COL749: Computational Social Choice

Lecture 8

Fair Allocation of Indivisible Goods

Jan 30, 2025 | Rohit Vaish


Vaccine
COVID-19
Vaccine
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The Model

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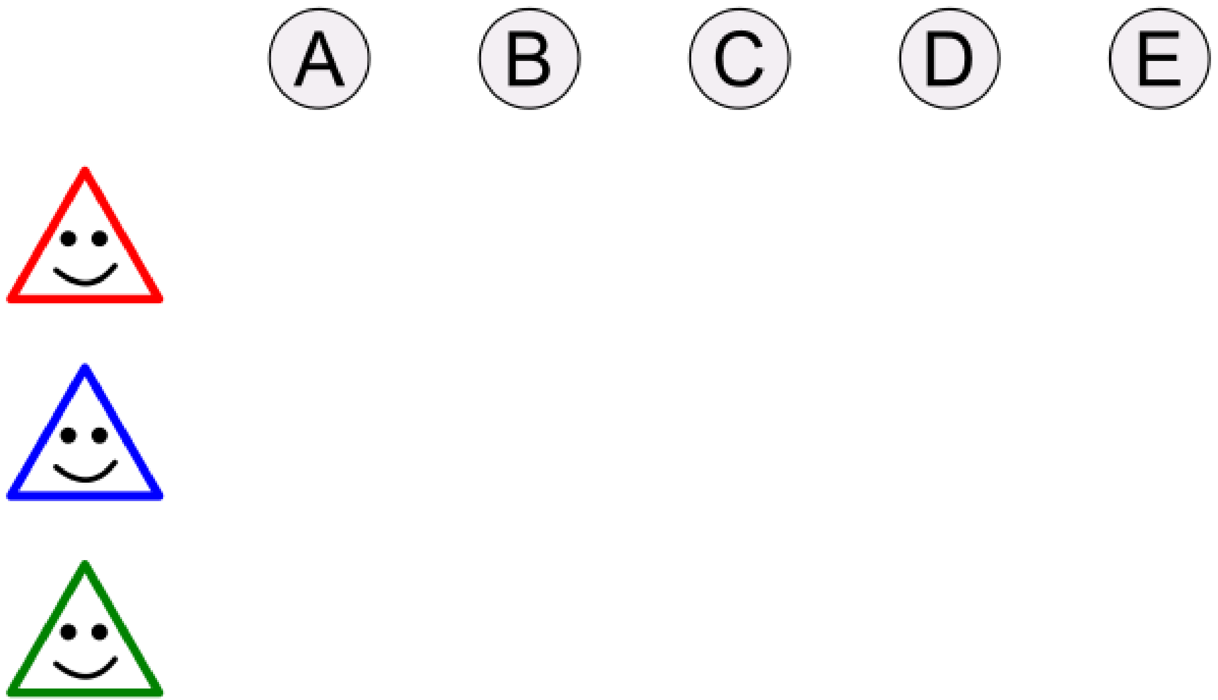

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Additive valuations

{B D E} = {B} + {D} + {E}

= 0+1+1 = 2

Envy-Freeness [Gamow and Stern, 1958; Foley, 1967]

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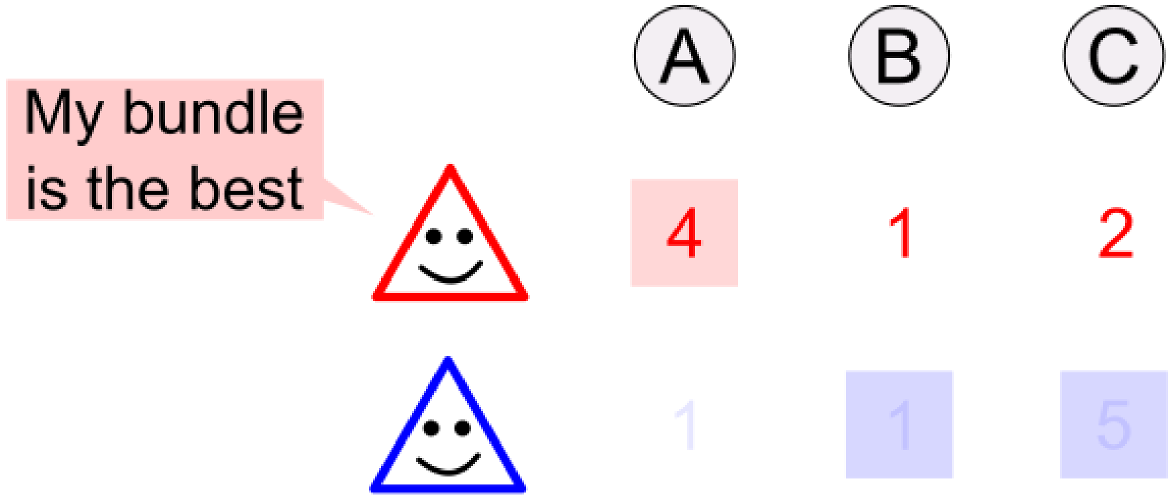

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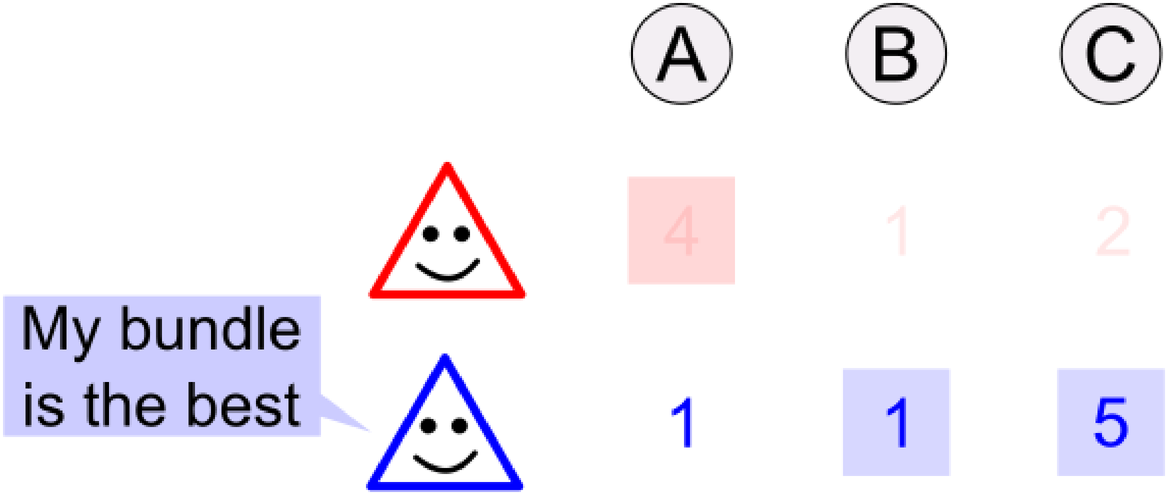

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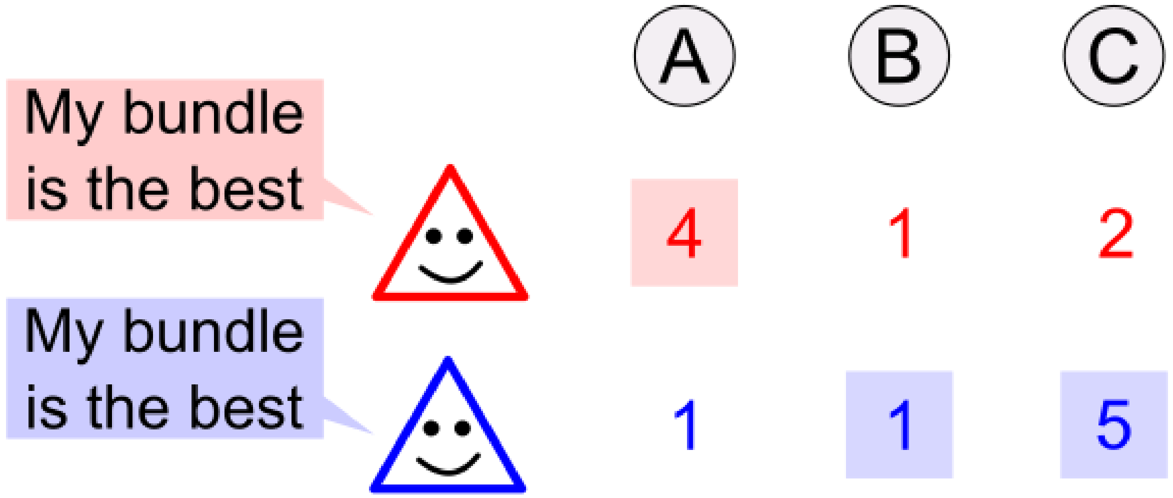

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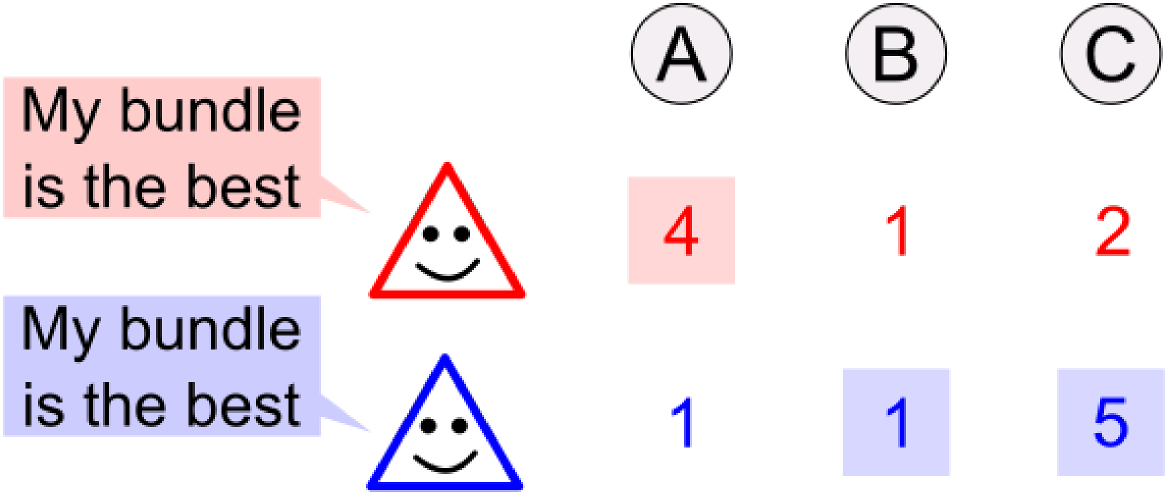

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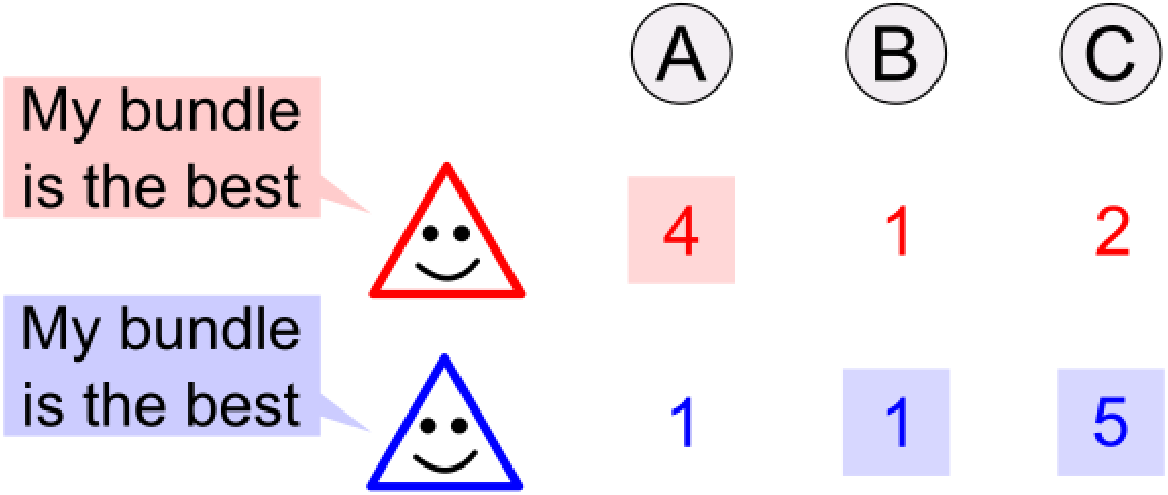

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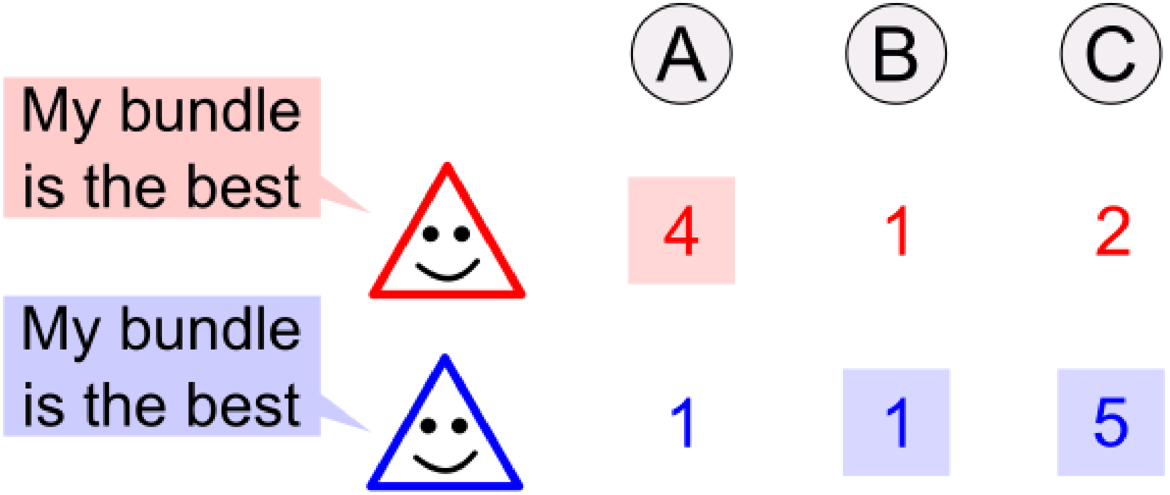

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- Checking whether an EF allocation exists is NP-complete

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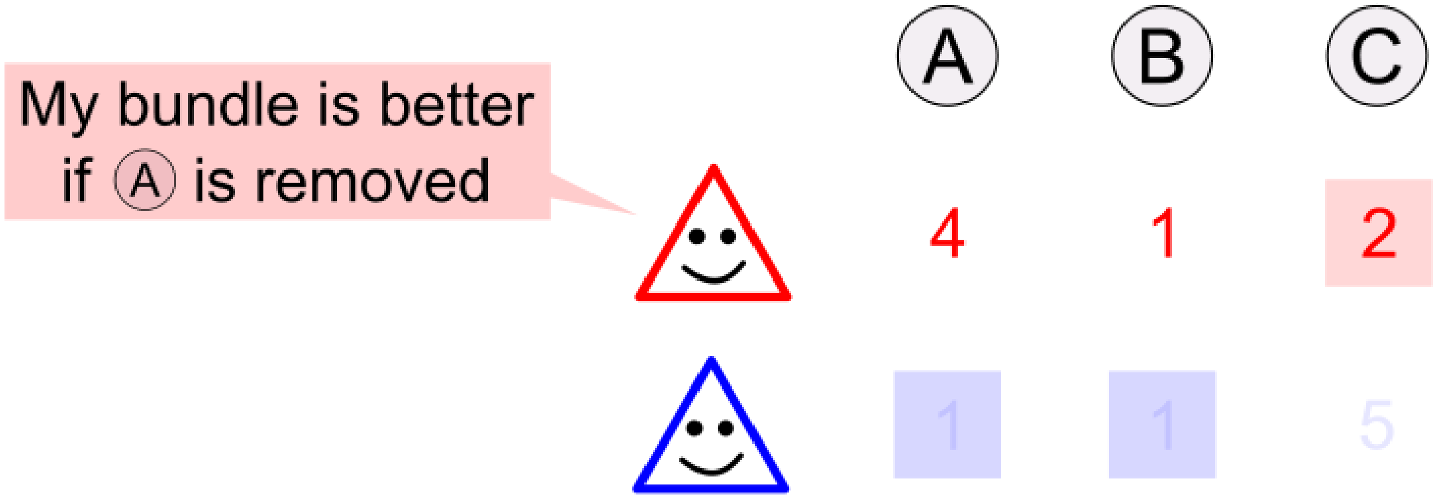

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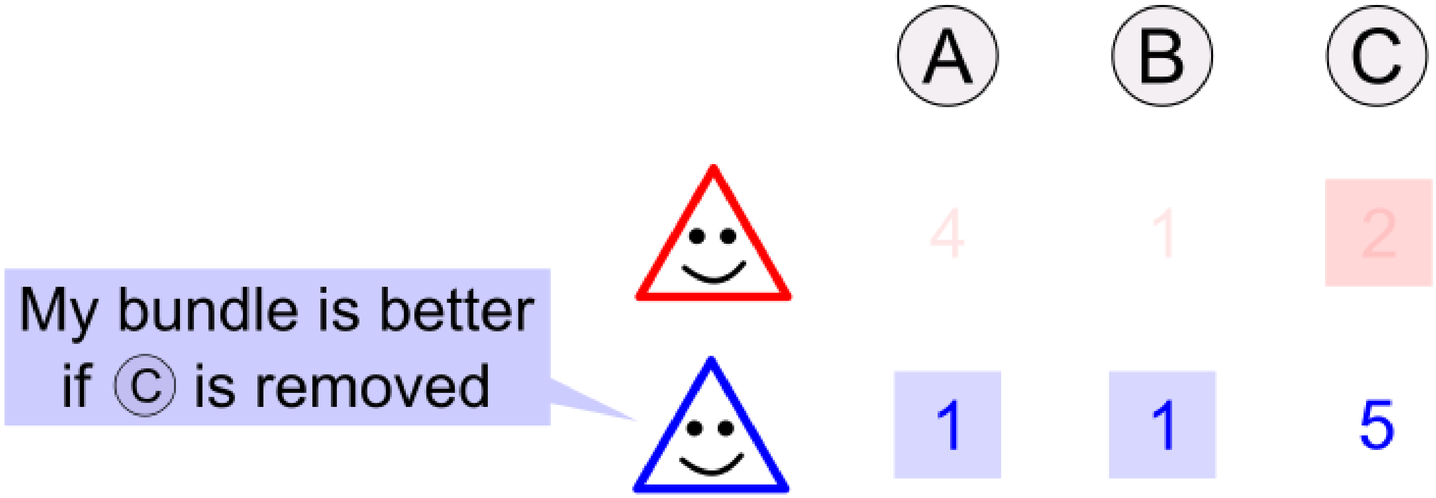

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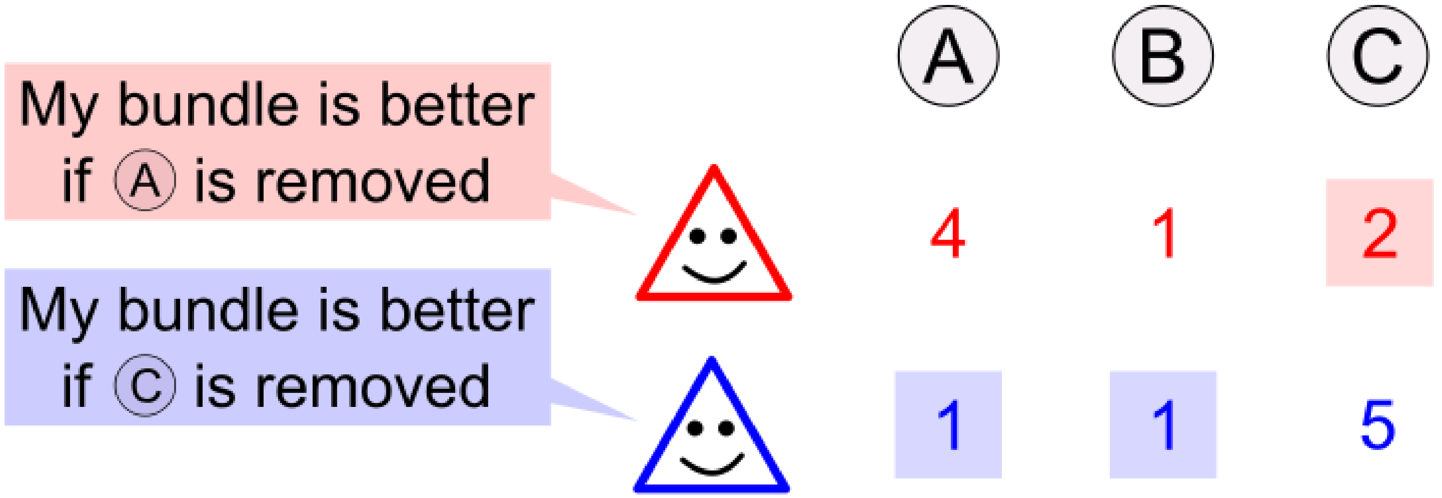

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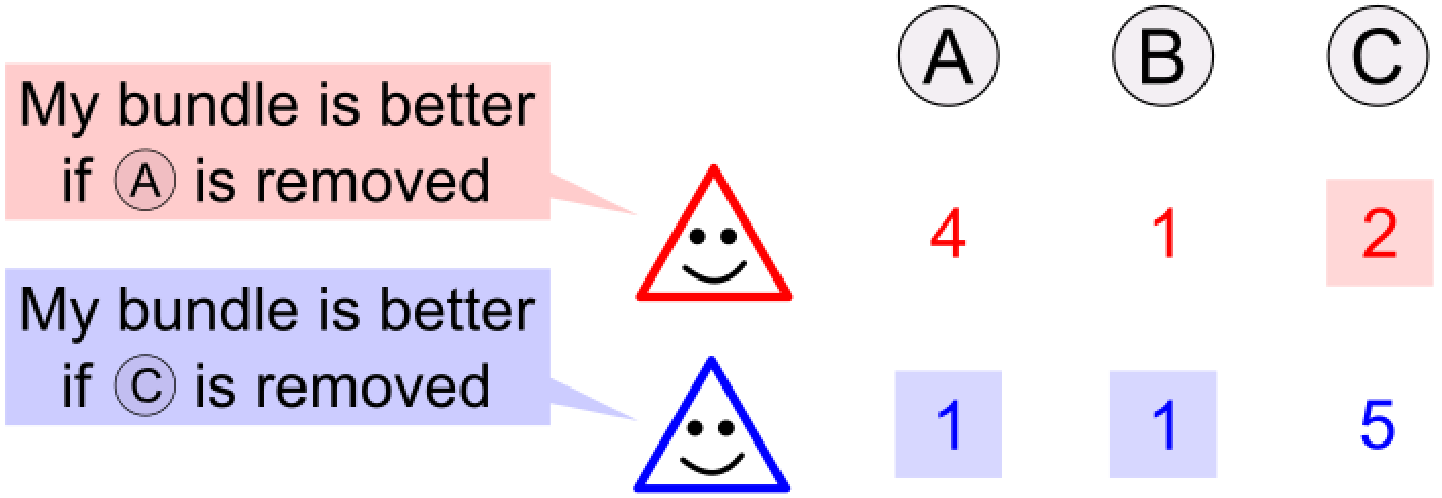

