COL749: Computational Social Choice

Lecture 9

Fairness and Efficiency

Feb 03, 2025 | Rohit Vaish

The Model

	A	B	C	D	E
	4	1	2	2	2
	1	0	5	1	1
	1	1	5	1	1

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Additive valuations

Envy-Freeness Up To One Good [Budish, 2011]

Envy can be eliminated by removing some good in the envied bundle.

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	4	1	2
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Envy can be eliminated by removing some good in the envied bundle.

	Ⓐ	Ⓑ	Ⓒ
My bundle is better if Ⓐ is removed	4	1	2
My bundle is better if Ⓒ is removed	1	1	5

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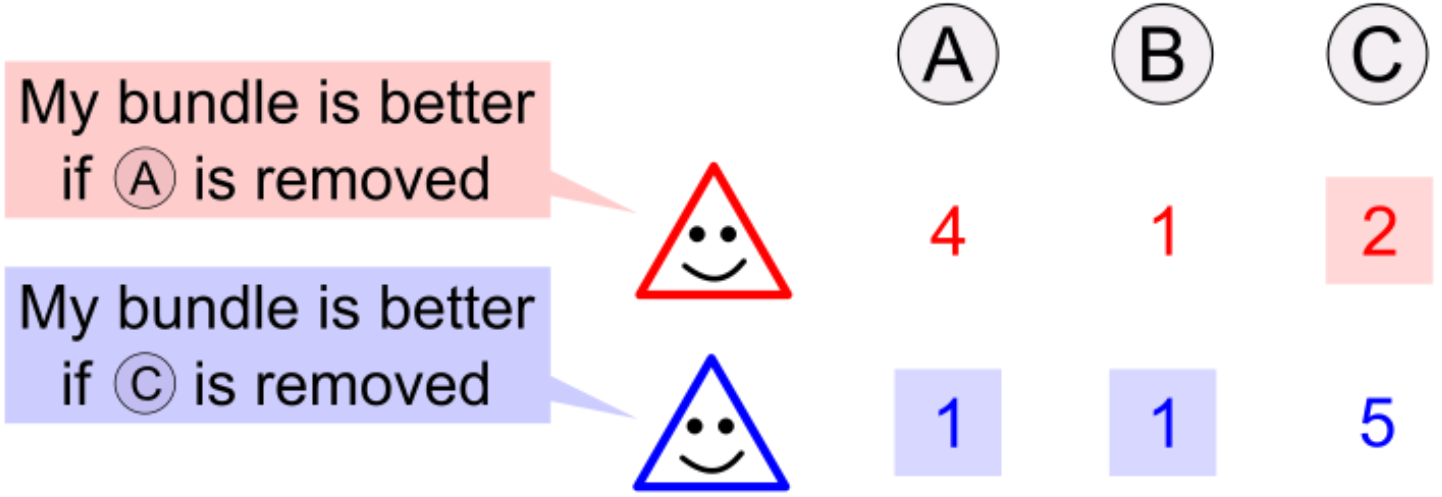

Allocation $A = (A_1, \ldots, A_n)$ is EF1 if for every pair of agents i, k, there exists a good $j \in A_k$ such that $v_i(A_i) \geq v_i(A_k \setminus \{j\})$.

