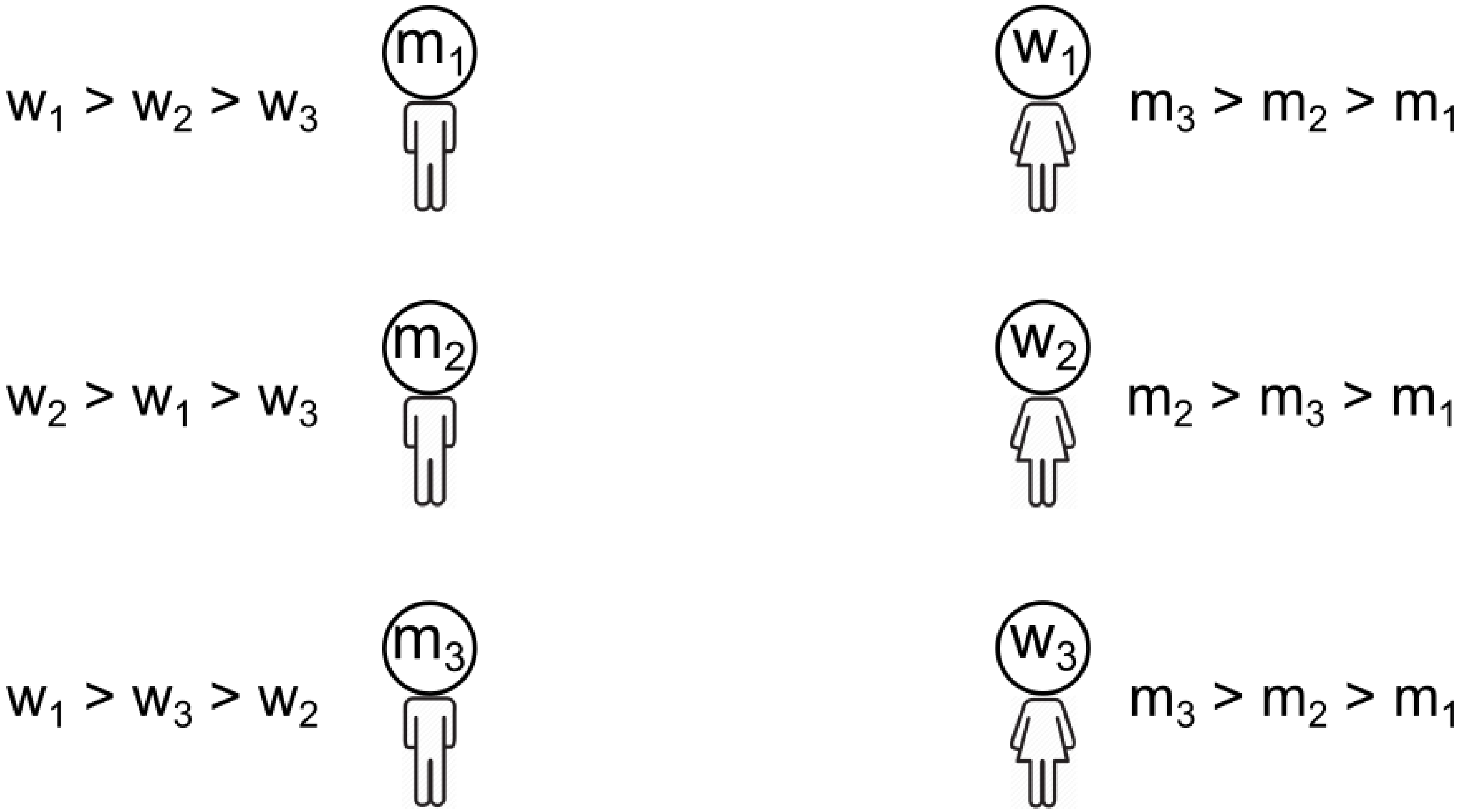
COL749: Computational Social Choice

Lecture 4

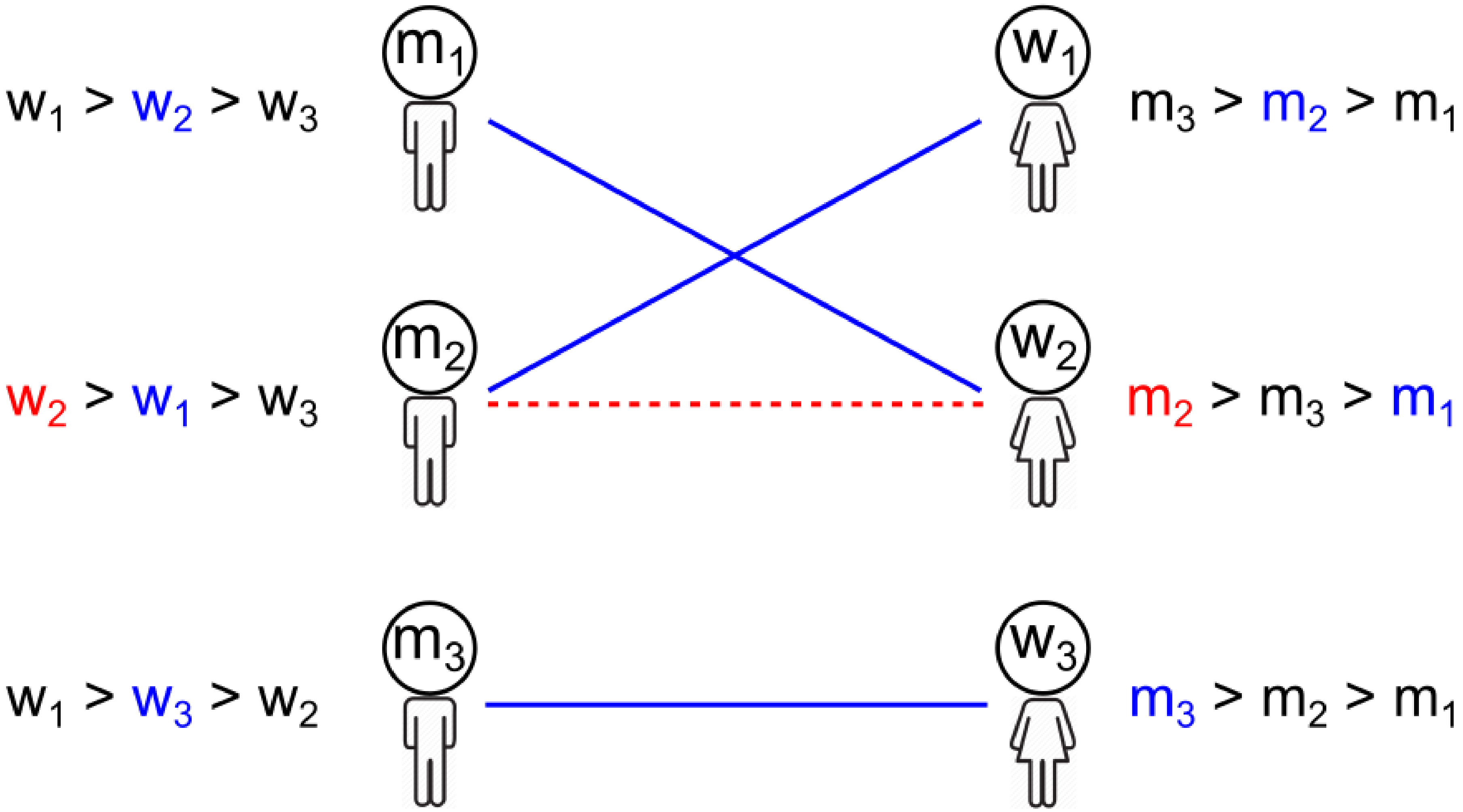
Fairness in Stable Matching Problem

Jan 13, 2025 | Rohit Vaish

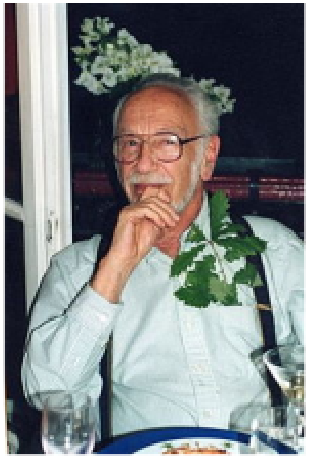
Stable Matching Problem



Stable Matching Problem



A matching is stable if there is no blocking pair.

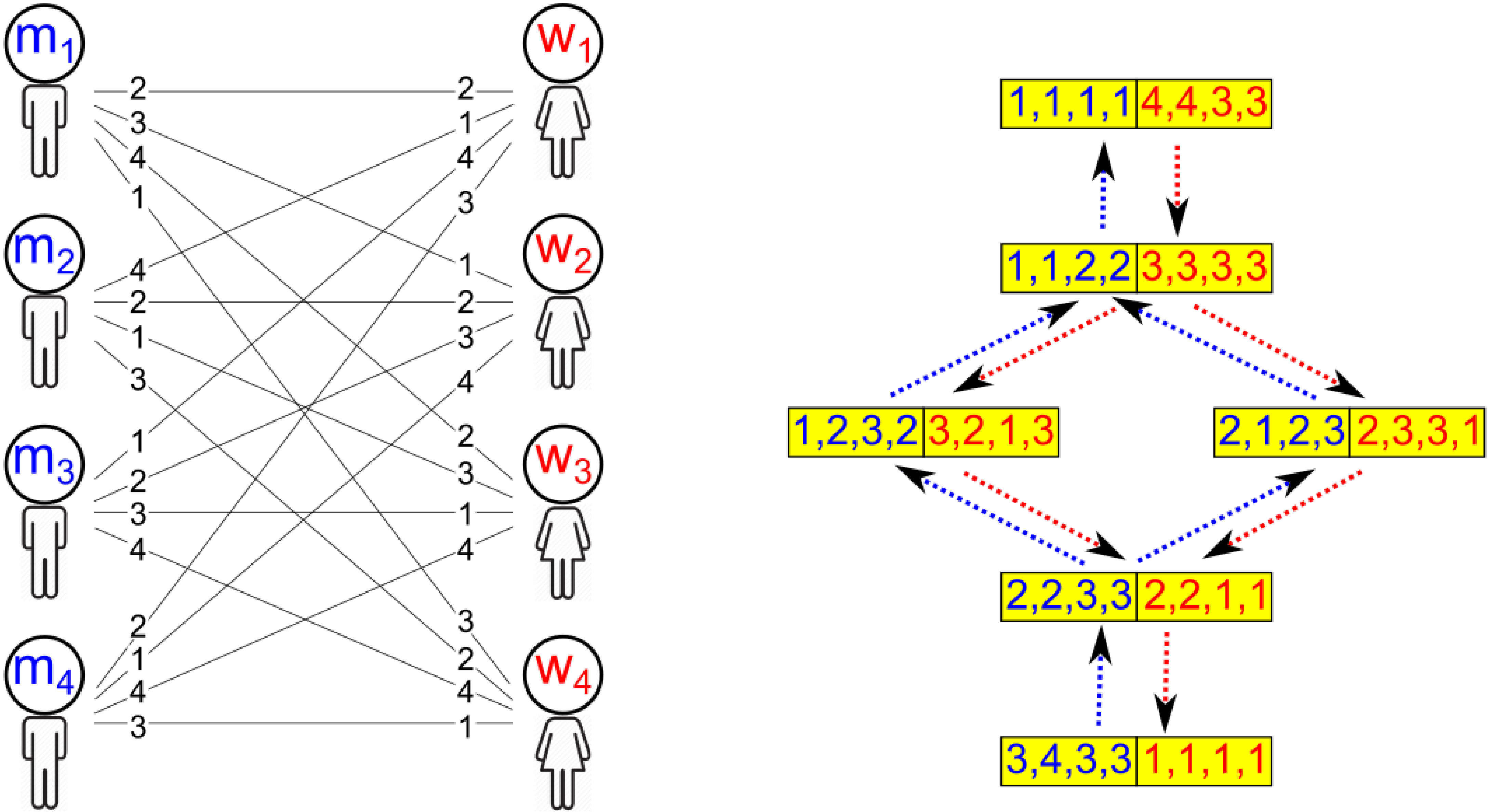


COLLEGE ADMISSIONS AND THE STABILITY OF MARRIAGE

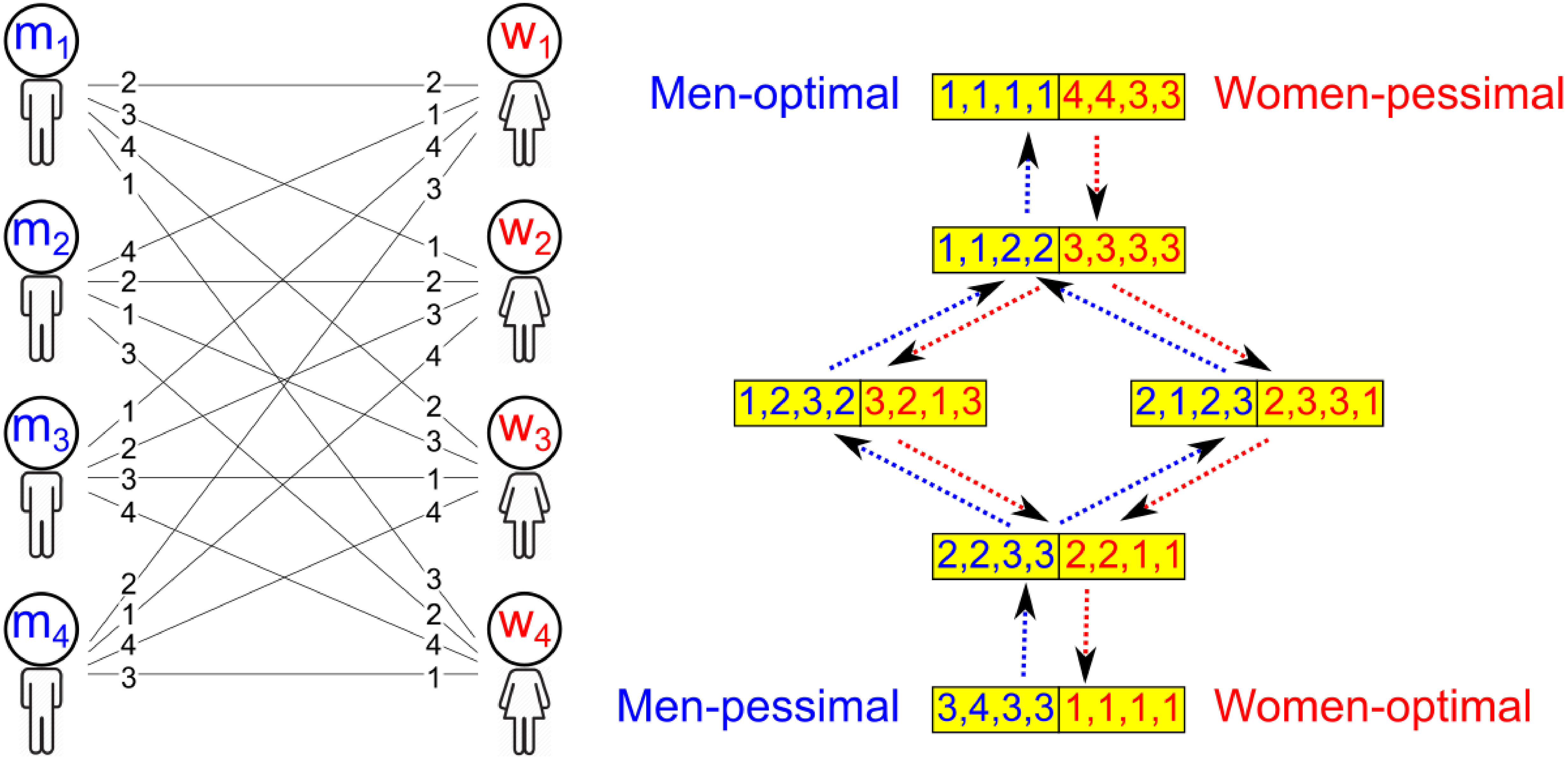
D. GALE* AND L. S. SHAPLEY, Brown University and the RAND Corporation