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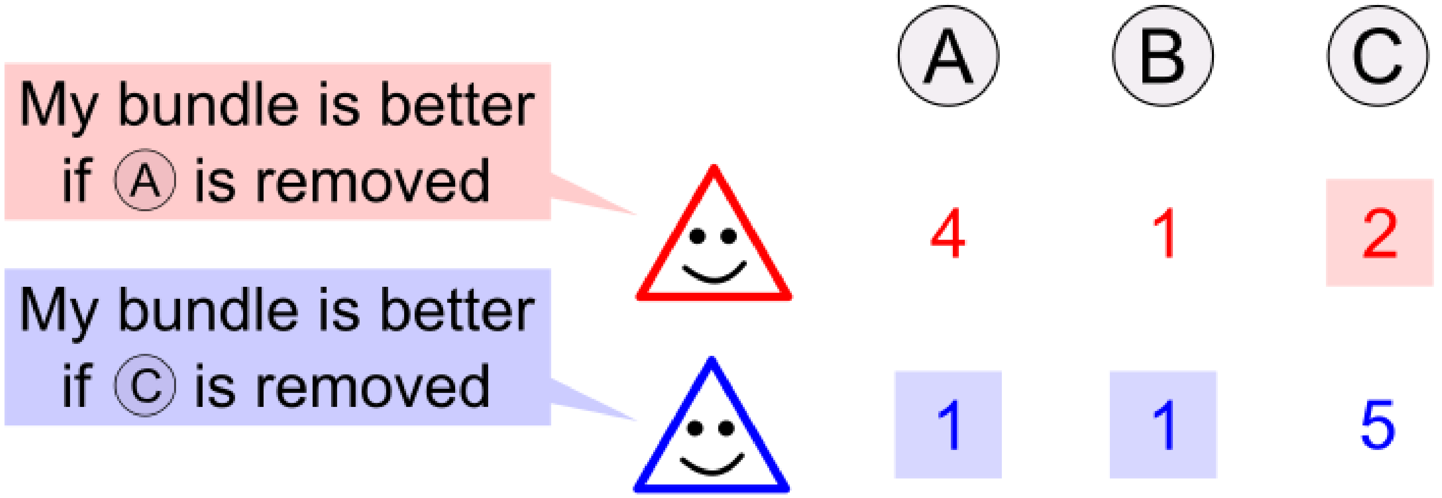

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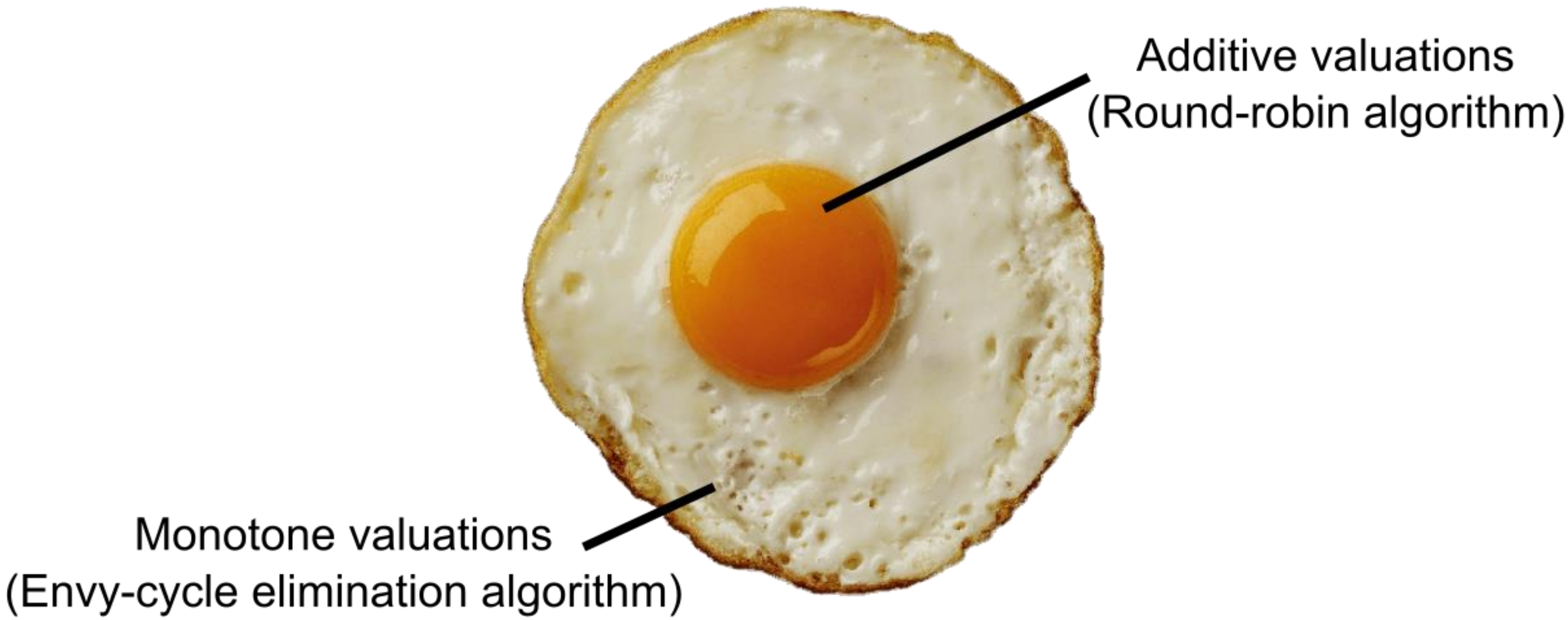
Guaranteed to exist and efficiently computable

Coming Up

Algorithms for finding an EF1 allocation



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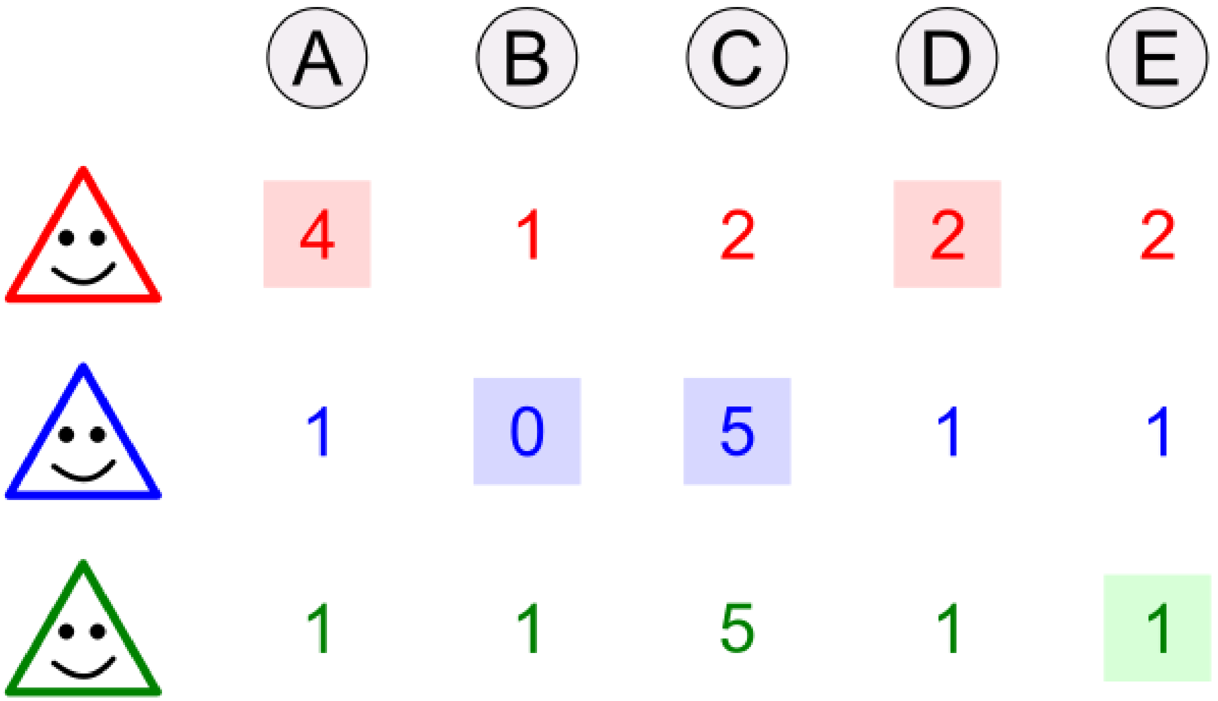

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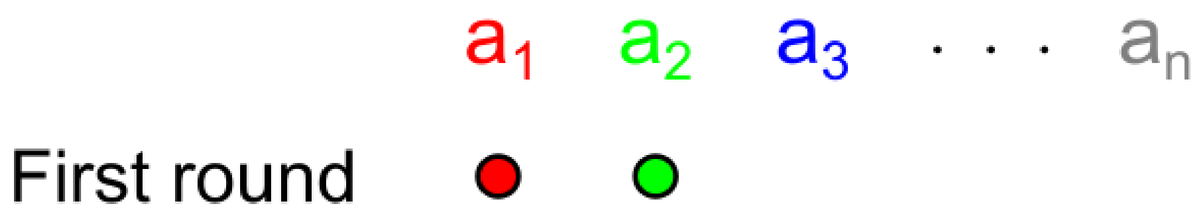

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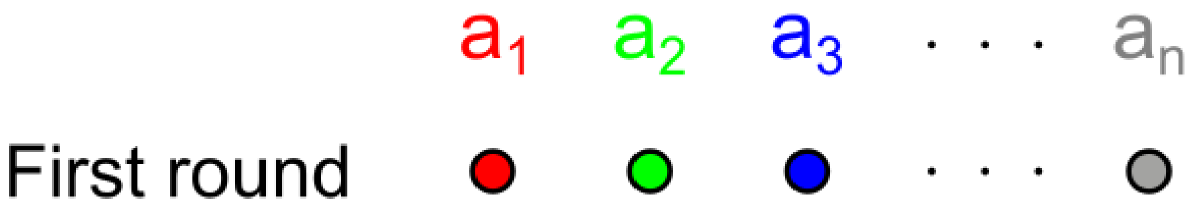

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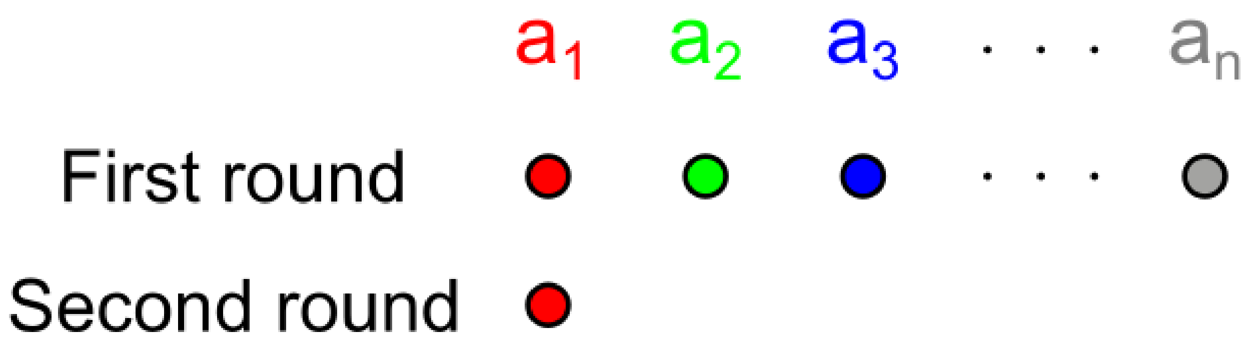

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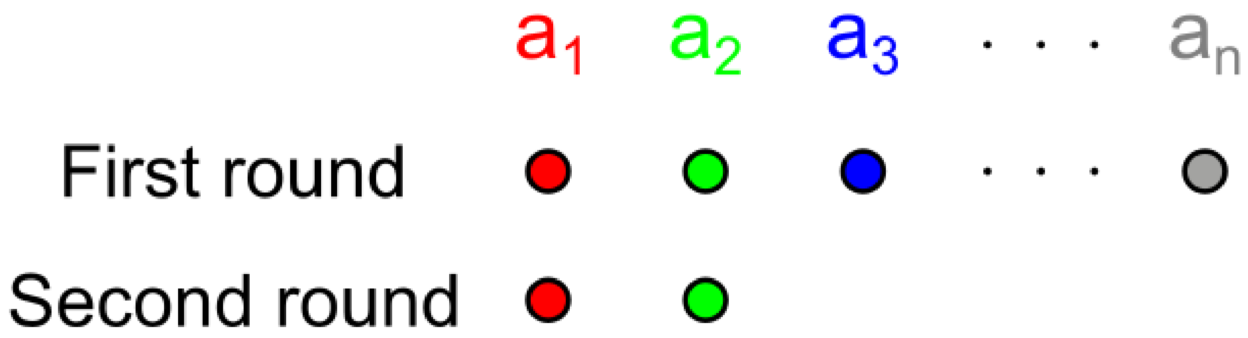

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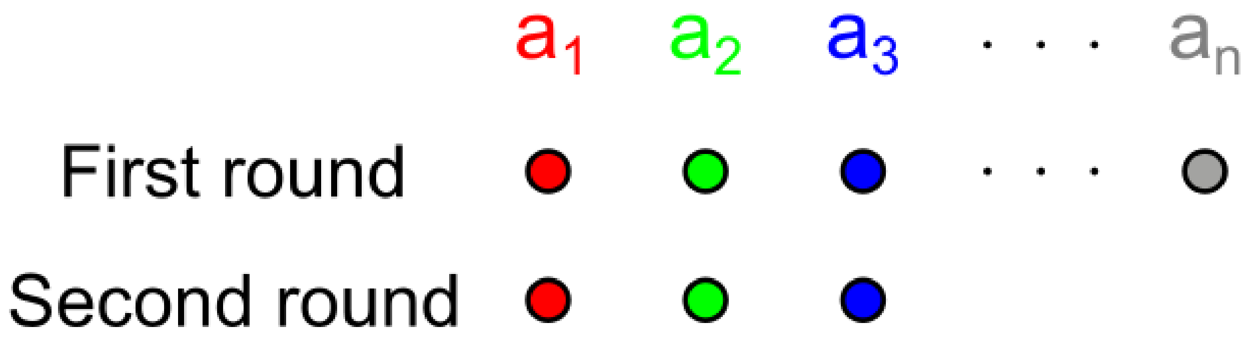

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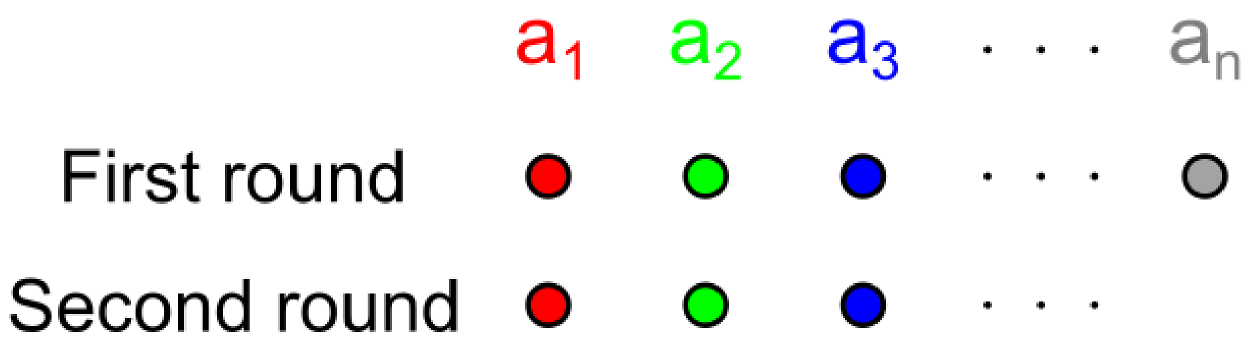

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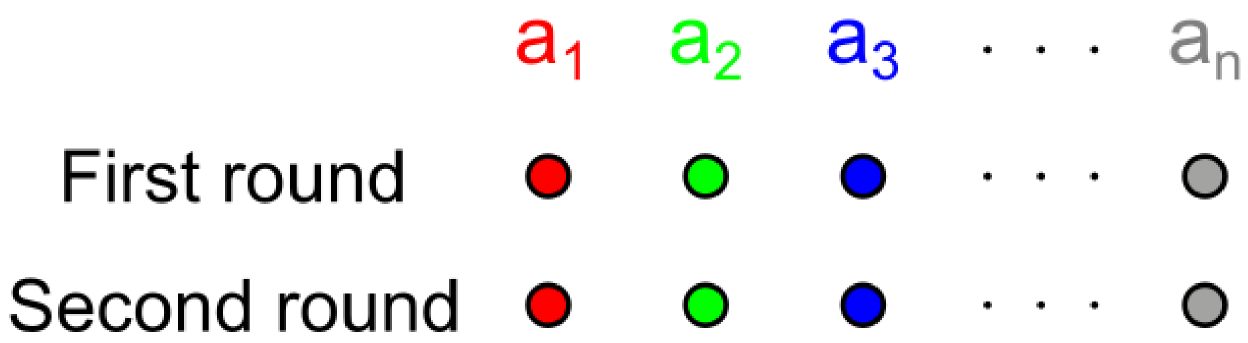

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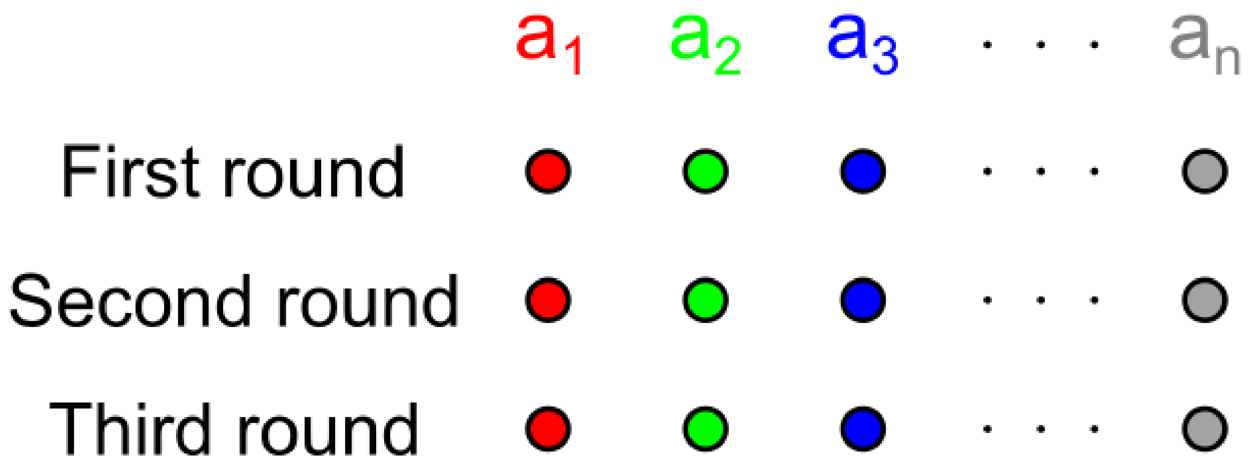

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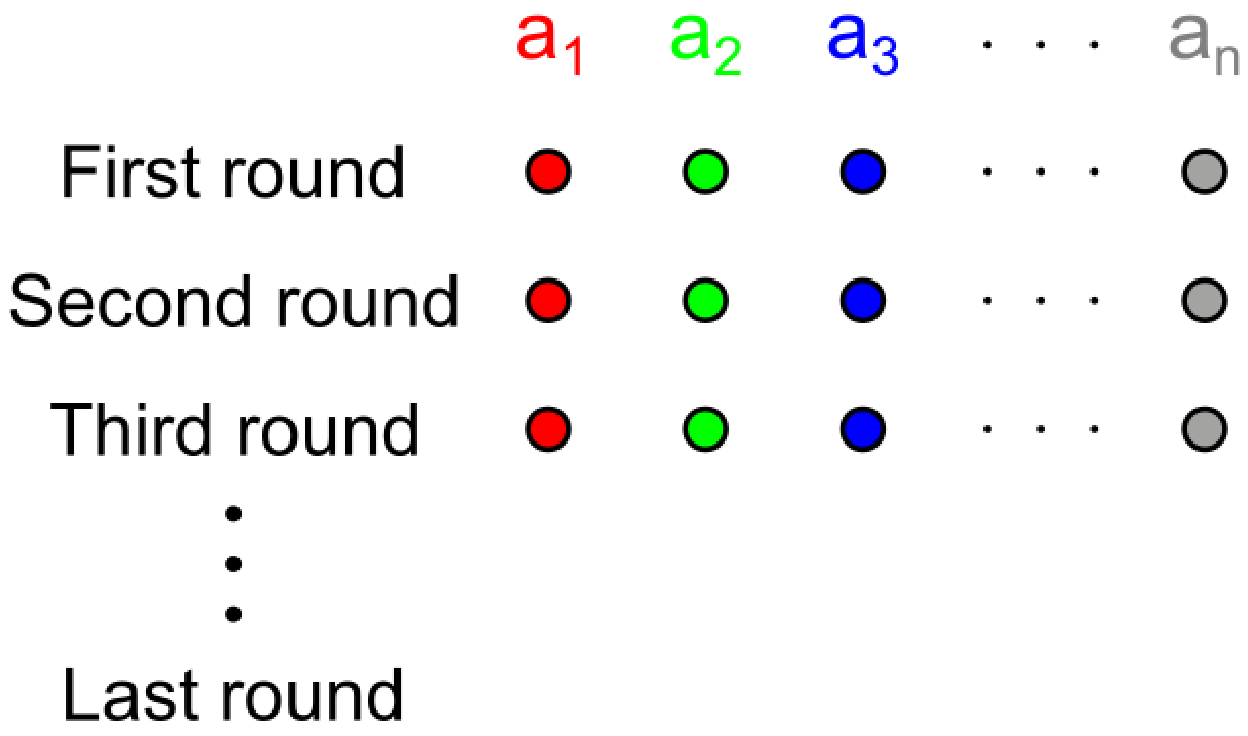

