COL749: Computational Social Choice

Lecture 6

House Allocation

Jan 23, 2025 | Rohit Vaish

Housing Market

[Shapley and Scarf, *JME* 1974]

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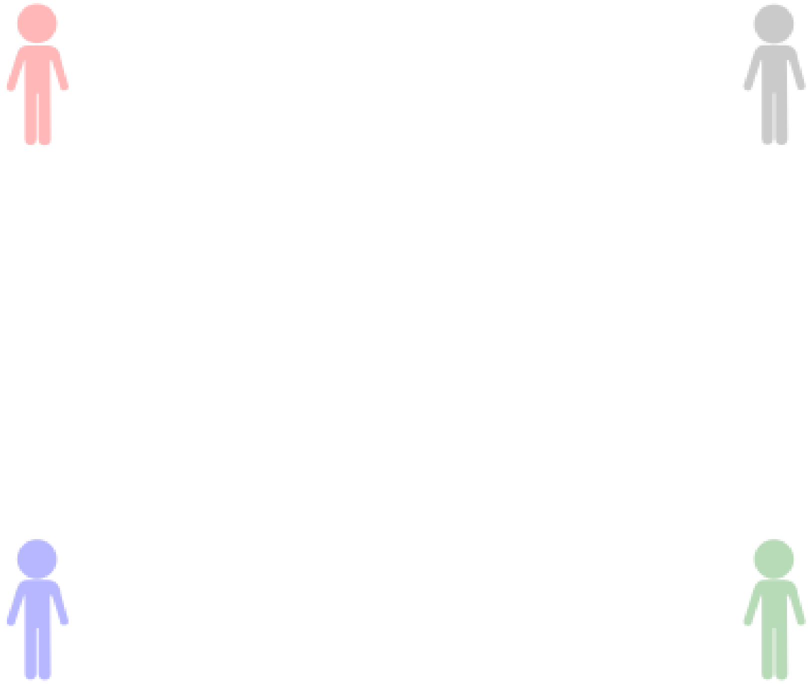

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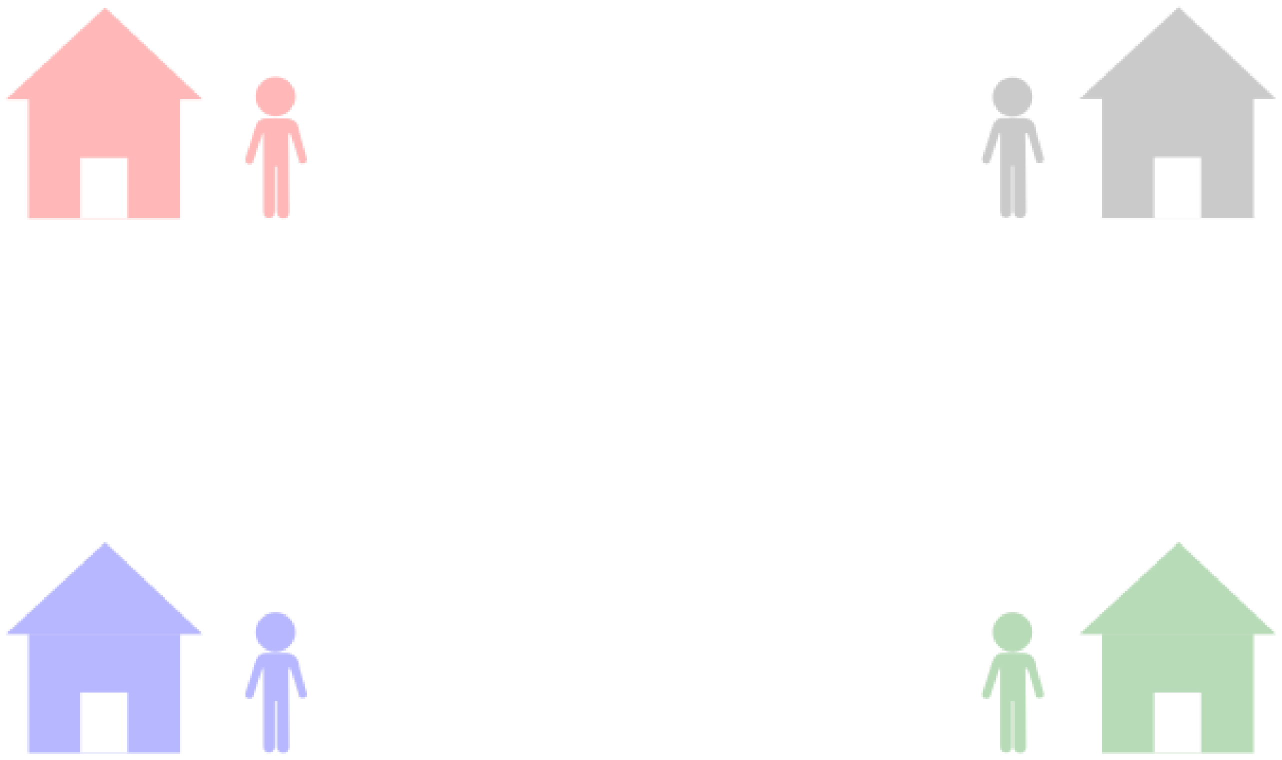

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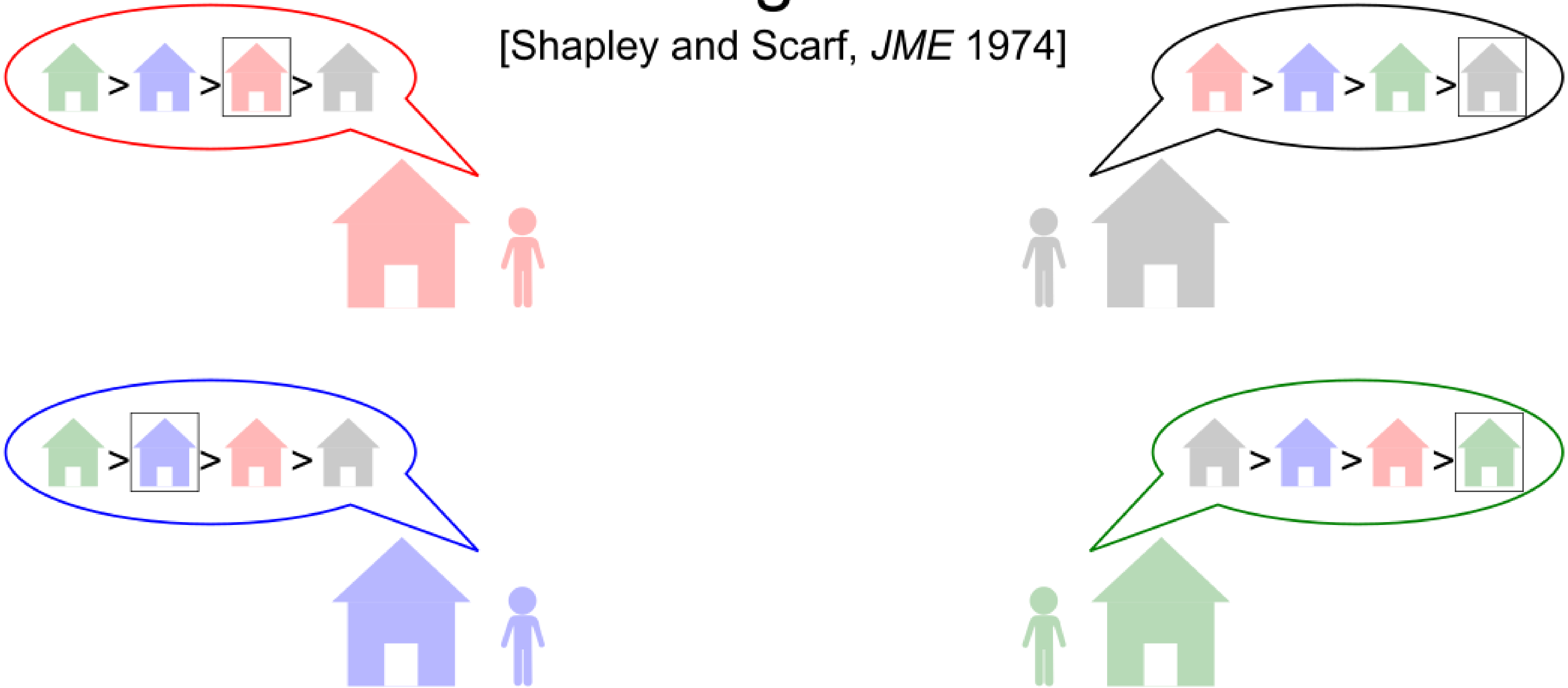

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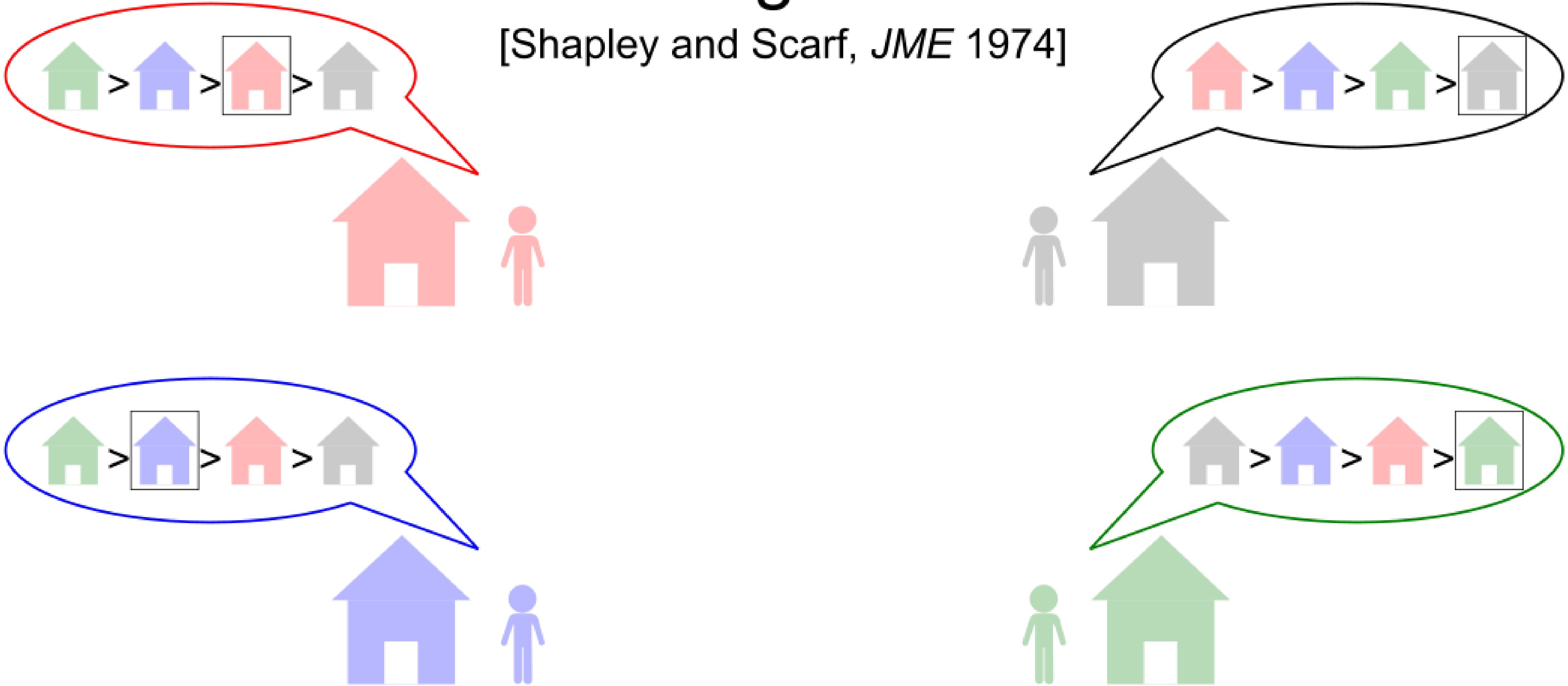

Can't use money. Only way to make agents happier is via *exchanges*.