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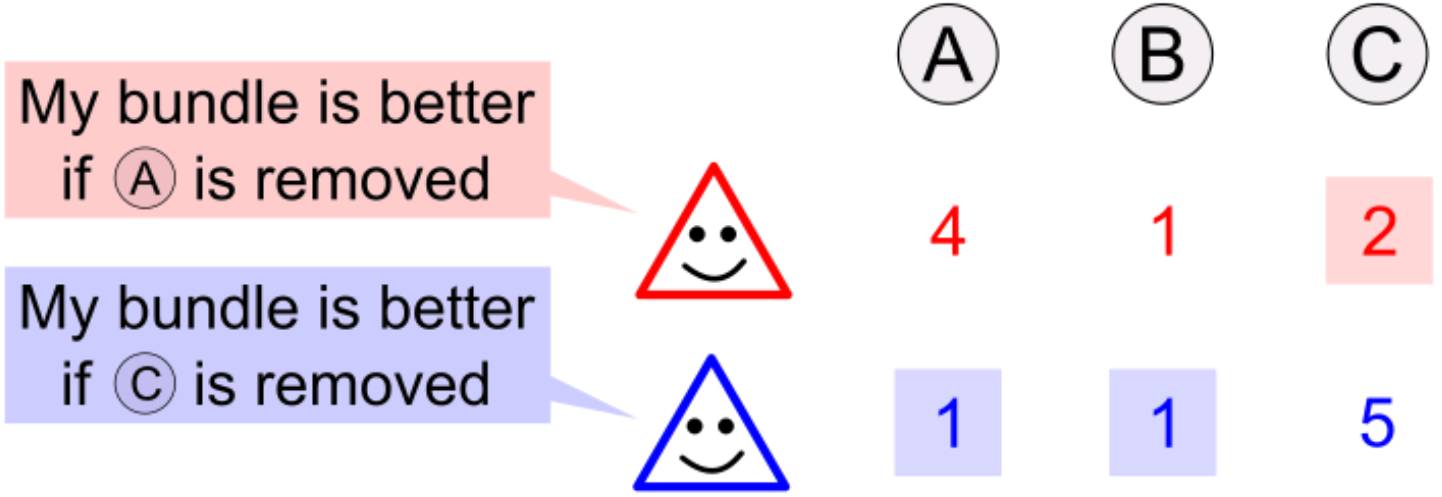

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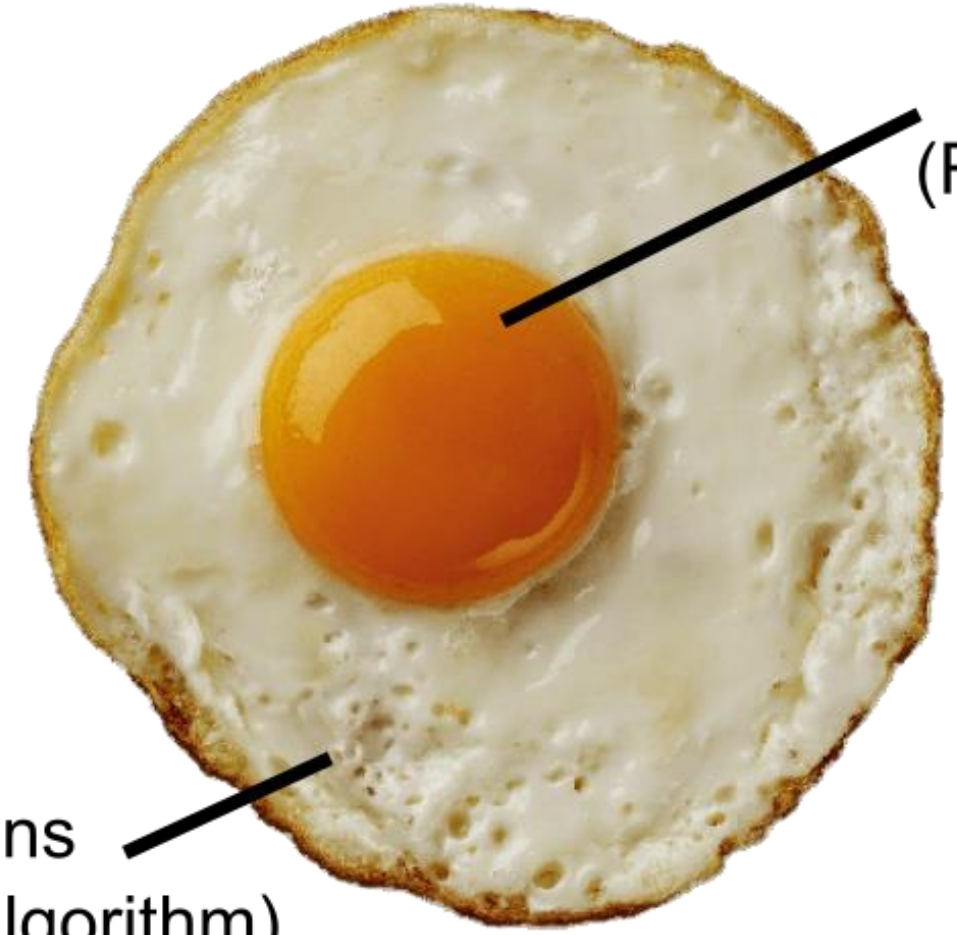
Guaranteed to exist and efficiently computable