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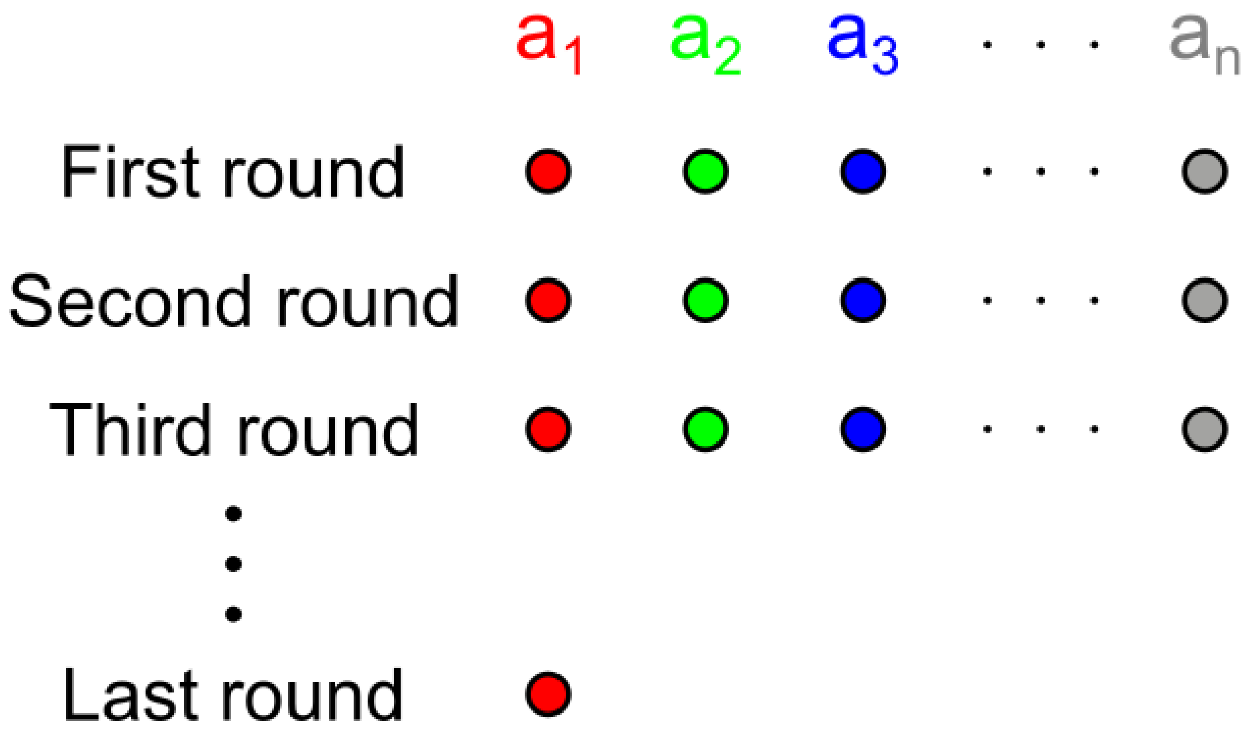

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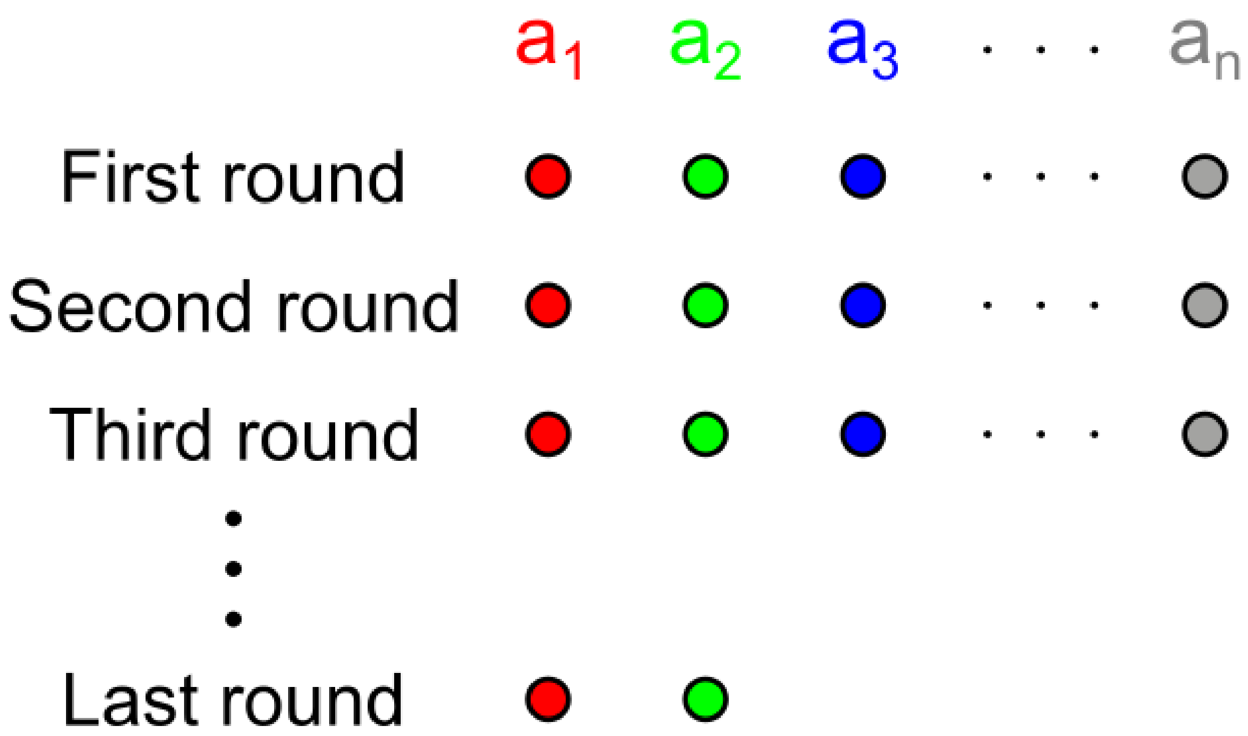

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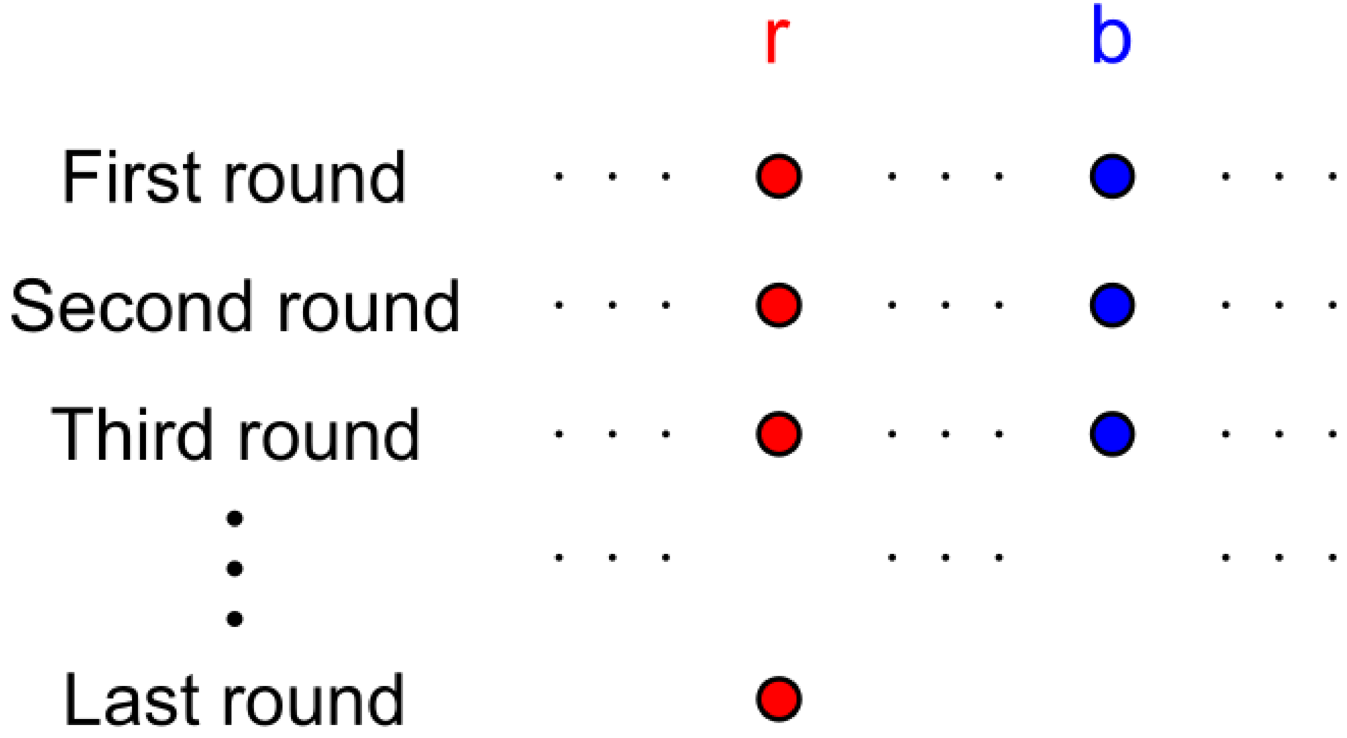

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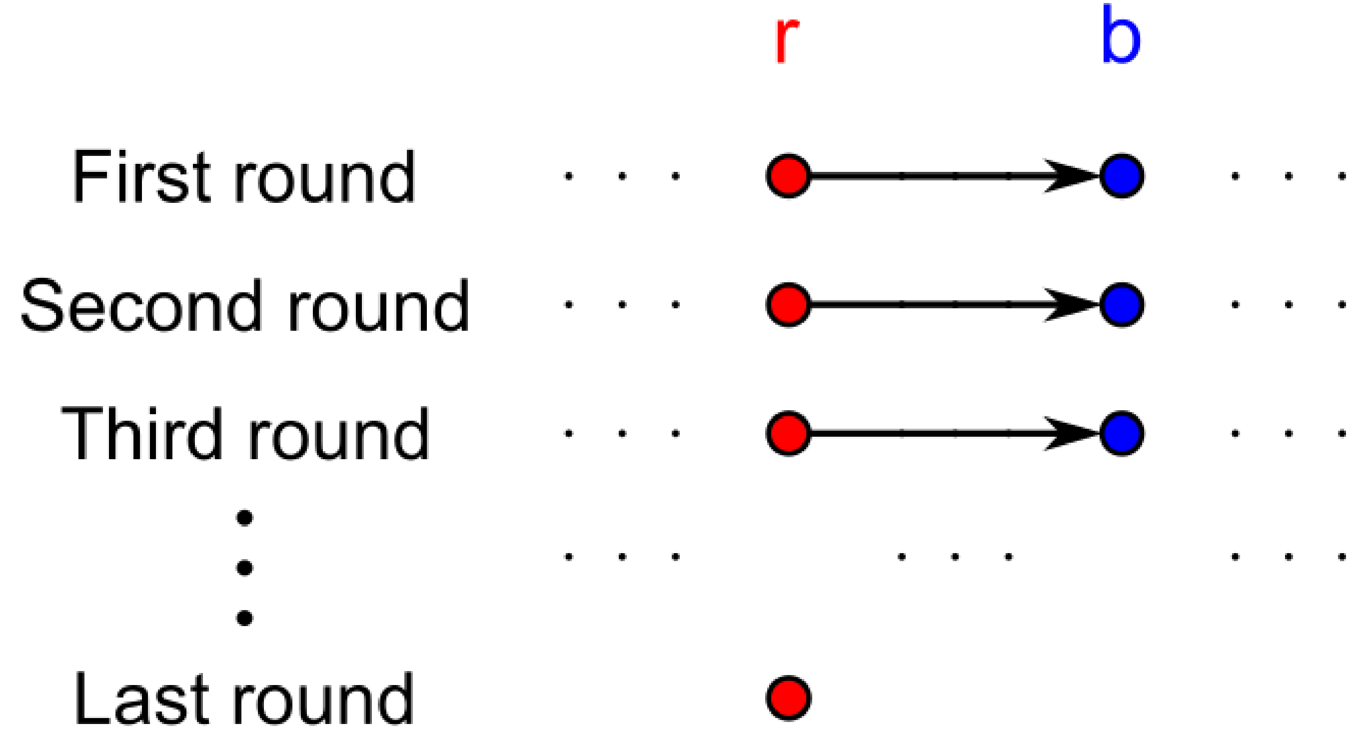

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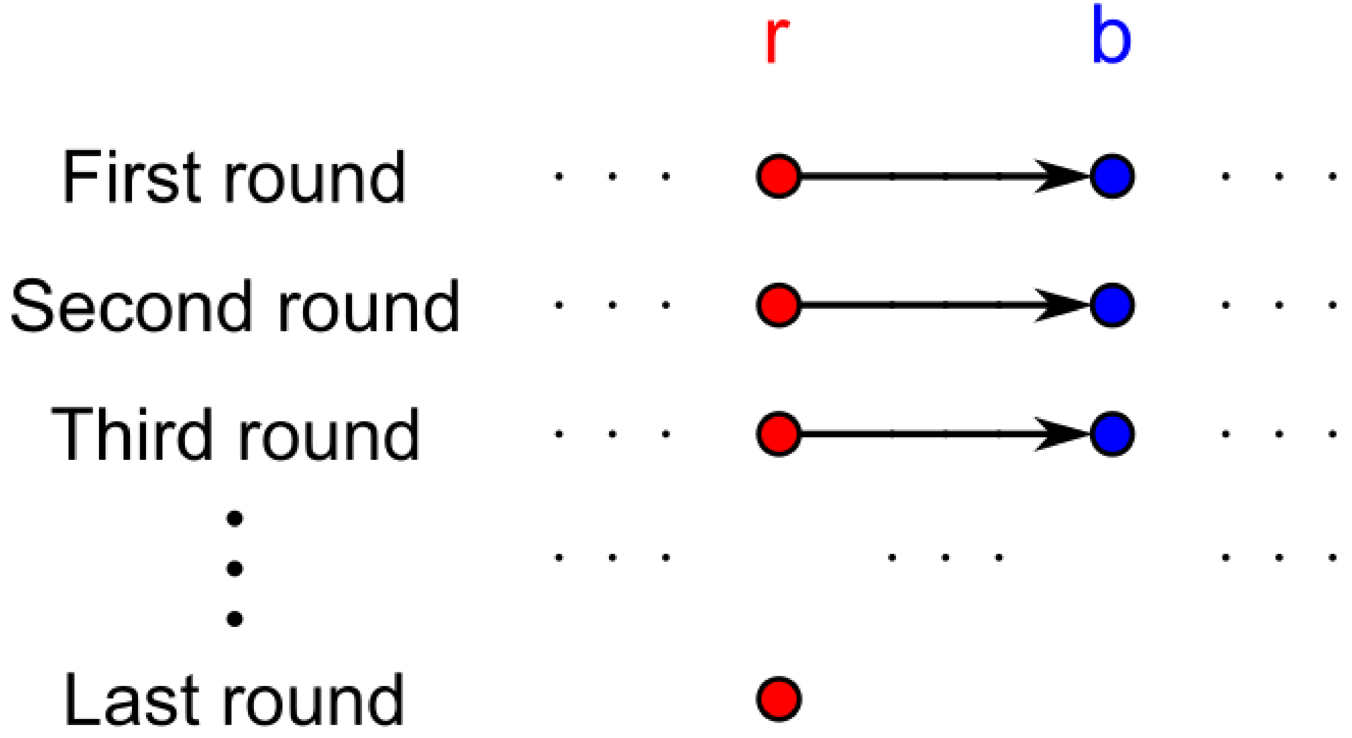

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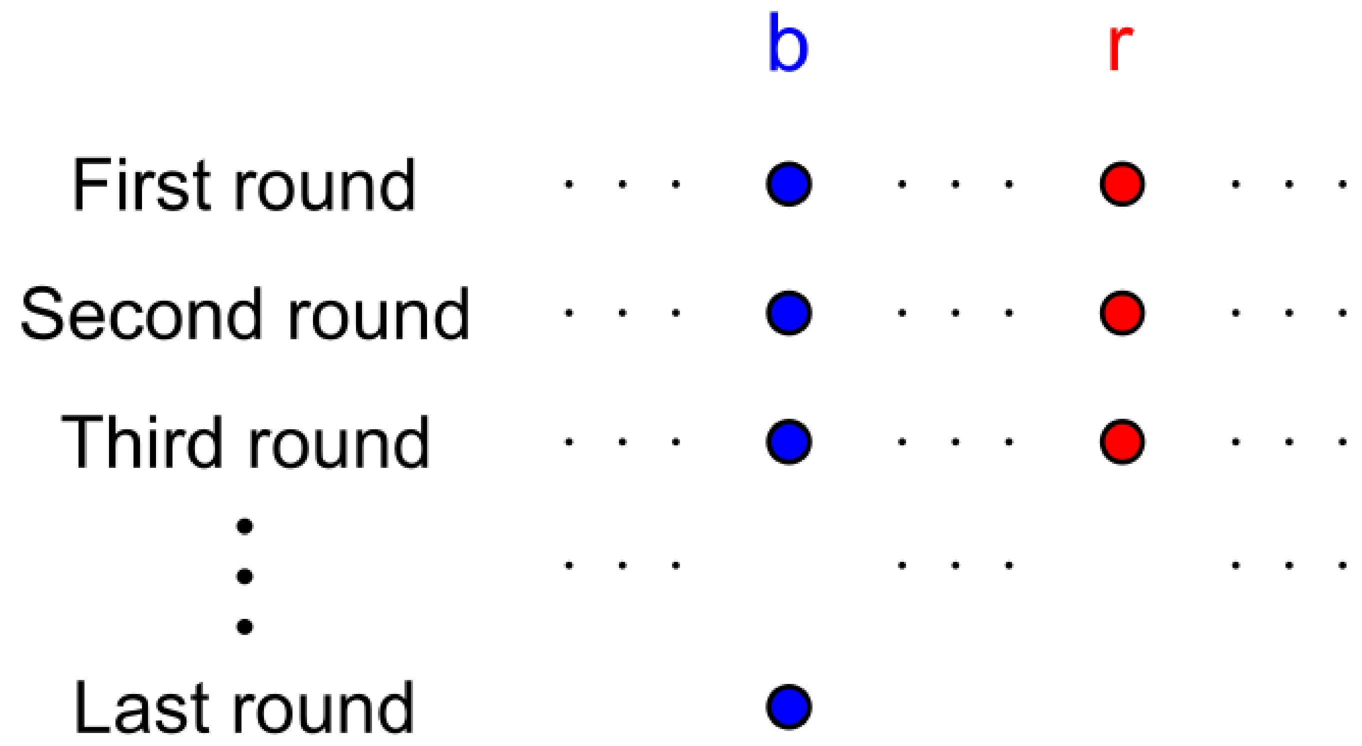

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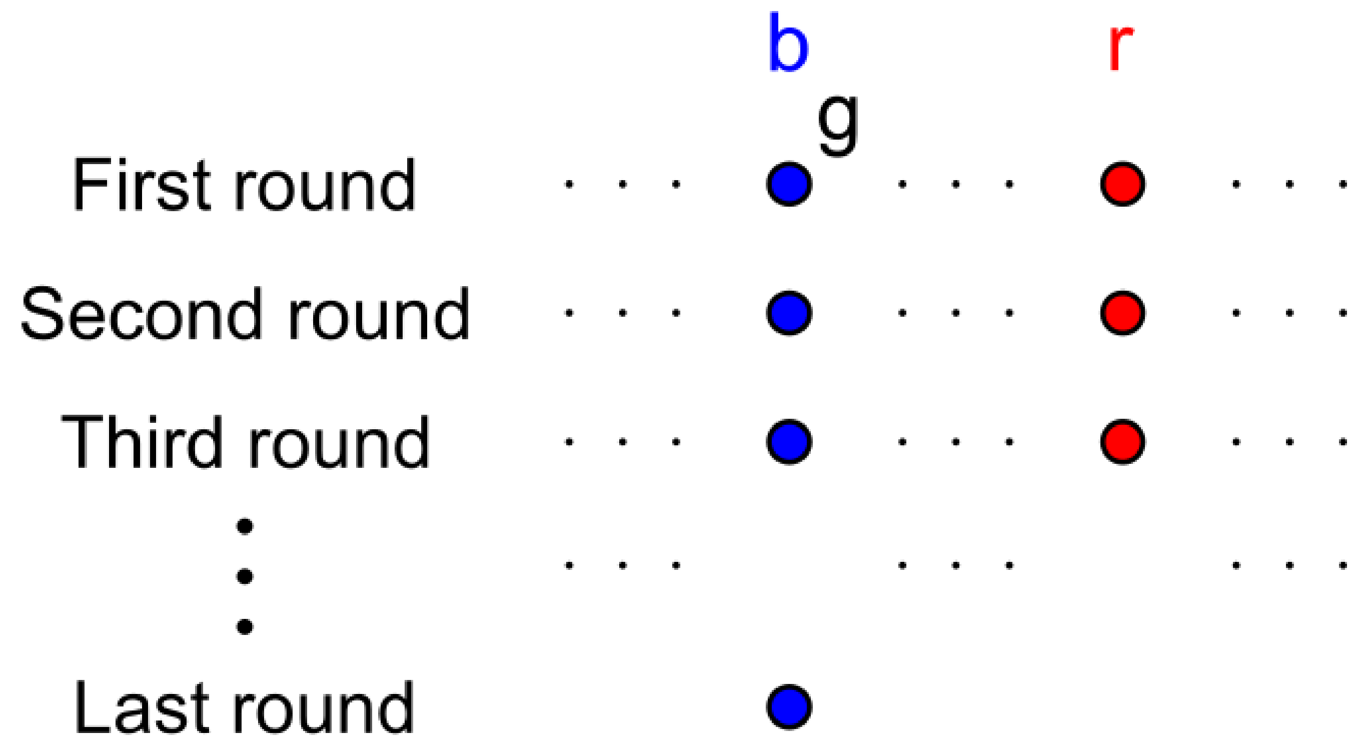