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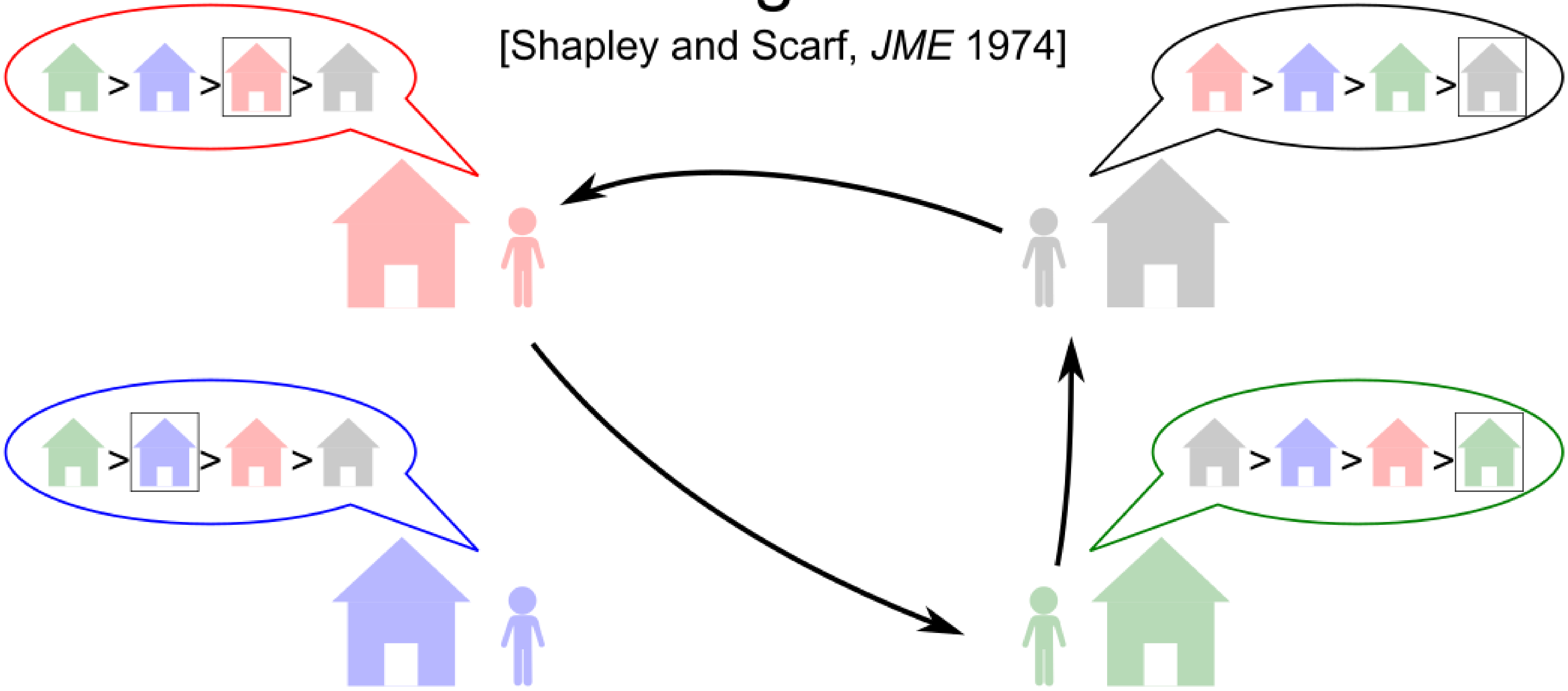

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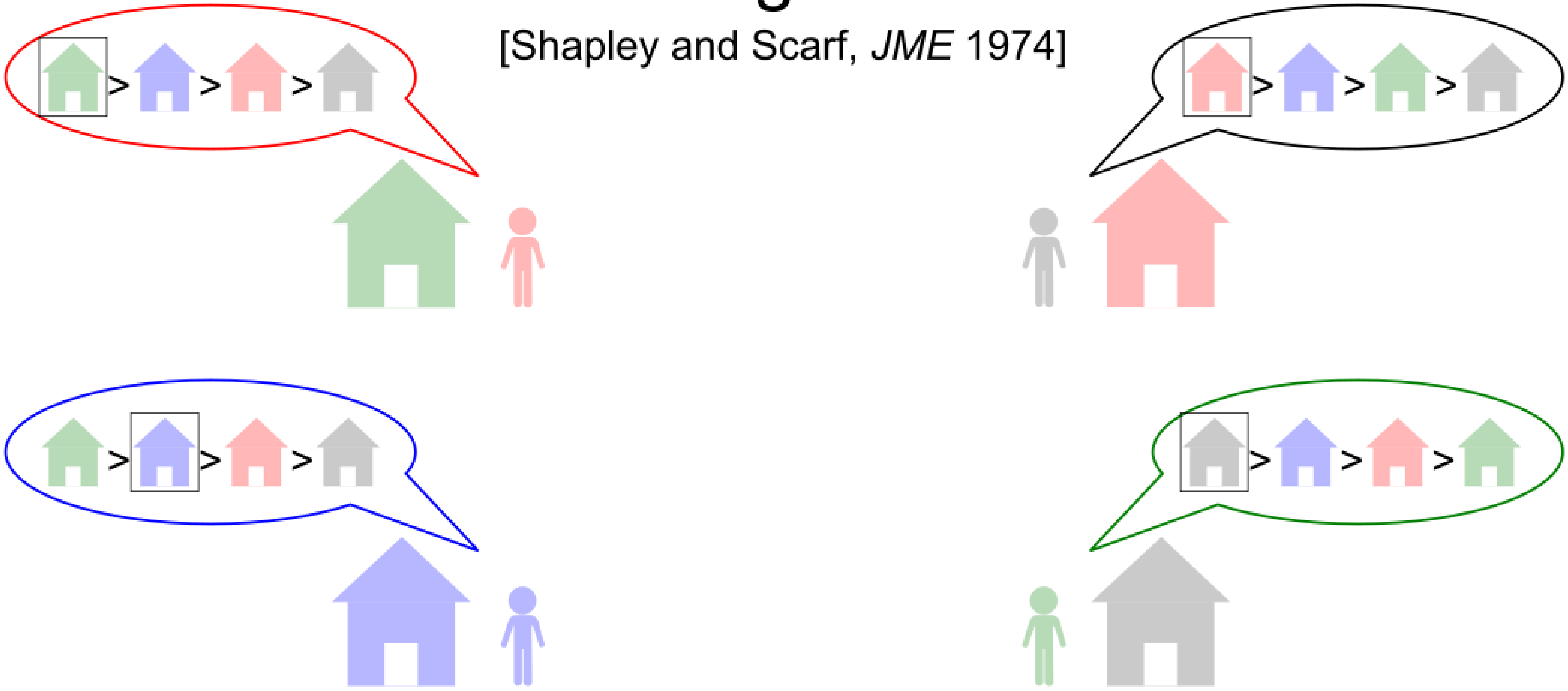

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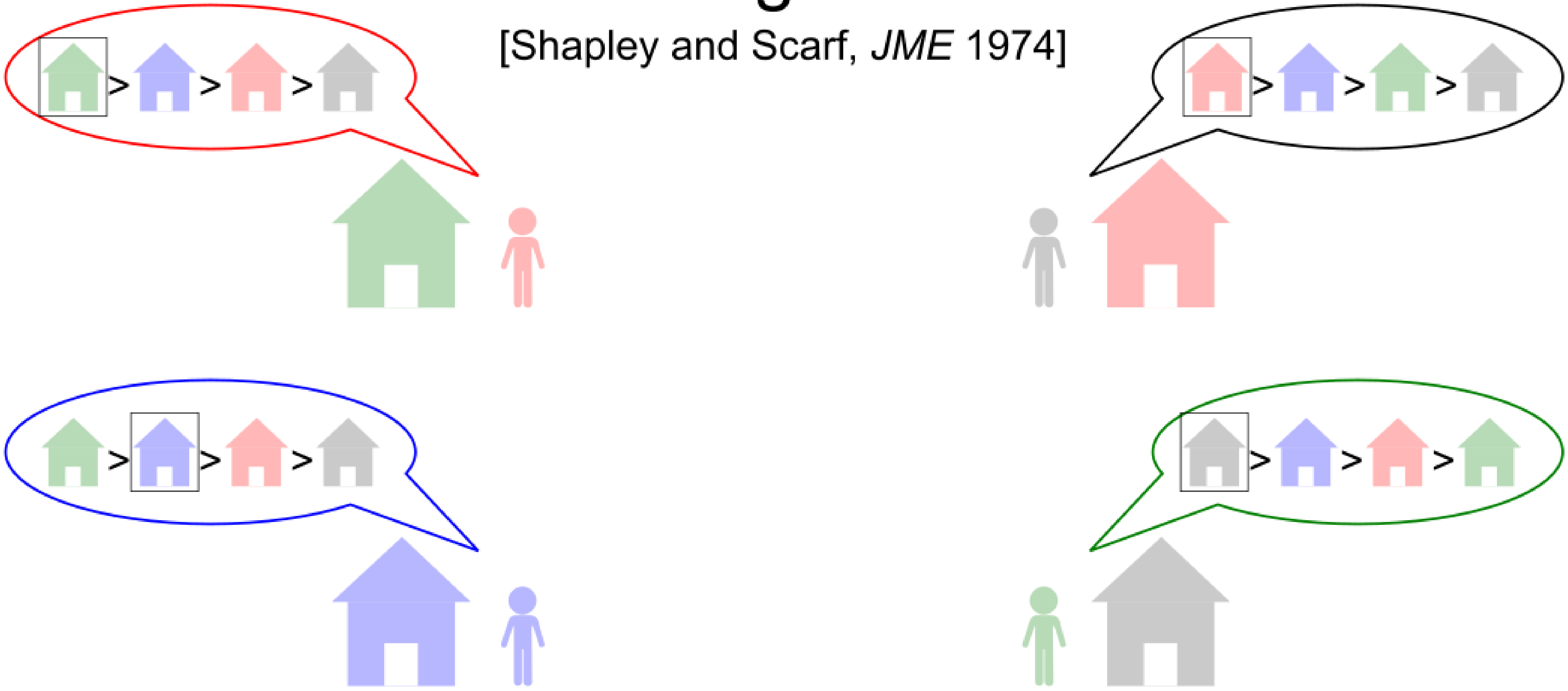

Is there a way to exchange houses to make the agents "maximally happy"?

Top-Trading Cycle Algorithm (TTCA)

[Shapley and Scarf, *JME* 1974 (attributed to David Gale)]

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While there is a remaining agent:

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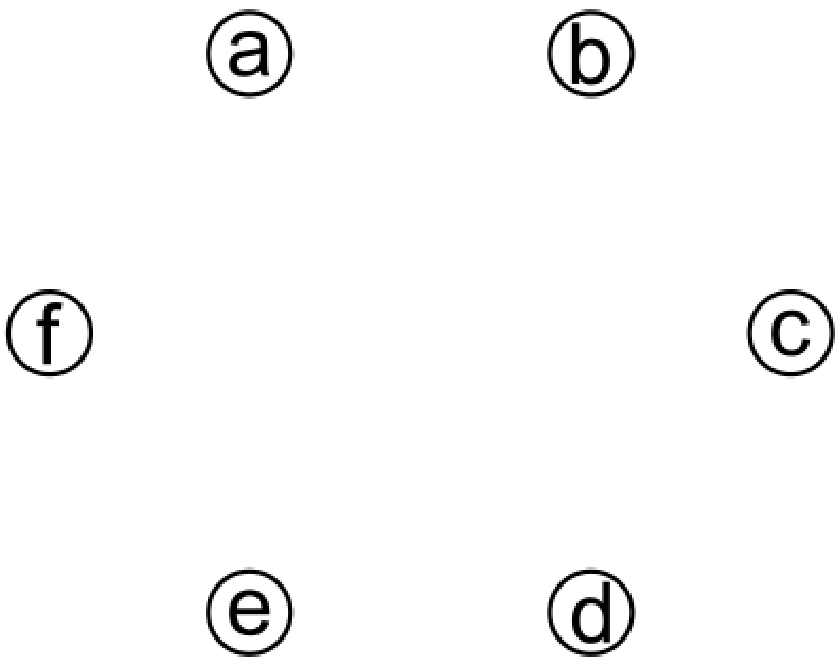