Last Time

Algorithms for finding an EF1 allocation



Additive valuations
(Round-robin algorithm)

Monotone valuations
(Envy-cycle elimination algorithm)

A trivial way of achieving fairness: Don't allocate anything!

A bare minimum efficiency requirement: Completeness

WHEN A COMPLETE ALLOCATION

SIMPLY ISN'T ENOUGH

"Obvious" Improvement

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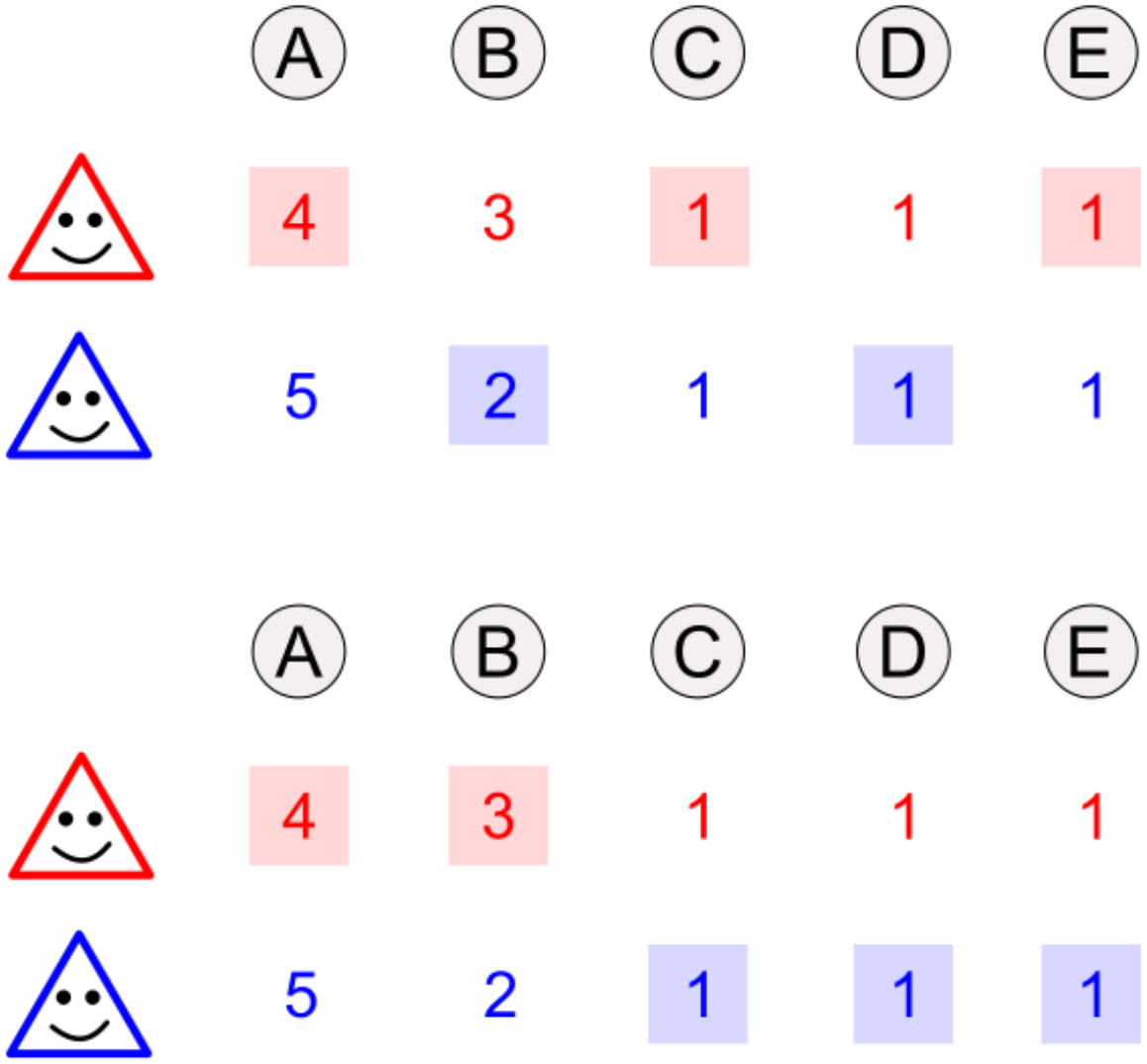
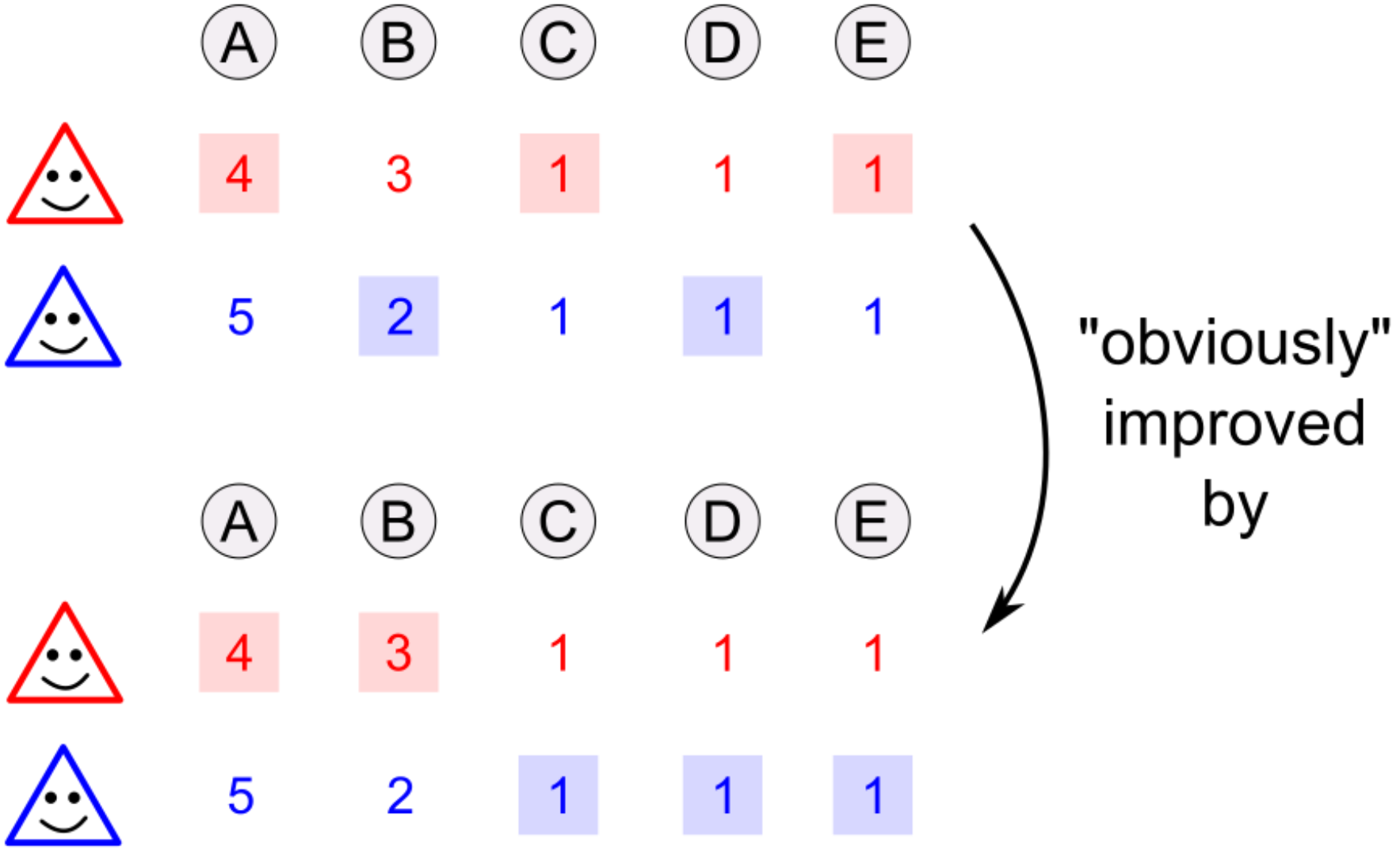
	A	B	C	D	E
	4	3	1	1	1
	5	2	1	1	1

