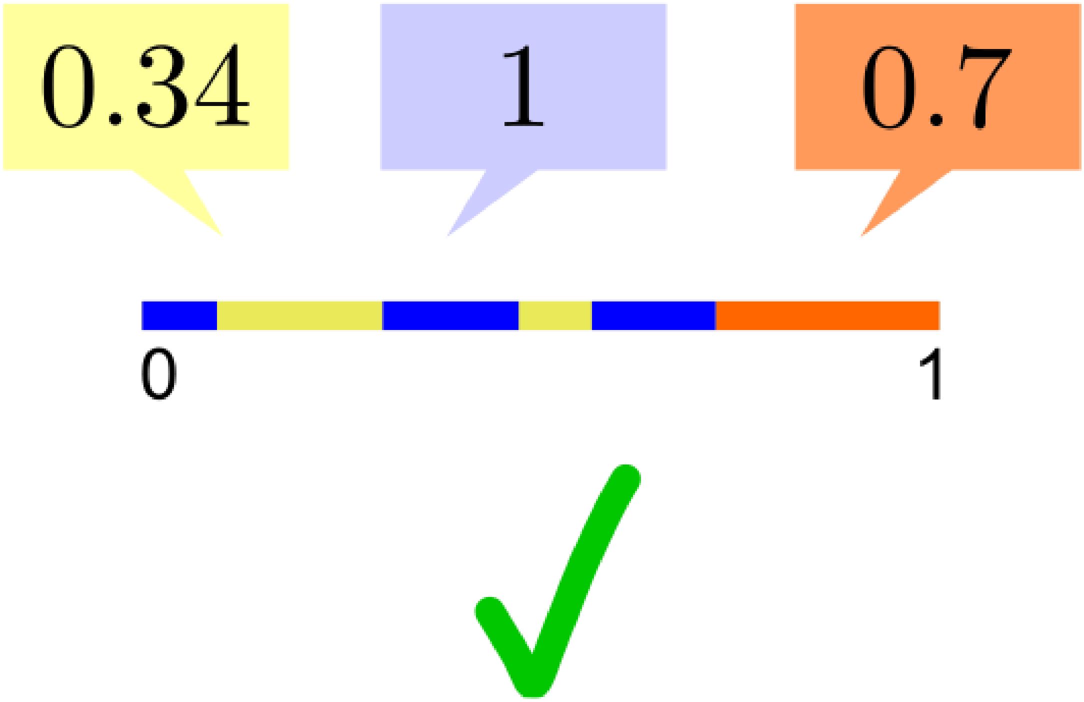

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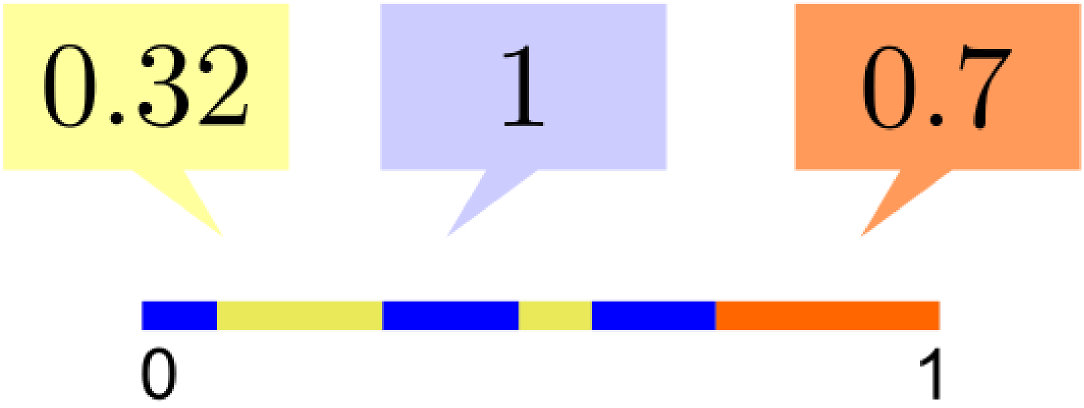