Source: *The American Mathematical Monthly*, Jan., 1962, Vol. 69, No. 1 (Jan., 1962), pp. 9–15

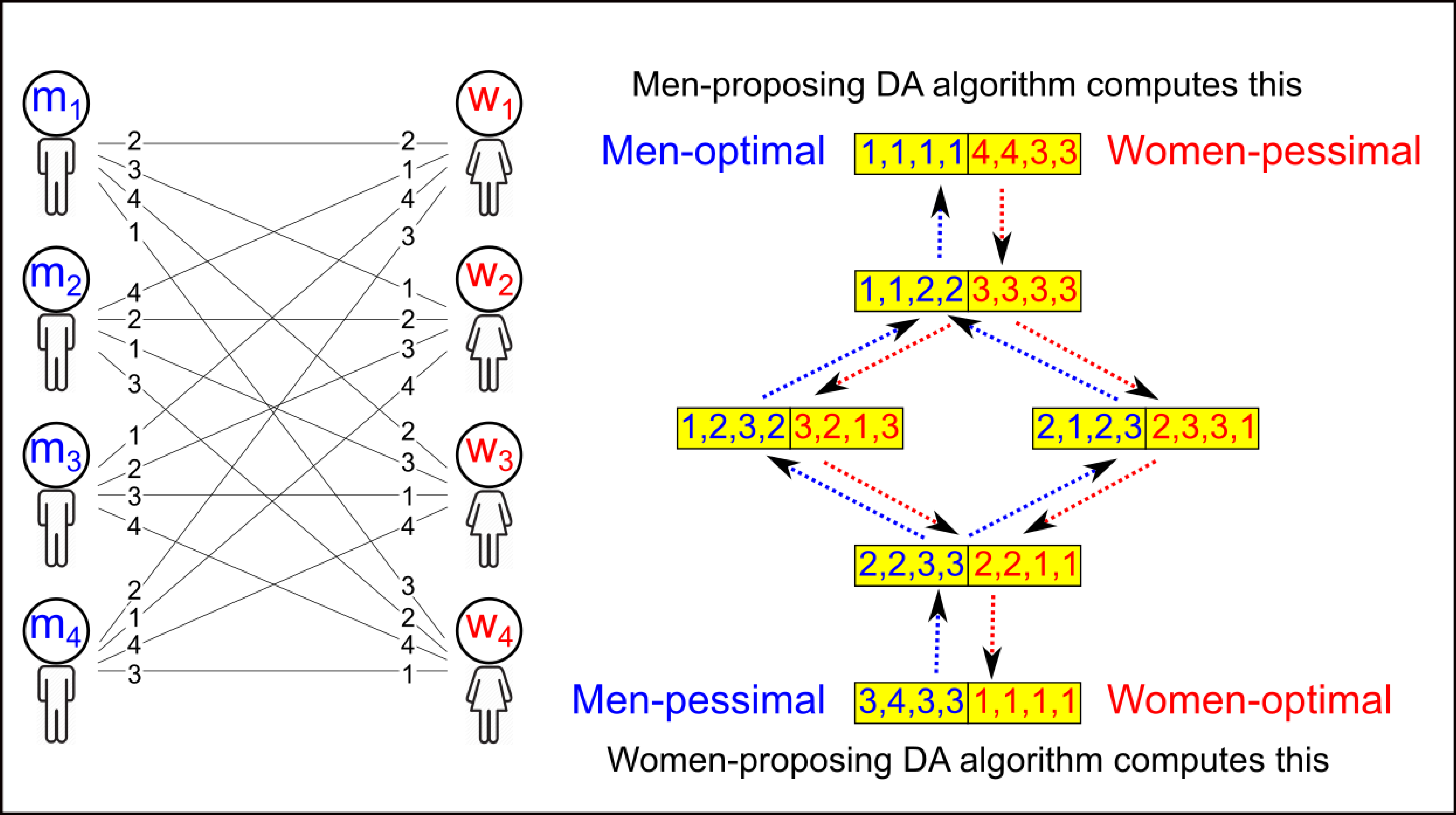
Given any preference profile, a stable matching for that profile always exists and can be computed in polynomial time.



m_1
m_2
m_3
m_4
w_1
w_2
w_3
w_4
1,1,1,1 4,4,3,3
1,1,2,2 3,3,3,3
1,2,3,2 3,2,1,3
2,1,2,3 2,3,3,1
2,2,3,3 2,2,1,1
3,4,3,3 1,1,1,1



m_1
m_2
m_3
m_4
w_1
w_2
w_3
w_4
Men-optimal
1,1,1,1 | 4,4,3,3
Women-pessimal
1,1,2,2 | 3,3,3,3
1,2,3,2 | 3,2,1,3
2,1,2,3 | 2,3,3,1
2,2,3,3 | 2,2,1,1
Men-pessimal
3,4,3,3 | 1,1,1,1
Women-optimal

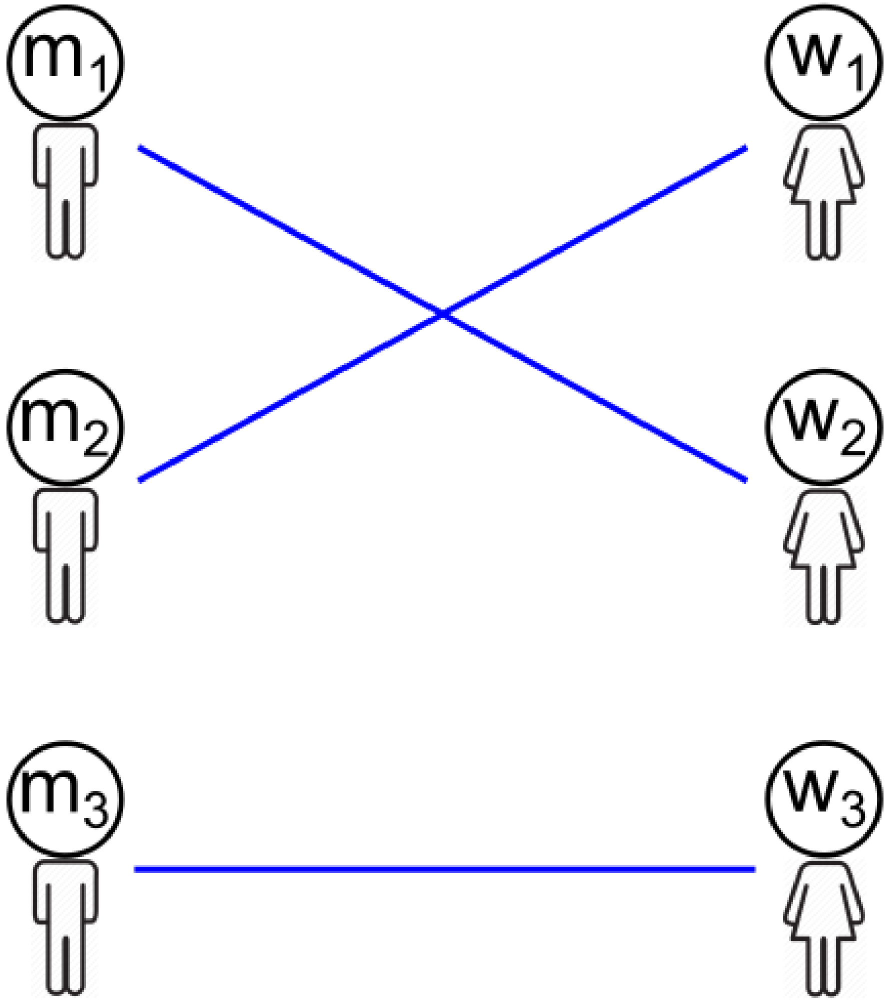


Men-proposing DA algorithm computes this
Men-optimal
1,1,1,1 4,4,3,3
Women-pessimal
1,1,2,2 3,3,3,3
1,2,3,2 3,2,1,3
2,1,2,3 2,3,3,1
2,2,3,3 2,2,1,1
Men-pessimal
3,4,3,3 1,1,1,1
Women-optimal
Women-proposing DA algorithm computes this
m_1
m_2
m_3
m_4
w_1
w_2
w_3
w_4

Goal for Today

Understanding the structure of the set of stable matchings through linear programming.

(This will guide us towards fair stable matchings.)



$$P = \begin{array}{c} \\ m_1 \\ m_2 \\ m_3 \end{array} \begin{array}{c} \begin{array}{ccc} w_1 & w_2 & w_3 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

Fractional Stable Matching

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Fractional Stable Matching

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$$X_{i,j} + \sum_{k: w_k \succ_{m_i} w_j} X_{i,k} + \sum_{k: m_k \succ_{w_j} m_i} X_{k,j} \geq 1 \text{ for all } i \in [n] \text{ and } j \in [n]$$ *Stability*

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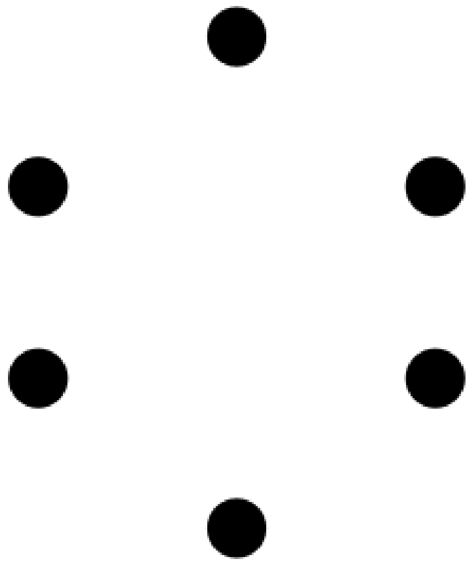
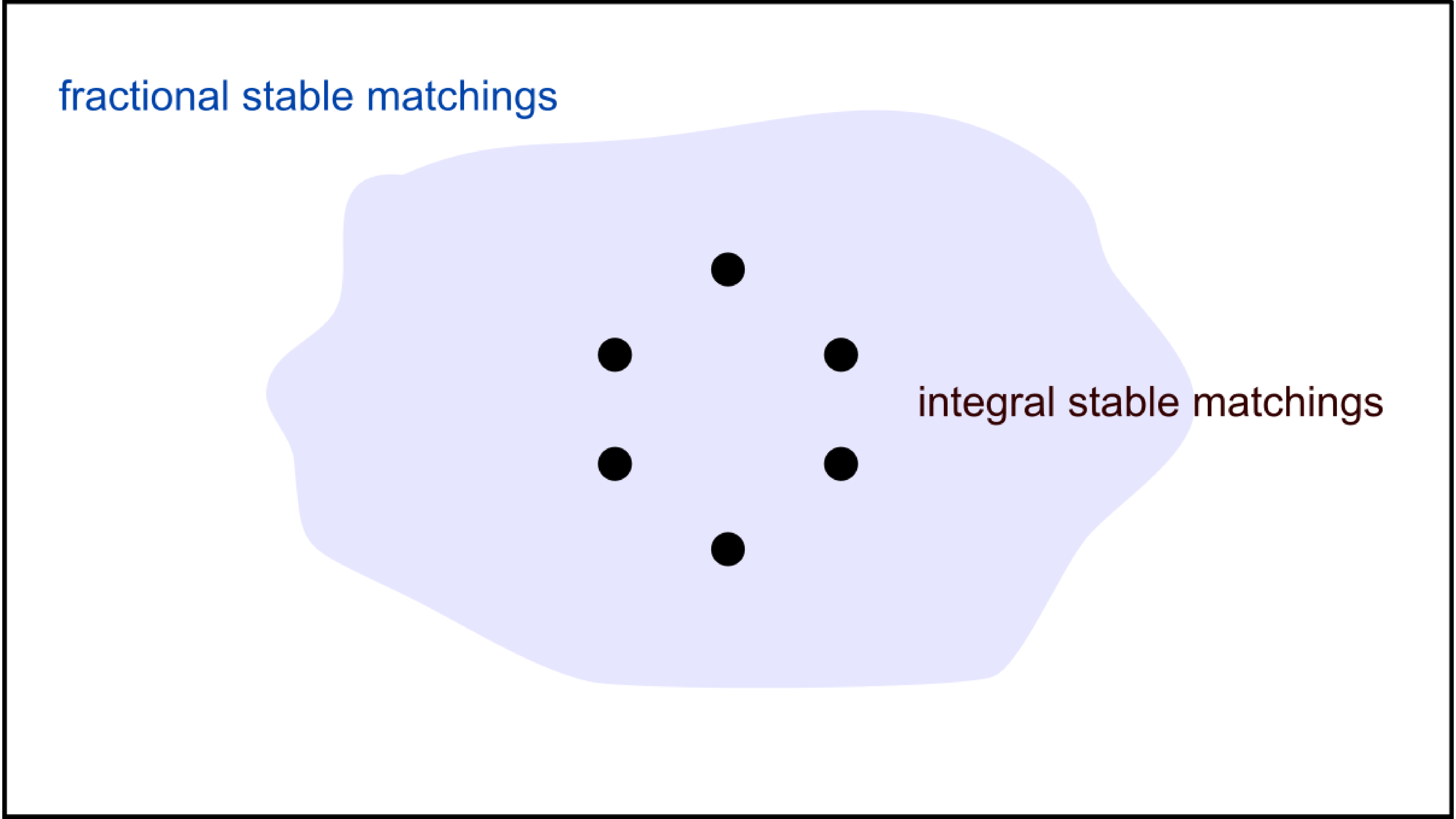
Any integral stable matching is also a fractional stable matching.