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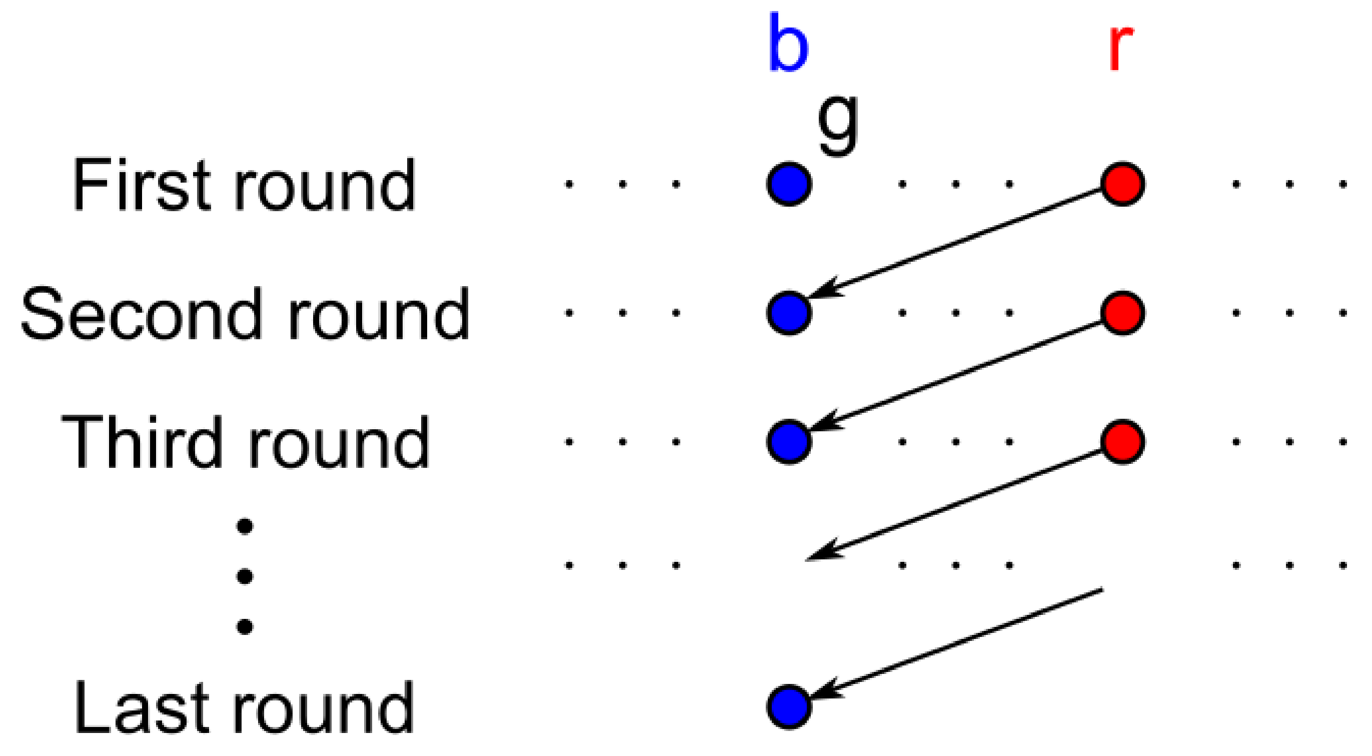

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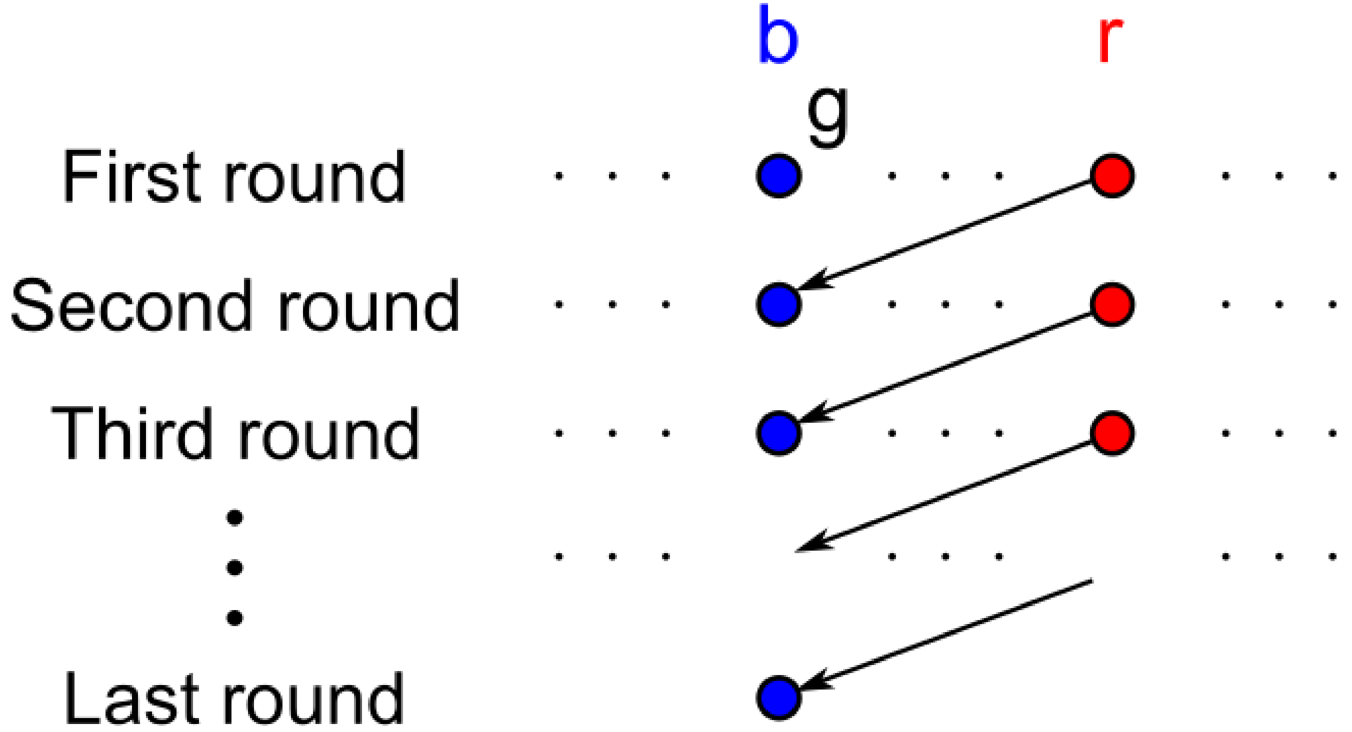

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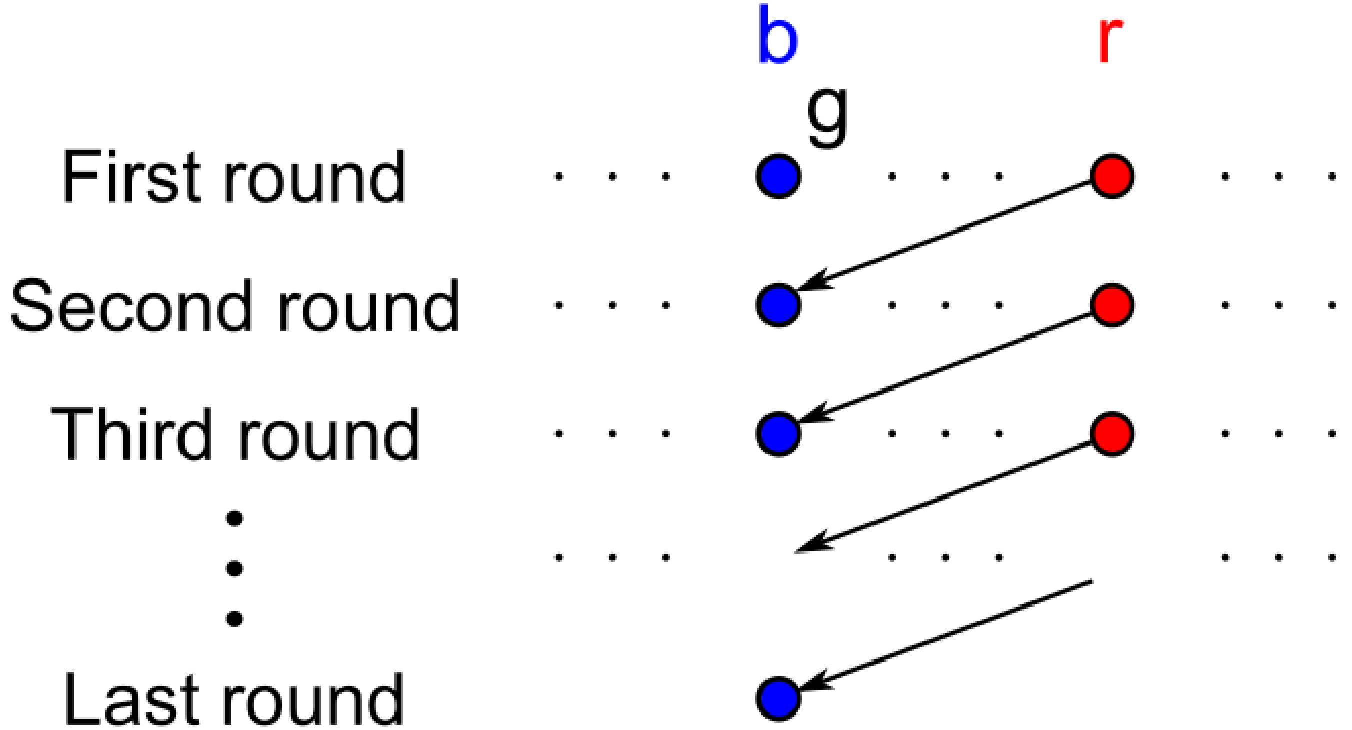

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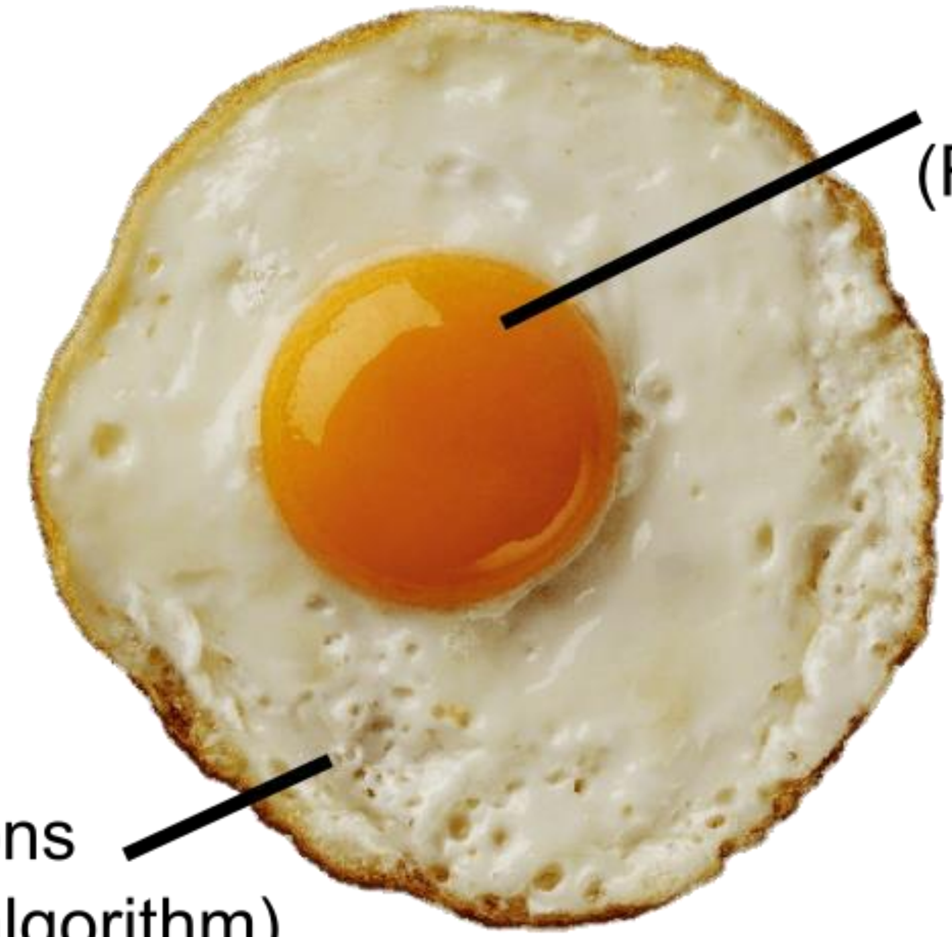
Algorithms for EF1



Envy graph of an allocation

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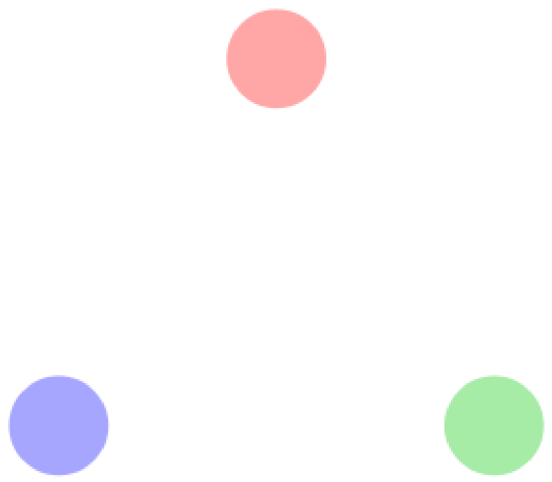

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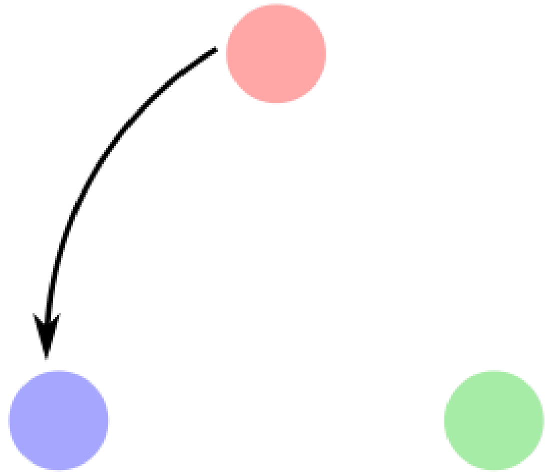

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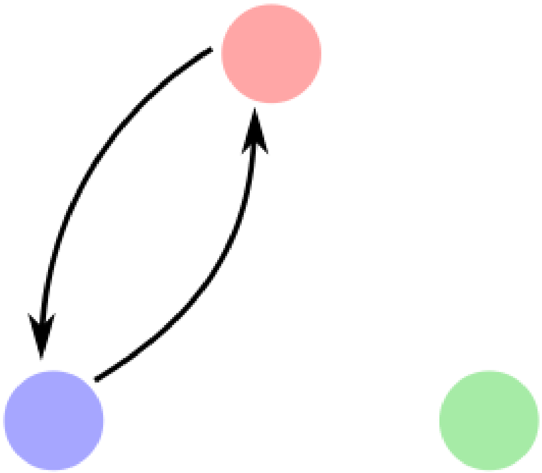

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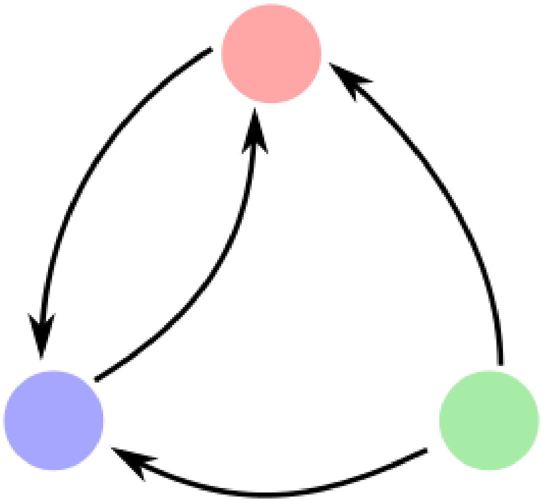

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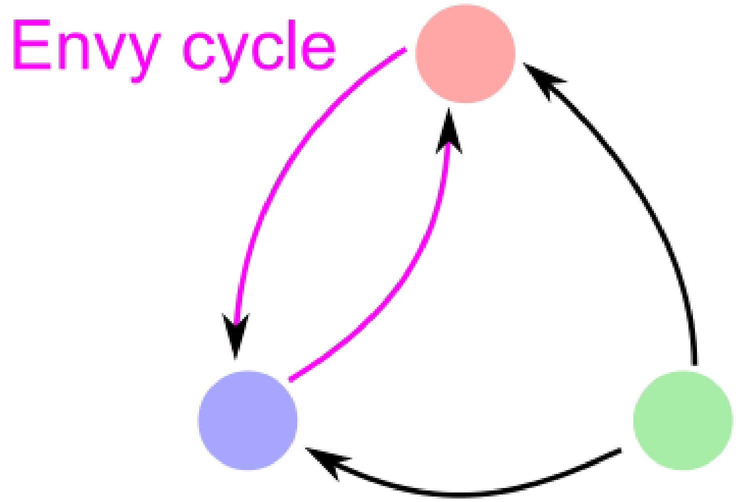

Envy-cycle elimination algorithm

[Lipton, Markakis, Mossel, and Saberi, *EC* 2004]

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While there is an unallocated good

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- Otherwise, resolve envy cycles until a source vertex shows up, and then assign the good to it.

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[Lipton, Markakis, Mossel, and Saberi, *EC* 2004]

While there is an unallocated good

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each agent in the cycle gets the bundle that it is pointing to

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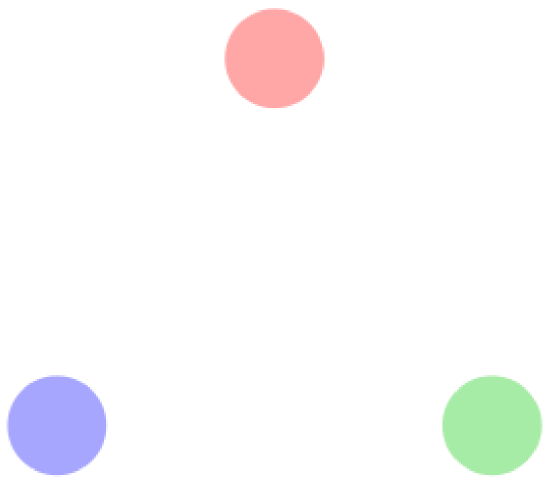

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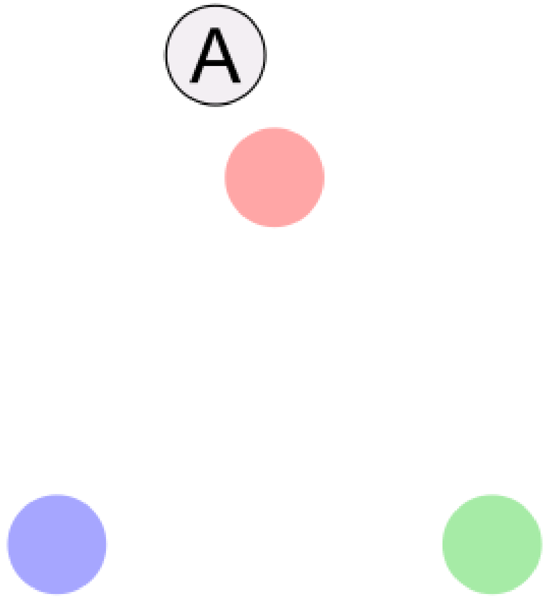

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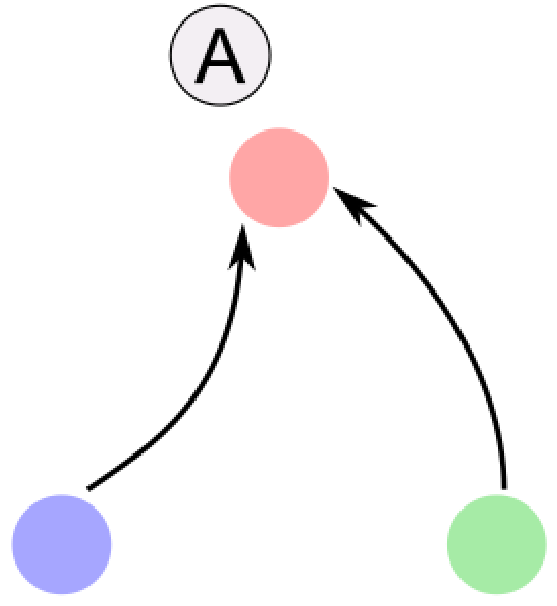

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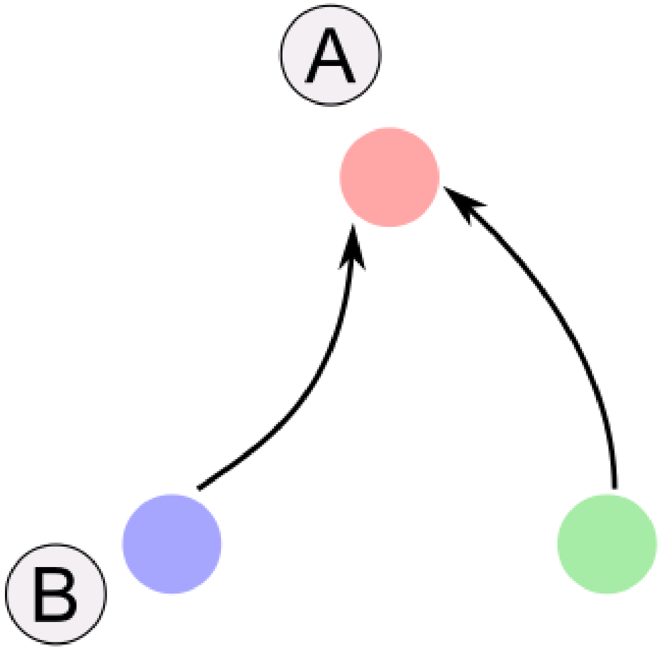

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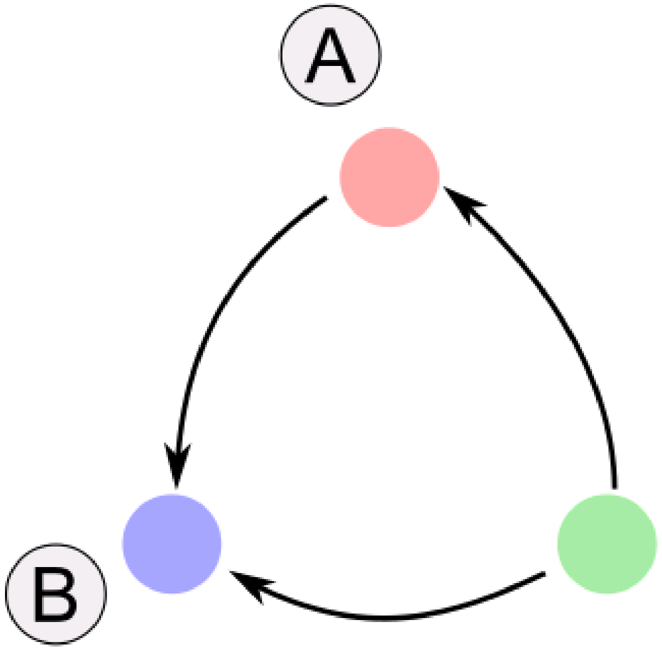