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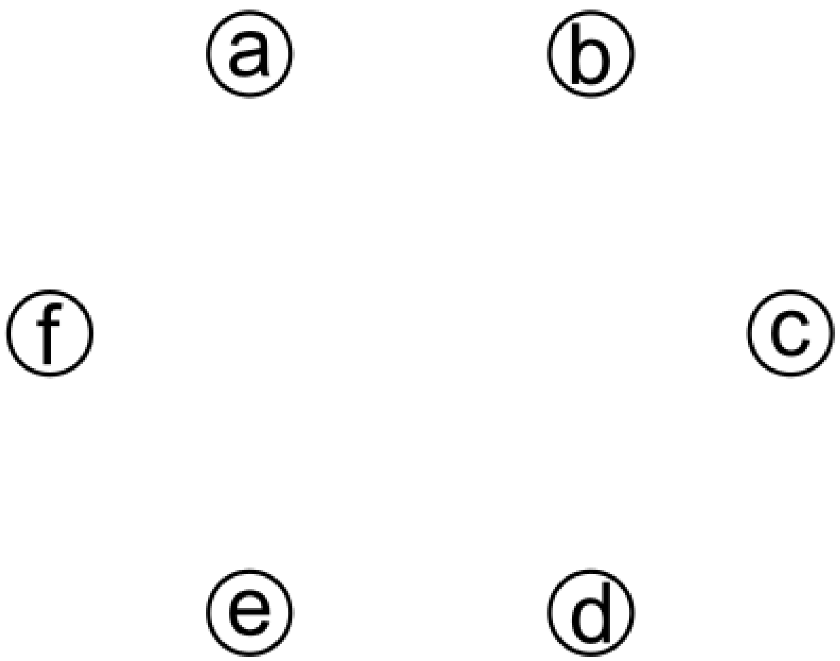

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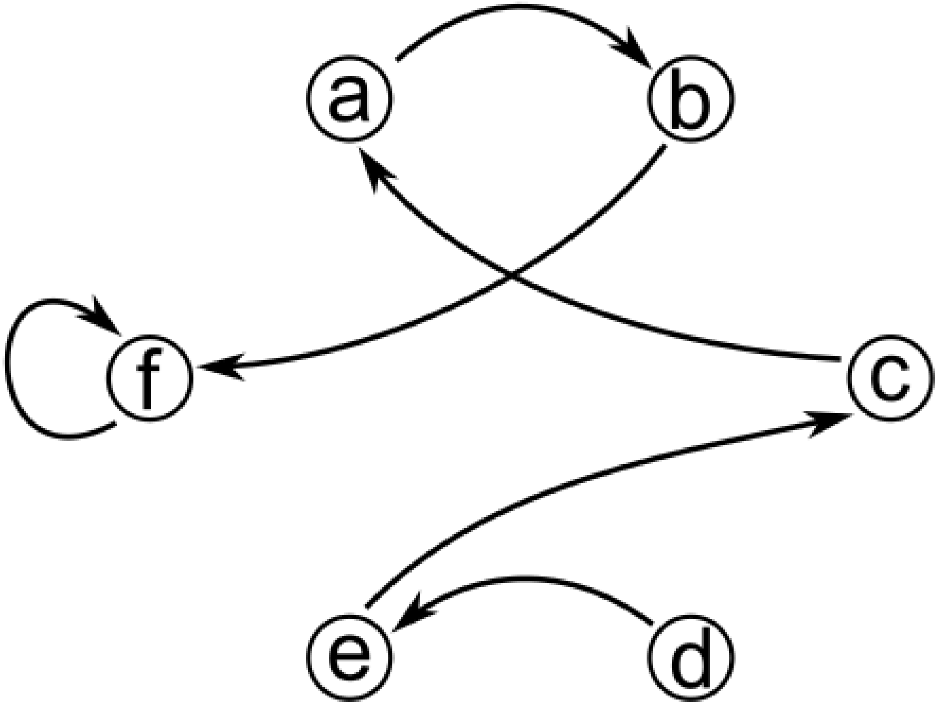

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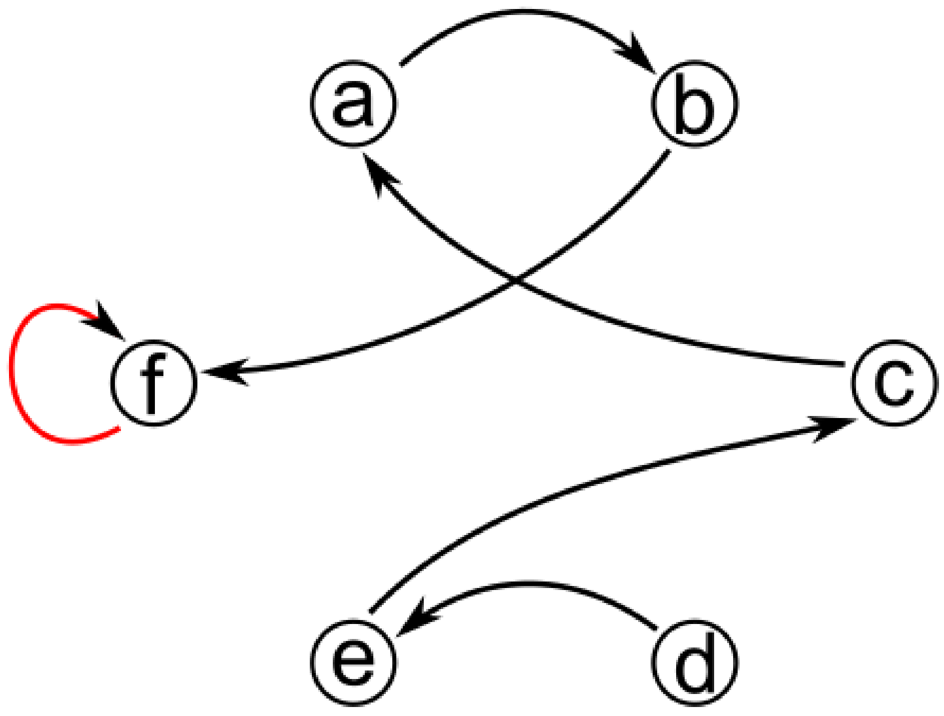

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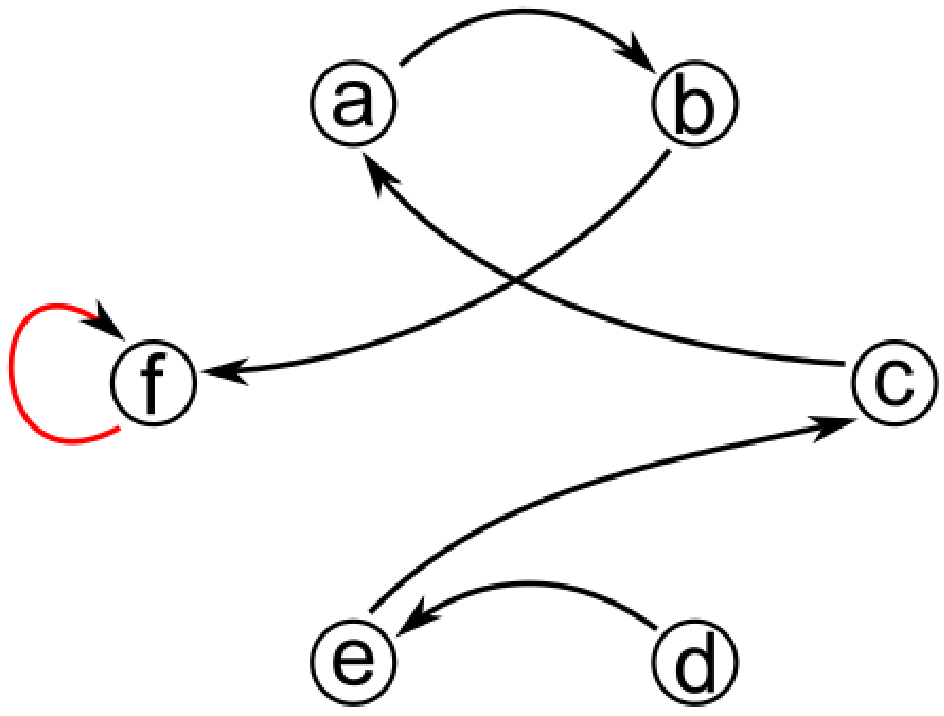

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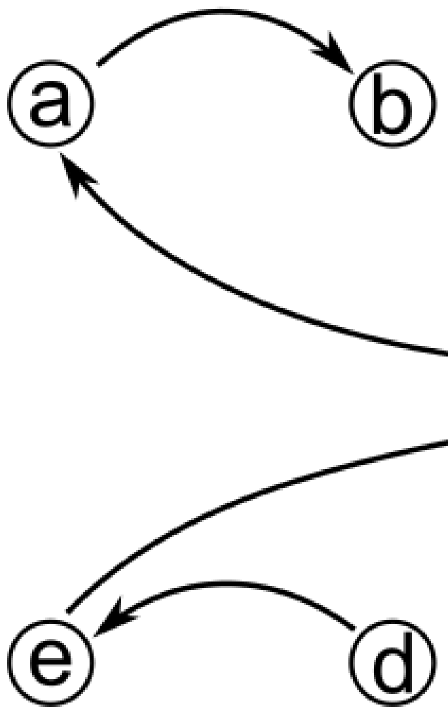

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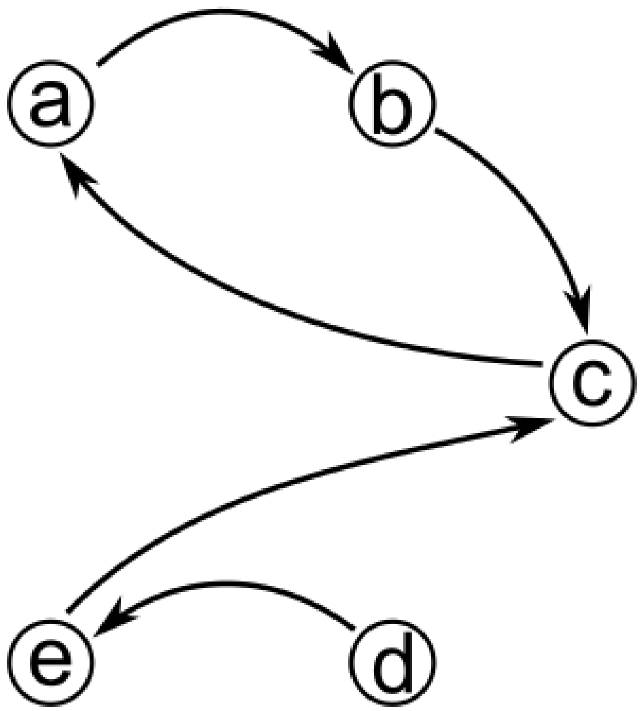

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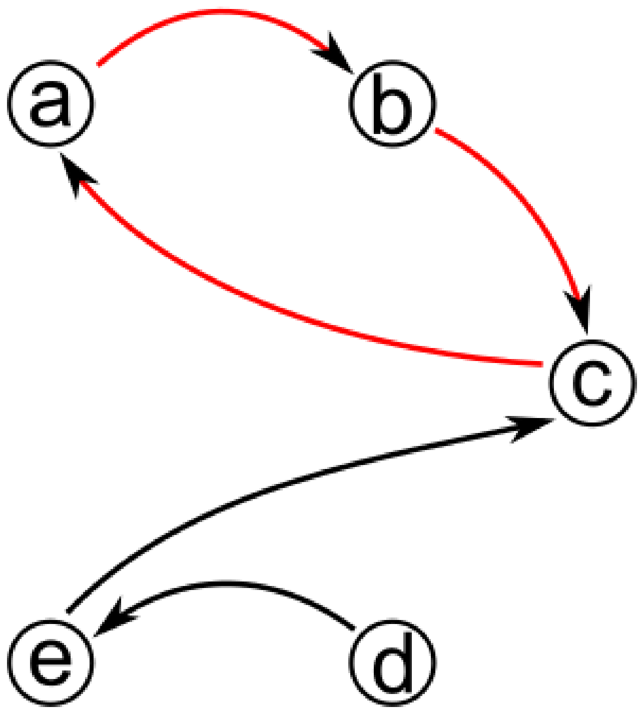

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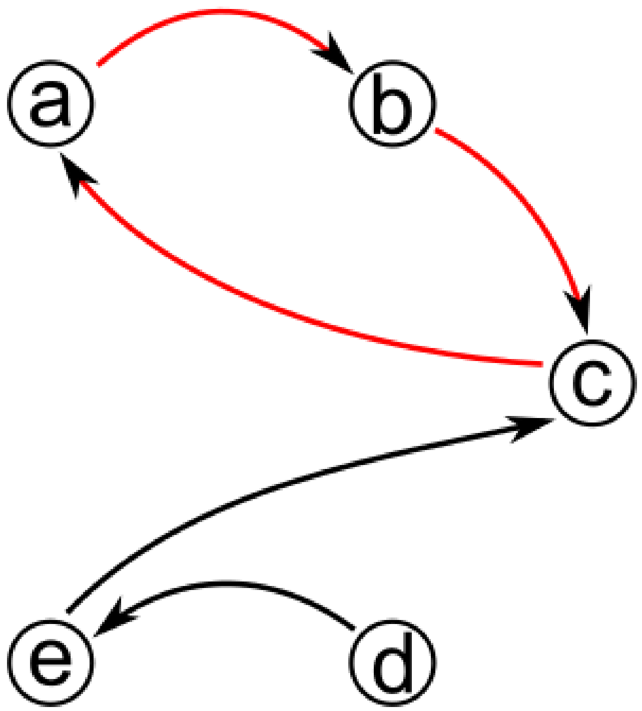

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Imagine a graph where the *vertices* are the (remaining) agents and the *edge* i → j means that agent i's favorite house is owned by agent j.