integral stable matchings


fractional stable matchings
integral stable matchings

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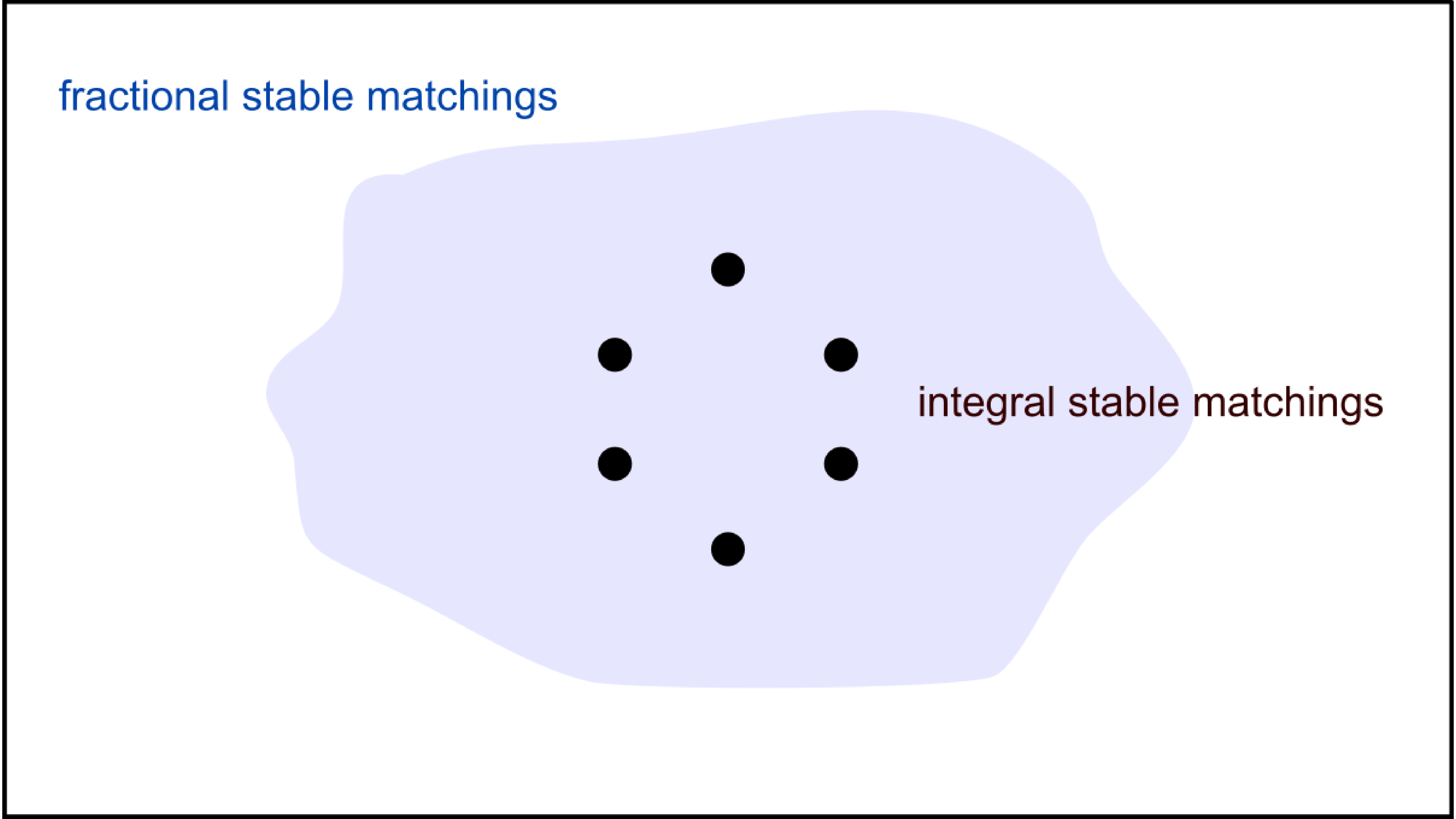
Any convex combination of integral stable matchings is also a fractional stable matching.

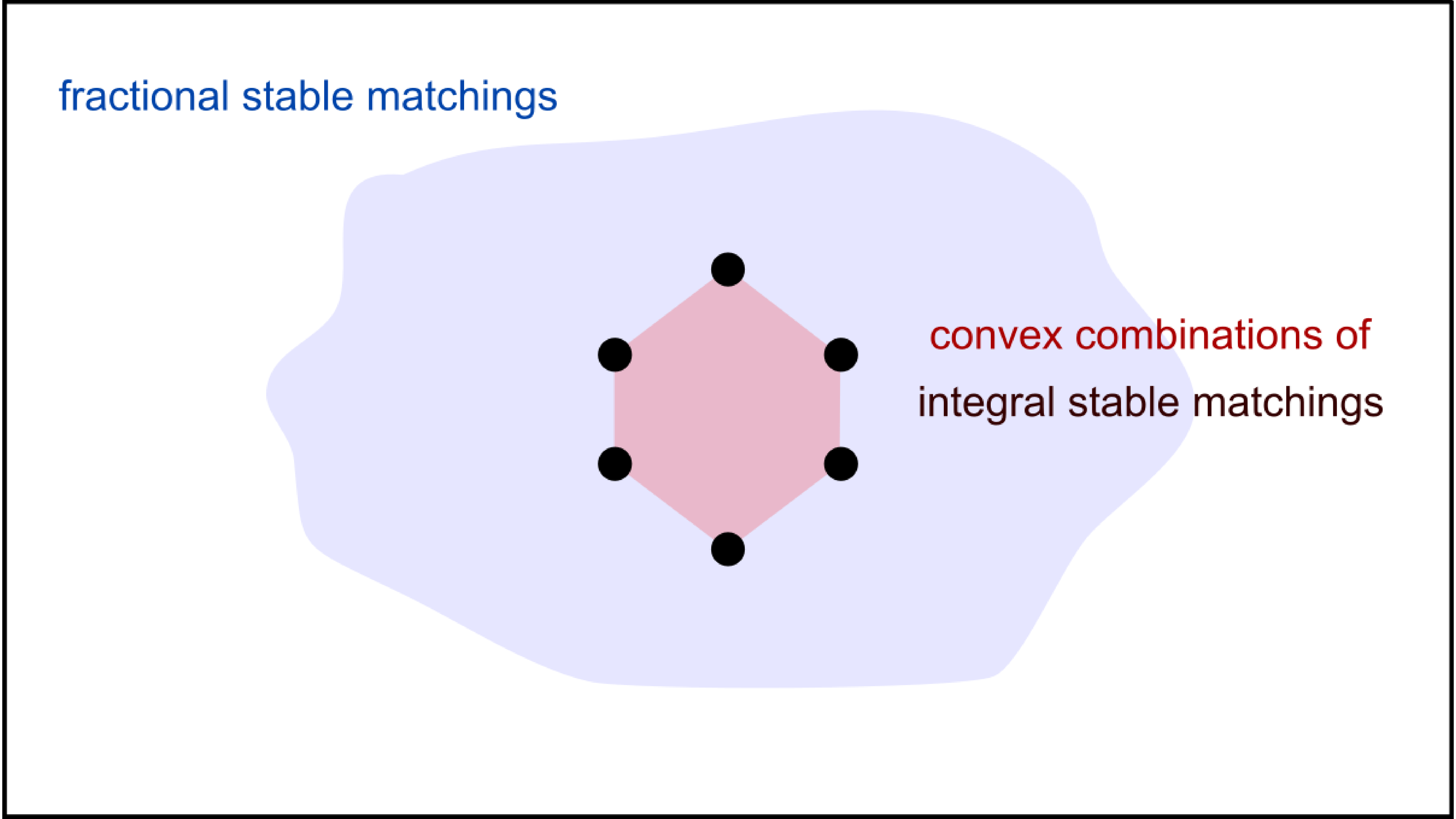
$$X = \sum_k \lambda_k P^k$$

integral stable matching

such that $\lambda_k \geq 0$ for all k and $\sum_k \lambda_k = 1$

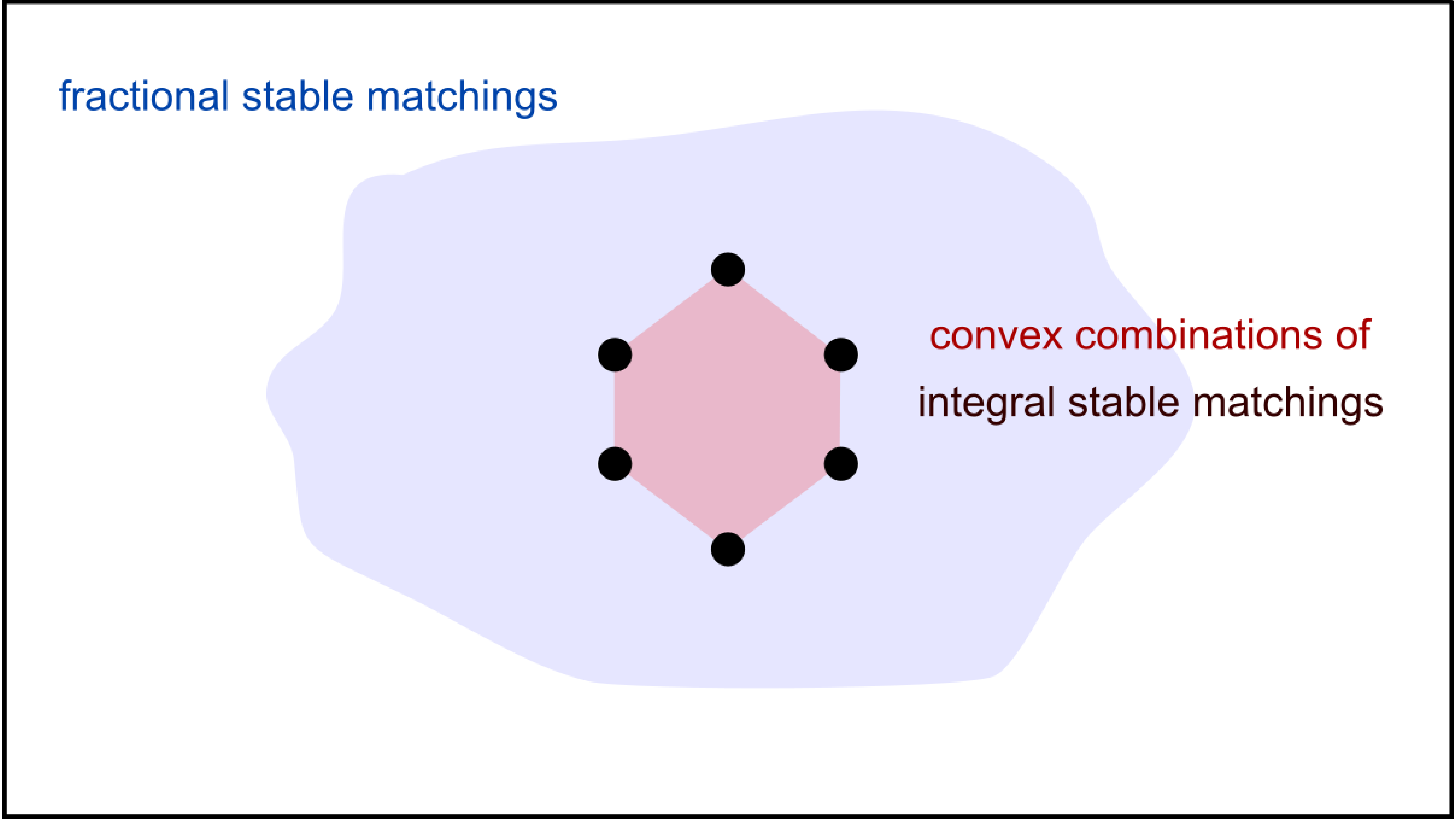
Any integral stable matching is also a fractional stable matching.


fractional stable matchings
integral stable matchings


fractional stable matchings
convex combinations of
integral stable matchings

[Vande Vate, *Oper. Res. Let.* 1989]

Any fractional stable matching can be expressed as a convex combination of integral stable matchings.


fractional stable matchings
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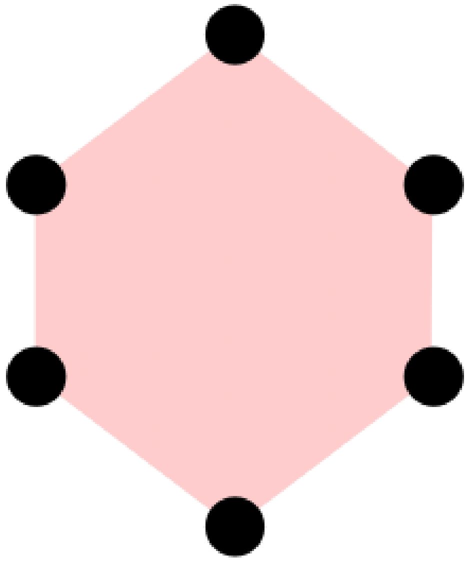


convex combinations of

integral stable matchings

=

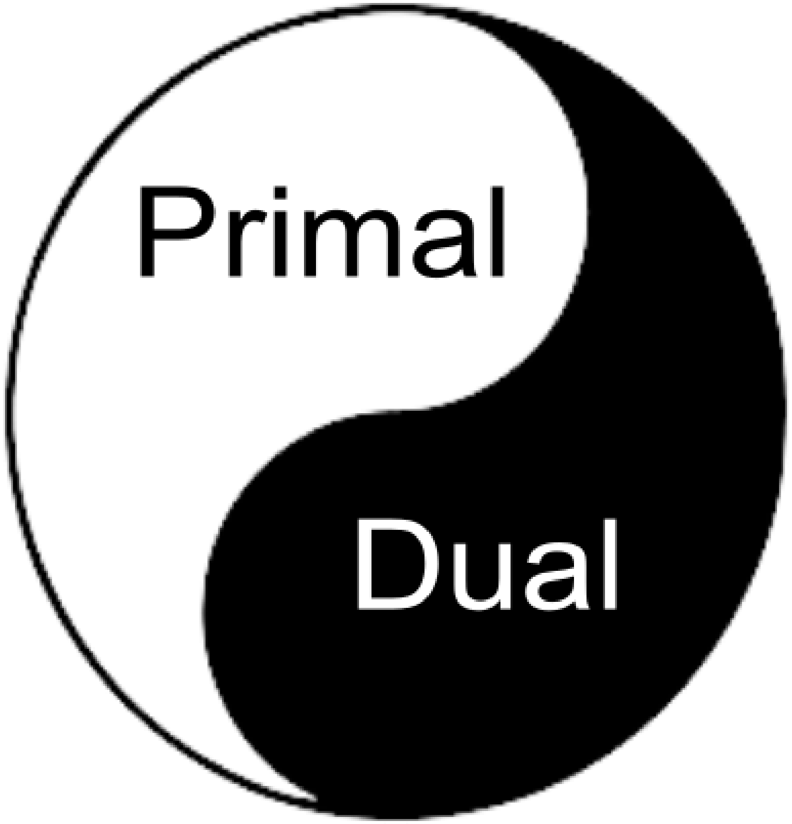
fractional stable matchings

[Vande Vate, *Oper. Res. Let.* 1989]

Any fractional stable matching can be expressed as a convex combination of integral stable matchings.

Coming up...

An elegant geometric proof that uses LP duality and its application in fair stable matchings.


Primal
Dual

Primal

$$\max \sum_{i,j} X_{i,j}$$

$$\sum_j X_{i,j} = 1 \; \forall i$$

$$\sum_i X_{i,j} = 1 \; \forall j$$

$$X_{i,j} + \sum_{k: w_k \succ_{m_i} w_j} X_{i,k} + \sum_{k: m_k \succ_{w_j} m_i} X_{k,j} \geq 1 \; \forall i, j$$

$$X_{i,j} \geq 0 \; \forall i, j$$

Primal

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Primal

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$$-X_{i,j} - \sum_{k: w_k \succ_{m_i} w_j} X_{i,k} - \sum_{k: m_k \succ_{w_j} m_i} X_{k,j} \leq -1 \; \forall i, j$$

$$X_{i,j} \geq 0 \; \forall i, j$$

Primal

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$$\gamma_{i,j}$$

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Don't worry about us.