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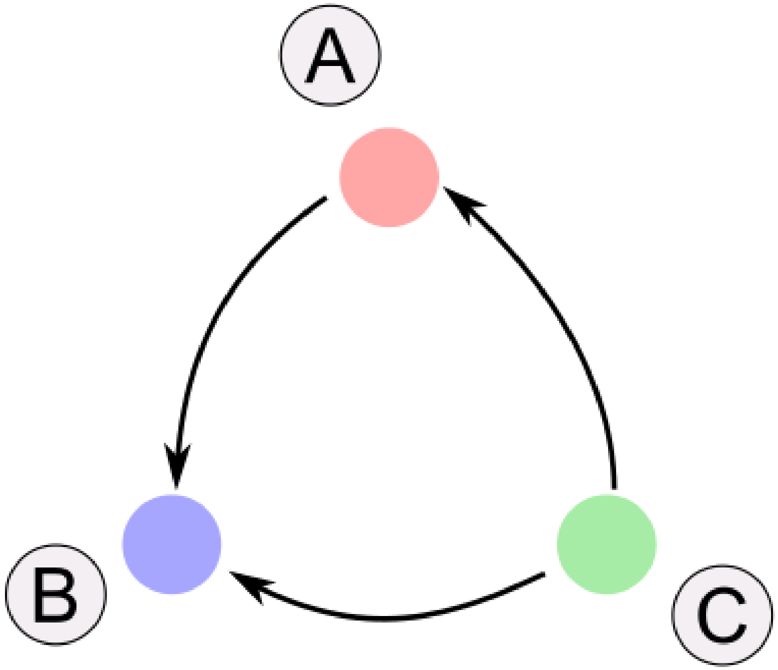

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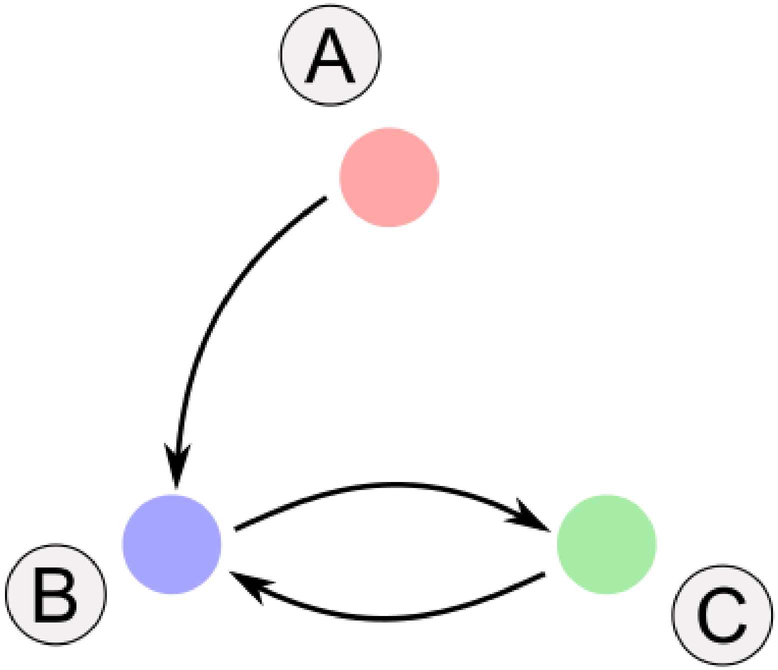

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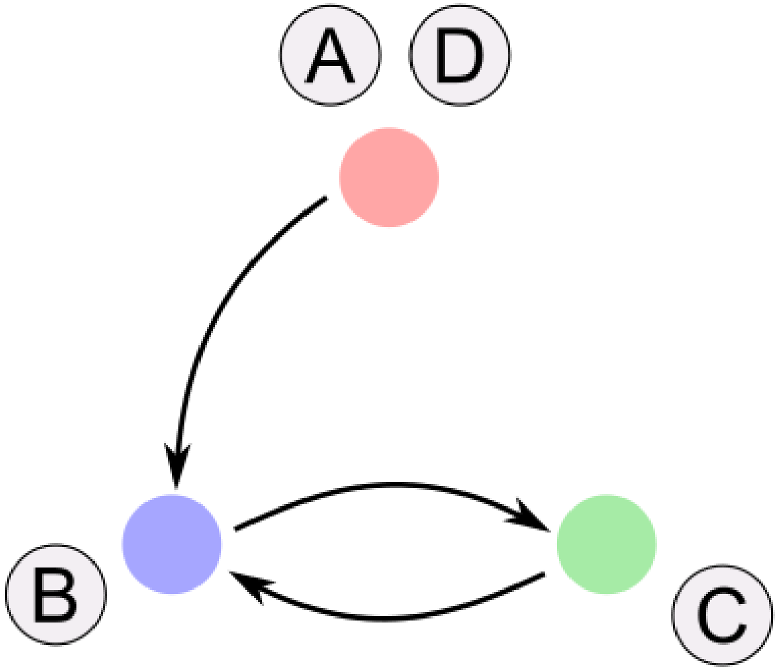

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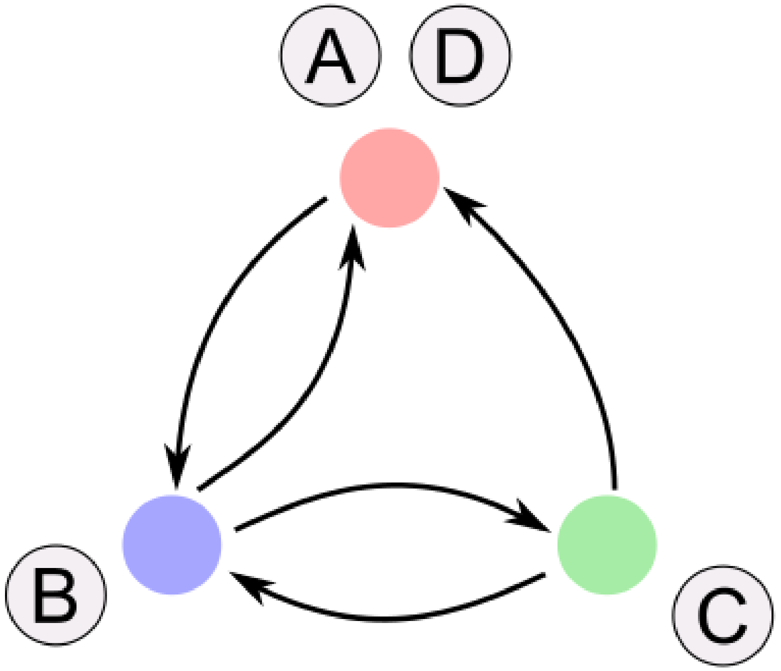

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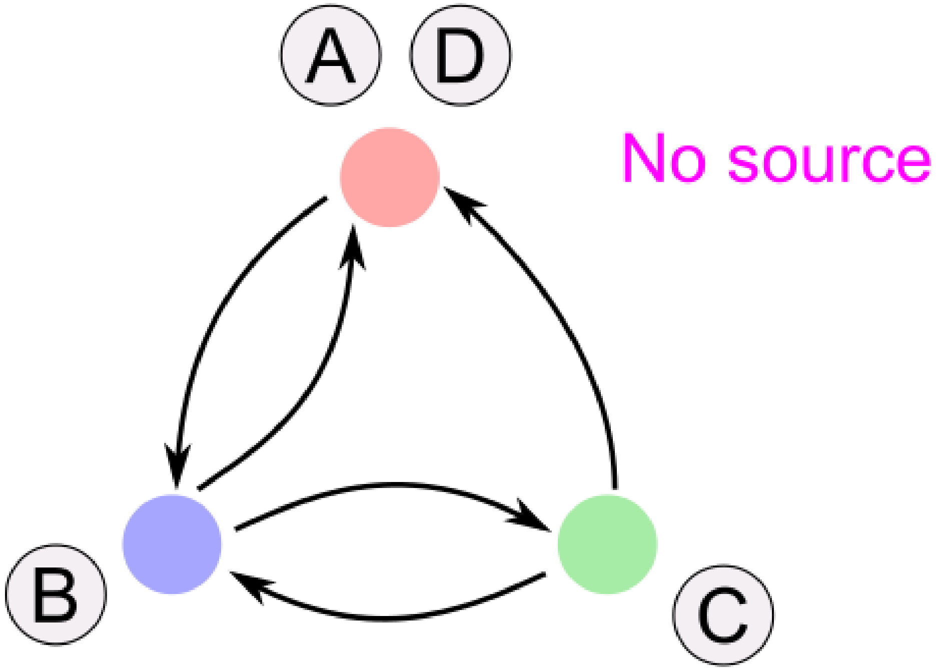

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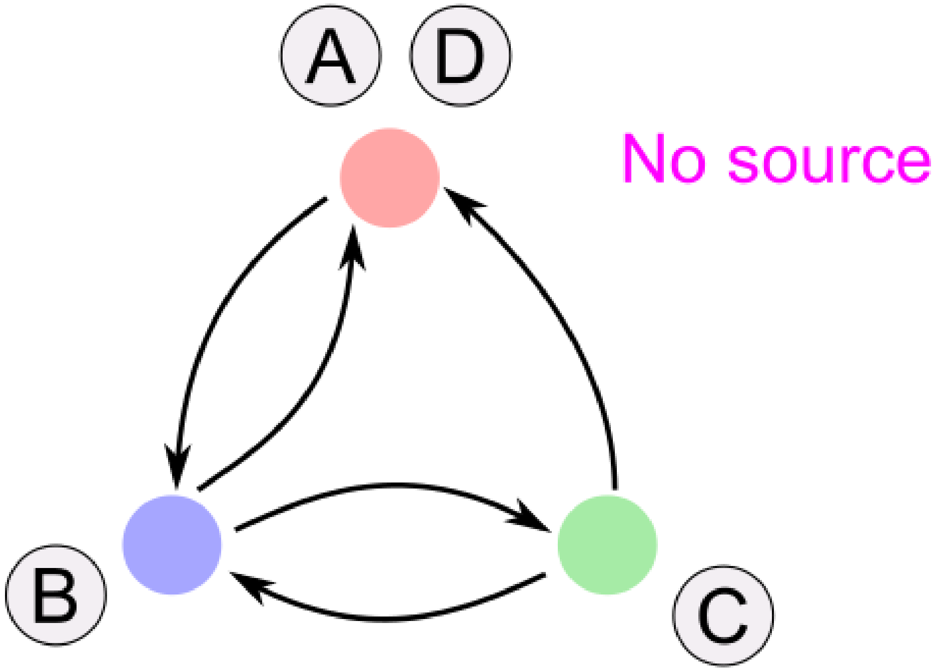

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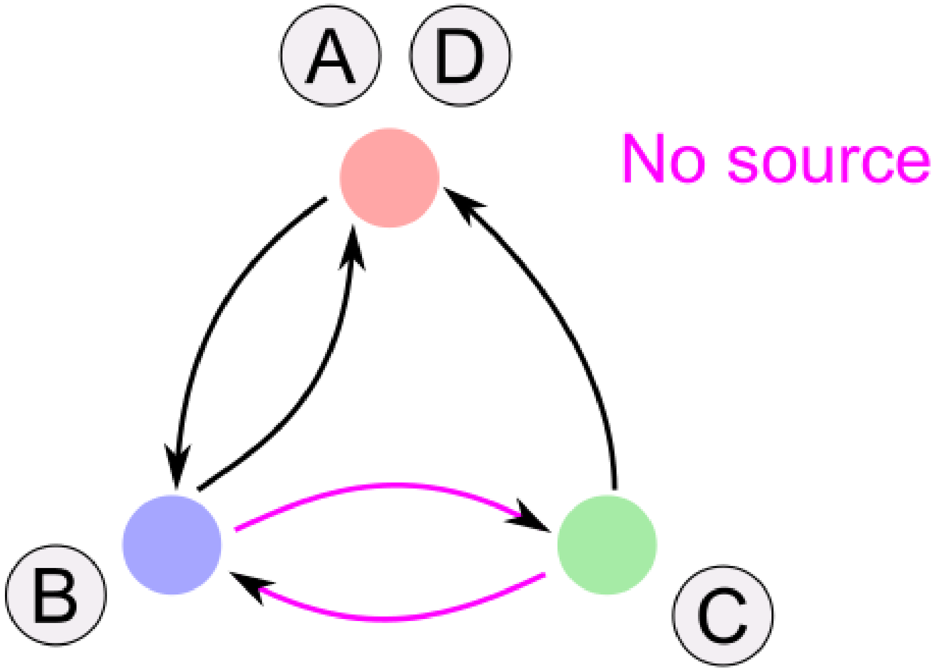

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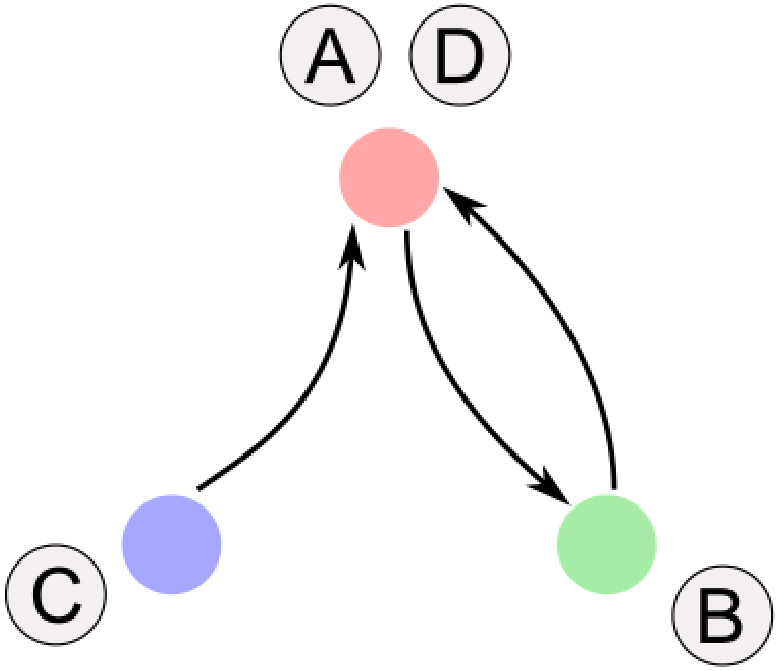

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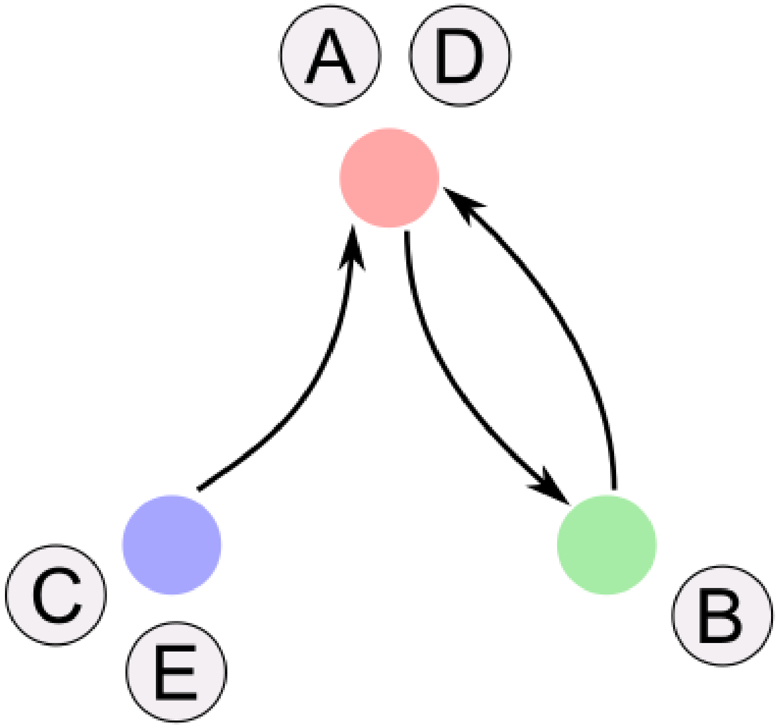

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- With n agents, at most $O(n^2)$ cycle resolutions required to create a source.
- Polynomial running time!

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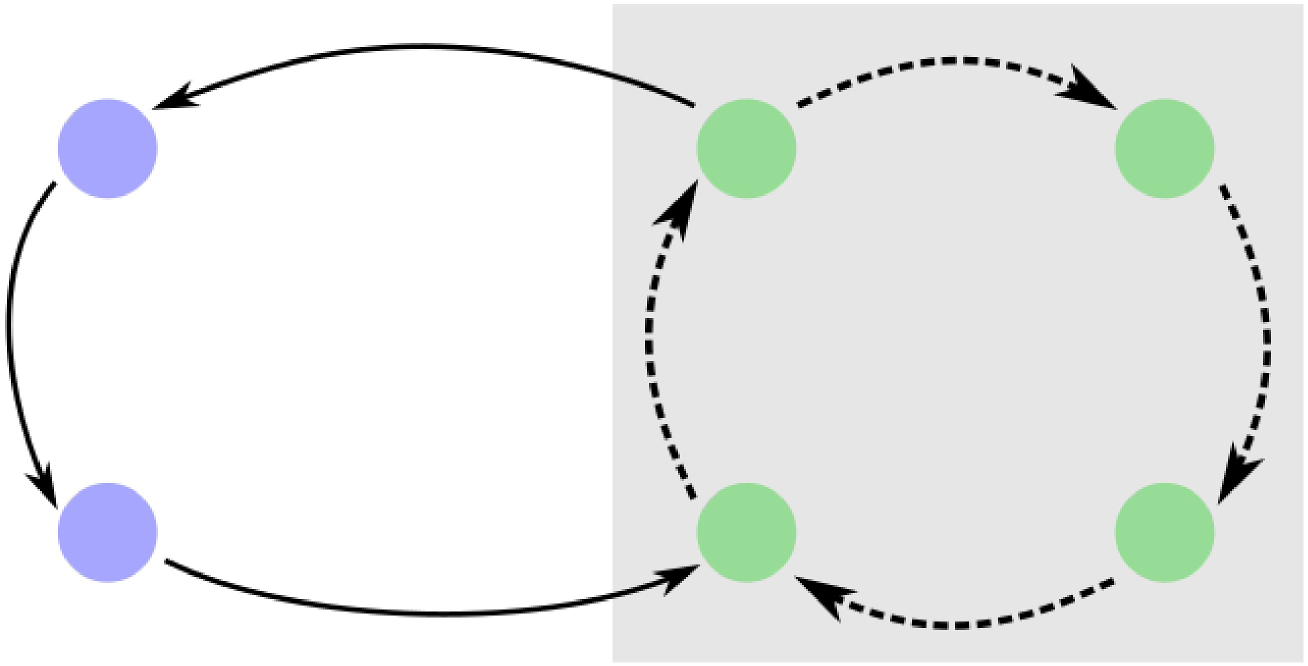

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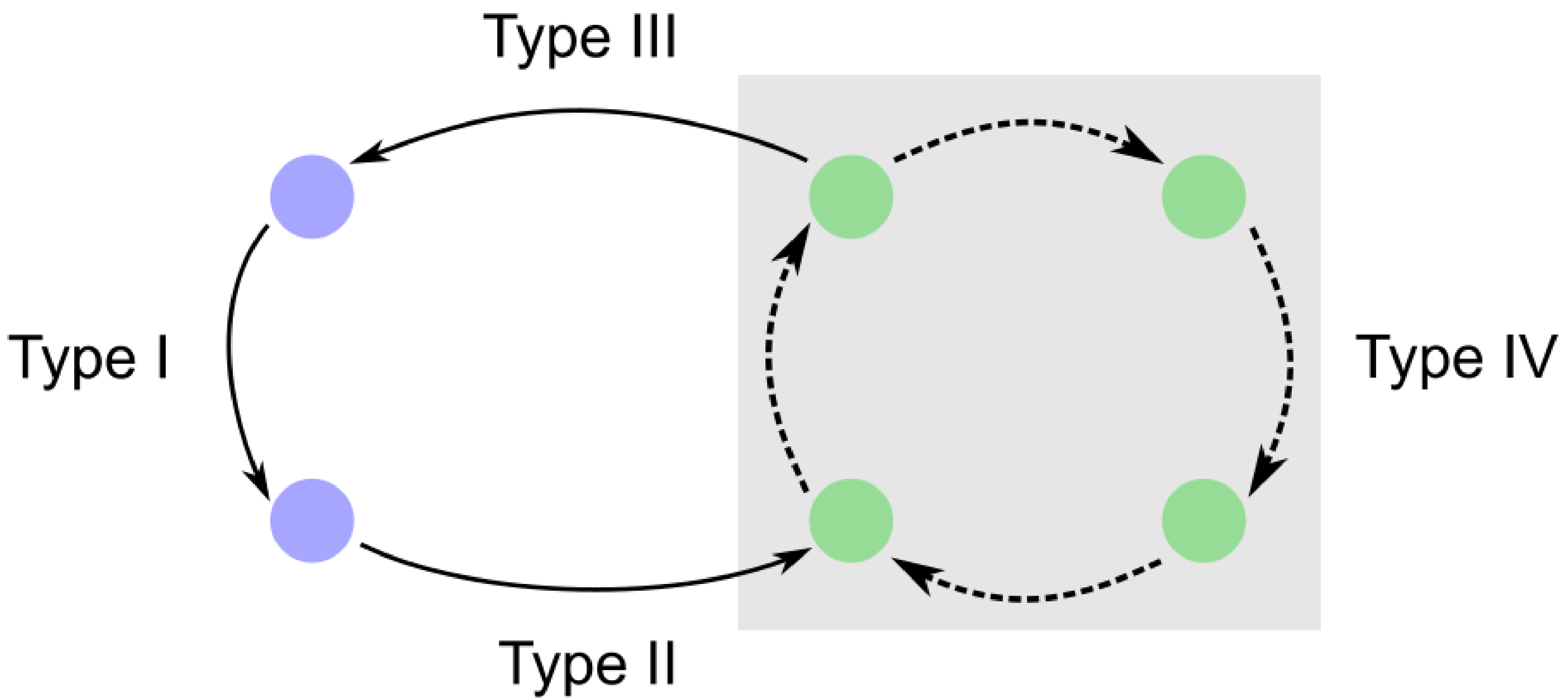