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At least one agent kicked out in each round.

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[Roth, *Econ. Letters* 1982]

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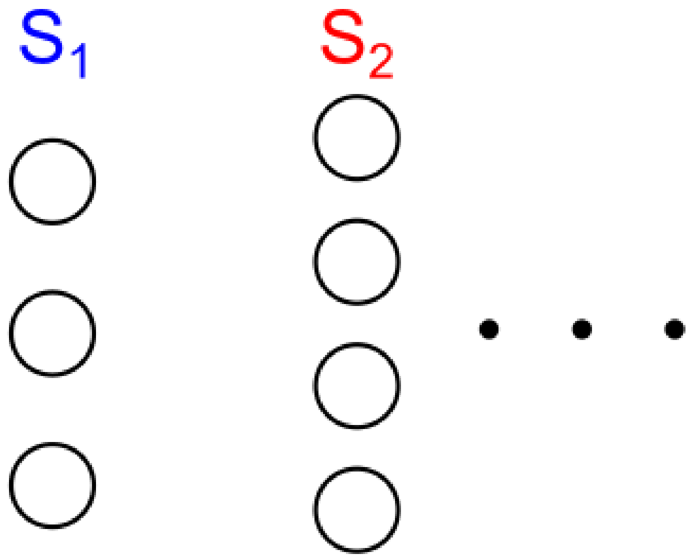

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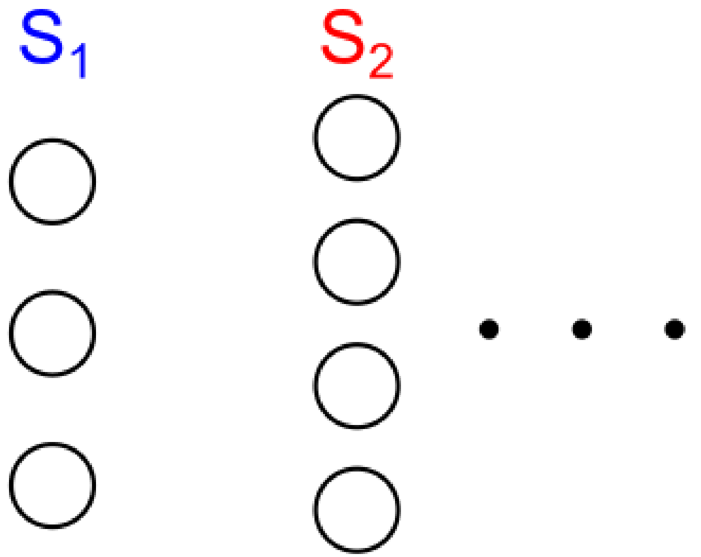

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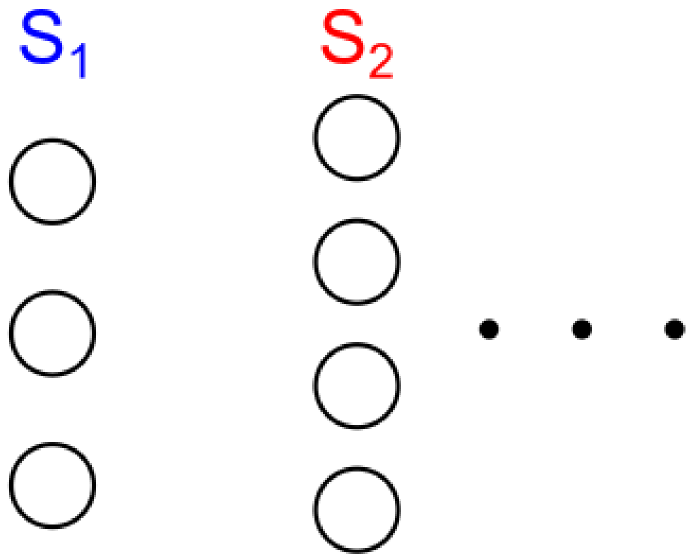

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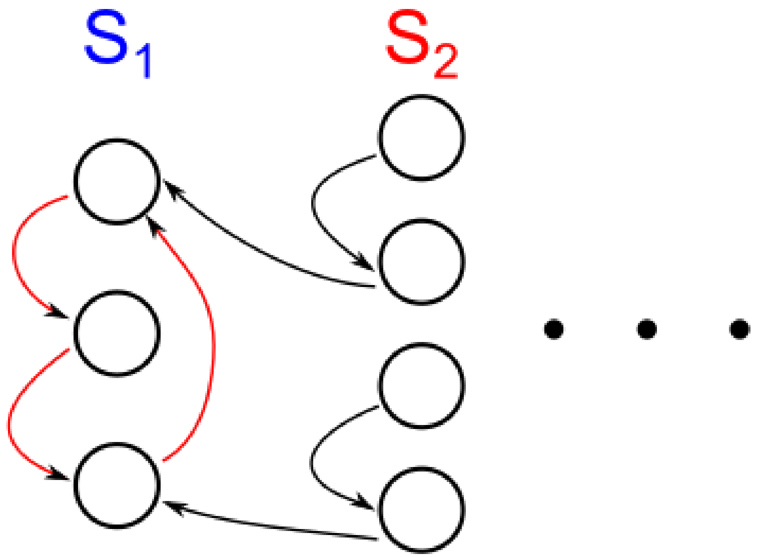

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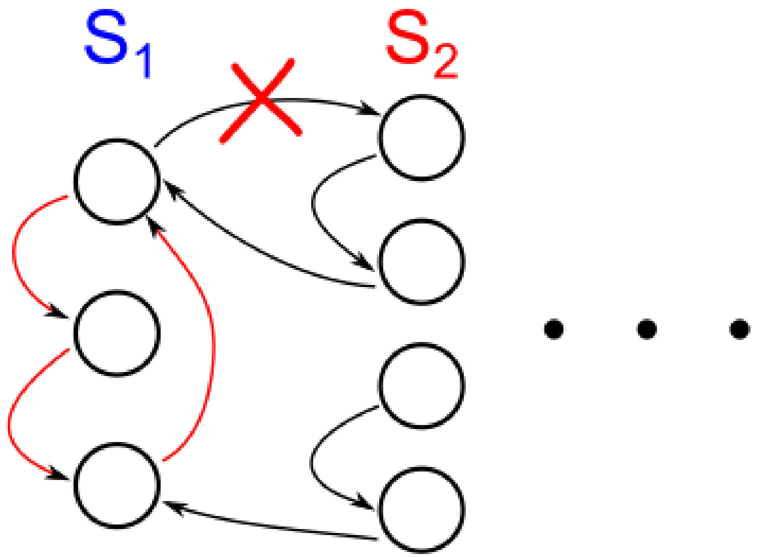

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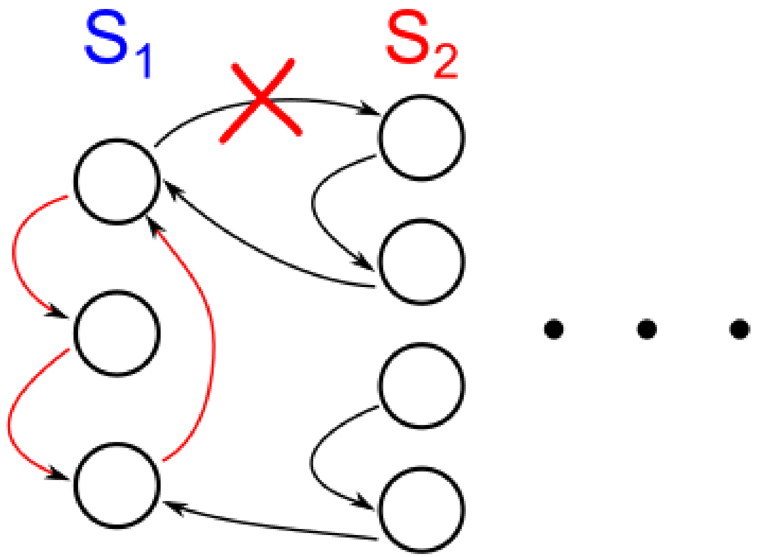

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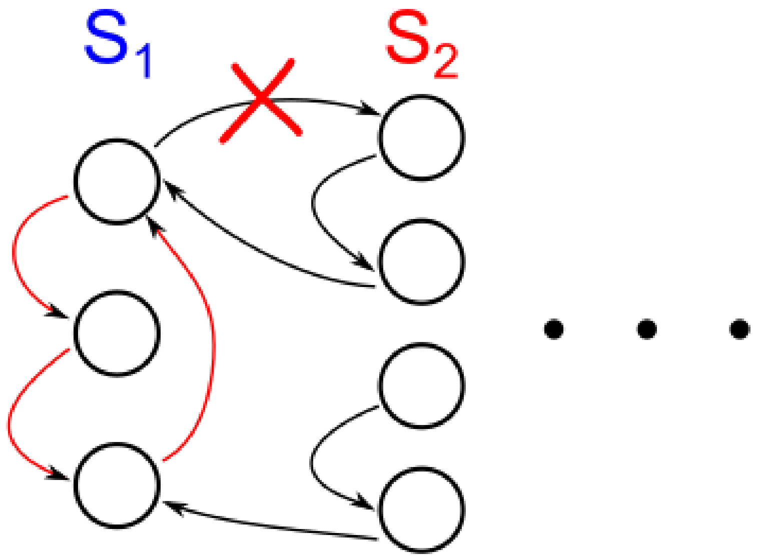

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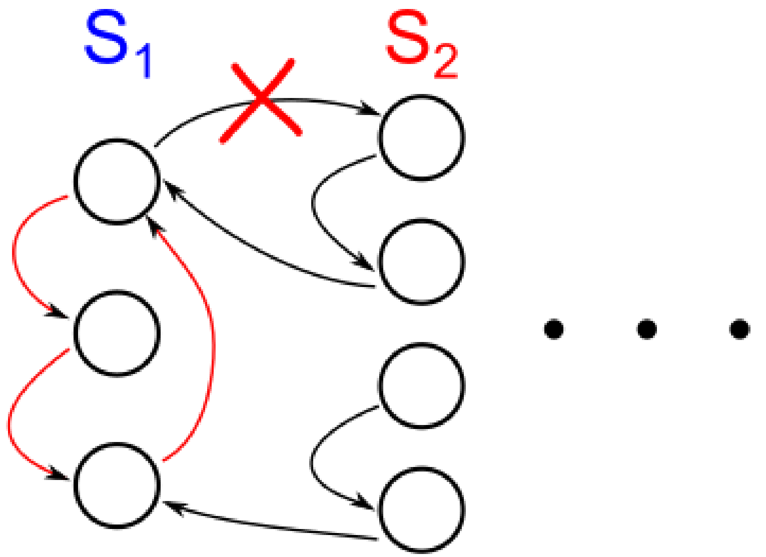

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Is TTCA group-strategyproof?

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Can a coalition of agents misreport their preferences
such that no one in the coalition is worse off
and someone is strictly better off w.r.t. their true preferences?

Is TTCA group-strategyproof?