Primal

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Combine the constraints in order to construct an upper bound on the objective.

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$$\sum_{i,j} \beta_j X_{i,j} = \sum_j \beta_j$$

$\gamma_{i,j}$

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$$-\sum_{i,j} \left(\gamma_{i,j} X_{i,j} + \sum_{k: w_k \succ_{m_i} w_j} \gamma_{i,j} X_{i,k} + \sum_{k: m_k \succ_{w_j} m_i} \gamma_{i,j} X_{k,j} \right) \leq -\sum_{i,j} \gamma_{i,j}$$

$$X_{i,j} \geq 0 \ \forall i, j$$

Primal

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$$X_{i,j} \geq 0 \ \forall i, j$$

as long as $\gamma_{i,j} \geq 0 \ \forall i, j$

Primal

Let's combine these.

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$$\sum_{i,j} X_{i,j} \left(\alpha_i + \beta_j - \gamma_{i,j} - \sum_{k: w_j \succ_{m_i} w_k} \gamma_{i,k} - \sum_{k: m_i \succ_{w_j} m_k} \gamma_{k,j} \right)$$

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as long as $\gamma_{i,j} \geq 0 \ \forall i, j$

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$$\leq \sum_i \alpha_i + \sum_j \beta_j - \sum_{i,j} \gamma_{i,j}$$

$$\textstyle\sum_{i,j} X_{i,j} \left(\alpha_i + \beta_j - \gamma_{i,j} - \displaystyle\sum_{k: w_j \succ_{m_i} w_k} \gamma_{i,k} - \sum_{k: m_i \succ_{w_j} m_k} \gamma_{k,j} \right)$$

$$\leq \sum_i \alpha_i + \sum_j \beta_j - \sum_{i,j} \gamma_{i,j}$$

subject to this being at least 1

minimize this

$$\textstyle\sum_{i,j} X_{i,j} \left(\alpha_i + \beta_j - \gamma_{i,j} - \displaystyle\sum_{k: w_j \succ_{m_i} w_k} \gamma_{i,k} - \sum_{k: m_i \succ_{w_j} m_k} \gamma_{k,j} \right)$$

$$\leq \sum_i \alpha_i + \sum_j \beta_j - \sum_{i,j} \gamma_{i,j}$$

as long as
$\gamma_{i,j} \geq 0 \ \forall i, j$

subject to this being at least 1

minimize this

$$\sum_{i,j} X_{i,j} \left(\alpha_i + \beta_j - \gamma_{i,j} - \sum_{k: w_j \succ_{m_i} w_k} \gamma_{i,k} - \sum_{k: m_i \succ_{w_j} m_k} \gamma_{k,j} \right)$$

$$\leq \sum_i \alpha_i + \sum_j \beta_j - \sum_{i,j} \gamma_{i,j}$$

Dual

$$\min \sum_i \alpha_i + \sum_j \beta_j - \sum_{i,j} \gamma_{i,j}$$

$$\alpha_i + \beta_j - \gamma_{i,j} - \sum_{k:w_j \succ_{m_i} w_k} \gamma_{i,k} - \sum_{k:m_i \succ_{w_j} m_k} \gamma_{k,j} \geq 1 \ \forall i,j$$

$$\gamma_{i,j} \geq 0 \ \forall i,j$$

Recall the Primal

$$\max \sum_{i,j} X_{i,j}$$

$$\sum_j X_{i,j} = 1 \ \forall i$$

$$\sum_i X_{i,j} = 1 \ \forall j$$

$$X_{i,j} + \sum_{k : w_k \succ_{m_i} w_j} X_{i,k} + \sum_{k : m_k \succ_{w_j} m_i} X_{k,j} \geq 1 \ \forall i, j$$

$$X_{i,j} \geq 0 \ \forall i, j$$

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$$\sum_j X_{i,j} = 1 \ \forall i$$

$$\sum_i X_{i,j} = 1 \ \forall j$$

By Gale and Shapley's result, primal is always feasible!

$$X_{i,j} + \sum_{k: w_k \succ_{m_i} w_j} X_{i,k} + \sum_{k: m_k \succ_{w_j} m_i} X_{k,j} \geq 1 \ \forall i, j$$

$$X_{i,j} \geq 0 \ \forall i, j$$

Let $X_{i,j}$ be a feasible primal solution.

Dual

$$\min \sum_i \alpha_i + \sum_j \beta_j - \sum_{i,j} \gamma_{i,j}$$

$$\alpha_i + \beta_j - \gamma_{i,j} - \sum_{k: w_j \succ_{m_i} w_k} \gamma_{i,k} - \sum_{k: m_i \succ_{w_j} m_k} \gamma_{k,j} \geq 1 \ \forall i, j$$

$$\gamma_{i,j} \geq 0 \ \forall i, j$$

Dual

$$\min \sum_i \alpha_i + \sum_j \beta_j - \sum_{i,j} \gamma_{i,j}$$

$$\alpha_i + \beta_j - \gamma_{i,j} - \sum_{k:w_j \succ_{m_i} w_k} \gamma_{i,k} - \sum_{k:m_i \succ_{w_j} m_k} \gamma_{k,j} \geq 1 \ \forall i, j$$

$$\gamma_{i,j} \geq 0 \ \forall i, j$$

Substitute $\alpha_i = \sum_j X_{i,j}$, $\beta_j = \sum_i X_{i,j}$, $\gamma_{i,j} = X_{i,j}$ for all i, j

Dual

$$\min \sum_i \alpha_i + \sum_j \beta_j - \sum_{i,j} \gamma_{i,j}$$

$$\alpha_i + \beta_j - \gamma_{i,j} - \sum_{k: w_j \succ_{m_i} w_k} \gamma_{i,k} - \sum_{k: m_i \succ_{w_j} m_k} \gamma_{k,j} \geq 1 \ \forall i, j$$

$$\gamma_{i,j} \geq 0 \ \forall i, j$$

Substitute $\alpha_i = \sum_j X_{i,j}$, $\beta_j = \sum_i X_{i,j}$, $\gamma_{i,j} = X_{i,j}$ for all i, j

Dual

$$\sum_{i,j} X_{i,j}$$

$$\alpha_i + \beta_j - \gamma_{i,j} - \sum_{k: w_j \succ_{m_i} w_k} \gamma_{i,k} - \sum_{k: m_i \succ_{w_j} m_k} \gamma_{k,j} \geq 1 \ \forall i, j$$

$$\gamma_{i,j} \geq 0 \ \forall i, j$$

Substitute $\alpha_i = \sum_j X_{i,j}$, $\beta_j = \sum_i X_{i,j}$, $\gamma_{i,j} = X_{i,j}$ for all i, j

Dual

$$\sum_{i,j} X_{i,j}$$

$$\alpha_i + \beta_j - \gamma_{i,j} - \sum_{k: w_j \succ_{m_i} w_k} \gamma_{i,k} - \sum_{k: m_i \succ_{w_j} m_k} \gamma_{k,j} \geq 1 \ \forall i, j$$

$$\gamma_{i,j} \geq 0 \ \forall i, j$$

Substitute $\alpha_i = \sum_j X_{i,j}$, $\beta_j = \sum_i X_{i,j}$, $\gamma_{i,j} = X_{i,j}$ for all i, j

Dual

$$\sum_{i,j} X_{i,j}$$

$$\sum_j X_{i,j} + \sum_i X_{i,j} - X_{i,j} - \sum_{k: w_j \succ_{m_i} w_k} X_{i,k} - \sum_{k: m_i \succ_{w_j} m_k} X_{k,j} \geq 1 \ \forall i, j$$

$$\gamma_{i,j} \geq 0 \ \forall i, j$$

Substitute $\alpha_i = \sum_j X_{i,j}$, $\beta_j = \sum_i X_{i,j}$, $\gamma_{i,j} = X_{i,j}$ for all i, j

Dual

$$\sum_{i,j} X_{i,j}$$

$$\sum_j X_{i,j} + \sum_i X_{i,j} - X_{i,j} - \sum_{k: w_j \succ_{m_i} w_k} X_{i,k} - \sum_{k: m_i \succ_{w_j} m_k} X_{k,j} \geq 1 \ \forall i, j$$

$$\gamma_{i,j} \geq 0 \ \forall i, j$$

Substitute $\alpha_i = \sum_j X_{i,j}$, $\beta_j = \sum_i X_{i,j}$, $\gamma_{i,j} = X_{i,j}$ for all i, j

Dual

$$\sum_{i,j} X_{i,j}$$

$$\sum_{k:w_k \succ_{m_i} w_j} X_{i,k} + \sum_{i} X_{i,j} - \sum_{k:m_i \succ_{w_j} m_k} X_{k,j} \geq 1 \ \forall i,j$$

$$\gamma_{i,j} \geq 0 \ \forall i,j$$

Substitute $\alpha_i = \sum_j X_{i,j}$, $\beta_j = \sum_i X_{i,j}$, $\gamma_{i,j} = X_{i,j}$ for all i, j

Dual

$$\sum_{i,j} X_{i,j}$$

$$\sum_{k:w_k \succ_{m_i} w_j} X_{i,k} + \sum_{i} X_{i,j} - \sum_{k:m_i \succ_{w_j} m_k} X_{k,j} \geq 1 \ \forall i, j$$

$$\gamma_{i,j} \geq 0 \ \forall i, j$$

Substitute $\alpha_i = \sum_j X_{i,j}$, $\beta_j = \sum_i X_{i,j}$, $\gamma_{i,j} = X_{i,j}$ for all i, j

Dual

$$\sum_{i,j} X_{i,j}$$

$$\sum_{k:w_k \succ_{m_i} w_j} X_{i,k} + \boxed{\sum_i X_{i,j} - \sum_{k:m_i \succ_{w_j} m_k} X_{k,j}} \geq 1 \ \forall i,j$$

$$\gamma_{i,j} \geq 0 \ \forall i,j$$

Substitute $\alpha_i = \sum_j X_{i,j}$, $\beta_j = \sum_i X_{i,j}$, $\gamma_{i,j} = X_{i,j}$ for all i, j

Dual

$$\sum_{i,j} X_{i,j}$$

$$\sum_{k:w_k \succ_{m_i} w_j} X_{i,k} + \boxed{\sum_{k:m_k \succ_{w_j} m_i} X_{k,j} + X_{i,j}} \geq 1 \ \forall i, j$$

$$\gamma_{i,j} \geq 0 \ \forall i, j$$

Substitute $\alpha_i = \sum_j X_{i,j}$, $\beta_j = \sum_i X_{i,j}$, $\gamma_{i,j} = X_{i,j}$ for all i, j

Dual

$$\sum_{i,j} X_{i,j}$$

$$\sum_{k:w_k \succ_{m_i} w_j} X_{i,k} + \sum_{k:m_k \succ_{w_j} m_i} X_{k,j} + X_{i,j} \geq 1 \; \forall i, j$$

$$\gamma_{i,j} \geq 0 \; \forall i, j$$

Substitute $\alpha_i = \sum_j X_{i,j}$, $\beta_j = \sum_i X_{i,j}$, $\gamma_{i,j} = X_{i,j}$ for all i, j

Dual

$$\sum_{i,j} X_{i,j}$$

$$\sum_{k:w_k \succ_{m_i} w_j} X_{i,k} + \sum_{k:m_k \succ_{w_j} m_i} X_{k,j} + X_{i,j} \geq 1 \; \forall i, j$$

$$\gamma_{i,j} \geq 0 \; \forall i, j$$

Substitute $\alpha_i = \sum_j X_{i,j}$, $\beta_j = \sum_i X_{i,j}$, $\gamma_{i,j} = X_{i,j}$ for all i, j

Dual

$$\sum_{i,j} X_{i,j}$$

$$\sum_{k:w_k \succ_{m_i} w_j} X_{i,k} + \sum_{k:m_k \succ_{w_j} m_i} X_{k,j} + X_{i,j} \geq 1 \ \forall i, j$$

$$X_{i,j} \geq 0 \ \forall i, j$$

Substitute $\alpha_i = \sum_j X_{i,j}$, $\beta_j = \sum_i X_{i,j}$, $\gamma_{i,j} = X_{i,j}$ for all i, j

Dual

$$\sum_{i,j} X_{i,j}$$

$$\sum_{k: w_k \succ_{m_i} w_j} X_{i,k} + \sum_{k: m_k \succ_{w_j} m_i} X_{k,j} + X_{i,j} \geq 1 \ \forall i, j$$

$$X_{i,j} \geq 0 \ \forall i, j$$

Substitute $\alpha_i = \sum_j X_{i,j}$, $\beta_j = \sum_i X_{i,j}$, $\gamma_{i,j} = X_{i,j}$ for all i, j

Dual

Stability constraint from primal

$$\sum_{i,j} X_{i,j}$$

$$\sum_{k:w_k \succ_{m_i} w_j} X_{i,k} + \sum_{k:m_k \succ_{w_j} m_i} X_{k,j} + X_{i,j} \geq 1 \ \forall i,j$$

$$X_{i,j} \geq 0 \ \forall i,j$$

Substitute $\alpha_i = \sum_j X_{i,j}$, $\beta_j = \sum_i X_{i,j}$, $\gamma_{i,j} = X_{i,j}$ for all i, j

Dual

Stability constraint from primal

$$\sum_{i,j} X_{i,j}$$

$$\sum_{k:w_k \succ_{m_i} w_j} X_{i,k} + \sum_{k:m_k \succ_{w_j} m_i} X_{k,j} + X_{i,j} \geq 1 \ \forall i,j$$

$$X_{i,j} \geq 0 \ \forall i,j$$

Dual feasible!

Substitute $\alpha_i = \sum_j X_{i,j}$, $\beta_j = \sum_i X_{i,j}$, $\gamma_{i,j} = X_{i,j}$ for all i, j

Dual

Objective is equal to that in the primal

$$\sum_{i,j} X_{i,j}$$

Stability constraint from primal

$$\sum_{k:w_k \succ_{m_i} w_j} X_{i,k} + \sum_{k:m_k \succ_{w_j} m_i} X_{k,j} + X_{i,j} \geq 1 \;\forall i,j$$

$$X_{i,j} \geq 0 \;\forall i,j$$

Dual feasible!

Substitute $\alpha_i = \sum_j X_{i,j}$, $\beta_j = \sum_i X_{i,j}$, $\gamma_{i,j} = X_{i,j}$ for all i,j

Dual

Stability constraint from primal

Objective is equal to that in the primal

$$\sum_{i,j} X_{i,j}$$

By strong duality:

$X_{i,j}$ must be *primal optimal*, and

$\alpha_i = \sum_j X_{i,j}$, $\beta_j = \sum_i X_{i,j}$, $\gamma_{i,j} = X_{i,j}$ must be *dual optimal*.

$$\sum_{k:w_k \succ_{m_i} w_j} X_{i,k} + \sum_{k:m_k \succ_{w_j} m_i} X_{k,j} + X_{i,j} \geq 1 \;\forall i,j$$

$$X_{i,j} \geq 0 \;\forall i,j$$

Dual feasible!