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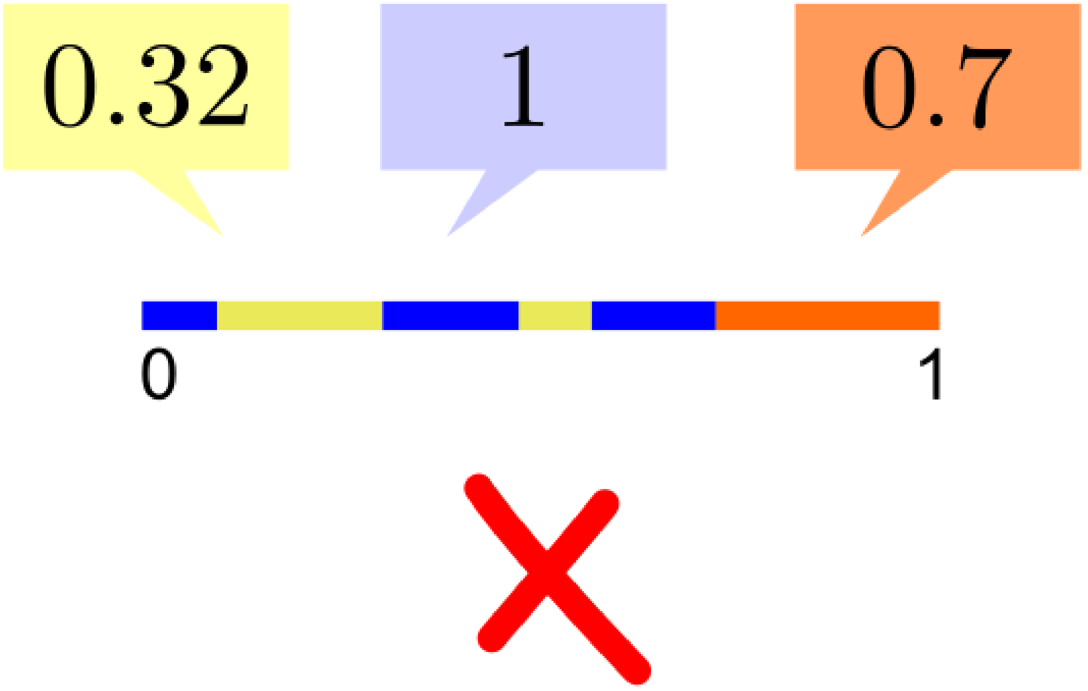

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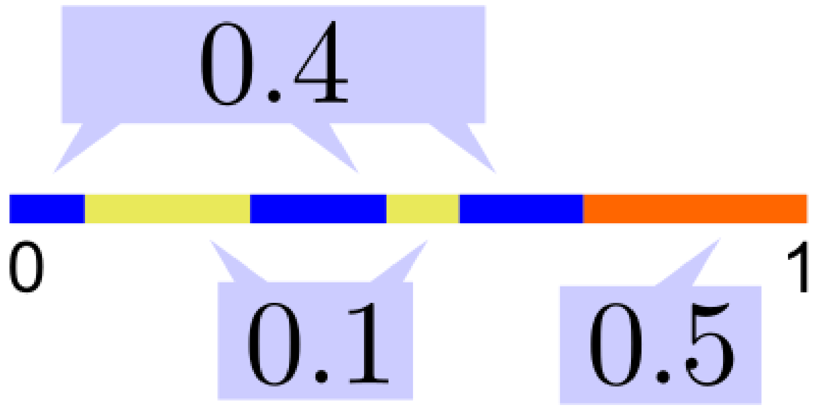