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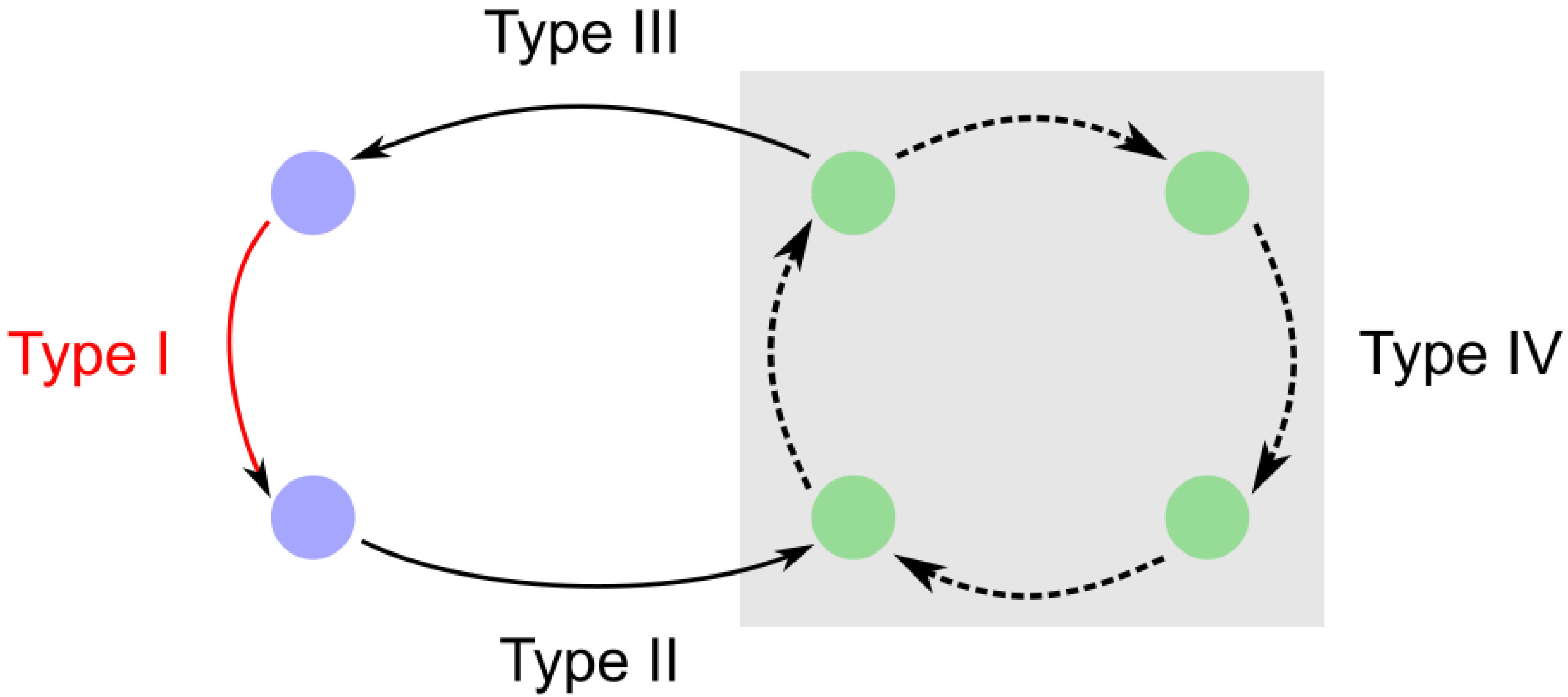

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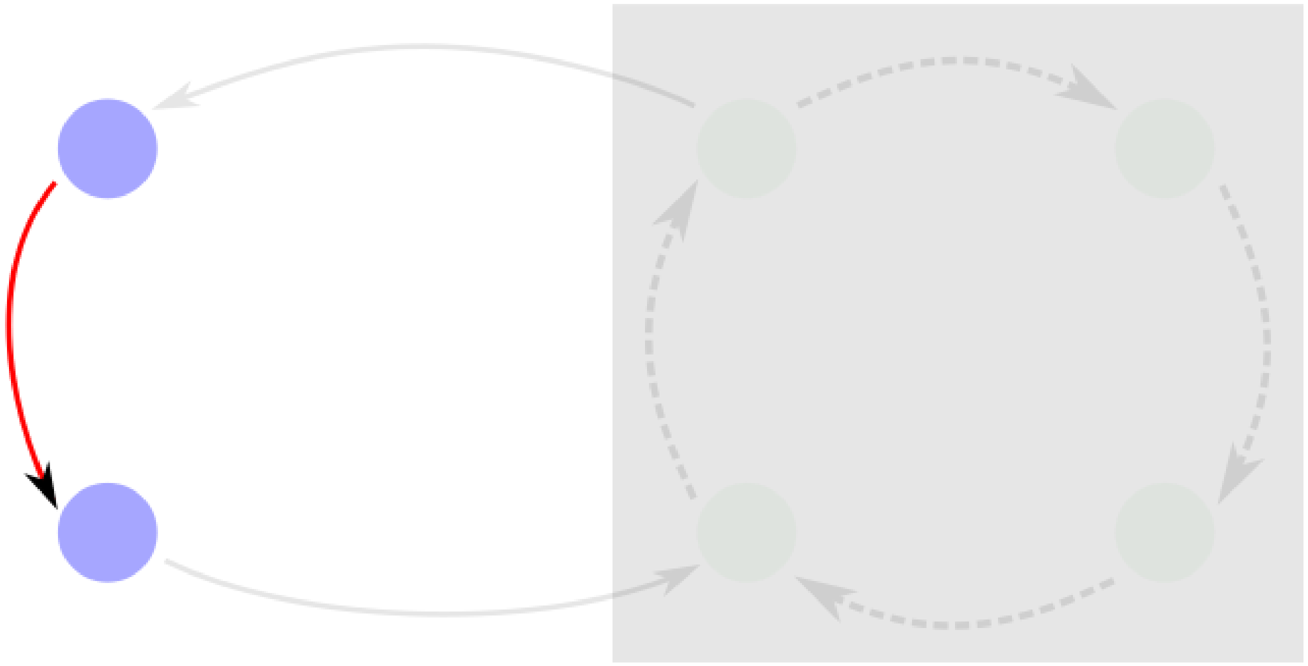

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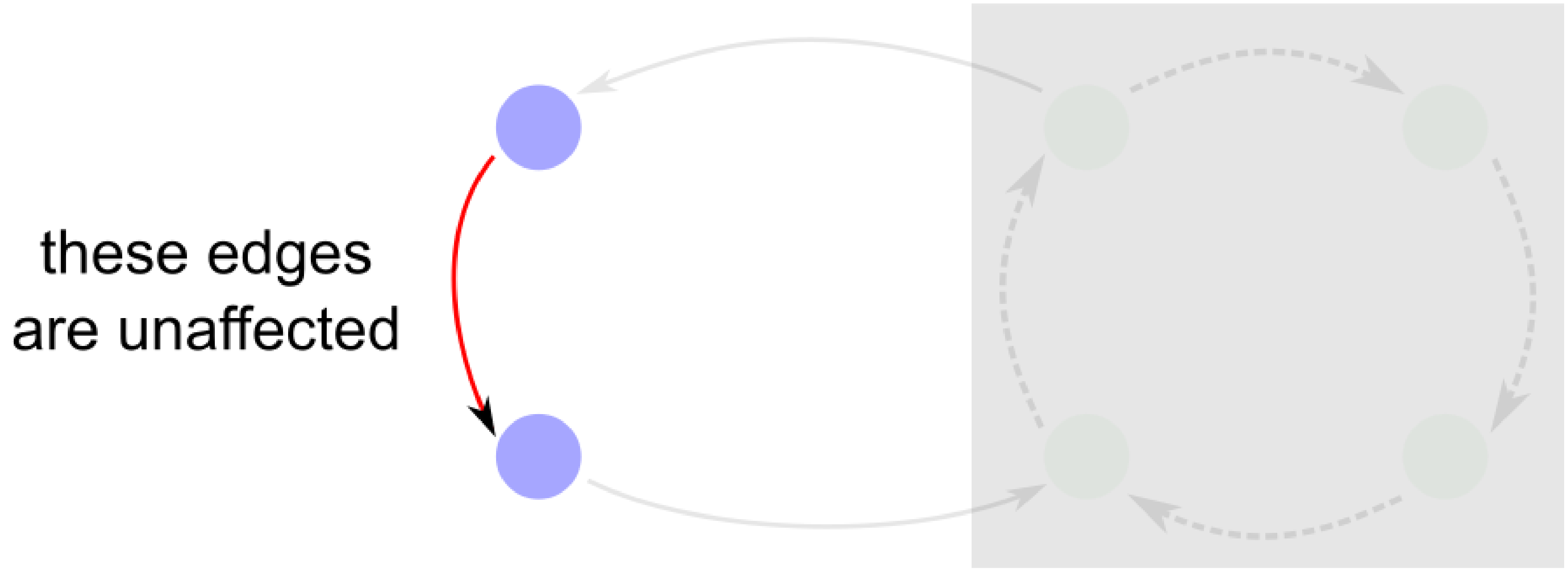

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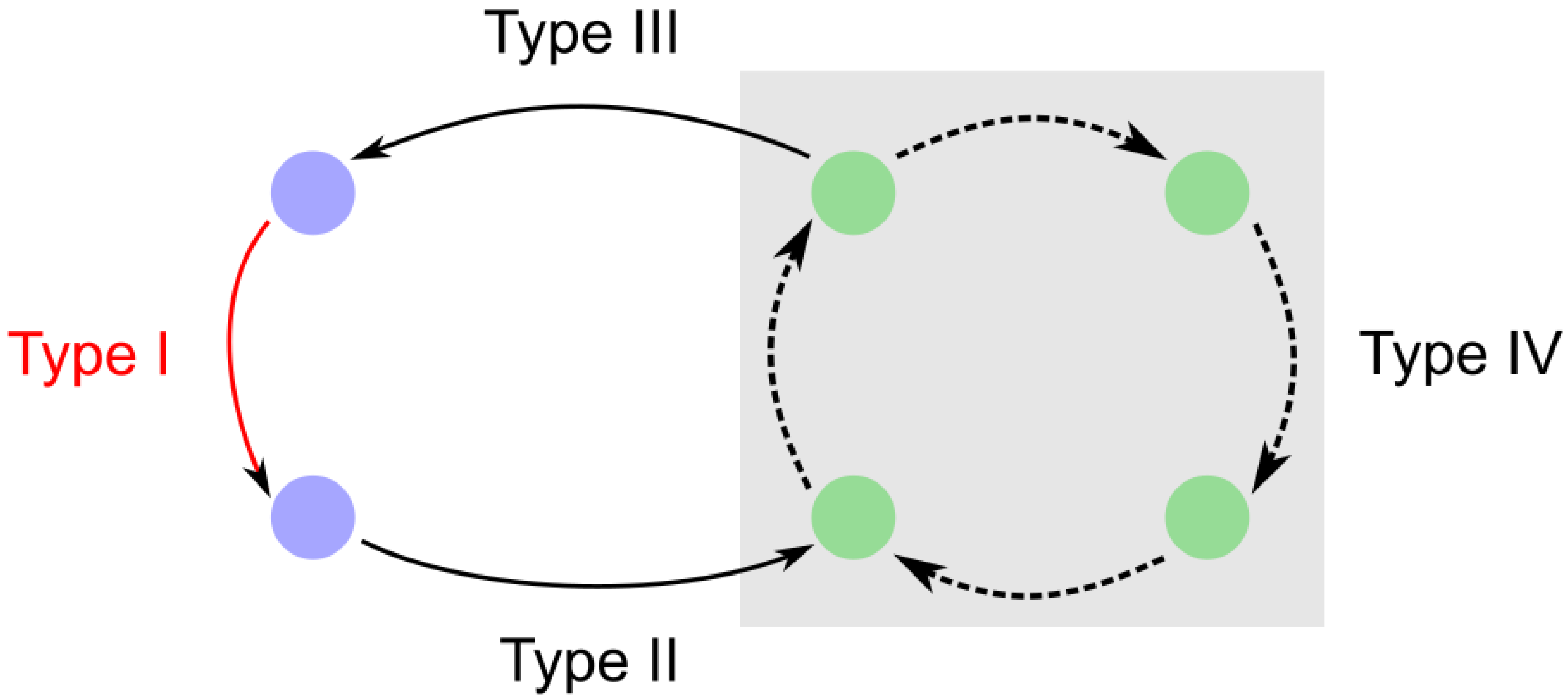

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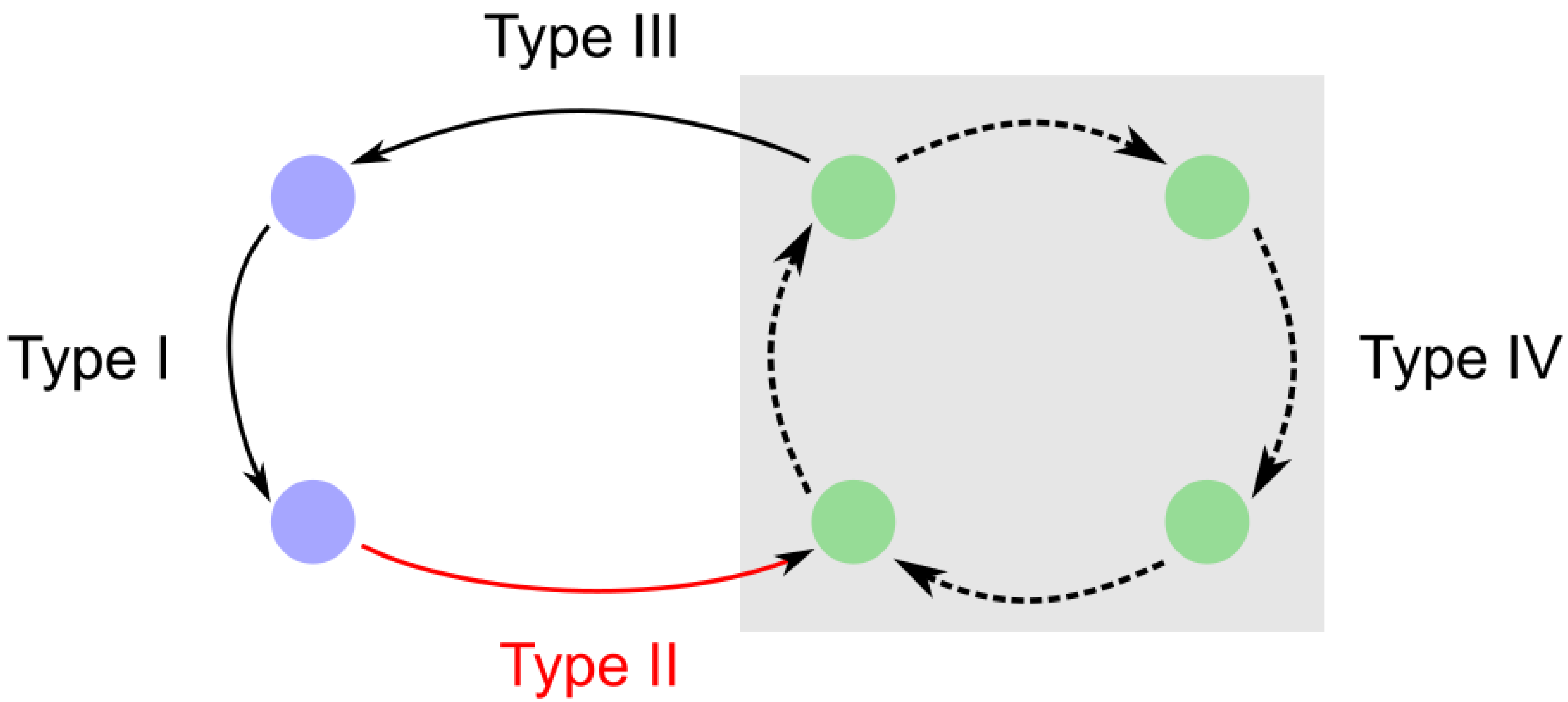

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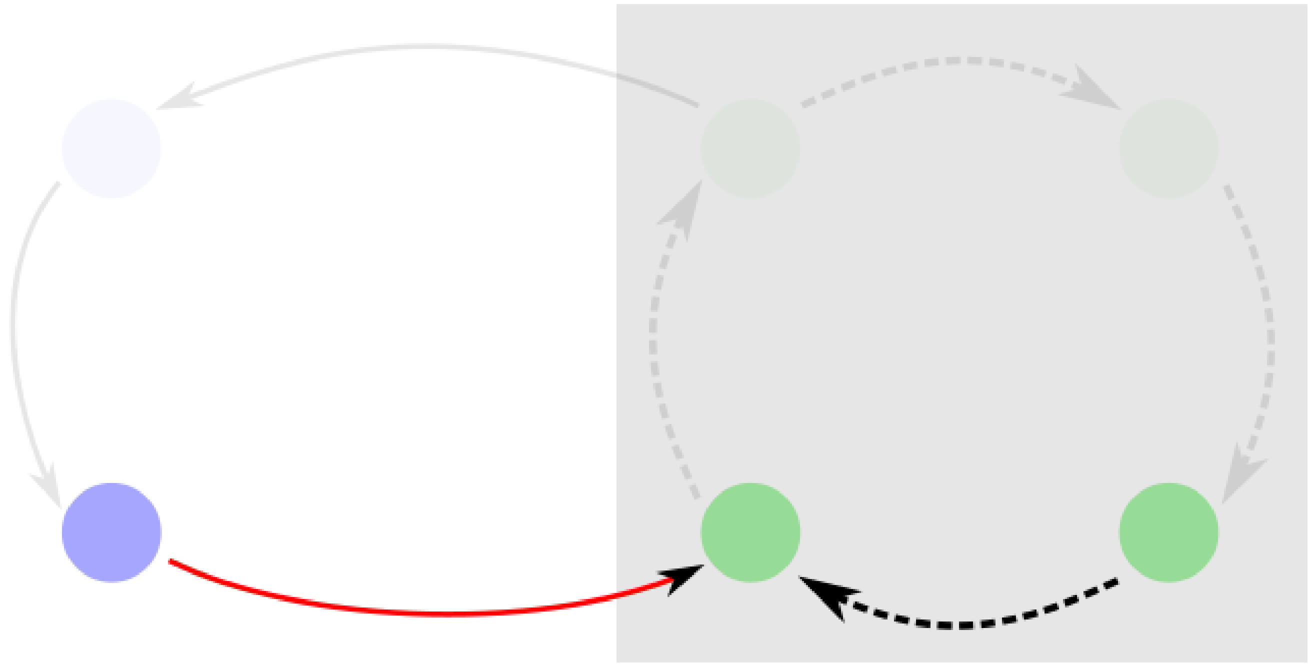

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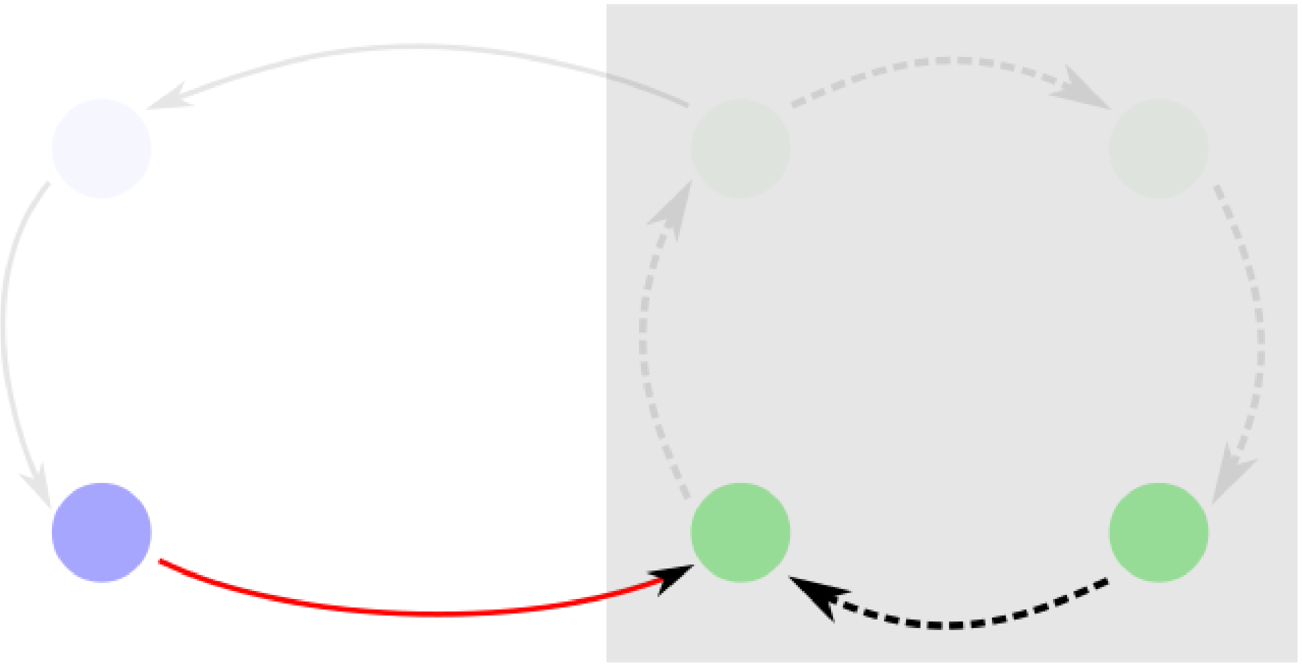

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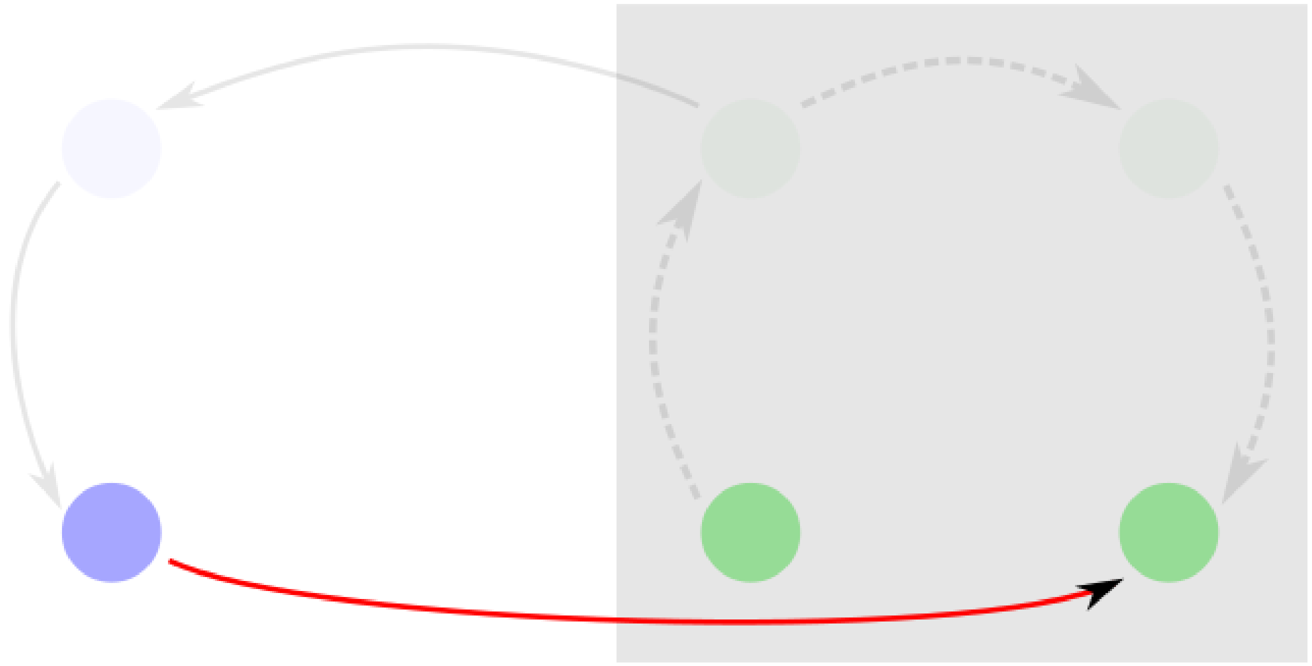

