Manipulation in Voting

Rohit Vaish

VOTING RULE

A mapping from preference profiles to candidates.

$$f: \quad \begin{array}{ccc} v_1 & v_2 & v_n \\ a & b & c \\ b & c & \cdots & a \\ c & a & b \end{array} \longrightarrow C$$
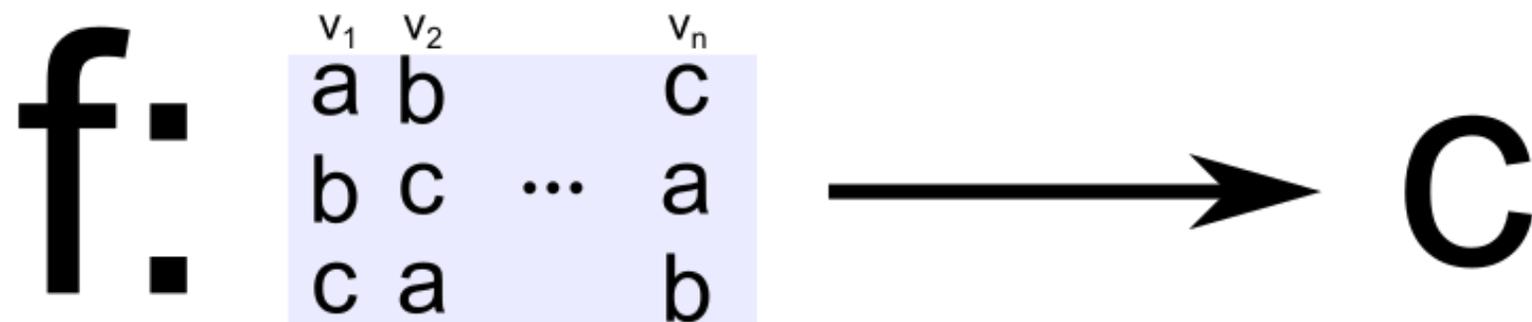
(also known as a *social choice function*)

Score-based	Runoff-based	Head-to-head election based
Plurality	Plurality with Runoff	Copeland
Borda Count	Single Transferable Vote	Schulze

Voting Rules are Everywhere!
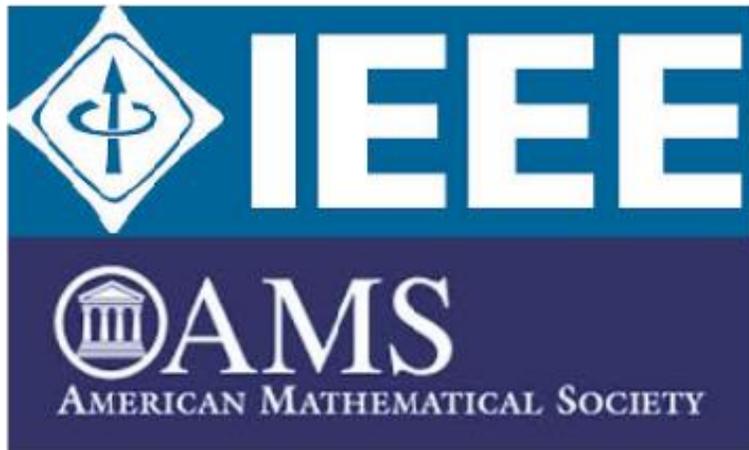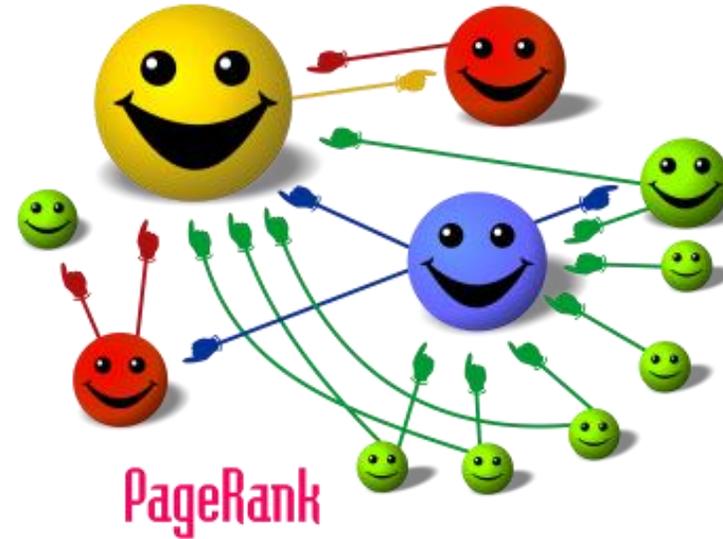
One Rule to Rule Them All?