Substitute $\alpha_i = \sum_j X_{i,j}$, $\beta_j = \sum_i X_{i,j}$, $\gamma_{i,j} = X_{i,j}$ for all i,j

Dual

Stability constraint from primal

Objective is equal to that in the primal

$$\sum X_{i,j}$$

By complementary slackness:

For any primal feasible X,

$$X_{i,j} > 0 \Rightarrow X_{i,j} + \sum_{k:w_k \succ_{m_i} w_j} X_{i,k} + \sum_{k:m_k \succ_{w_j} m_i} X_{k,j} = 1.$$

$$\sum_{k:w_k \succ_{m_i} w_j} X_{i,k} + \sum_{k:m_k \succ_{w_j} m_i} X_{k,j} + X_{i,j} \geq 1 \ \forall i,j$$

$$X_{i,j} \geq 0 \ \forall i,j$$

Dual feasible!

Substitute $\alpha_i = \sum_j X_{i,j}$, $\beta_j = \sum_i X_{i,j}$, $\gamma_{i,j} = X_{i,j}$ for all i, j

[Vande Vate, *Oper. Res. Let.* 1989]

Any fractional stable matching can be expressed as a convex combination of integral stable matchings.

[Vande Vate, *Oper. Res. Let.* 1989]

Any fractional stable matching can be expressed as a convex combination of integral stable matchings.

Proof by picture (and LP duality)

[Teo and Sethuraman, *MOR* 1998].

[Vande Vate, *Oper. Res. Let.* 1989]

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Let X be any fractional stable matching.

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Any fractional stable matching can be expressed as a convex combination of integral stable matchings.

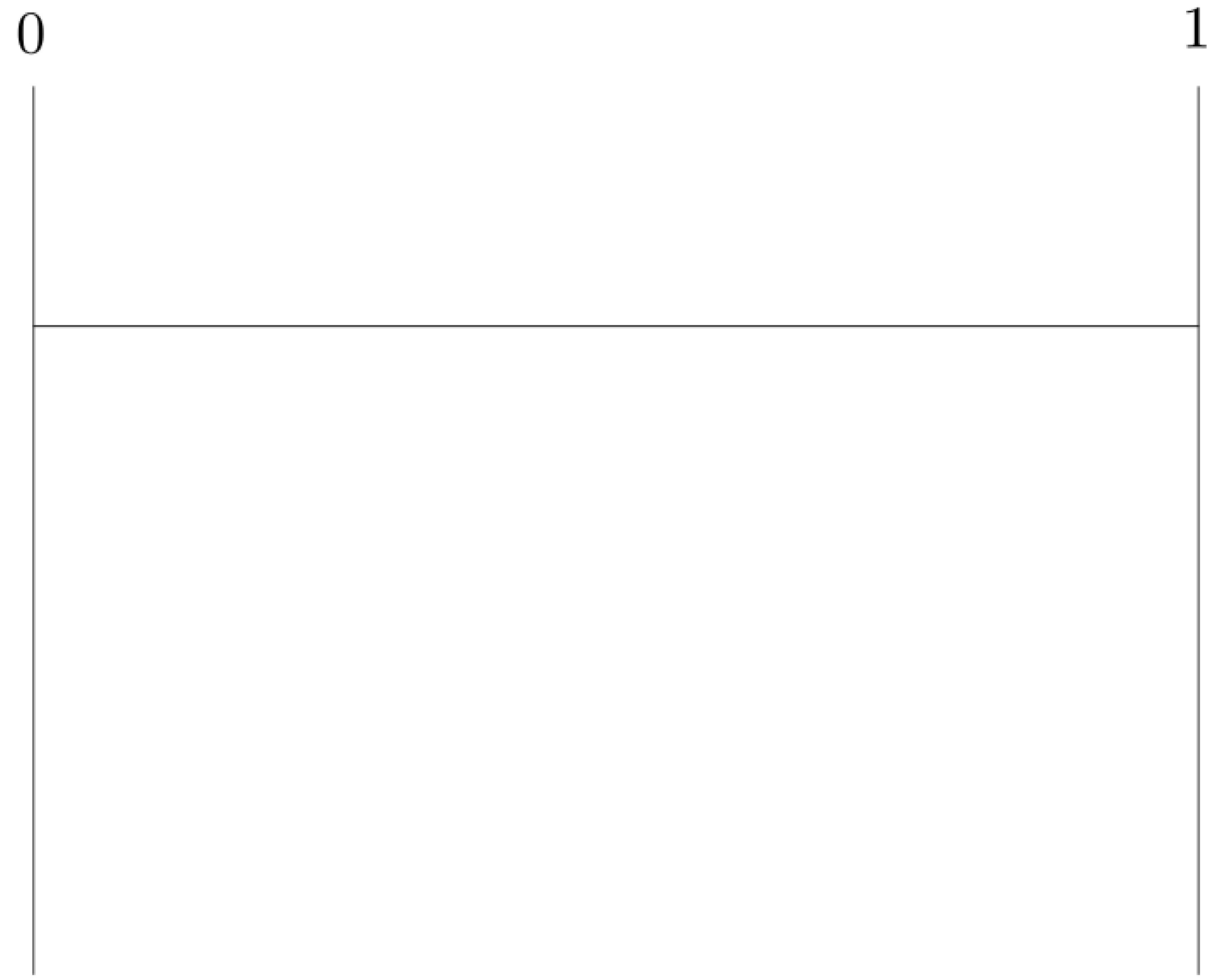
Let X be any fractional stable matching.

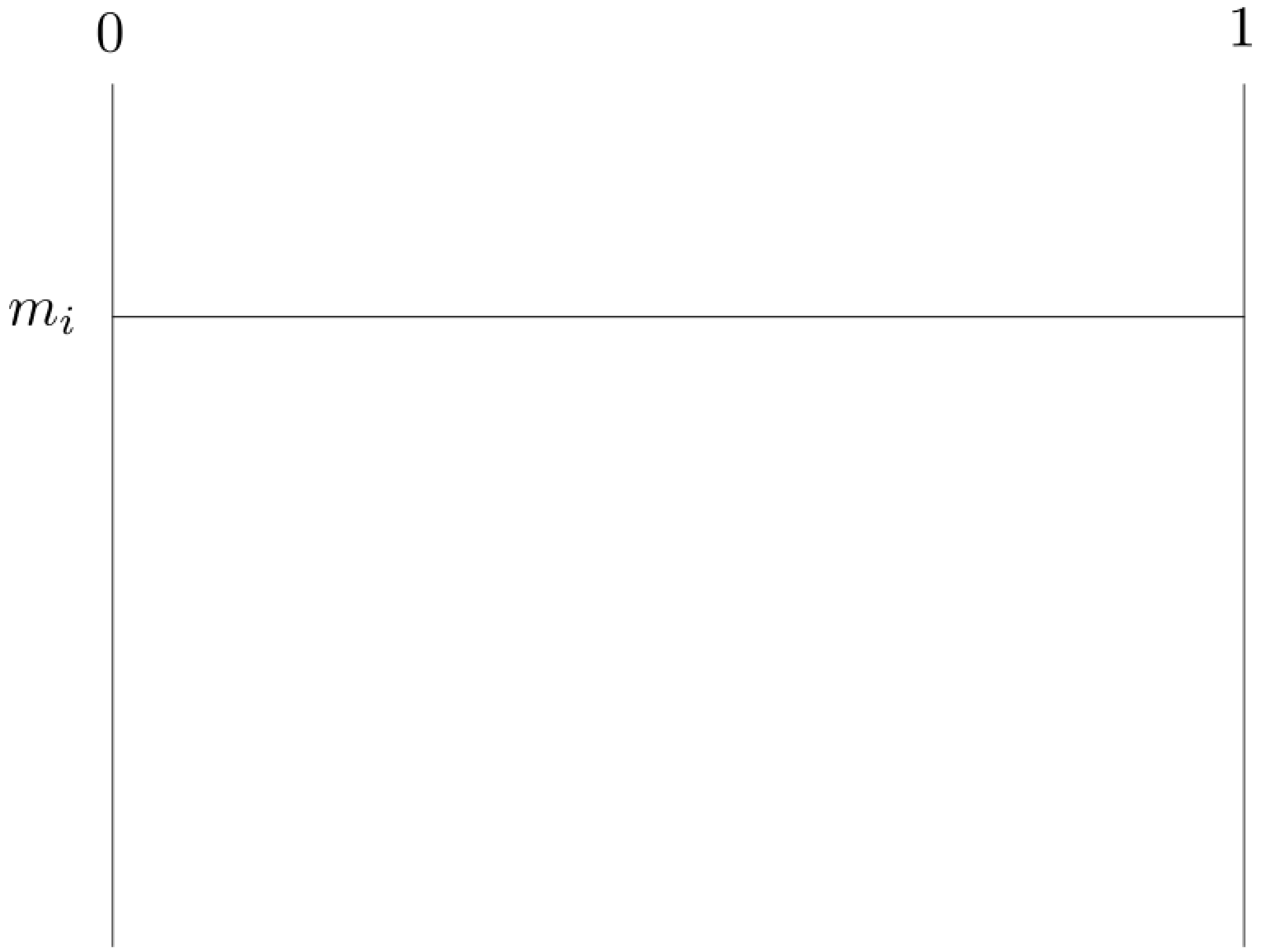
Then, X is primal feasible.

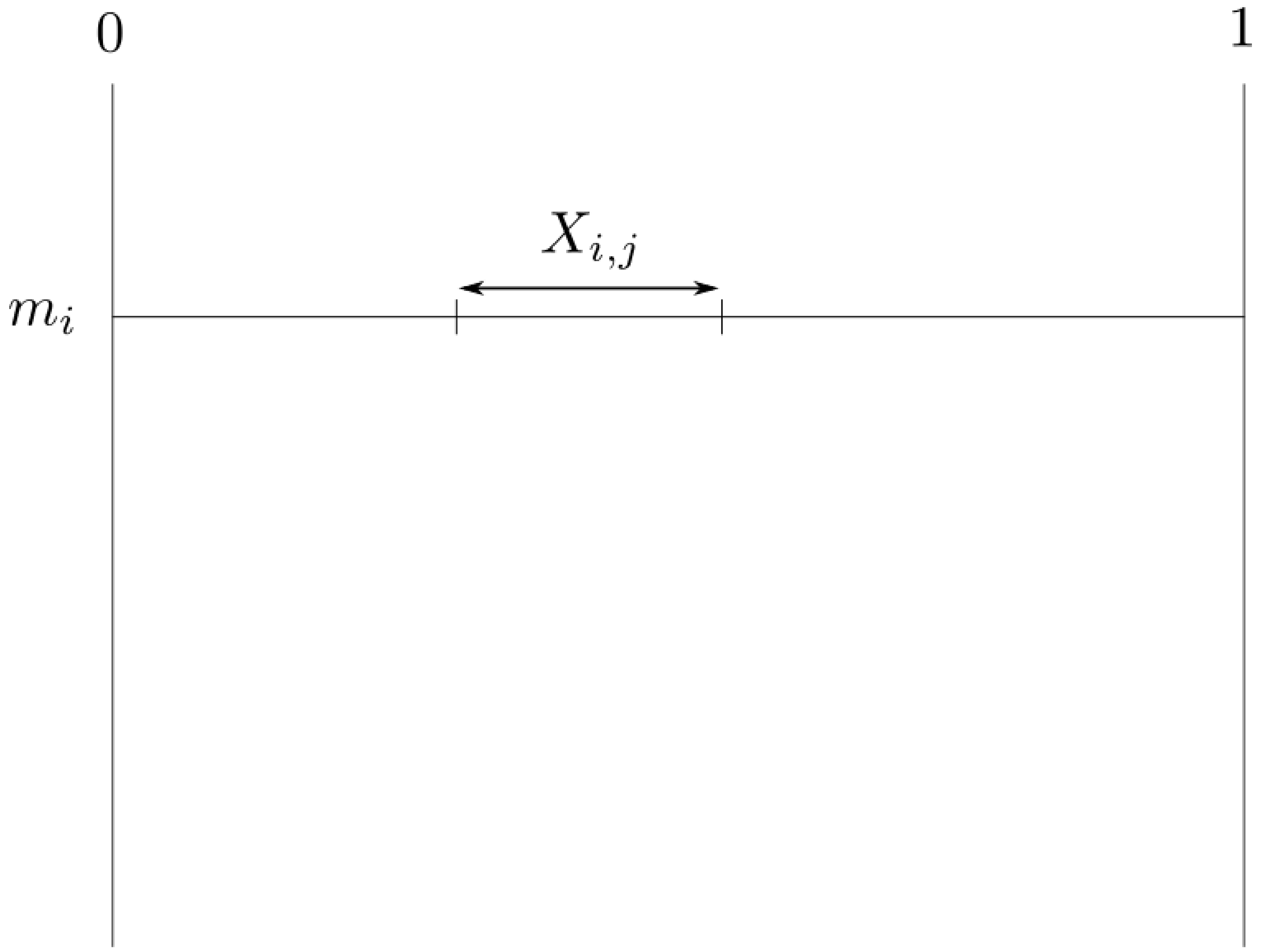
Recall complementary slackness:

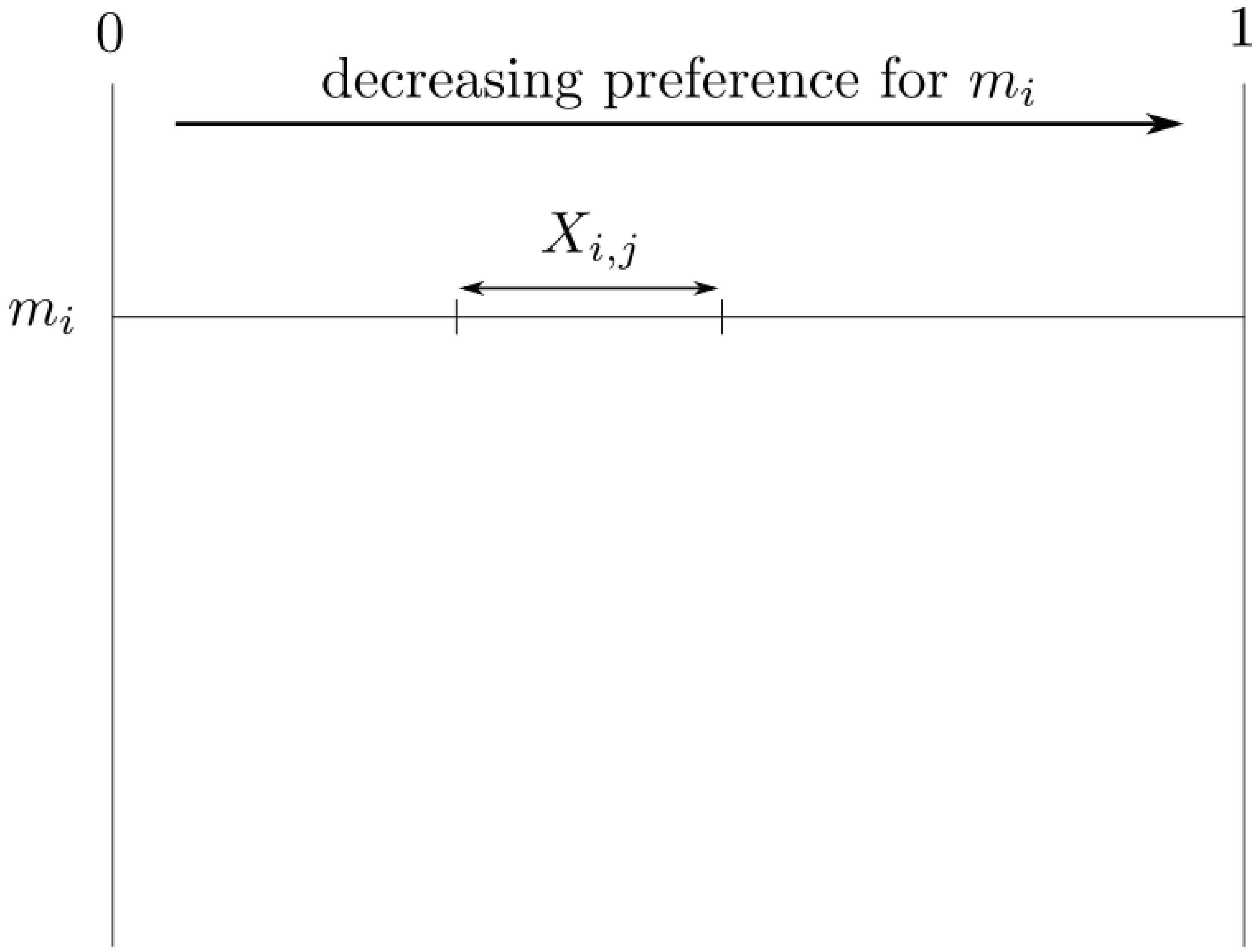
For any primal feasible X,

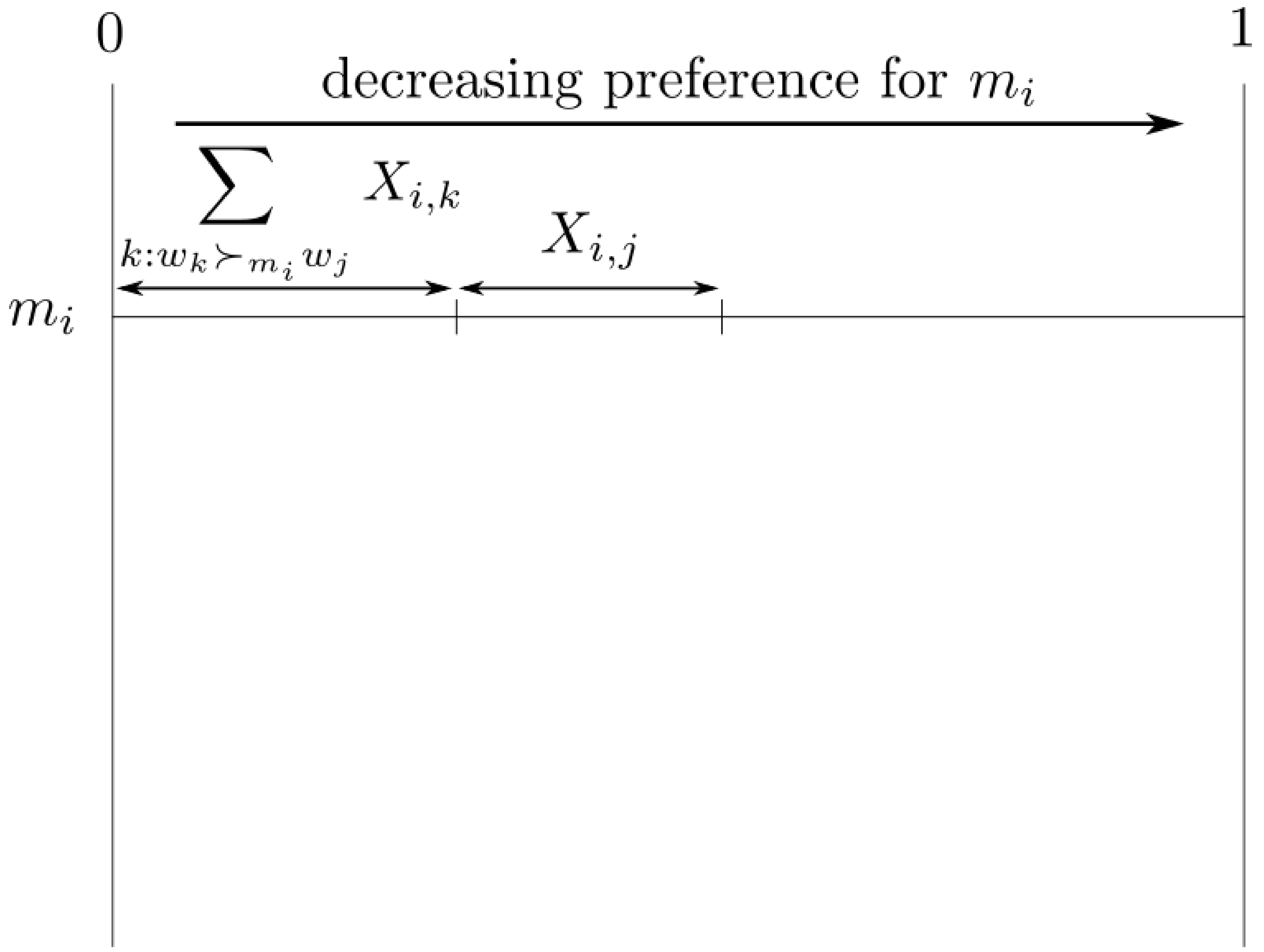
$$X_{i,j} > 0 \Rightarrow X_{i,j} + \sum_{k: w_k \succ_{m_i} w_j} X_{i,k} + \sum_{k: m_k \succ_{w_j} m_i} X_{k,j} = 1.$$