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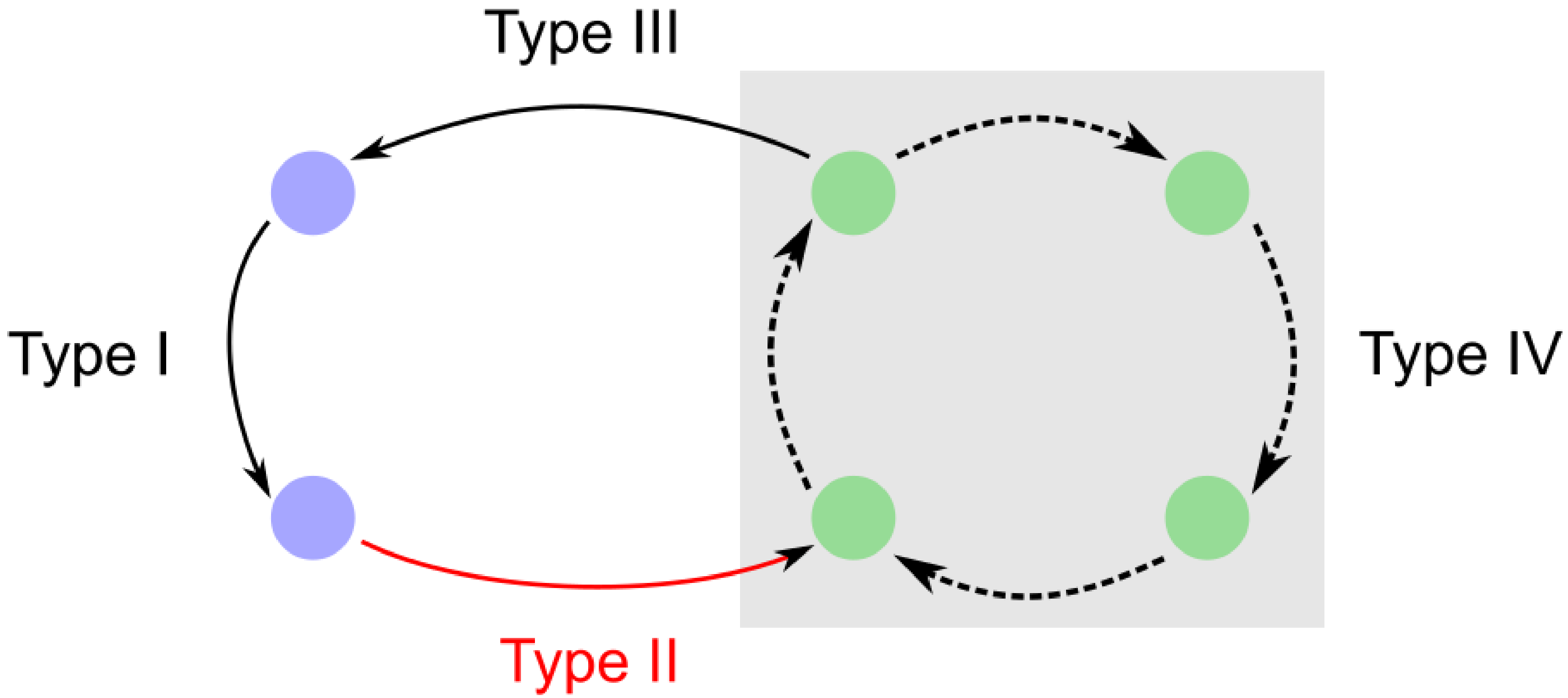

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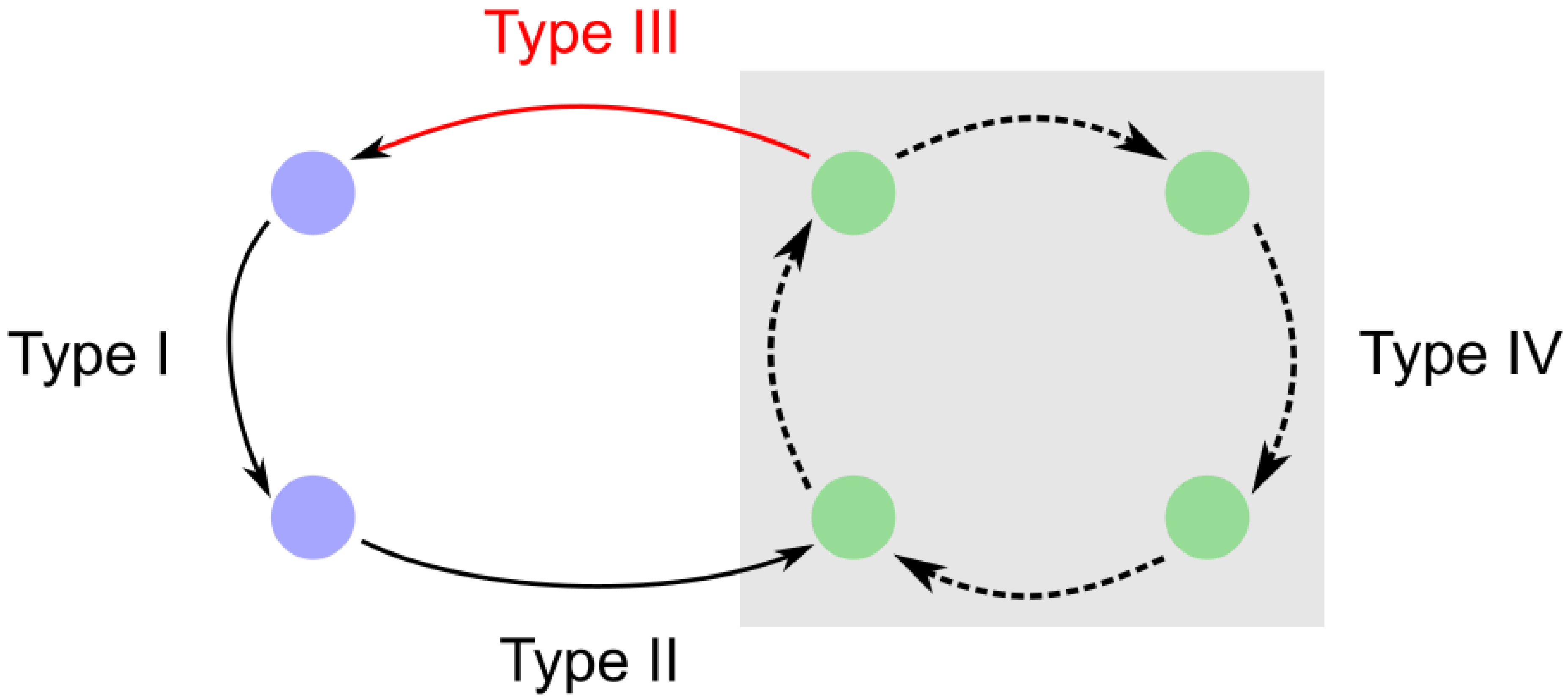

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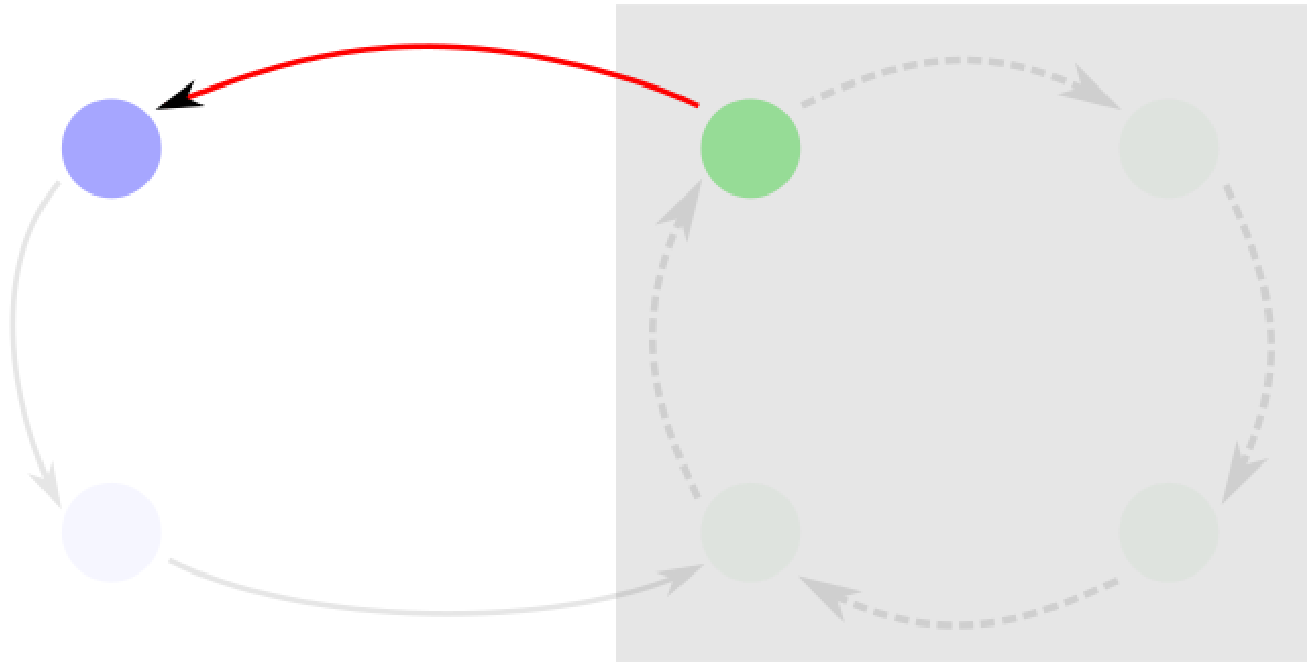

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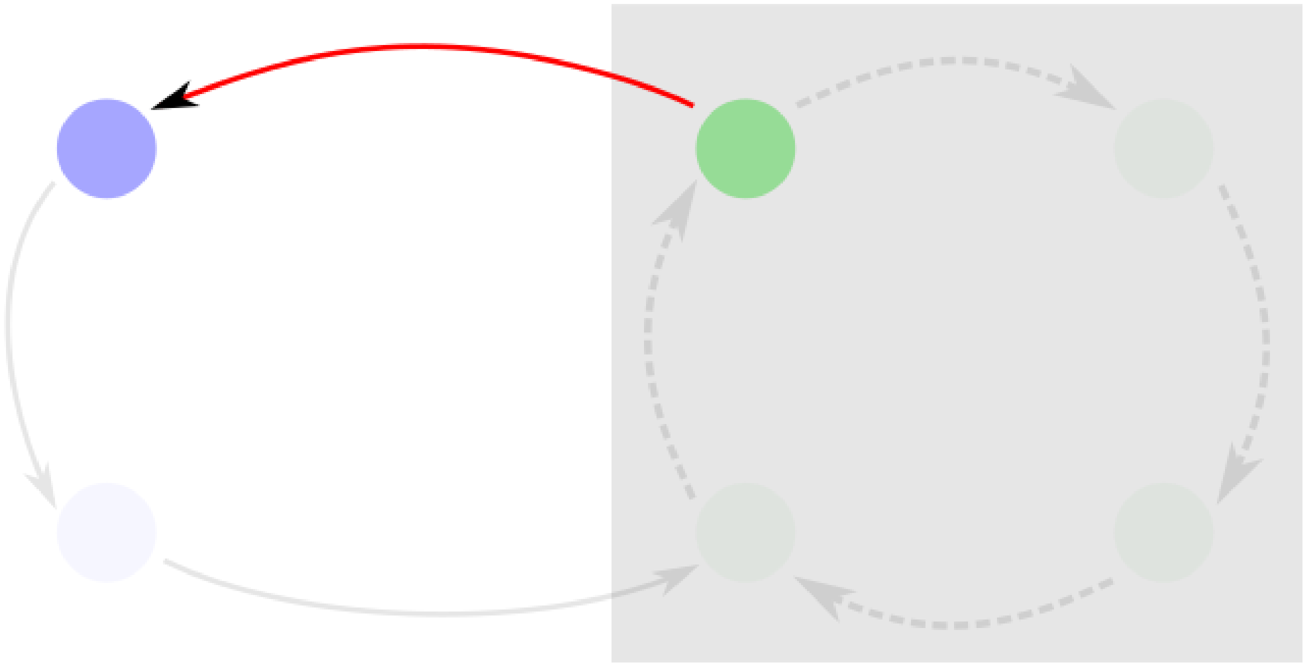

these edges can either stay or disappear (no new such edges are added)

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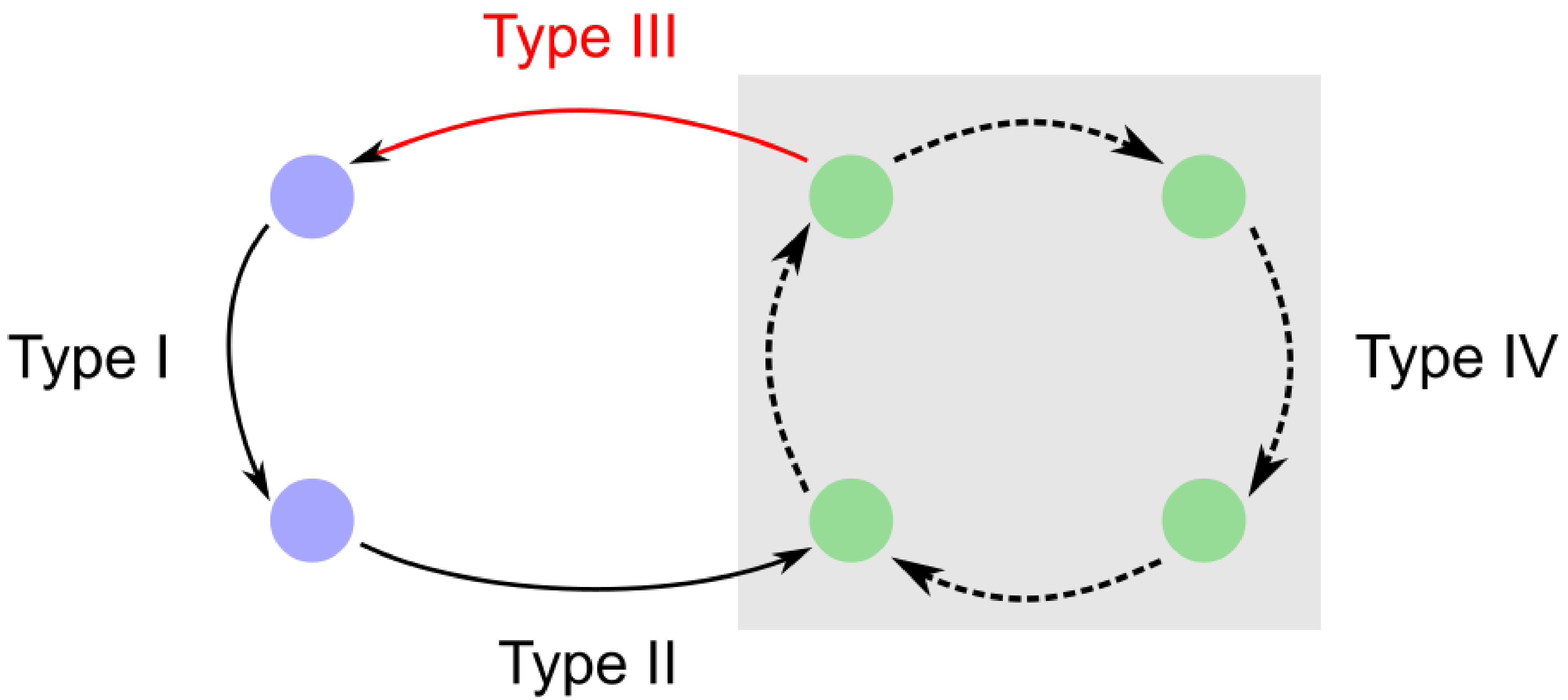

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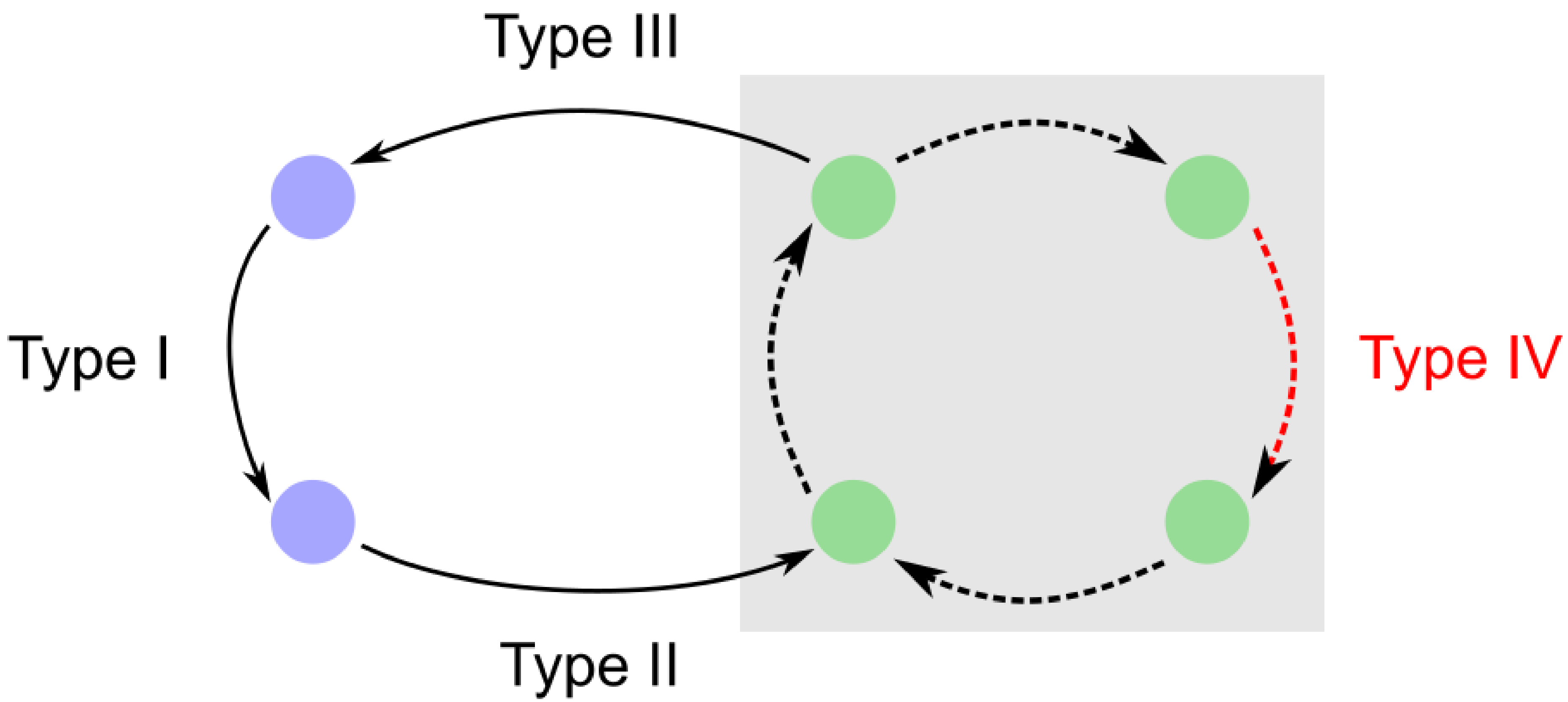

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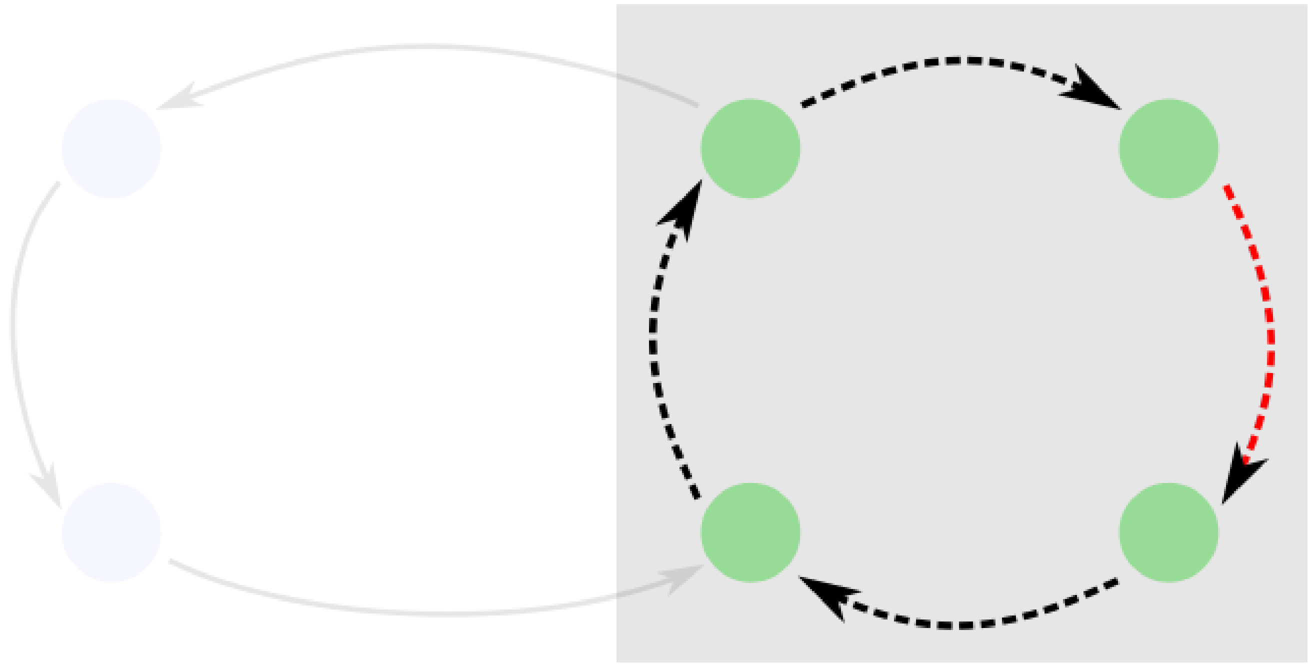