Monday, September 23, 2024

A 40 year old proof about top trading cycles is corrected (by two Berkeley grad students)

Science (and math) can be self-correcting, sometimes slowly. Here's an article that corrects the first proof that the top trading cycles algorithm is group strategy proof. That's a true result, with multiple subsequent proofs. But apparently the first proof presented wasn't the best one. That's good to know.

One reason this may have taken a long time to spot is that the result is correct, and that there are subsequent proofs that connect the result to properties of other mechanisms.

Will Sandholtz and Andrew Tai, the authors, did this work as Ph.D. students at UC Berkeley. (good for them!)

Group incentive compatibility in a market with indivisible goods: A comment by Will Sandholtz and Andrew Tai

"Highlights

• Bird (1984), first to show top trading cycles is group strategy-proof, has errors.

•We present corrected results and proofs.

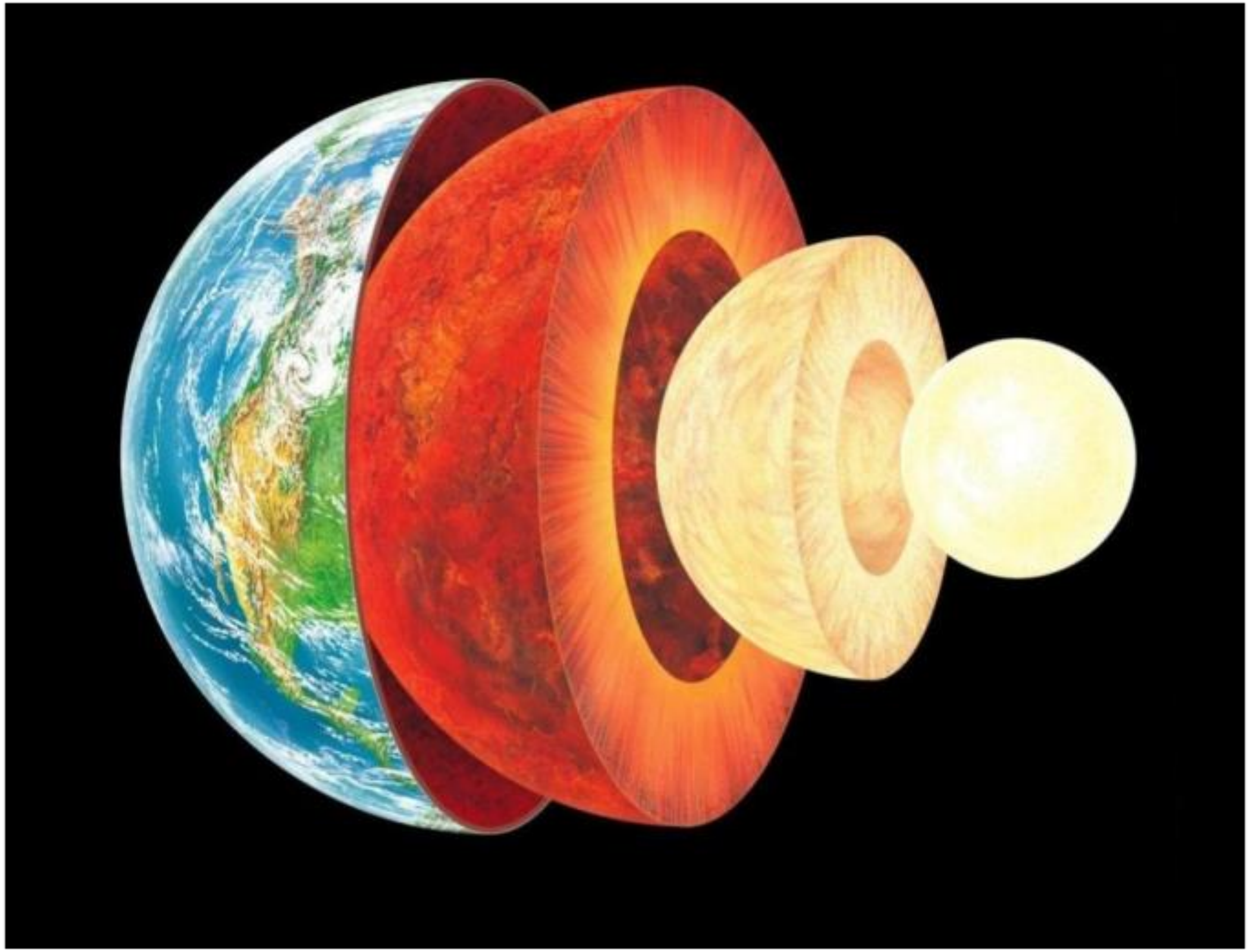
•We present a novel proof of strong group strategy-proofness without non-bossiness.

"Abstract: We note that the proofs of Bird (1984), the first to show group strategy-proofness of top trading cycles (TTC), require correction. We provide a counterexample to a critical claim and present corrected proofs in the spirit of the originals. We also present a novel proof of strong group strategy-proofness using the corrected results."

Posted by Al Roth at 5:58 AM

Labels: mathematics, papers, peer review, science, TTC

The Core



The Core

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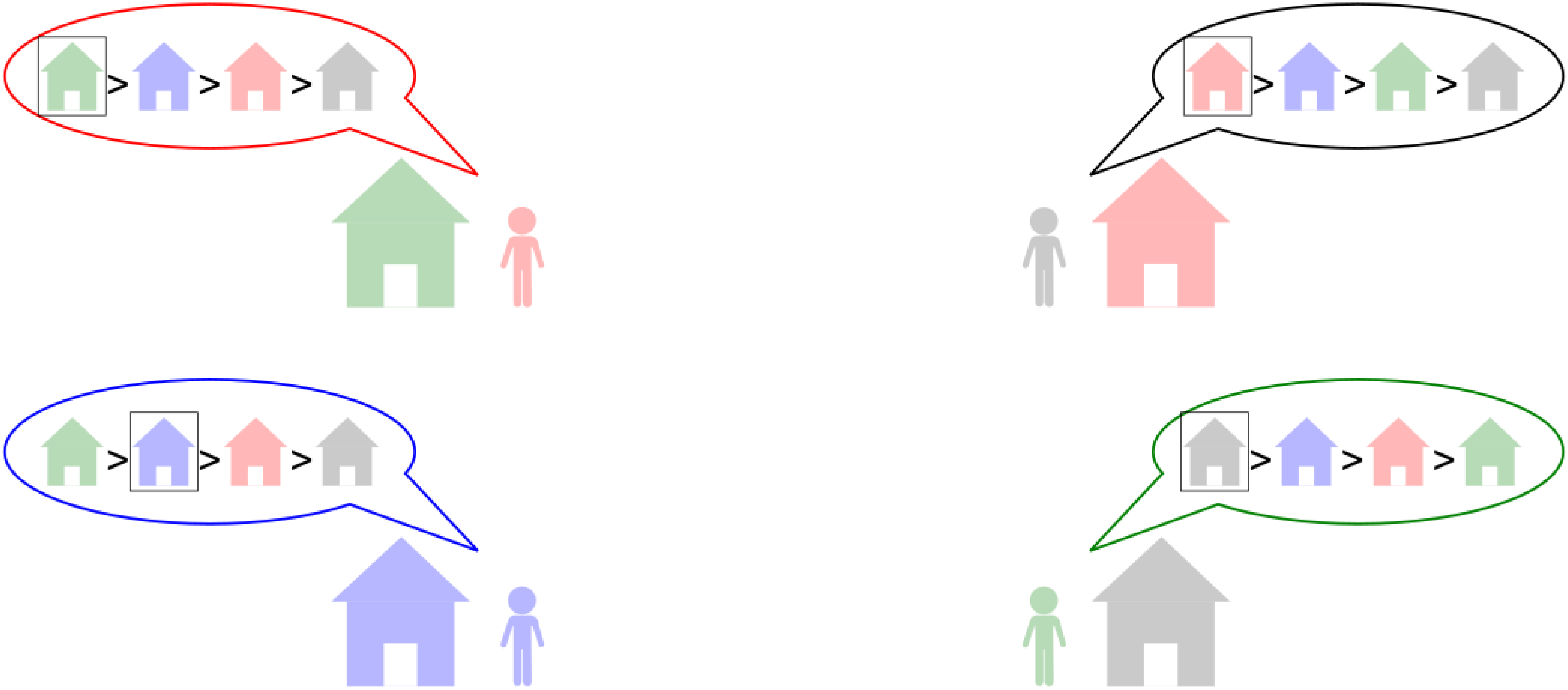
A coalition of agents blocks an allocation A if:

they can redistribute their **endowed houses** among themselves such that, compared to A, none of them is worse off and at least one of them is strictly better off (i.e., redistributing endowments is a Pareto improvement over A).

Example of a Core Allocation



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Suppose not. Then, a coalition C of agents must block the TTCA allocation T. Let R be a redistribution of endowments among agents in C that they find Pareto better than T.

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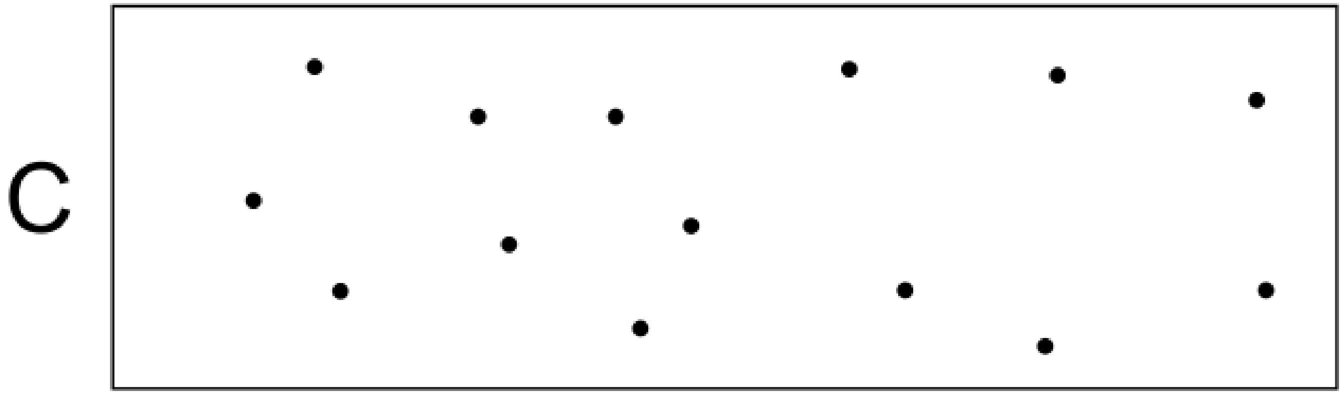

T= TTCA outcome, R= redistribution of endowments within C

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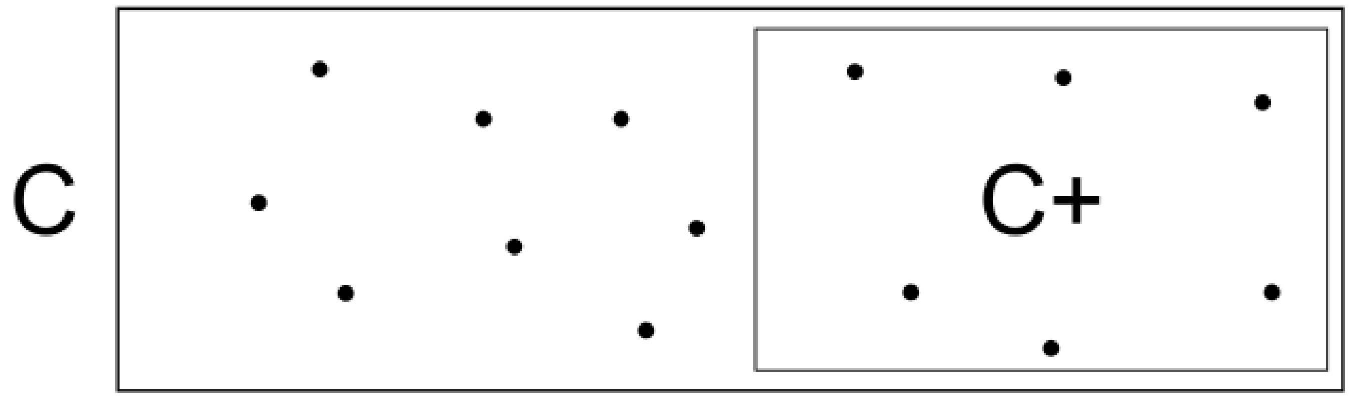
Let C+ ⊆ C be the agents in C who *strictly prefer* R over T.