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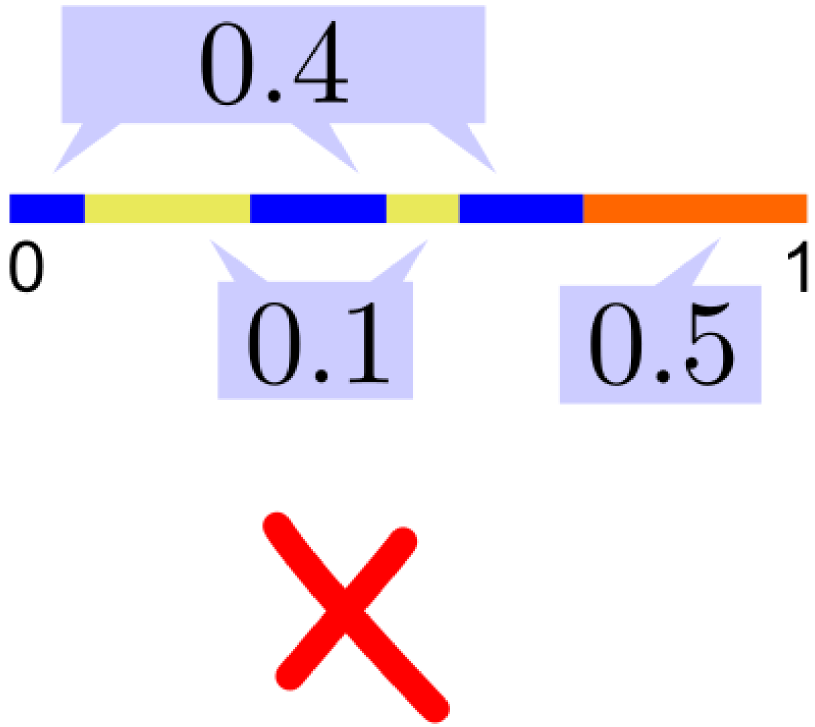

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For two agents (n=2), is one property stronger than the other?

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What about three or more agents?

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Prop implies EF for *two* agents (but no more)

Robertson-Webb Query Model

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Types of queries that can be used to access the valuation functions

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$\texttt{eval}_i(x, y)$: returns $v_i([x, y])$

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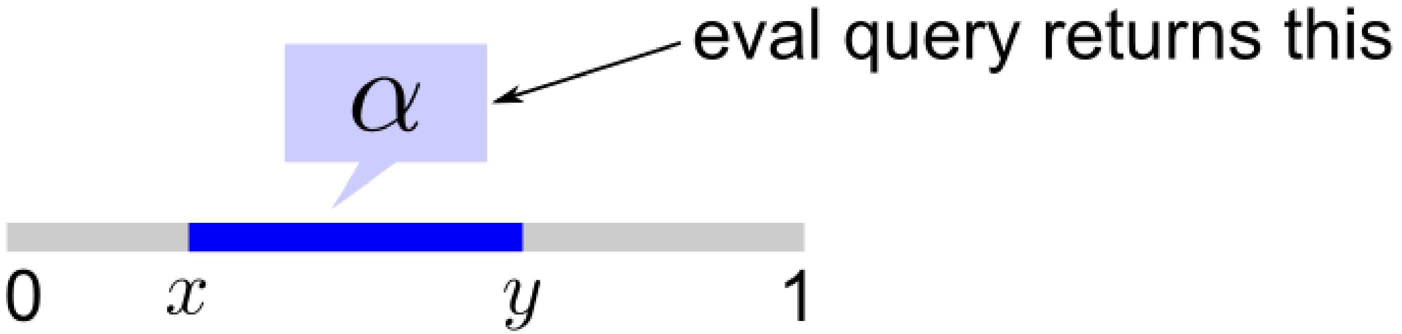

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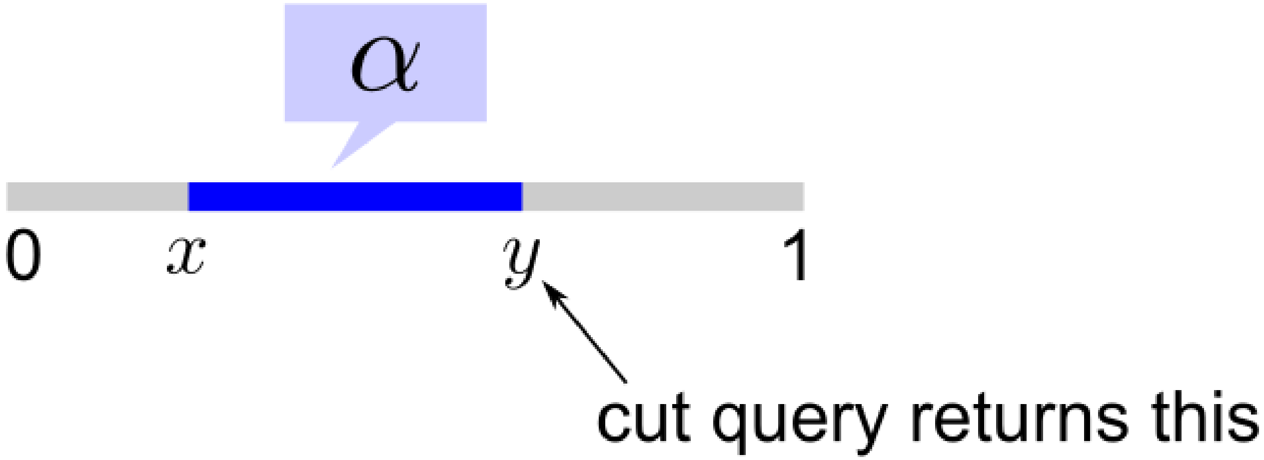