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Axiomatic Approach

- Formulate a set of "reasonable" properties
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Unique voting rule with all the desirable properties

Reality

No such voting rule

ONTO

For any candidate "a", there exists a profile where "a" wins.

$$f(\boxed{\overset{v_1 \quad v_2 \qquad\qquad v_n}{}}) = a$$

STRATEGYPROOF

No voter can improve by misreporting its preferences.

For any profile $\boxed{}$, any voter v_i, and any misreport $\textcolor{red}{|}$, it must be that

$$f\left(\begin{array}{ccc} v_1 & v_i & v_n \\ & & \end{array}\right) \succeq_i f\left(\begin{array}{ccc} v_1 & v_i & v_n \\ & & \end{array}\right)$$

An Obviously Bad Voting Rule

A voting rule is called a dictatorship if there exists a voter v_i such that for any preferences of the other voters, the voting outcome is the favorite candidate of voter v_i.

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A dictatorship is onto and strategyproof.

[Gibbard'73; Satterthwaite'75]

With three or more candidates, a voting rule is onto and strategyproof if and only if it is a dictatorship.

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Quiz!

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Onto + strategyproof but two candidates:

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Onto + strategyproof but two candidates:

≥3 candidates + strategyproof but not onto:

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Onto + strategyproof but two candidates: Majority

≥3 candidates + strategyproof but not onto:

≥3 candidates + onto but not strategyproof:

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Onto + strategyproof but two candidates: Majority

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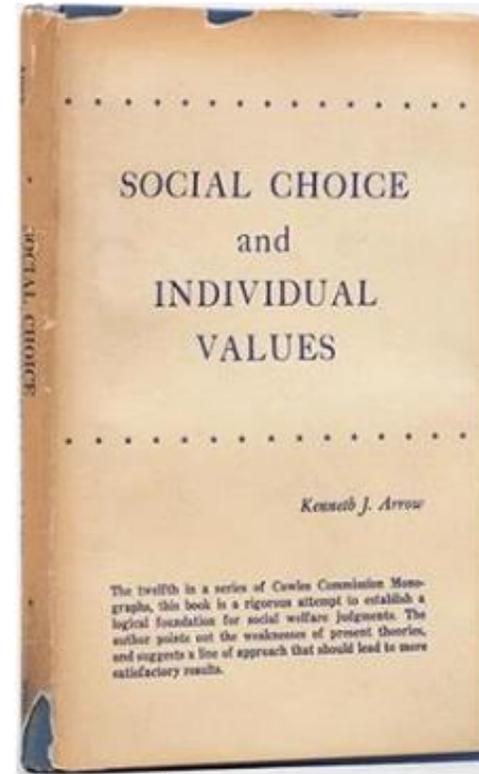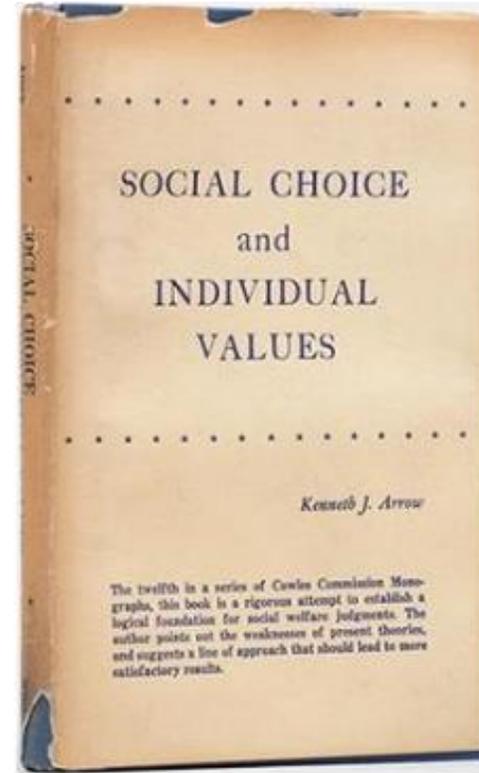|	
Onto + strategyproof but two candidates:	Majority
≥3 candidates + strategyproof but not onto:	Constant/Restricted Majority
≥3 candidates + onto but not strategyproof:	Plurality/Borda/...

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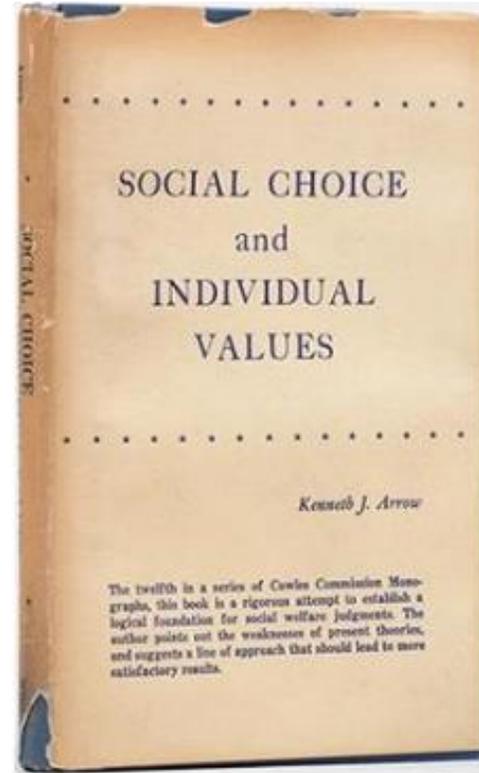