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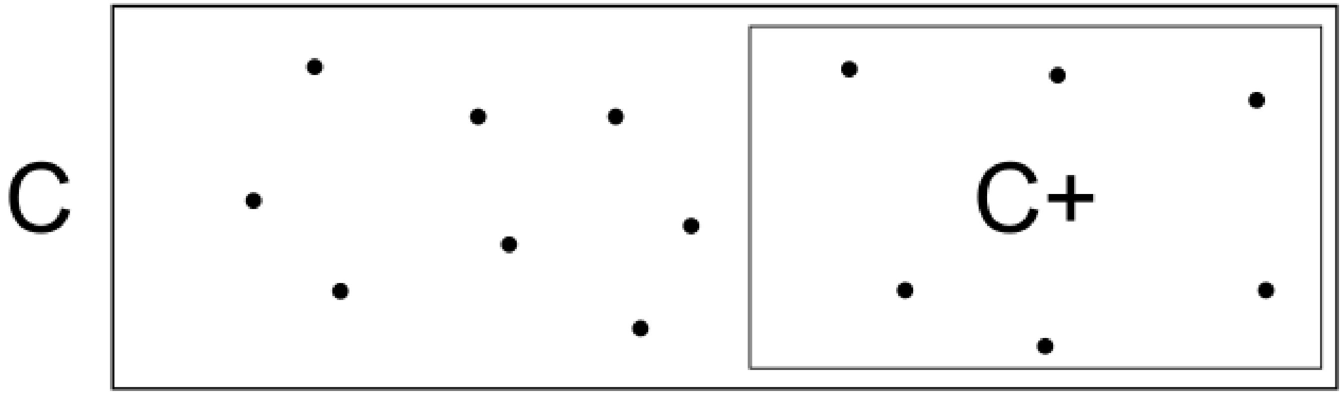

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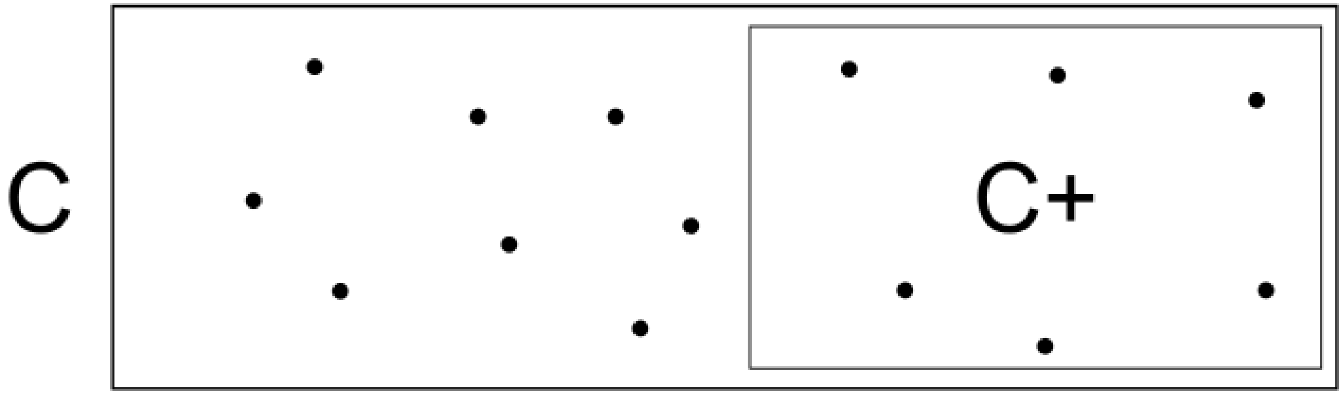

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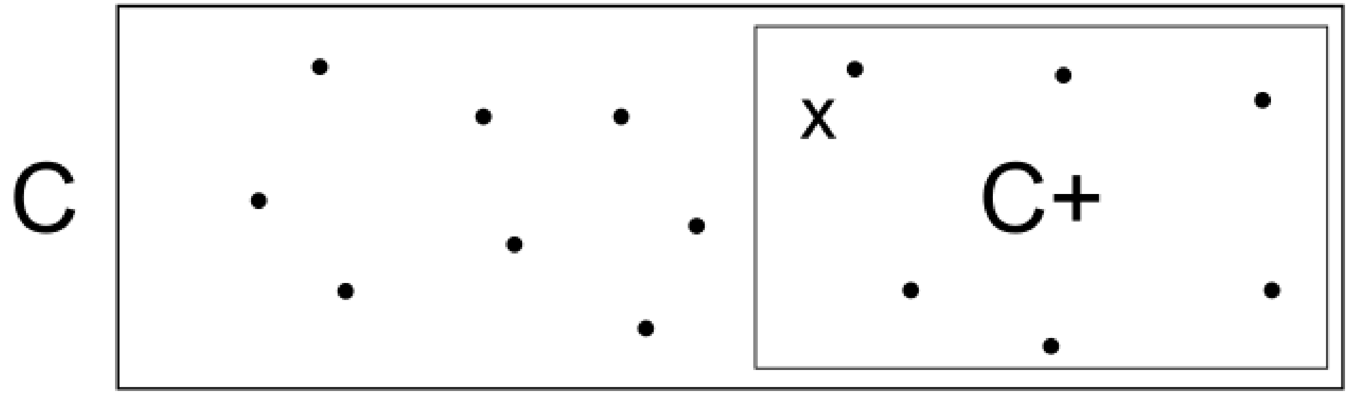

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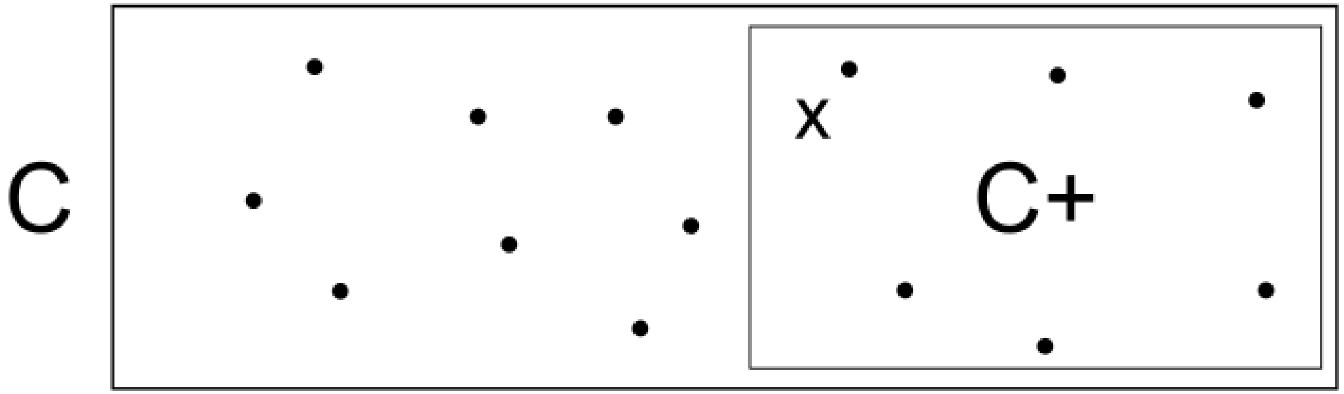
[Shapley and Scarf, *JME* 1974 (attributed to David Gale)]

At the time x is eliminated (round r), all agents in C+ are still available under TTCA along with their endowed houses.



C+: R>T

C\C+: R=T

T= TTCA outcome, R= redistribution of endowments within C

TTCA outcome is in the core.

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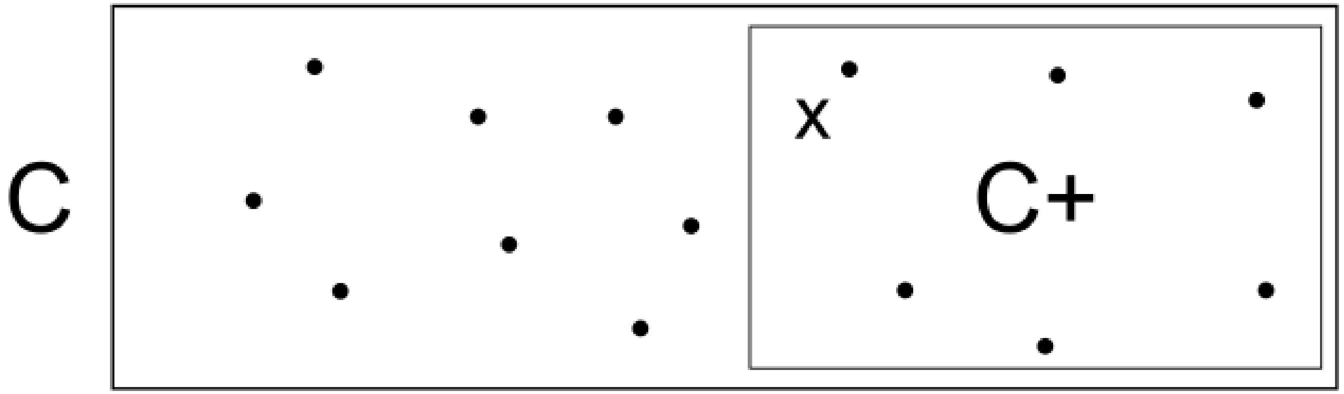
The house x gets under T is at least as good (according to x's preference) as the endowed house of any agent in C+.



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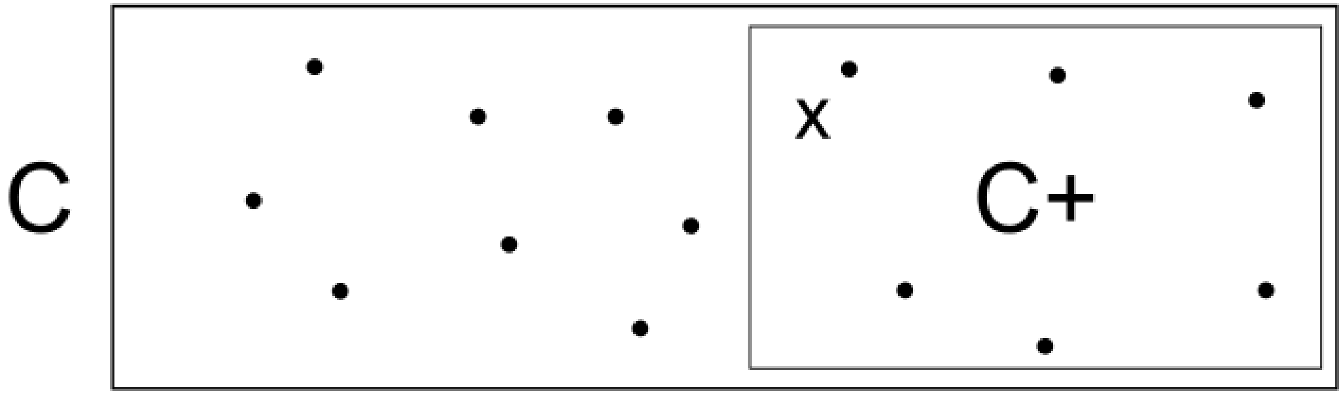
The house x gets under R is strictly better (according to x's preference) than the endowed house of any agent in C+.



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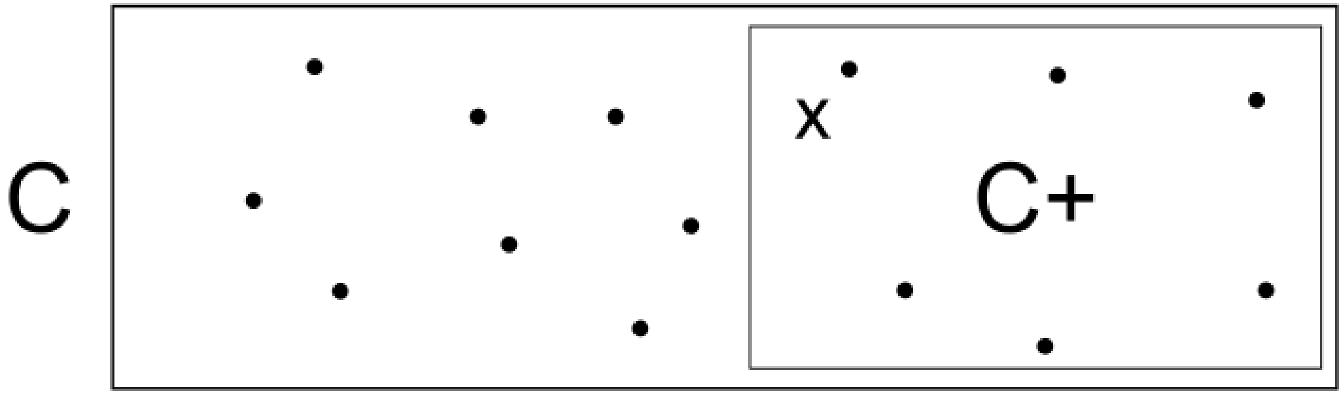
The house x gets under R must be the endowed house of some agent y ∈ C\C+.