Cake-Cutting Algorithms

Let's start by thinking about proportionality for two agents.

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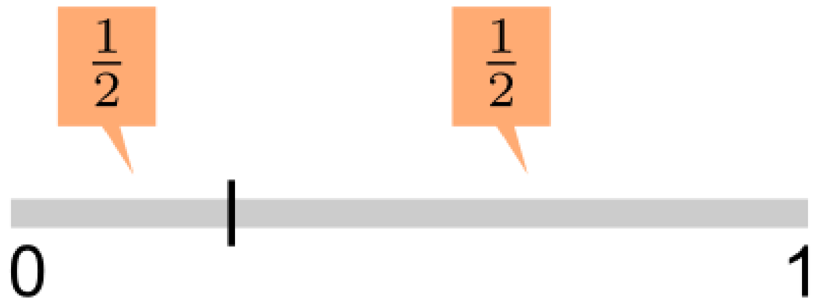

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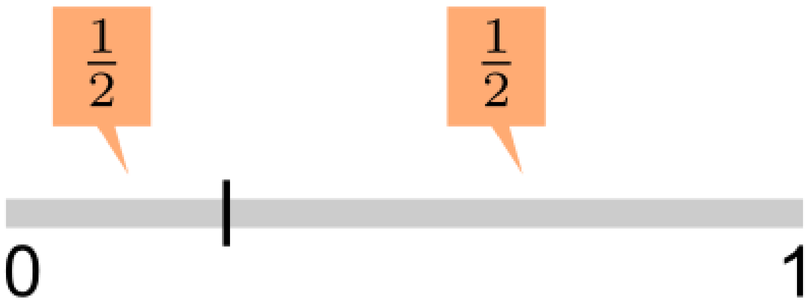

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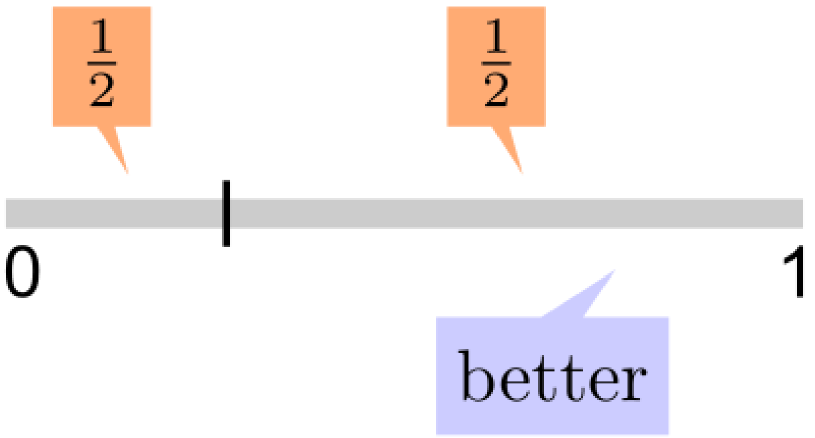