"Obvious" Improvement



"obviously" improved by

"Obvious" Improvement



Strictly improving someone without hurting anyone else

Pareto Optimality

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Is EF1 compatible with Pareto optimality?

Round Robin Fails Pareto Optimality

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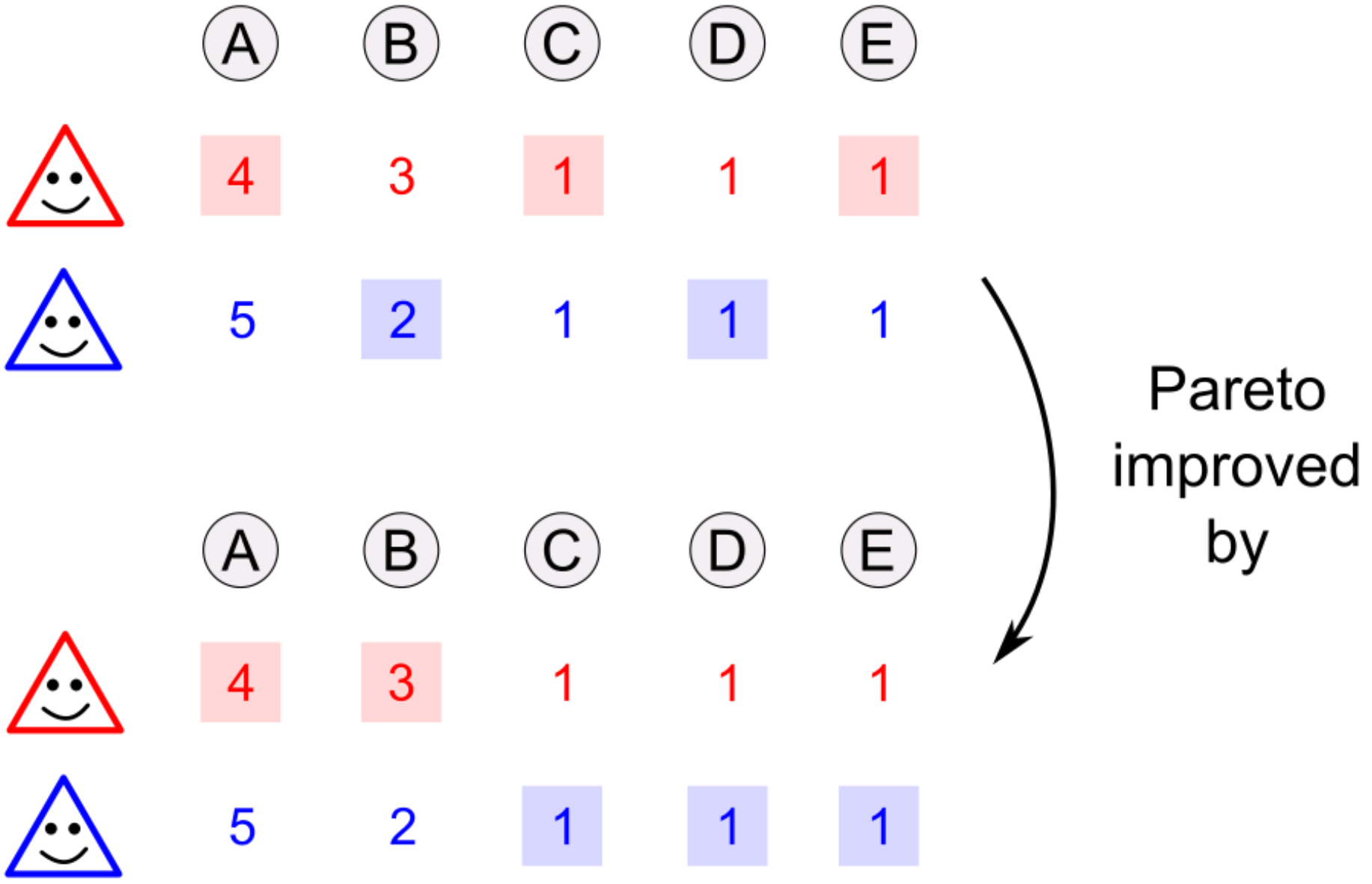