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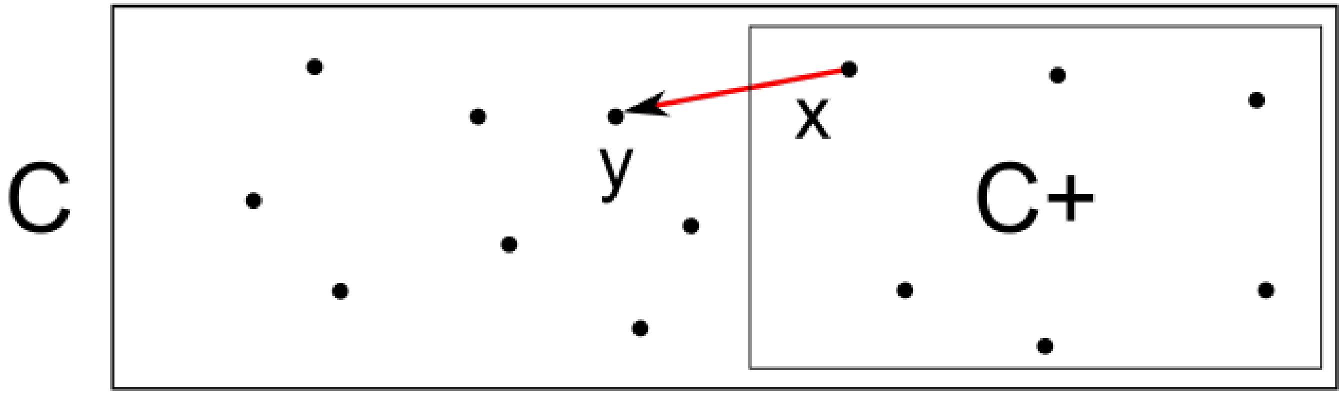

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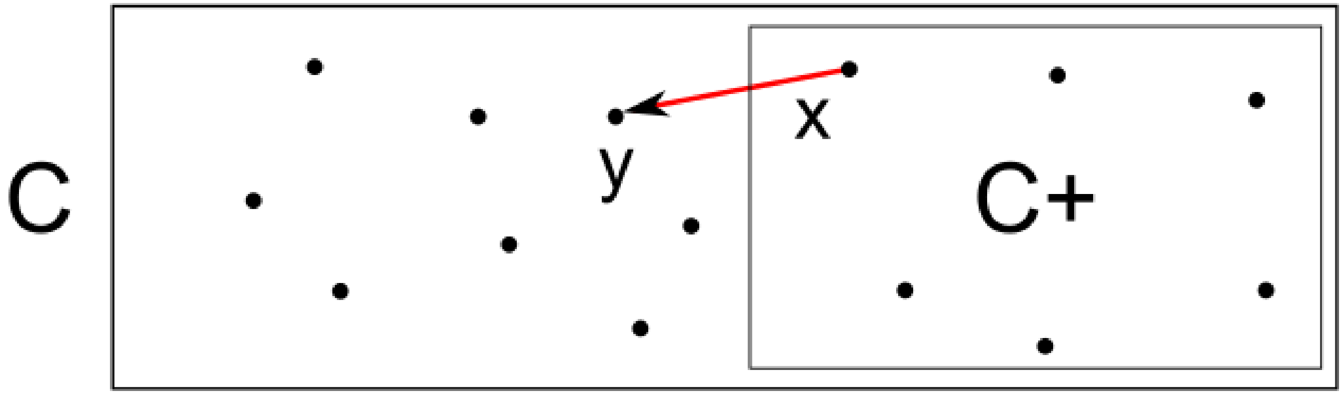
Then, under TTCA, agent y must have been eliminated in round r-1 or earlier (i.e., strictly before x).



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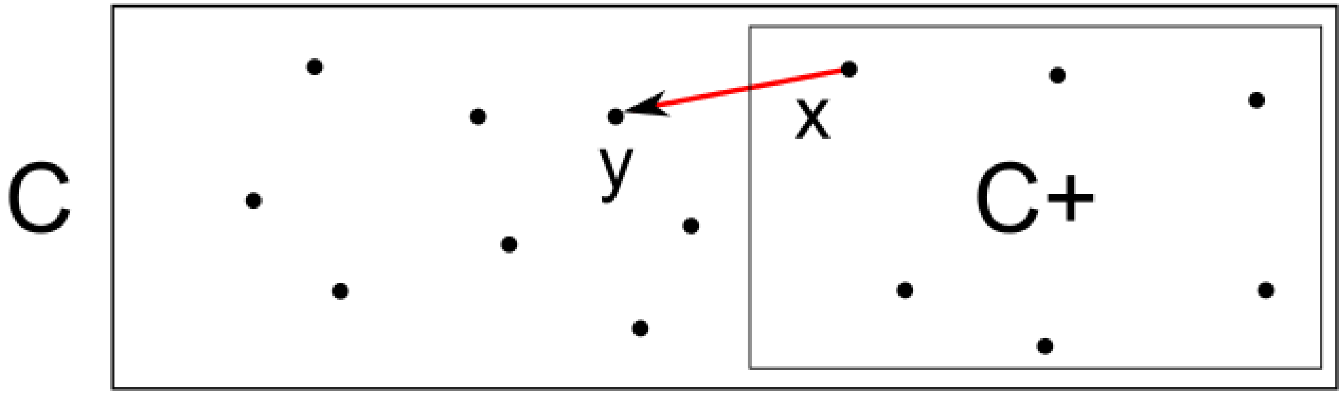
Agent y cannot get its own endowed house under the TTCA outcome T.



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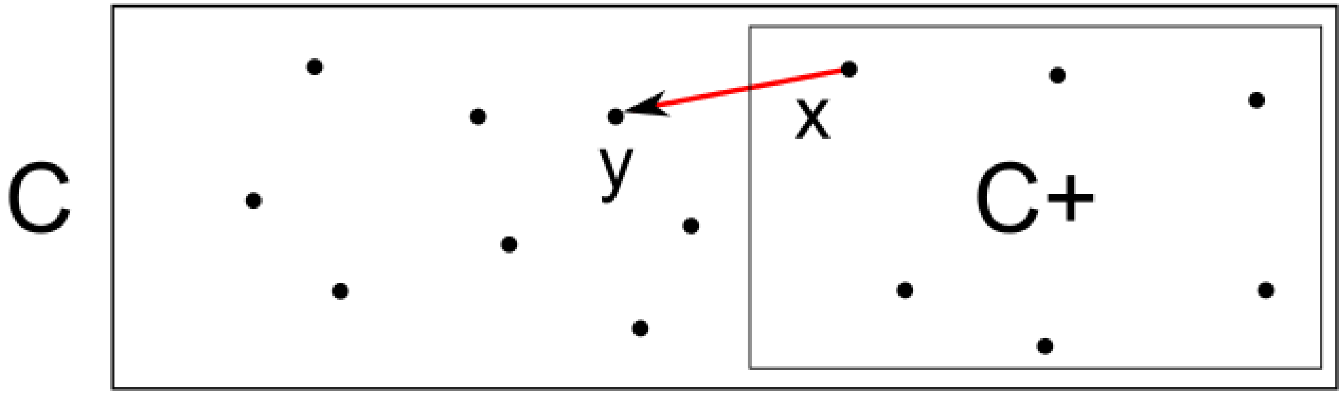
So, under T (as well as R), y gets the endowed house of agent z ∈ C.



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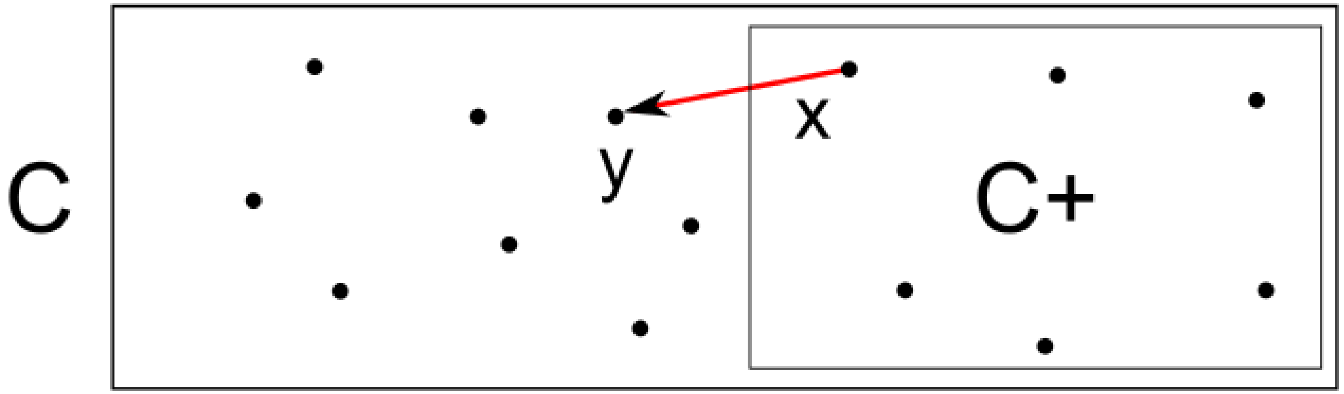

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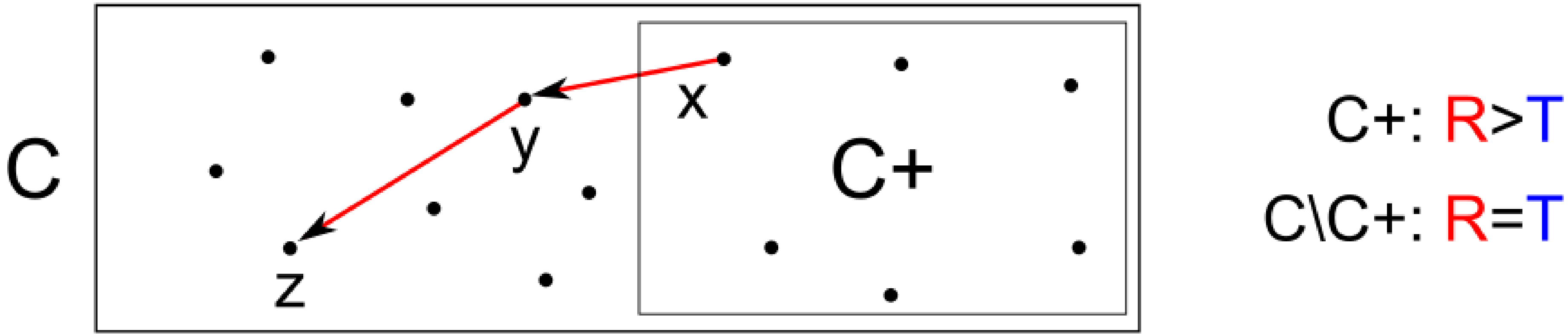

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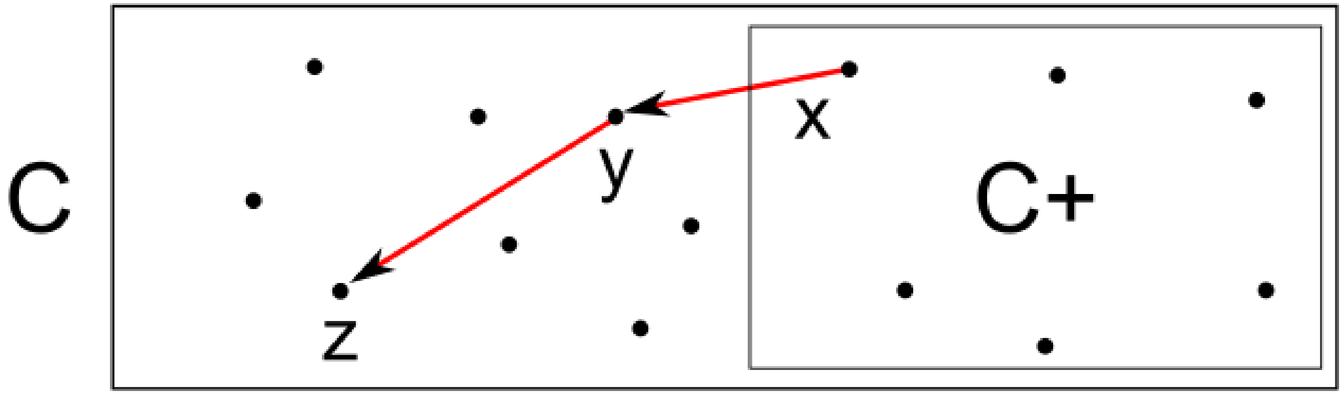
Then, under TTCA, agent z must be eliminated in the *same round* as y.