0
1

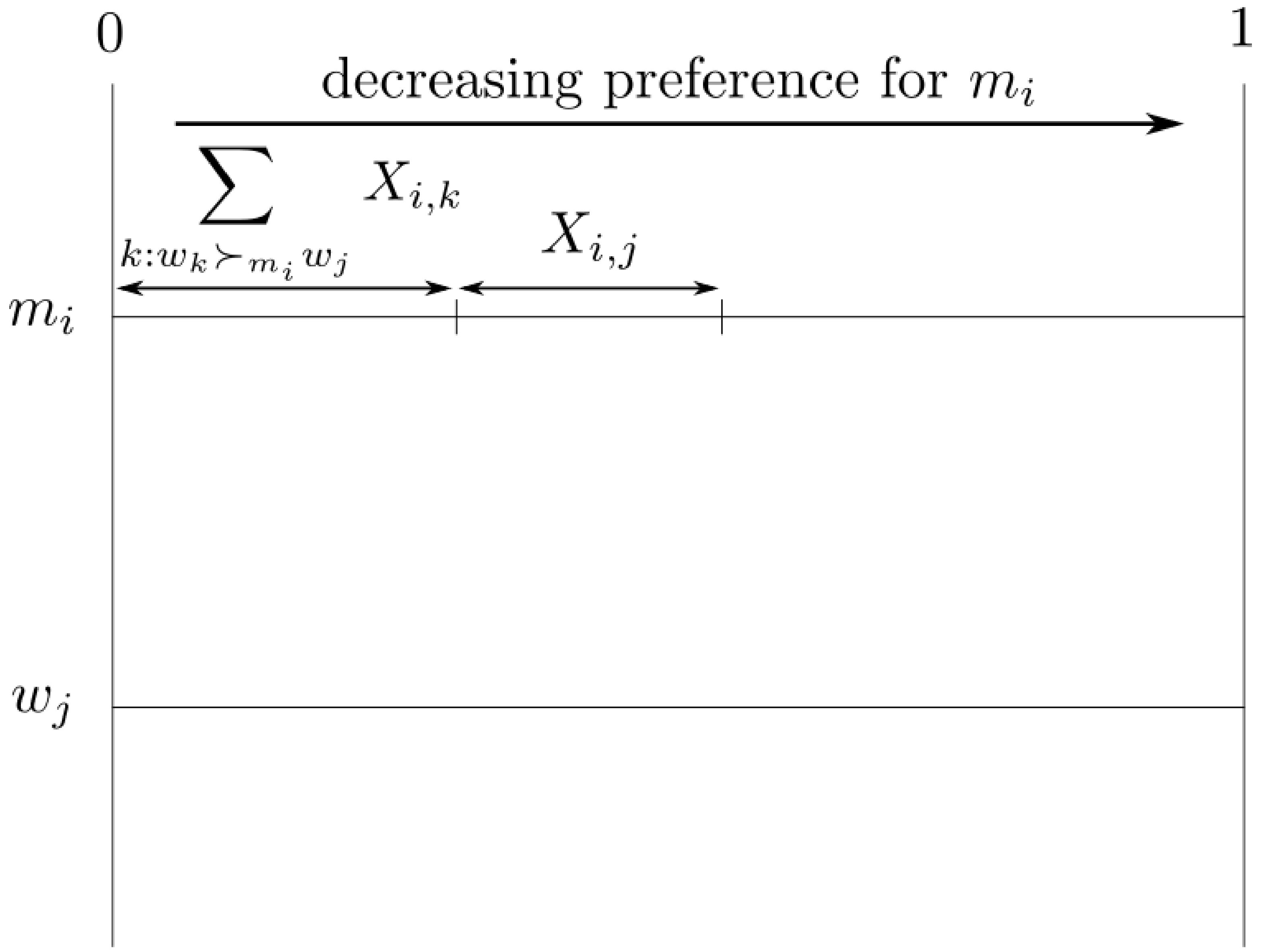

0
1
m_i

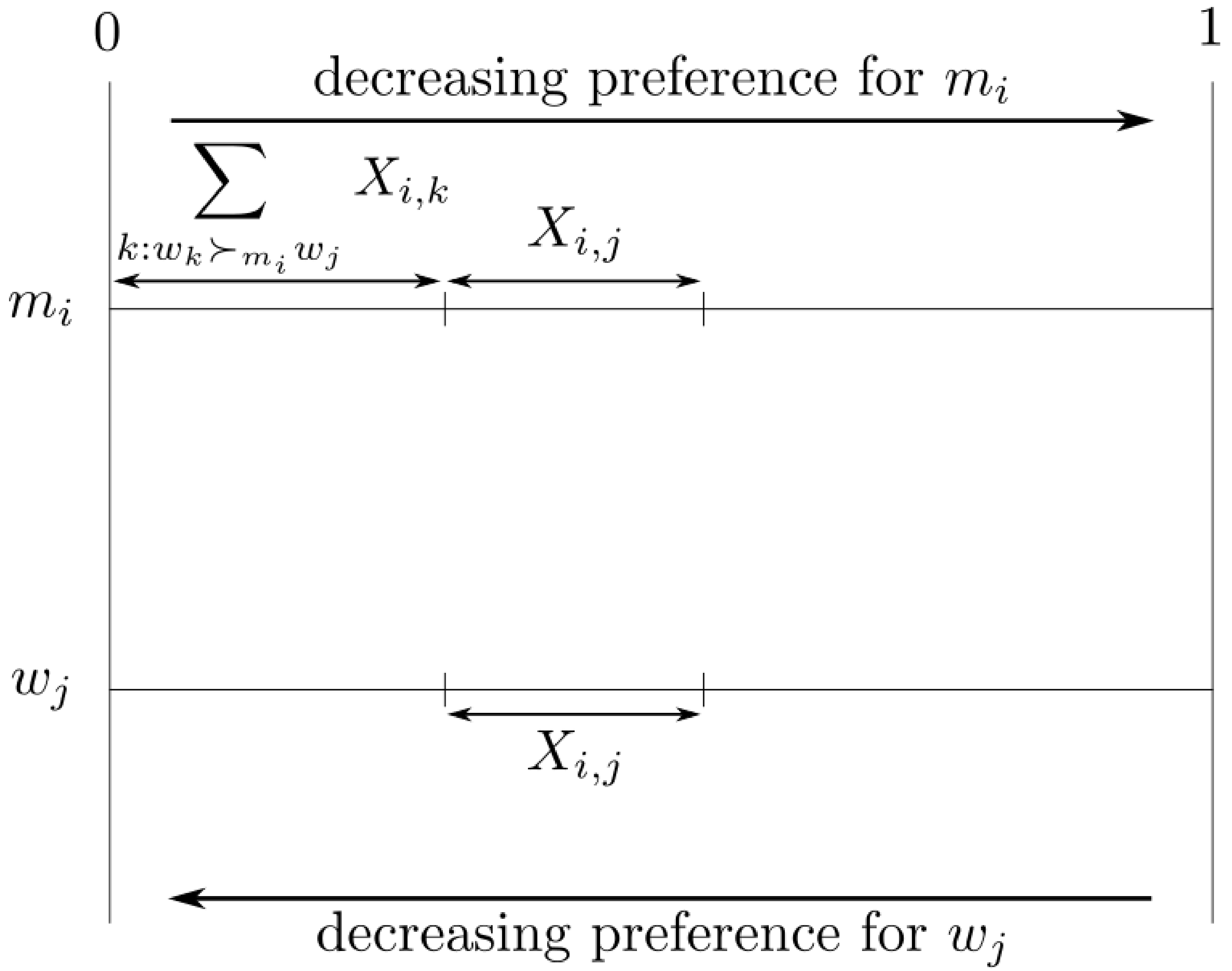

0
1
$X_{i,j}$
m_i

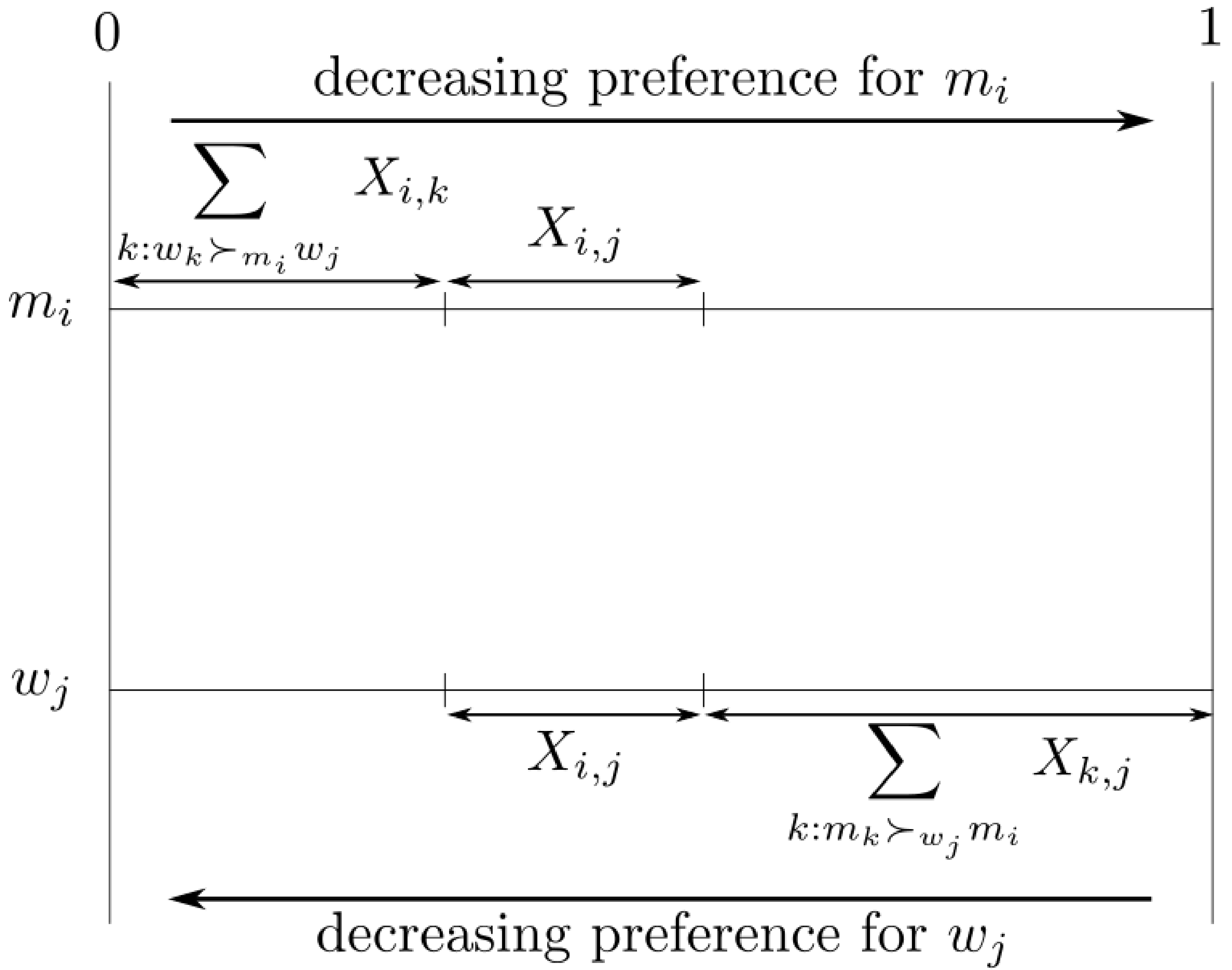

0
1
decreasing preference for m_i
$X_{i,j}$
m_i



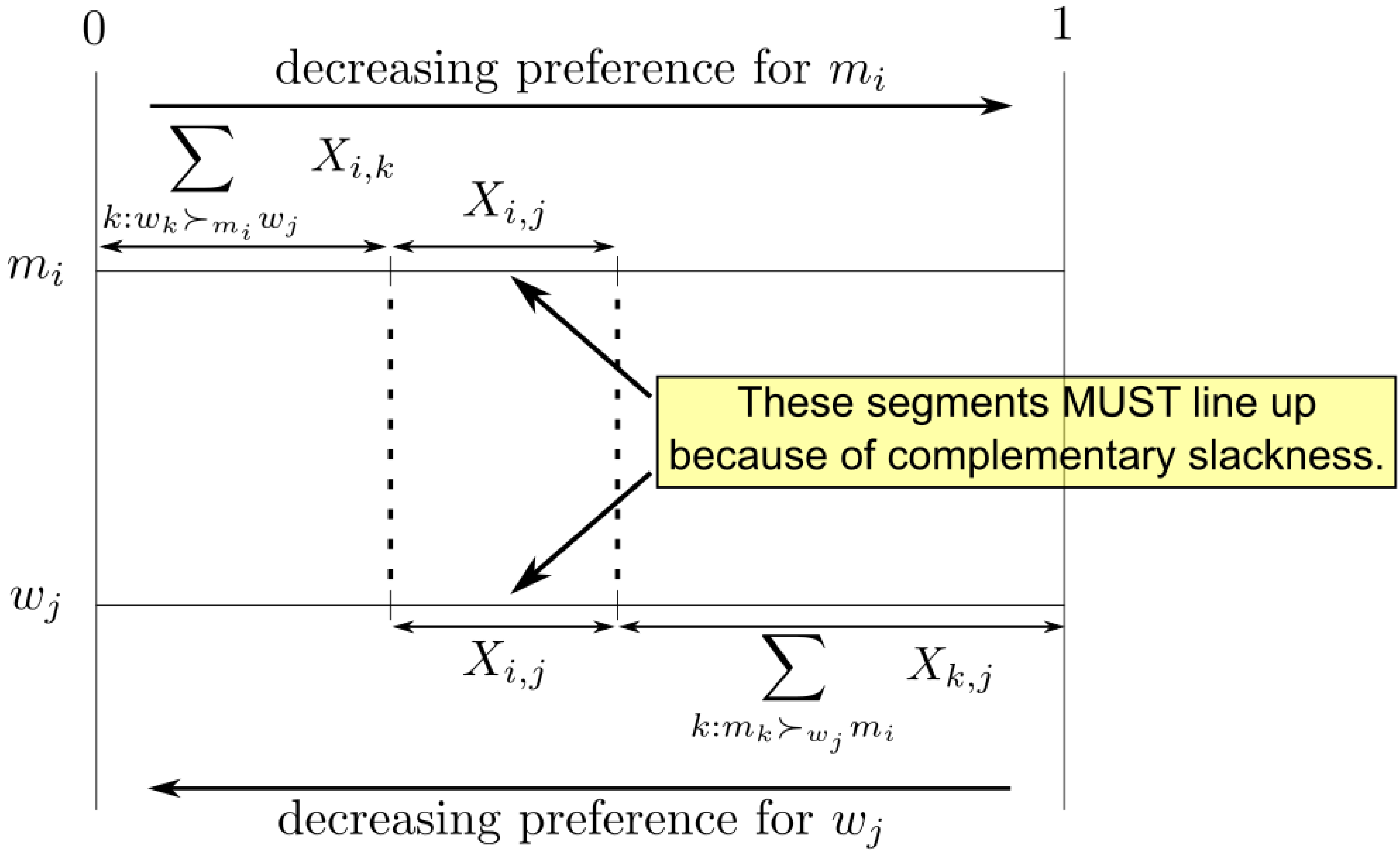
0
1
decreasing preference for m_i
$\sum_{k:w_k \succ_{m_i} w_j} X_{i,k}$
$X_{i,j}$
m_i

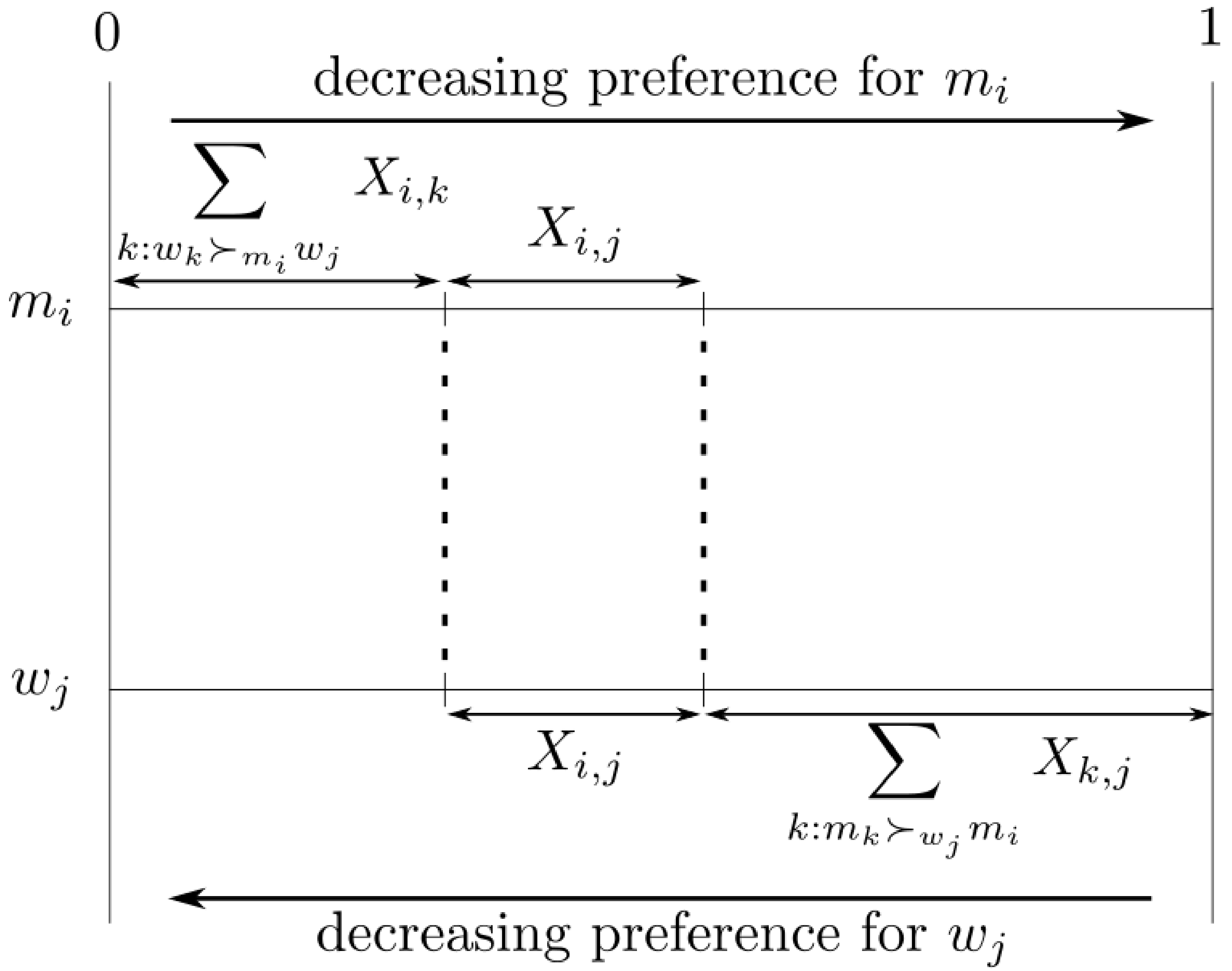

0
1
decreasing preference for m_i
$\sum_{k:w_k \succ_{m_i} w_j} X_{i,k}$
$X_{i,j}$
m_i
w_j

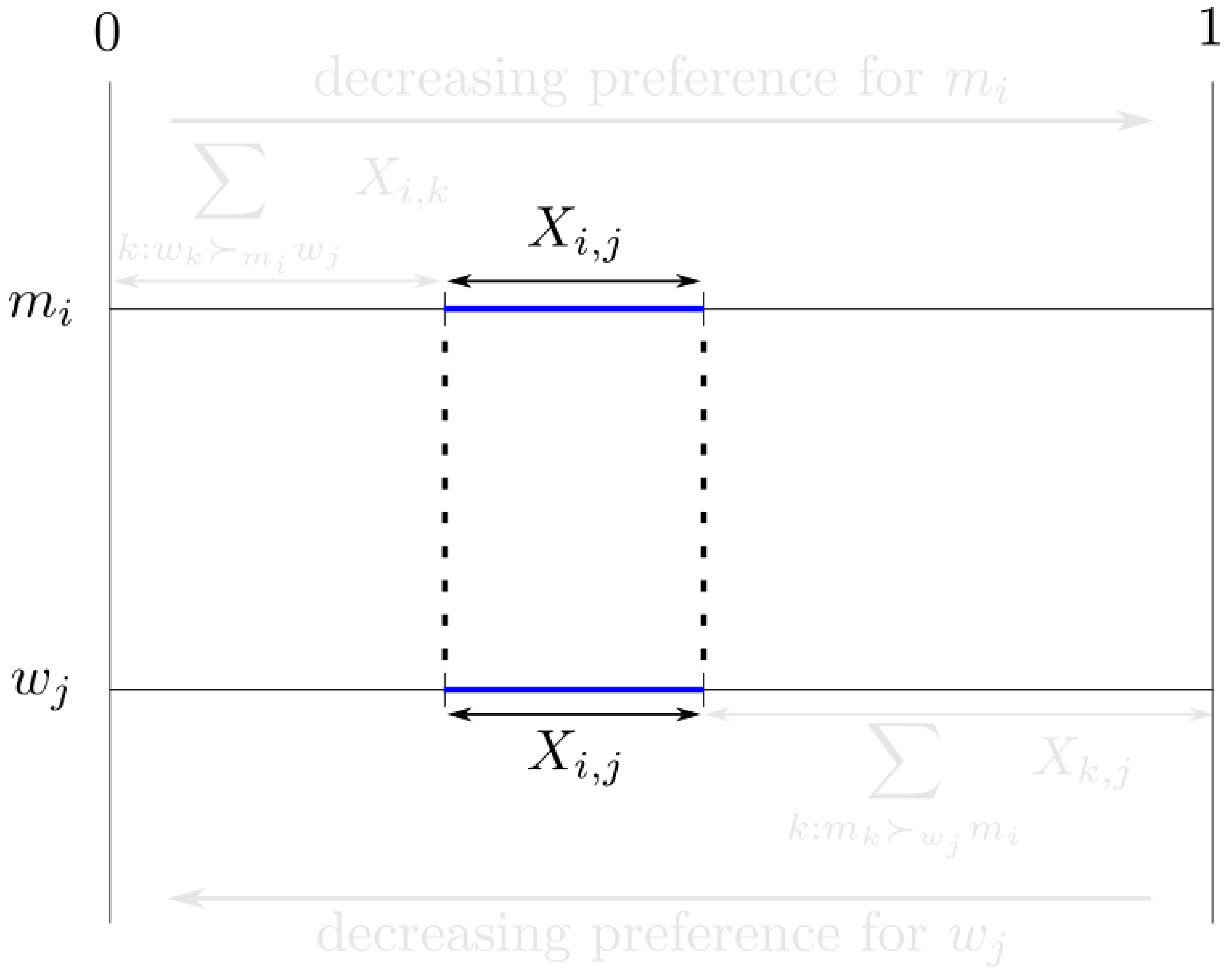

0
1
decreasing preference for m_i
$\sum_{k:w_k \succ_{m_i} w_j} X_{i,k}$
$X_{i,j}$
m_i
w_j
$X_{i,j}$
decreasing preference for w_j