Envy-Cycle Elimination Fails Pareto Optimality

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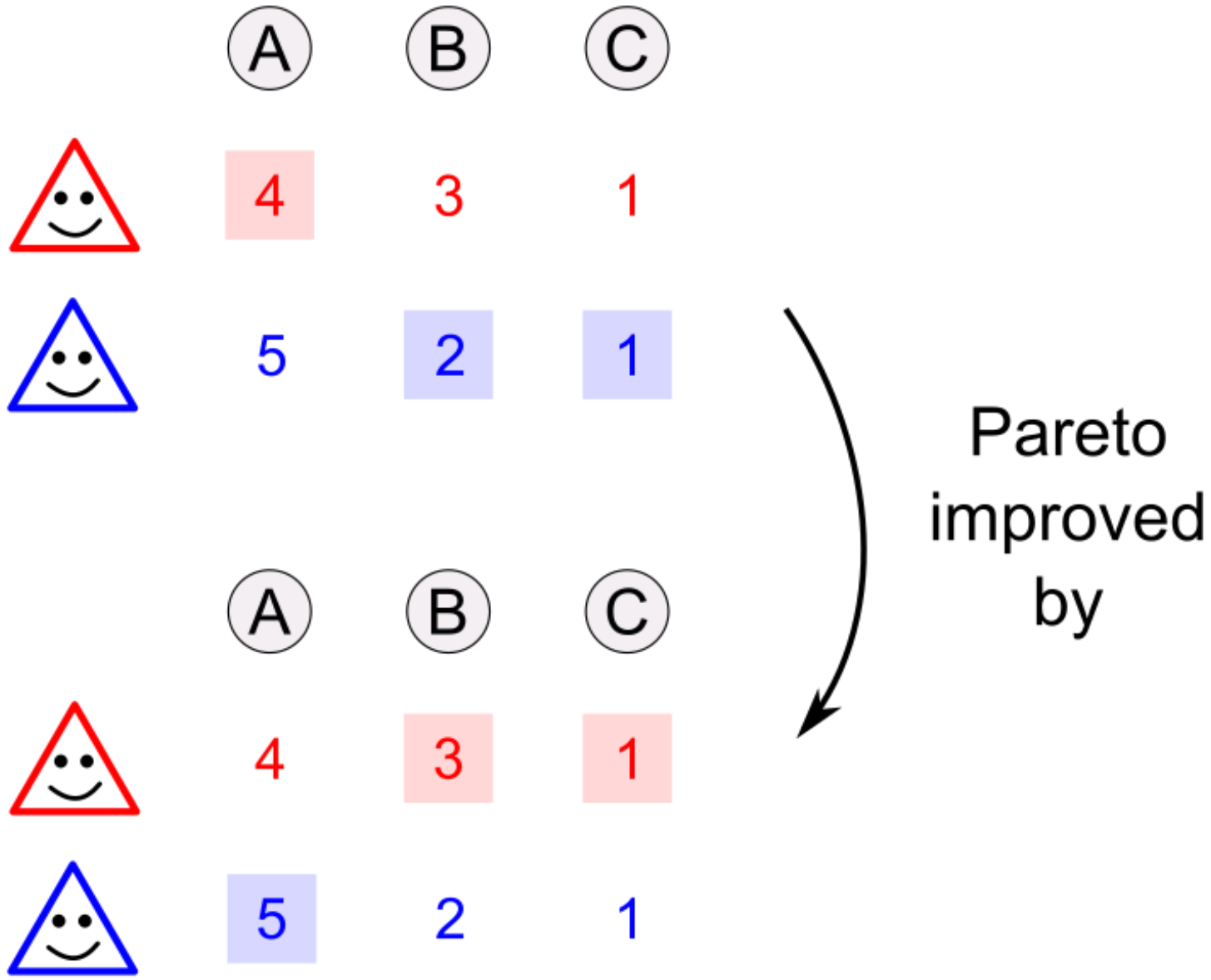