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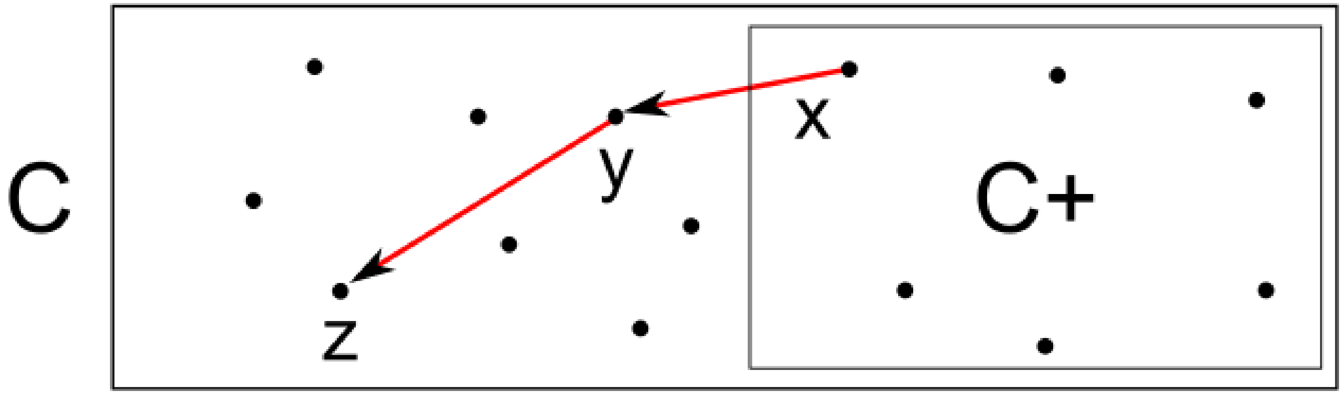
That means agent z gets the endowed house of $w \in C \backslash C+$ such that $w \neq y$.



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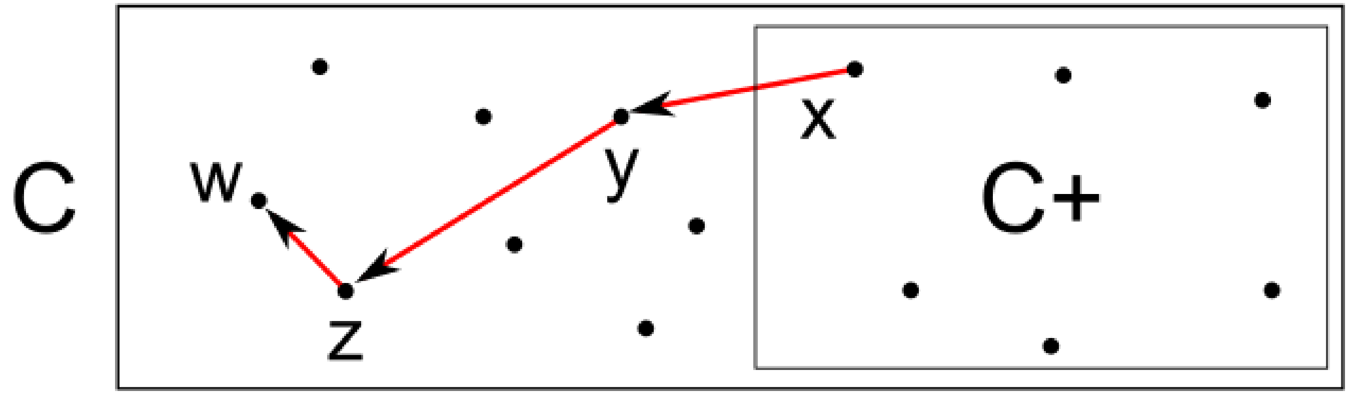

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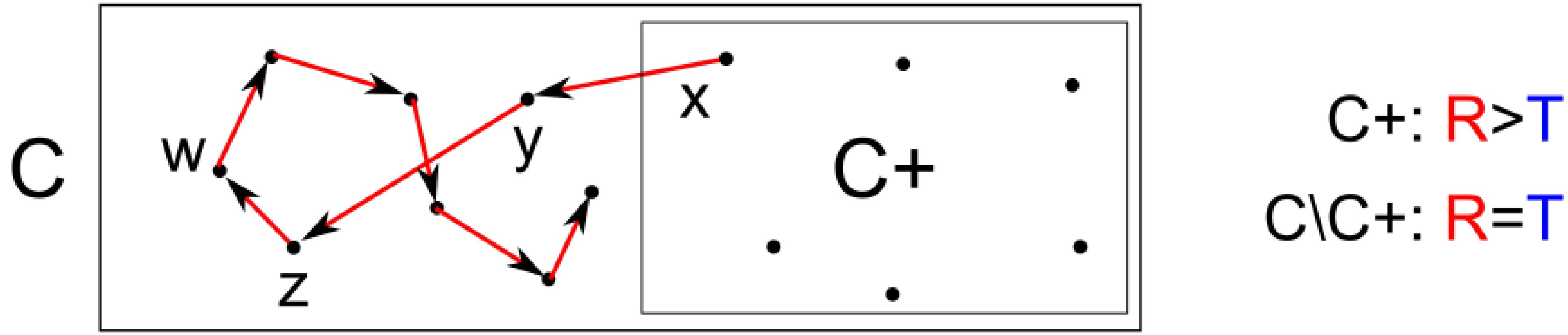
Continuing in this manner, we get an "unending" chain within C\C+.



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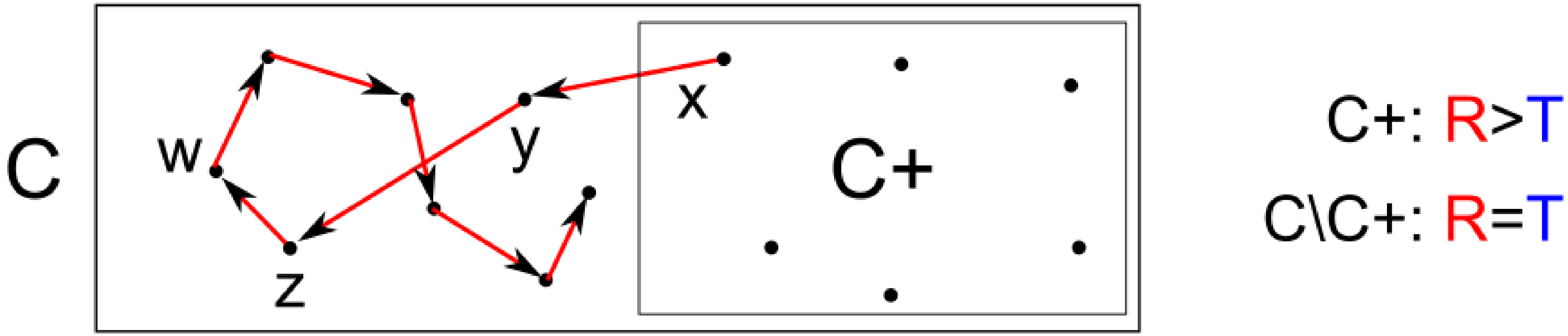

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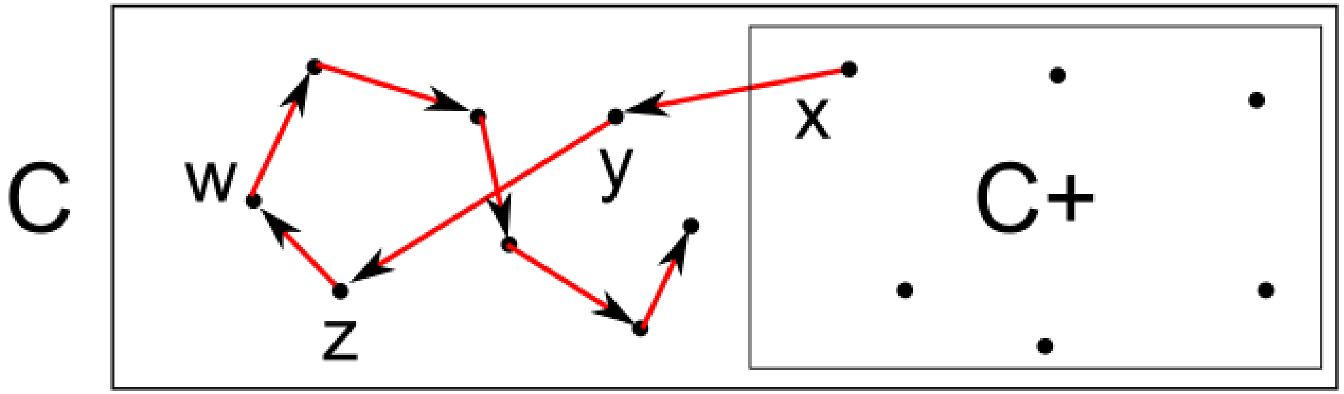

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Awesome Properties of TTCA

- Polynomial running time
- (Group) strategyproof
- Core
- Pareto optimal (extra credit!)

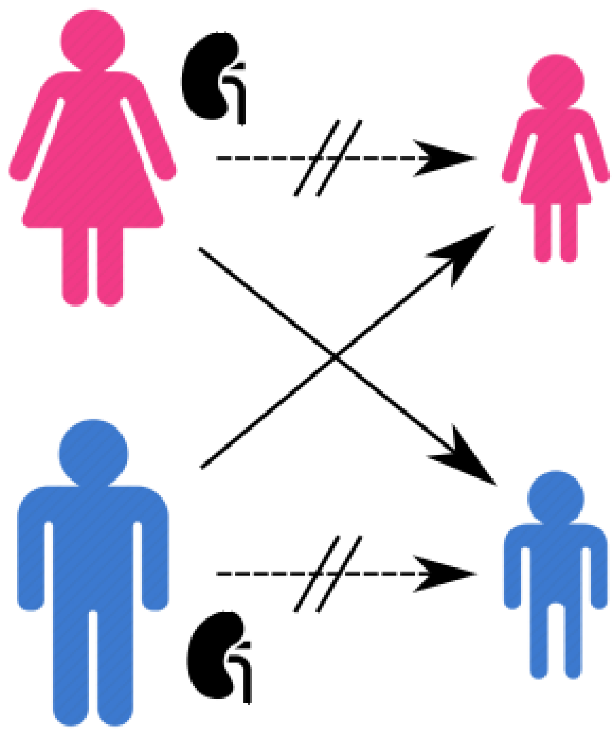
no other allocation where all agents are weakly better off and someone is strictly better off

Reminders

Project groups: Due Jan 26

Assignment 1: Due Jan 28

Next Time: Application of TTCA



Quiz

Quiz

Prove that the core outcome is unique.

References

- Housing markets and TTCA

Lloyd Shapley and Herbert Scarf
"*On Cores and Indivisibility*"
Journal of Mathematical Economics,1(1), 1974 pg 23-37
https://www.sciencedirect.com/science/article/pii/0304406874900330

- Truthfulness under TTCA

Alvin E. Roth
"*Incentive Compatibility in a Market with Indivisible Goods*"
Economics Letters, 9(2), 1982, pg 127-132
https://www.sciencedirect.com/science/article/pii/0165176582900039