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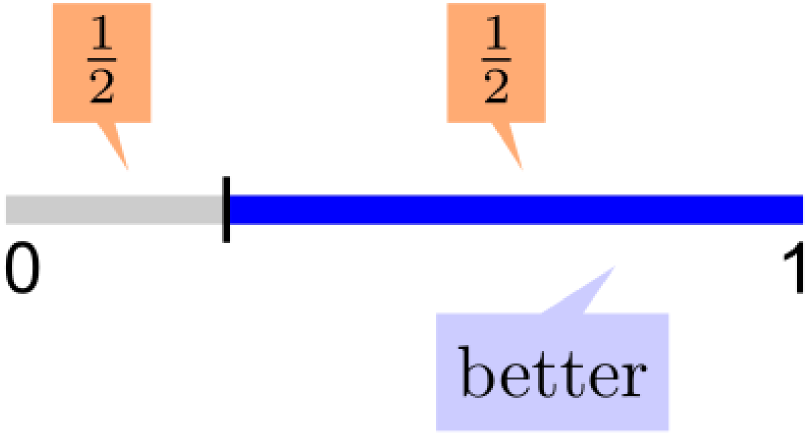

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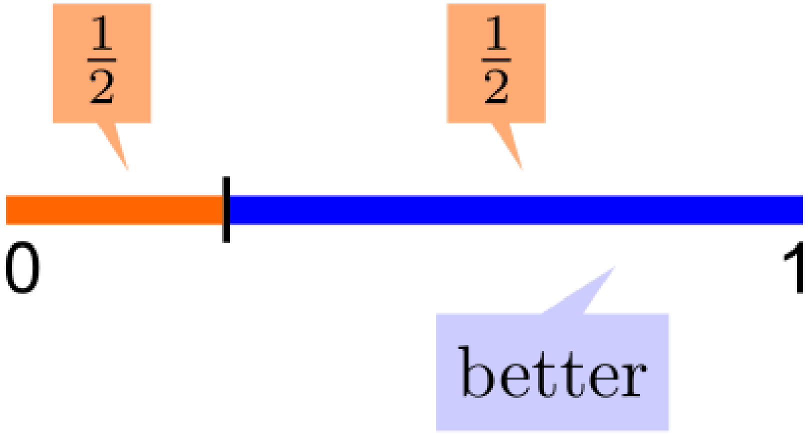

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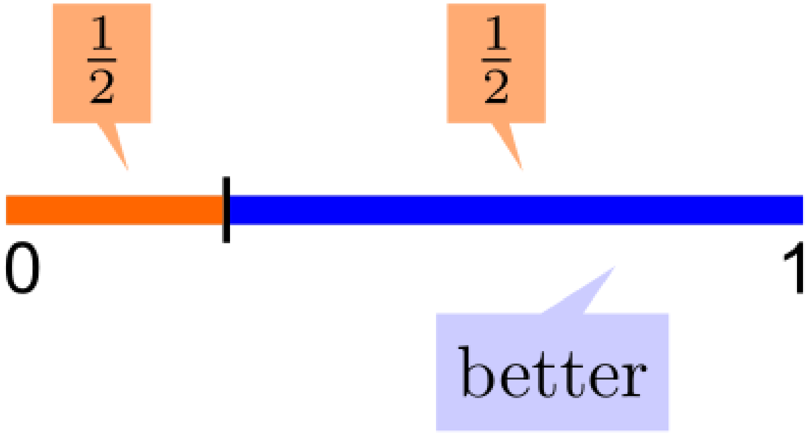

Is the cut-and-choose outcome proportional?

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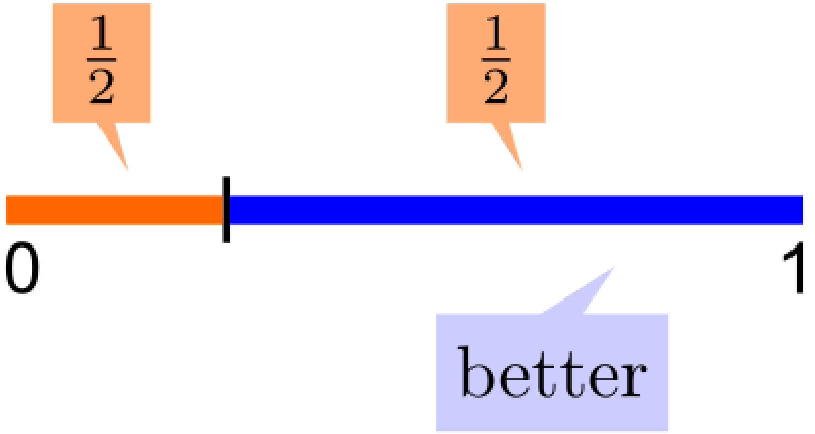

Is the cut-and-choose outcome proportional?