Another natural strategy for EF1+PO

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Start with an EF1 allocation, and repeatedly make Pareto improvements to it.

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Exercise: Pareto improvement can fail to preserve EF1.

Nash Social Welfare

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*If optimal is 0, then find *any* largest set of agents who can simultaneously be given positive utility and maximize the geometric mean with respect to only those agents.

[Caragiannis, Kurokawa, Moulin, Procaccia, Shah, and Wang, *TEAC* 2019]

Any Nash optimal allocation satisfies Pareto optimality (PO) and envy-freeness up to one good (EF1).

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Why PO?

Pareto improvement strictly improves NSW.

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Does such a good g* always exist?

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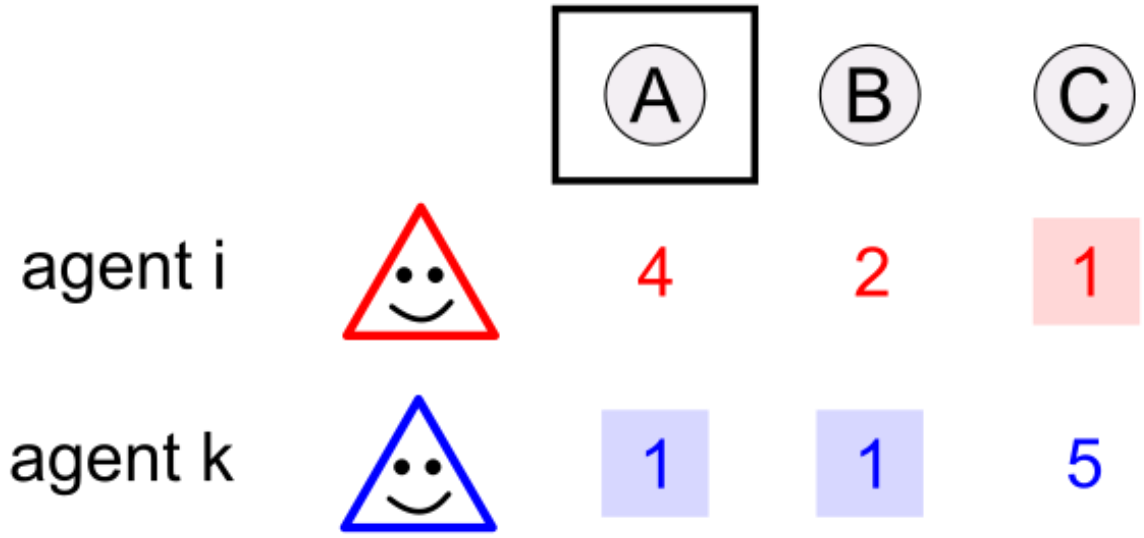

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We will show that transferring g* from agent k to agent i improves NSW.