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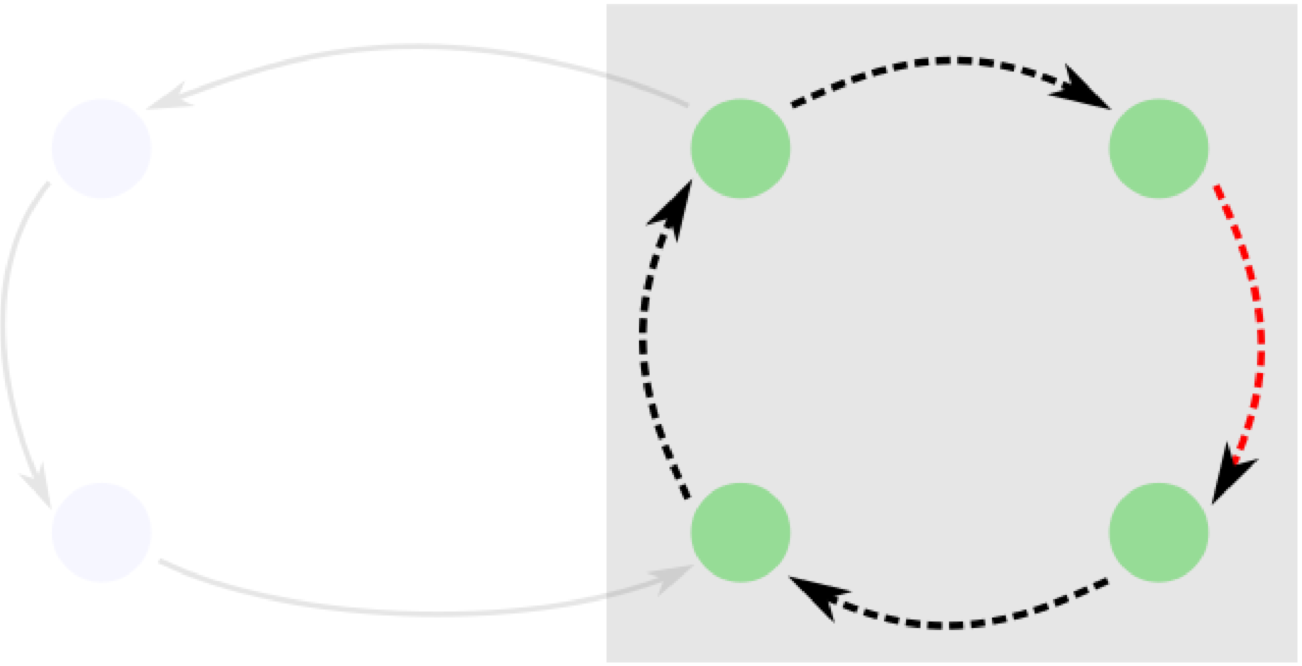

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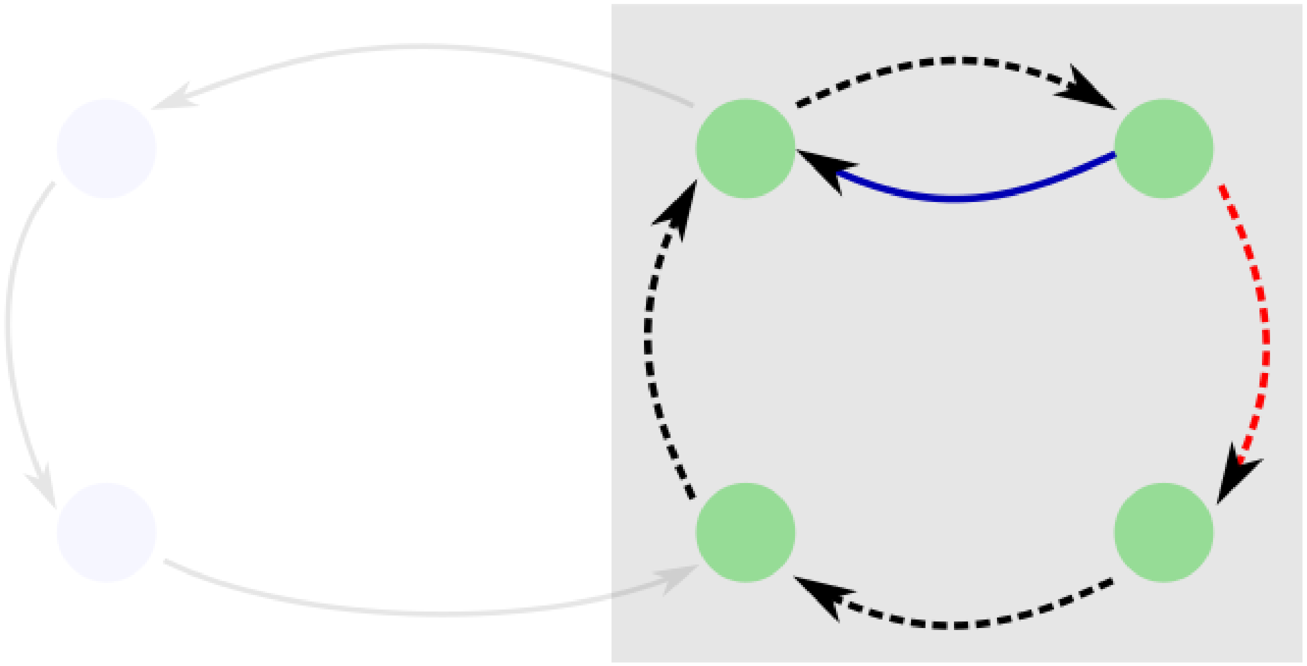

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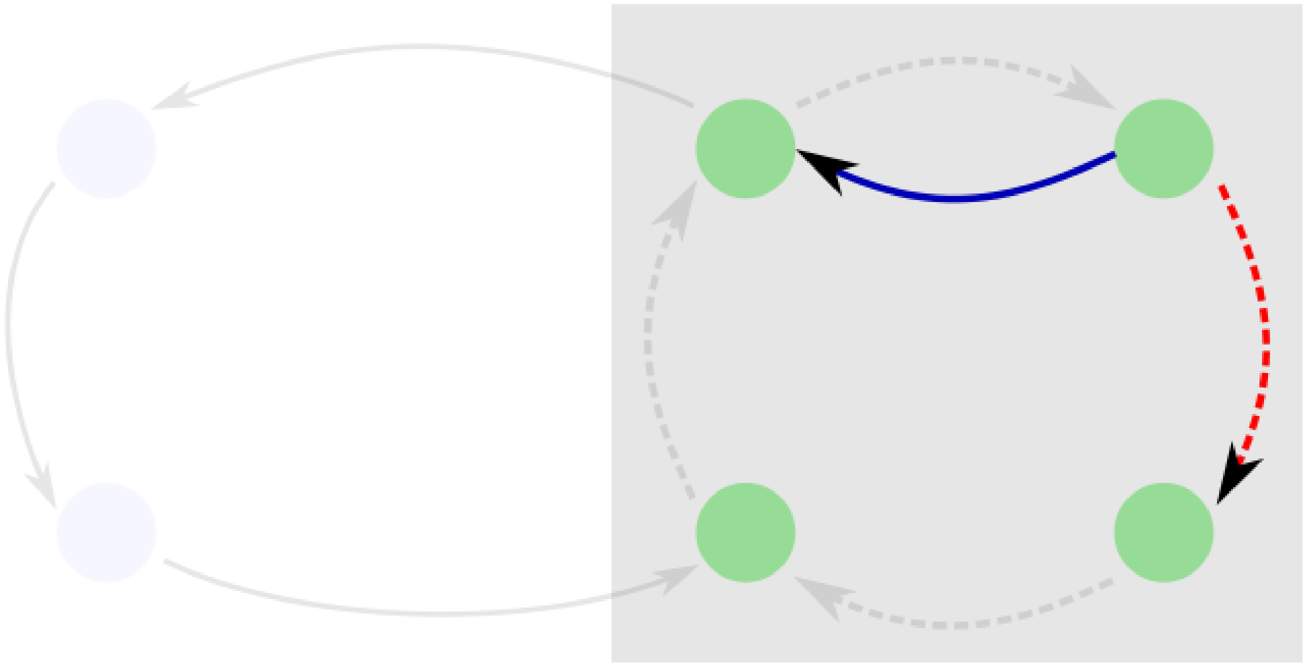

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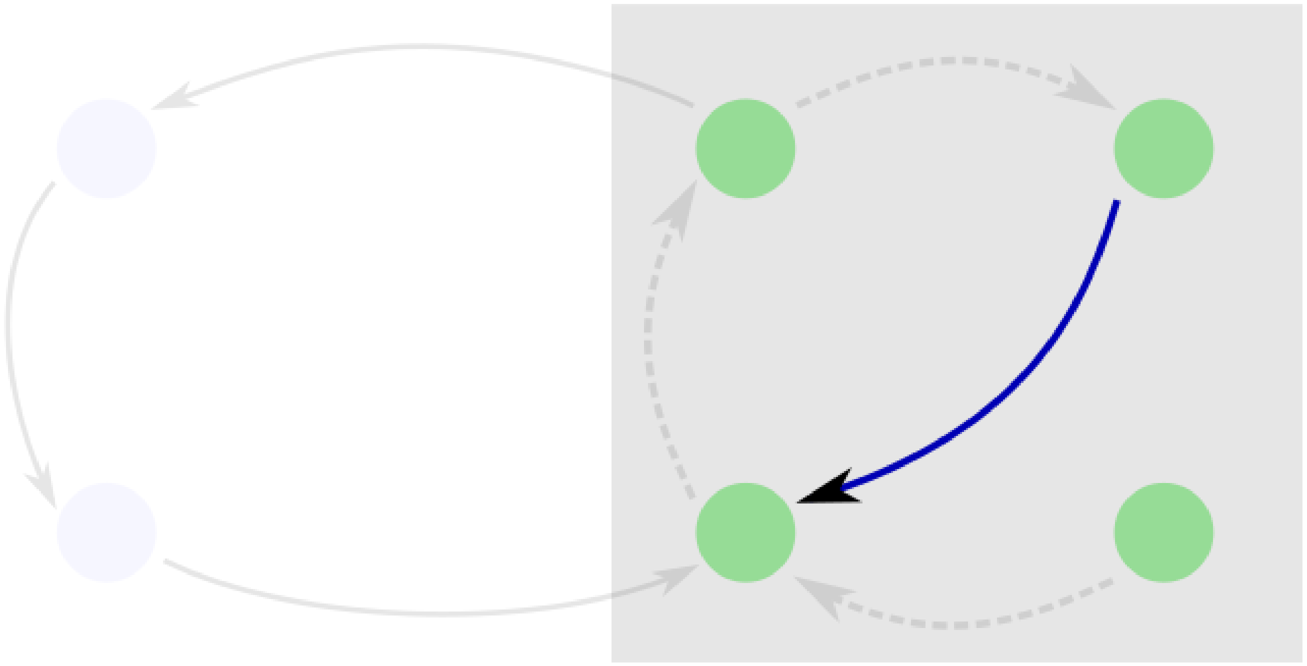

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Allocation A is EF1 if for every pair of agents i, k, there exists a good $j \in A_k$ such that $v_i(A_i) \geq v_i(A_k \setminus \{j\})$.

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> If the partial allocation at the beginning of an iteration is EF1, then the partial allocation at the end of that iteration is also EF1.

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Suppose good g is assigned to the source agent s. Then,

$$v_i(A_i) \geq v_i(A_s \cup \{g\} \setminus \{g\})$$

which means that EF1 is preserved.

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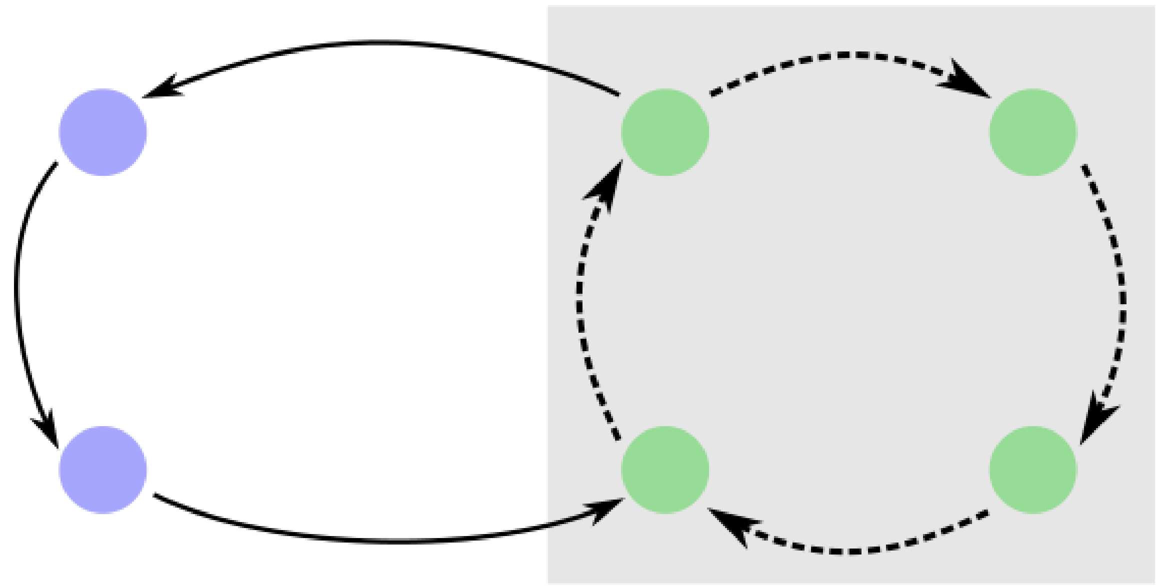

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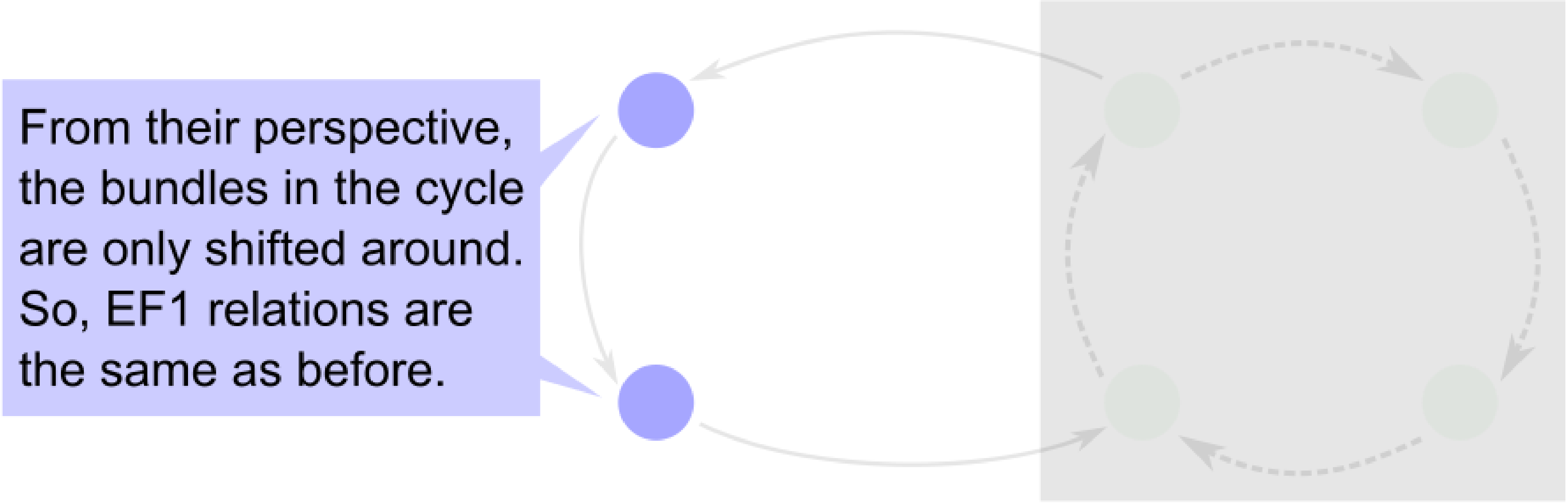

Does envy-cycle elimination algorithm return an EF1 allocation?

We will argue that each iteration of the algorithm "preserves" EF1.

While there is an unallocated good

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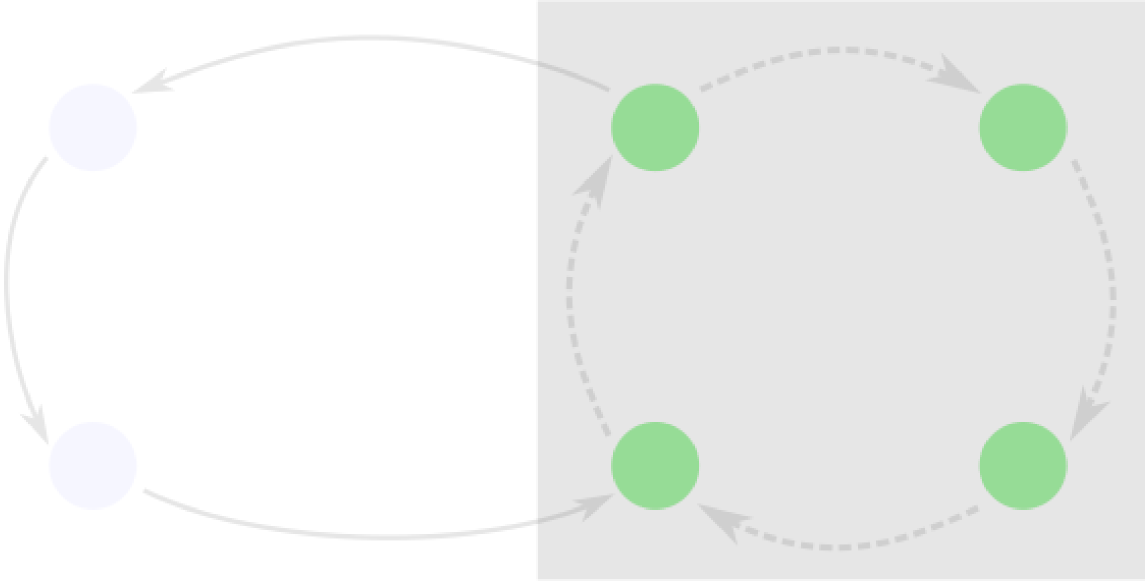

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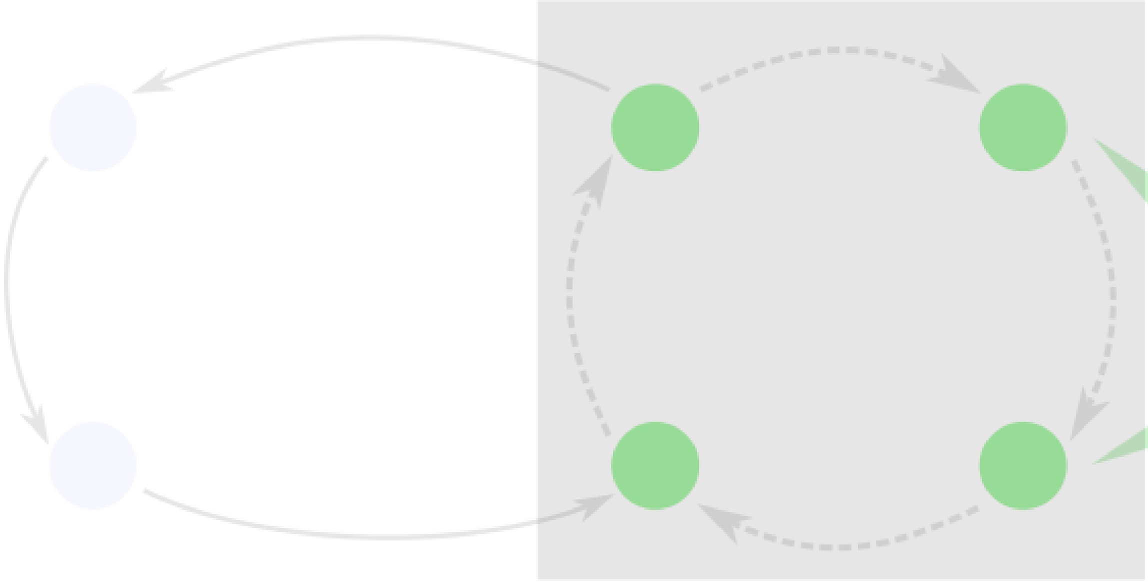

These agents are strictly better off, and any envied bundles are only shifted around. So, again, EF1 is maintained.

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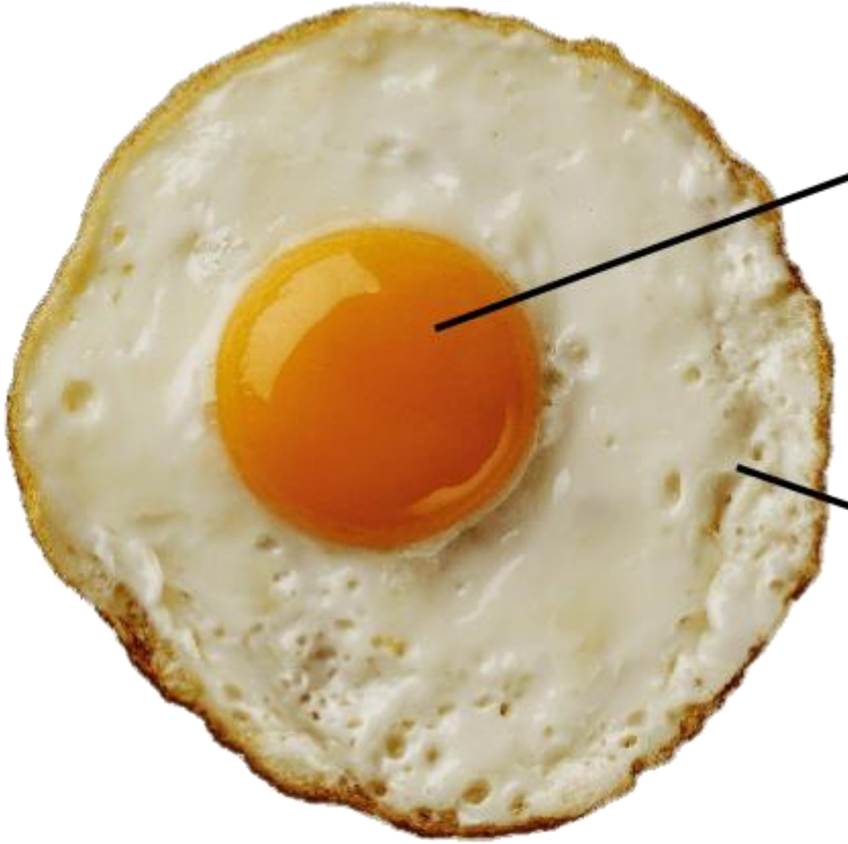

Additive: $v_i(S) = \sum_{j \in S} v_i(\{j\})$

Monotone: $S \subseteq T \Rightarrow v_i(S) \leq v_i(T)$

Next Time

Fairness and Efficiency

Quiz

Quiz

Prove or disprove:

For two agents, the round robin allocation is Pareto optimal.

An allocation A is Pareto optimal if there is no other allocation B such that:

- every agent is weakly better off under B, and
- some agent is strictly better off under B.

References

- Envy-cycle elimination algorithm

Richard Lipton, Evangelos Markakis, Elchanan Mossel, and Amin Saberi
“On Approximately Fair Allocations of Indivisible Goods”
EC 2004, pg 125-131
https://dl.acm.org/doi/10.1145/988772.988792