Yes! Agent 2's value is at least 1/2. Agent 1's value is exactly 1/2.

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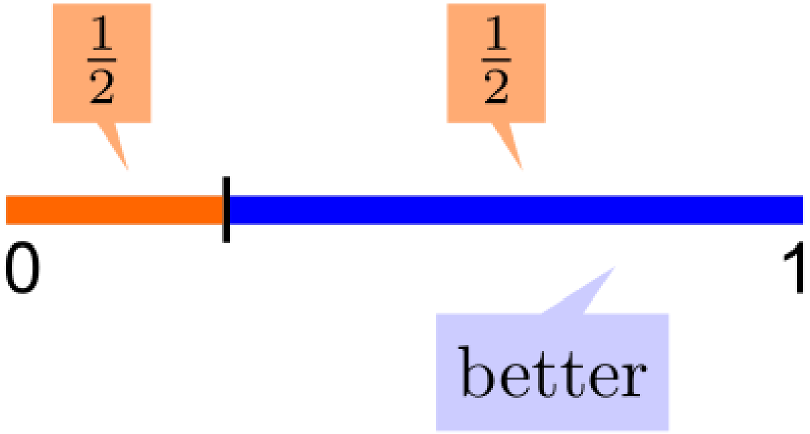

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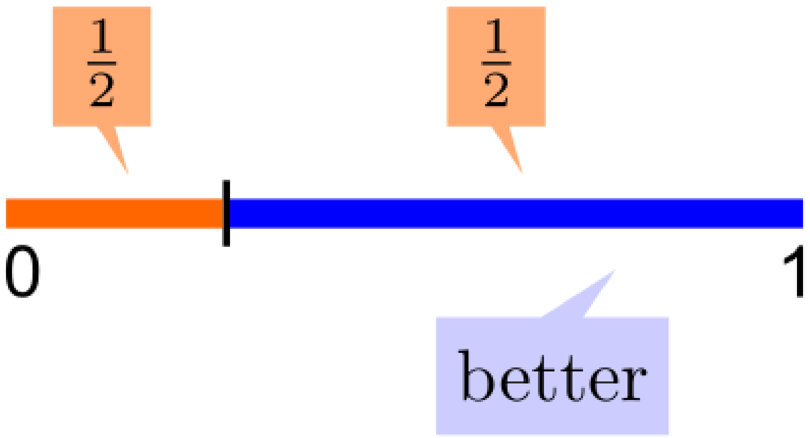

Is the cut-and-choose outcome envy-free?

Yes! EF and Prop are equivalent for two agents.

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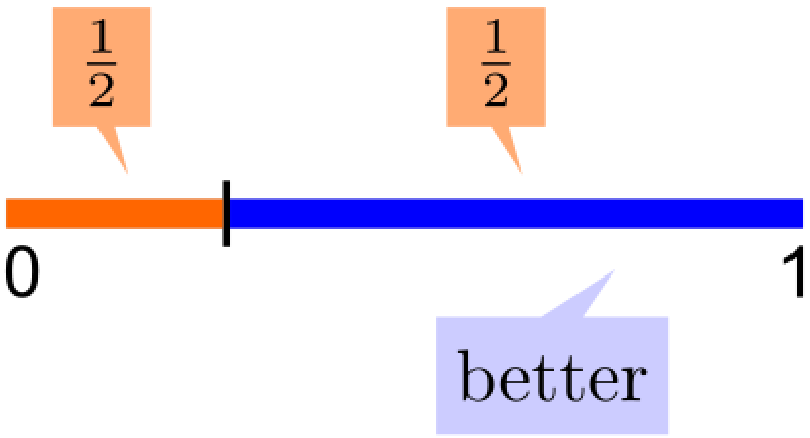

Can cut-and-choose be implemented in the Robertson-Webb model?