1951 PhD thesis of Kenneth Arrow

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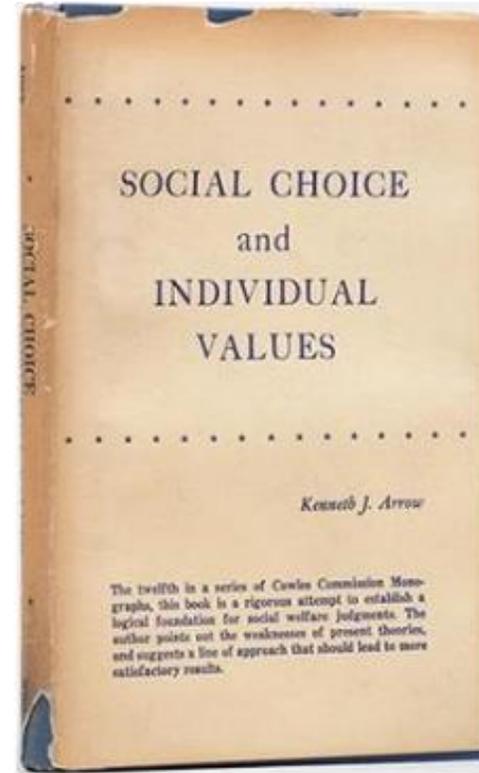
1951 PhD thesis of Kenneth Arrow

Voting Rules that Output Rankings

A mapping from preference profiles to rankings over candidates.

$$f: \quad \begin{matrix} v_1 & v_2 & & v_n \\ a & b & & c \\ b & c & \cdots & a \\ c & a & & b \end{matrix} \quad \longrightarrow \quad \begin{matrix} a \\ c \\ b \end{matrix}$$
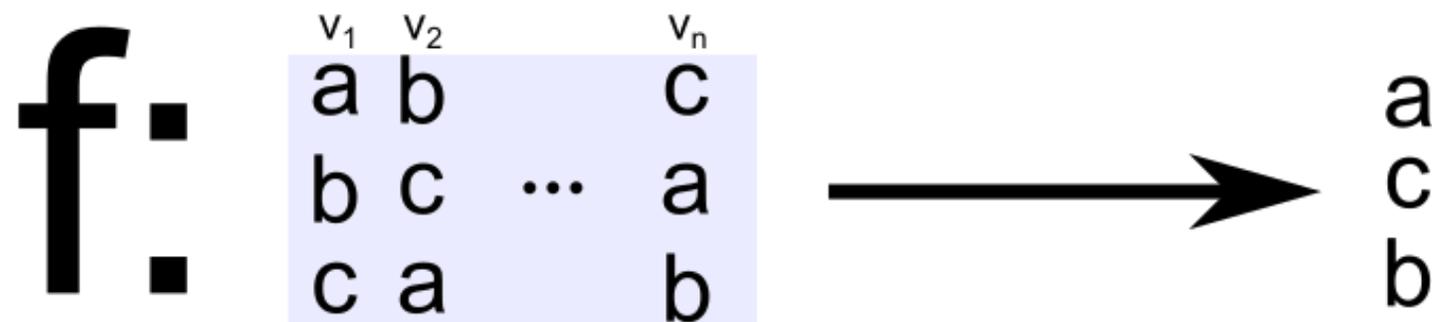
(also known as a *social welfare function* or SWF)

UNANIMOUS

If all voters prefer "a" over "b", then so does the SWF.

$$f\left(\begin{array}{cccc} v_1 & v_2 & & v_n \\ \vdots & a & & \vdots \\ a & b & \ldots & a \\ \vdots & \vdots & & b \\ b & \vdots & & \vdots \end{array}\right) = \begin{array}{c} \vdots \\ a \\ \vdots \\ b \\ \vdots \end{array}$$

DICTATORSHIP

An SWF that mimics the preferences of a fixed voter on all inputs.

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A dictatorship is unanimous and IIA.

[Arrow'51]

With three or more candidates,
an SWF is unanimous and IIA
if and only if it is a dictatorship.