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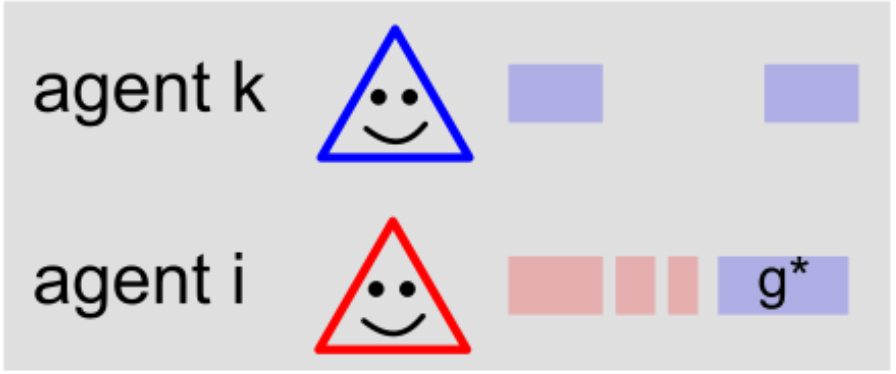

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We will show that NSW(B) > NSW(A).

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$$\Leftrightarrow \left(1 - \frac{v_k(g^*)}{v_k(A_k)}\right) \cdot \left(1 + \frac{v_i(g^*)}{v_i(A_i)}\right) > 1$$

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$$\Leftrightarrow v_k(A_k) > \frac{v_k(g^*)}{v_i(g^*)} [v_i(A_i) + v_i(g^*)]$$

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$$\text{Let } g^* \in \arg\min_{g \in A_k : v_i(\{g\}) > 0} v_k(\{g\}) / v_i(\{g\}).$$

$$\frac{\mathrm{NSW}(B)}{\mathrm{NSW}(A)} > 1 \Leftrightarrow v_k(A_k) > \frac{v_k(g^*)}{v_i(g^*)} [v_i(A_i) + v_i(g^*)]$$

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Ok, so an EF1+PO allocation always exists.

But what about computation?

Can a Nash optimal allocation be efficiently computed?

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[Nguyen, Nguyen, Roos, and Rothe, *JAAMAS 2014*; Lee, *IPL. 2017*]

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- A 0.69-approximation to Nash social welfare objective

splíddít

Share Rent

Moving into a new apartment with roommates? Create harmony by fairly assigning rooms and sharing the rent.

START >

Split Fare

Fairly split taxi fare, or the cost of an Uber or Lyft ride, when sharing a ride with friends.

START >

Assign Credit

Determine the contribution of each individual to a school project, academic paper, or business endeavor.

START >

Divide Goods

Distribute Tasks

Suggest an App

Next Time

Envy-freeness up to *any* good (EFX)

Quiz

Quiz

Construct an instance and an allocation A for that instance such that A maximizes Nash welfare but not egalitarian welfare.

Egalitarian welfare of an allocation = utility of the least-happy agent

References

- Maximum Nash Welfare is EF1 and PO.

  Ioannis Caragiannis, David Kurokawa, Hervé Moulin, Ariel D. Procaccia, Nisarg Shah, and Junxing Wang
  "The Unreasonable Fairness of Maximum Nash Welfare"
  ACM Transactions on Economics and Computation, 7(3), 2019 pg 1-32
  https://dl.acm.org/doi/10.1145/3355902

- APX-hardness of maximizing Nash social welfare

  Euiwoong Lee
  "*APX-Hardness of Maximizing Nash Social Welfare with Indivisible Items*"
  Information Processing Letters, 122, 2017 pg 17-20
  https://www.sciencedirect.com/science/article/pii/S0020019017300212

References

- Finding an EF1 and PO allocation in pseudopolynomial time.

  Siddharth Barman, Sanath Kumar Krishnamurthy, and Rohit Vaish
  “Finding Fair and Efficient Allocations”
  EC 2018, pg 557-574
  https://dl.acm.org/doi/10.1145/3219166.3219176