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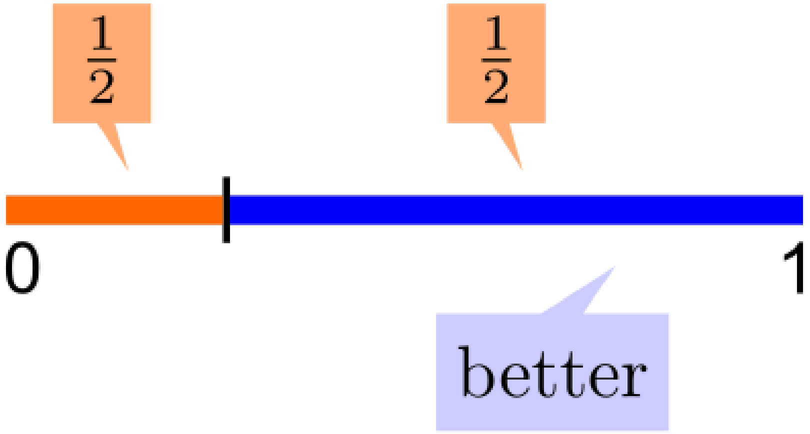

Can cut-and-choose be implemented in the Robertson-Webb model?

$$y = \texttt{cut}_1(0, 1/2)$$

Cut and Choose

1. Agent 1 cuts the cake into two equally-valued pieces (as per v_1).

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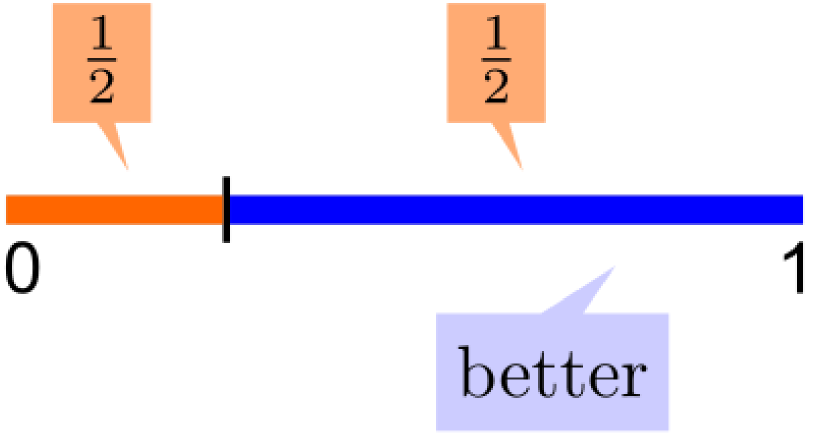


Can cut-and-choose be implemented in the Robertson-Webb model?

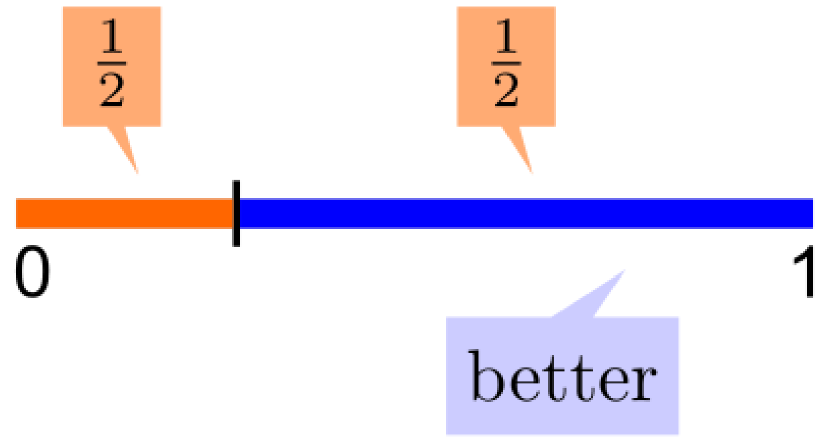
$$y = \texttt{cut}_1(0, 1/2)$$

$$\texttt{eval}_2(0, y)$$

Cut and Choose

1. Agent 1 cuts the cake into two equally-valued pieces (as per v_1).

2. Agent 2 chooses its preferred piece (as per v_2), and agent 1 gets the remaining piece.



For two agents, an envy-free/proportional cake division can be computed using two queries.