Method	Majority	Maj. loser	Mutual maj.	Condorcet	Cond. loser	Smith/ISDA	LIIA	IIA	Cloneproof	Monotone	Consistency	Participation	Reversal symmetry	Polytime/resolvable		Summable	Later-no-Harm	Later-no-Help	No favorite betrayal	Ballot type	Ranks =	Ranks >2
Approval	Rated[a]	No	No	No[b][c]	No	No[b]	Yes	Yes[d]	Yes[e]	Yes	Yes	Yes	Yes	O(N)	Yes	O(N)	No	Yes[f]	Yes	Approvals	Yes	No
Borda count	No	Yes	No	No[b]	Yes	No	No	No	Teams	Yes	Yes	Yes	Yes	O(N)	Yes	O(N)	No	Yes	No	Ranking	Yes	Yes
Bucklin	Yes	Yes	Yes	No	No	No	No	No	No	Yes	No	No	No	O(N)	Yes	O(N)	No	Yes	If equal preferences	Ranking	Yes	Yes
Copeland	Yes	Yes	Yes	Yes	Yes	Yes	No	No[b]	Teams, crowds	Yes	No[b]	No[b]	Yes	O(N²)	No	O(N²)	No[b]	No	No[b]	Ranking	Yes	Yes
IRV (AV)	Yes	Yes	Yes	No[b]	Yes	No[b]	No	No	Yes	No	No	No	No	O(N²)	Yes[g]	O(N!)[h]	Yes	Yes	No	Ranking	No	Yes
Kemeny–Young	Yes	Yes	Yes	Yes	Yes	Yes	Yes	No[b]	Spoilers	Yes	No[b][i]	No[b]	Yes	O(N!)	Yes	O(N²)[j]	No[b]	No	No[b]	Ranking	Yes	Yes
Highest median/Majority judgment[k]	Rated[l]	Yes[m]	No[n]	No[b][c]	No	No[b]	Yes	Yes[d]	Yes	Yes	No[o]	No[p]	Depends[q]	O(N)	Yes	O(N)[r]	No[s]	Yes	Yes	Scores[t]	Yes	Yes
Minimax	Yes	No	No	Yes[u]	No	No	No	No[b]	Spoilers	Yes	No[b]	No[b]	No	O(N²)	Yes	O(N²)	No[b][u]	No	No[b]	Ranking	Yes	Yes
Plurality/FPTP	Yes	No	No	No[b]	No	No[b]	No	No	Spoilers	Yes	Yes	Yes	No	O(N)	Yes	O(N)	N/A[v]	N/A[v]	No	Single mark	N/A	No
Score voting	No	No	No	No[b][c]	No	No[b]	Yes	Yes[d]	Yes	Yes	Yes	Yes	Yes	O(N)	Yes	O(N)	No	Yes	Yes	Scores	Yes	Yes
Ranked pairs	Yes	Yes	Yes	Yes	Yes	Yes	Yes	No[b]	Yes	Yes	No[b]	No[p][b]	Yes	O(N³)	Yes	O(N²)	No[b]	No	No[p][b]	Ranking	Yes	Yes
Runoff voting	Yes	Yes	No	No[b]	Yes	No[b]	No	No	Spoilers	No	No	No	No	O(N)[w]	Yes	O(N)[w]	Yes	Yes[x]	No	Single mark	N/A	No[y]
Schulze	Yes	Yes	Yes	Yes	Yes	Yes	No	No[b]	Yes	Yes	No[b]	No[p][b]	Yes	O(N³)	Yes	O(N²)	No[b]	No	No[p][b]	Ranking	Yes	Yes
STAR voting	No[z]	Yes	No[aa]	No[b][c]	Yes	No[b]	No	No	No	Yes	No	No	Depends[ab]	O(N)	Yes	O(N²)	No	No	No[ac]	Scores	Yes	Yes
Sortition, arbitrary winner[ad]	No	No	No	No[b]	No	No[b]	Yes	Yes	No	Yes	Yes	Yes	Yes	O(1)	No	O(1)	Yes	Yes	Yes	None	N/A	N/A
Random ballot[ae]	No	No	No	No[b]	No	No[b]	Yes	Yes	Yes	Yes	Yes	Yes	Yes	O(N)	No	O(N)	Yes	Yes	Yes	Single mark	N/A	No

"*Most systems are not going to work badly all of the time. All I proved is that all can work badly at times.*"

References

- A shared proof of Arrow's and Gibbard-Satterthwaite theorems: "Arrow's Theorem and the Gibbard-Satterthwaite Theorem: A Unified Approach" by Philip J. Reny. https://www.sciencedirect.com/science/article/pii/S0165176500003323

- Another nice proof of Arrow's theorem: https://www.youtube.com/watch?v=QLi_5LCwJ20

- The "big table" of voting rules is from the Wikipedia article "Comparison of Electoral Systems".

- A tribute to Arrow (see the end of the article for a nice anecdote): https://www.nytimes.com/2017/02/21/business/economy/kenneth-arrow-dead-nobel-laureate-in-economics.html