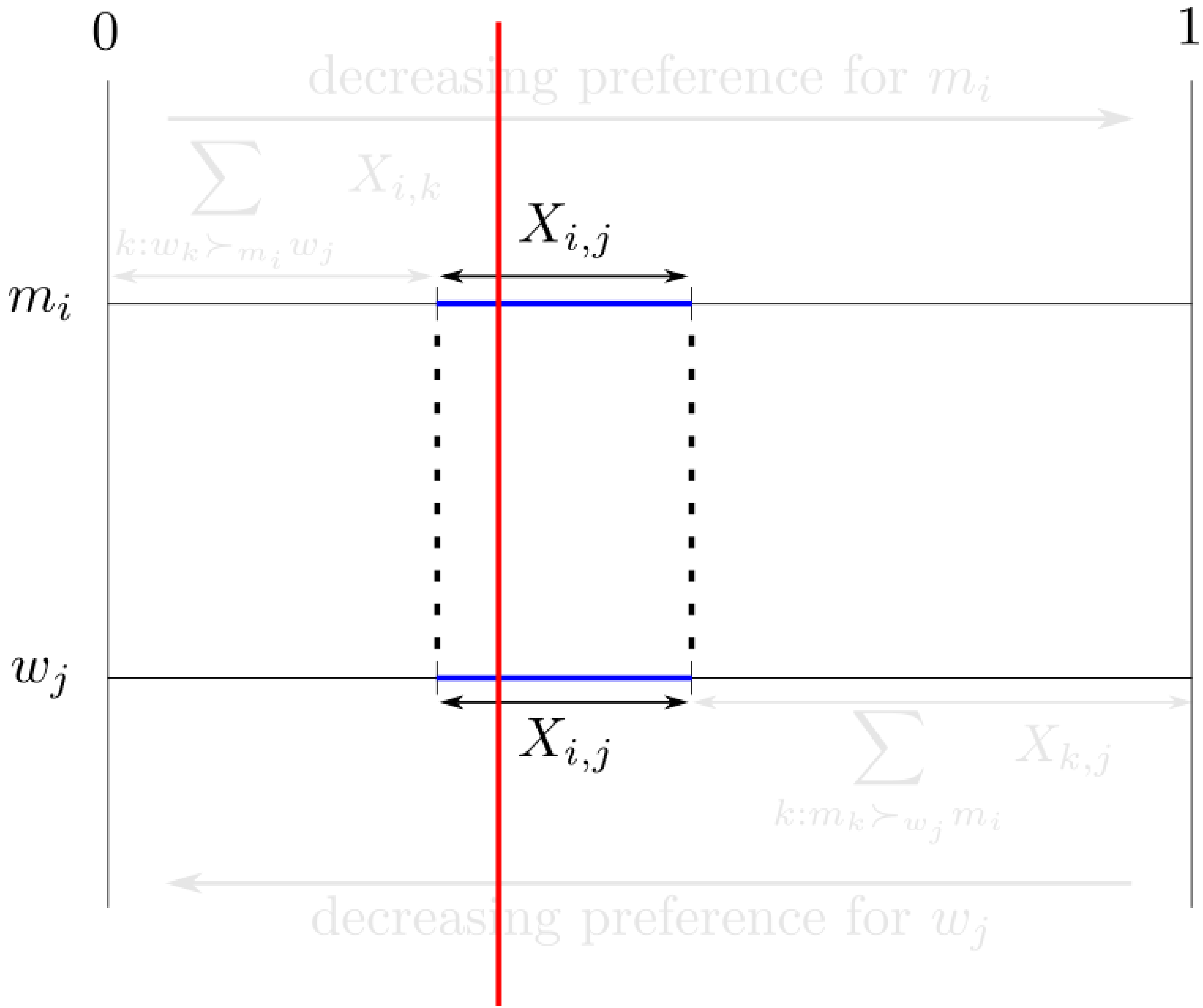


0
1
decreasing preference for m_i
$\sum_{k: w_k \succ_{m_i} w_j} X_{i,k}$
$X_{i,j}$
m_i
w_j
$X_{i,j}$
$\sum_{k: m_k \succ_{w_j} m_i} X_{k,j}$
decreasing preference for w_j

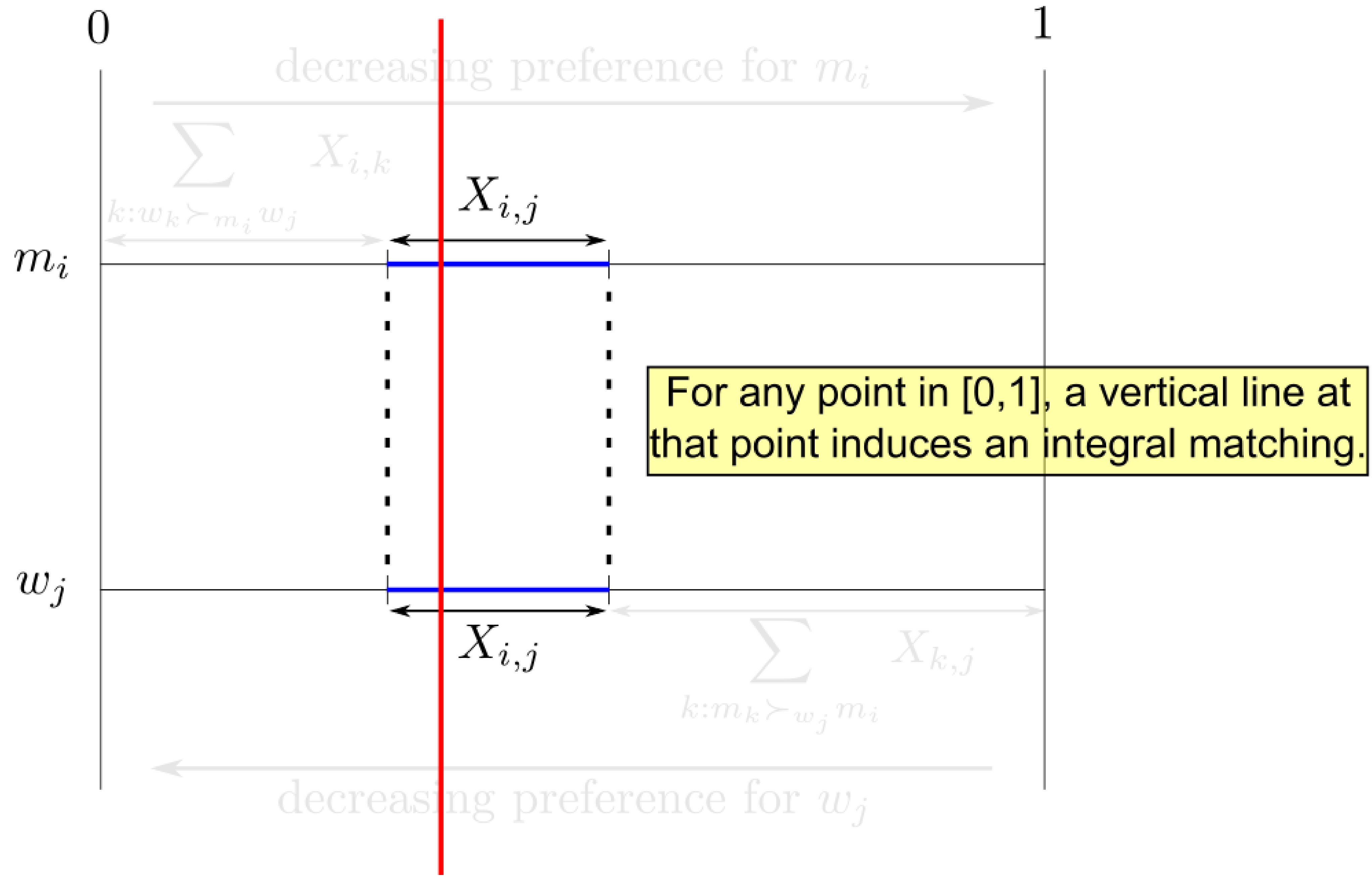

0
1
decreasing preference for m_i
$\sum_{k: w_k \succ_{m_i} w_j} X_{i,k}$
$X_{i,j}$
m_i
These segments MUST line up
because of complementary slackness.
w_j
$X_{i,j}$
$\sum_{k: m_k \succ_{w_j} m_i} X_{k,j}$
decreasing preference for w_j


0
1
decreasing preference for m_i
$\sum_{k: w_k \succ_{m_i} w_j} X_{i,k}$
$X_{i,j}$
m_i
w_j
$X_{i,j}$
$\sum_{k: m_k \succ_{w_j} m_i} X_{k,j}$
decreasing preference for w_j

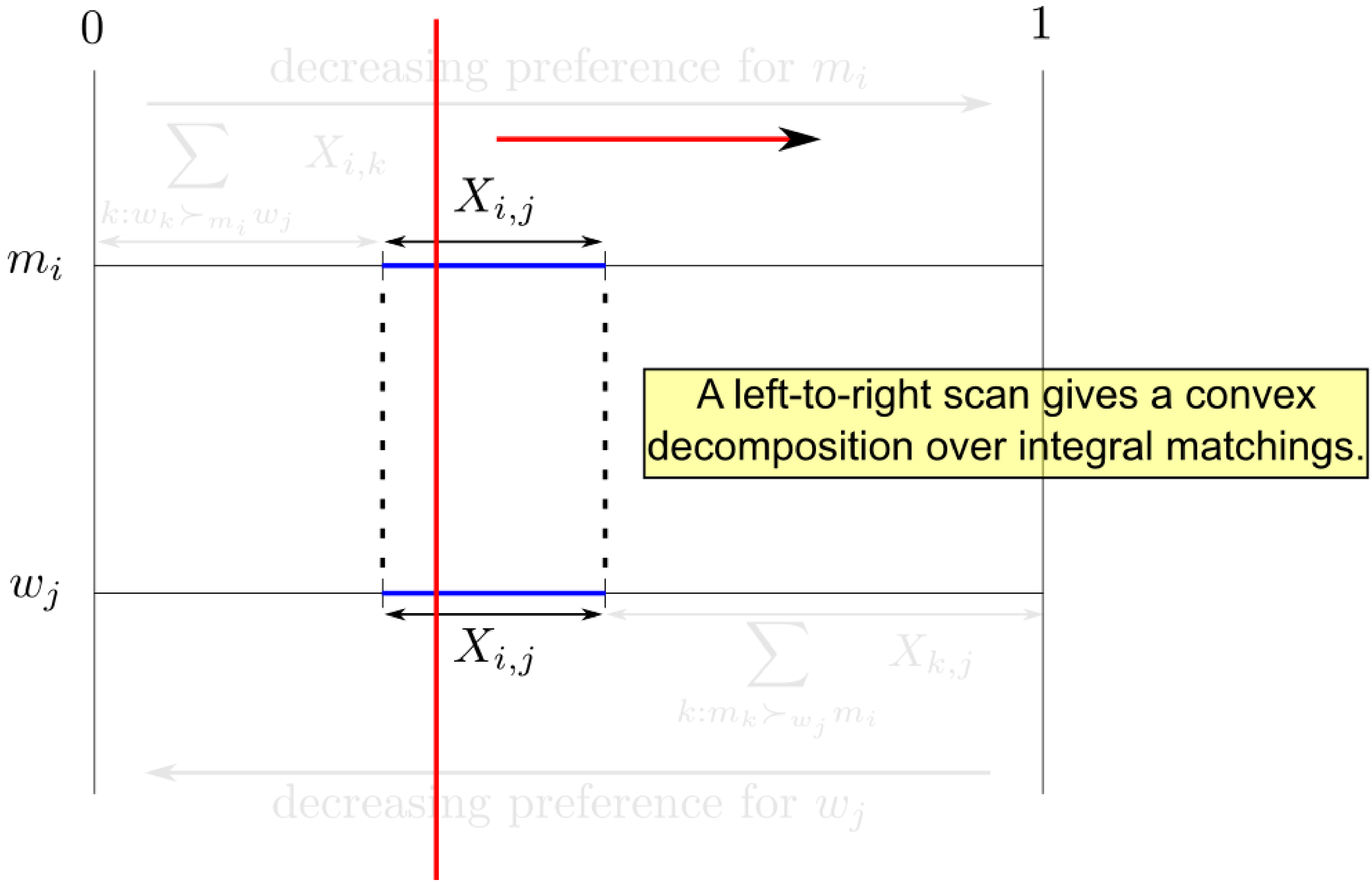

0
1
decreasing preference for m_i
$\sum_{k:w_k \succ_{m_i} w_j} X_{i,k}$
$X_{i,j}$
m_i
w_j
$X_{i,j}$
$\sum_{k:m_k \succ_{w_j} m_i} X_{k,j}$
decreasing preference for w_j

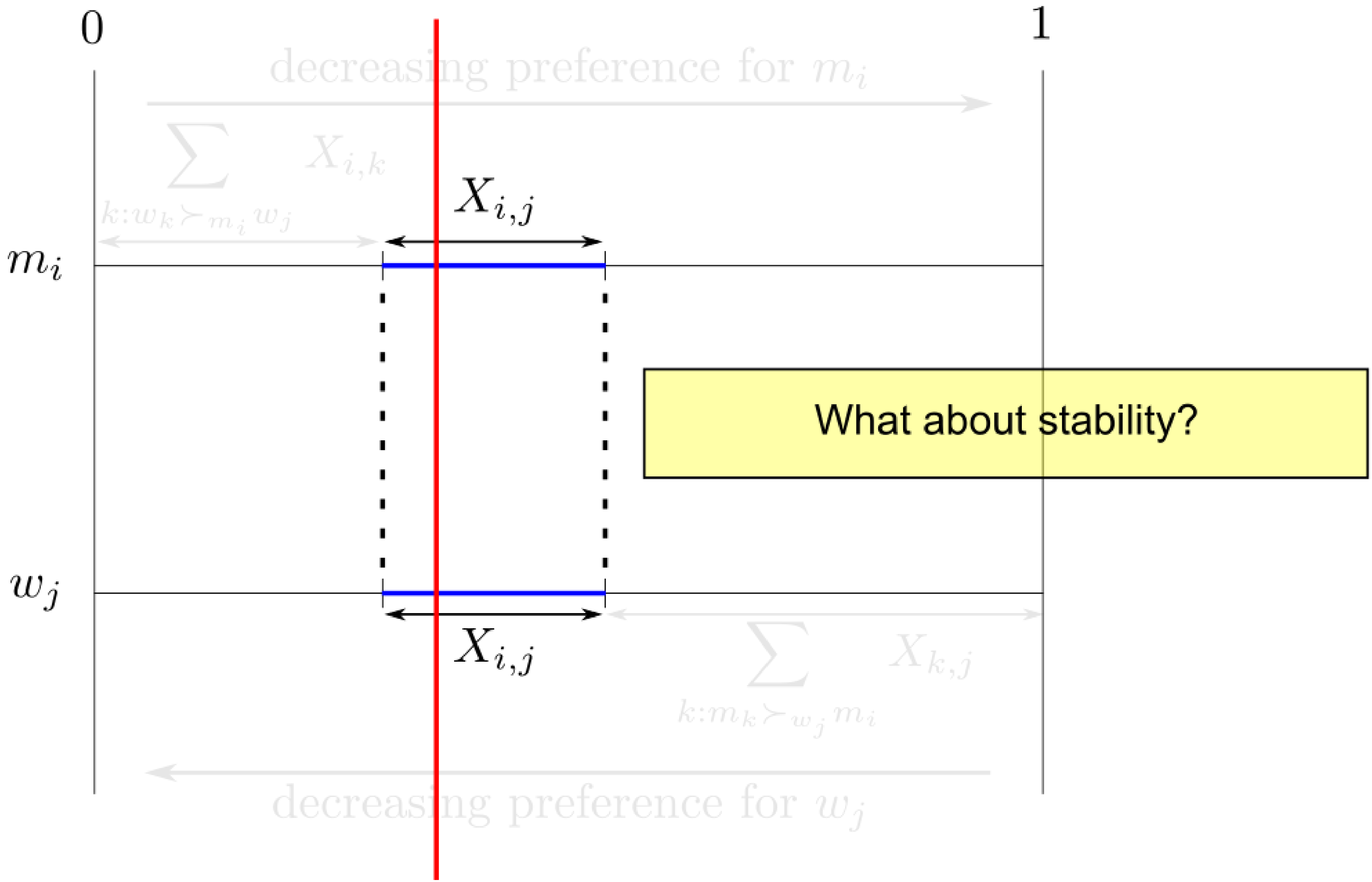


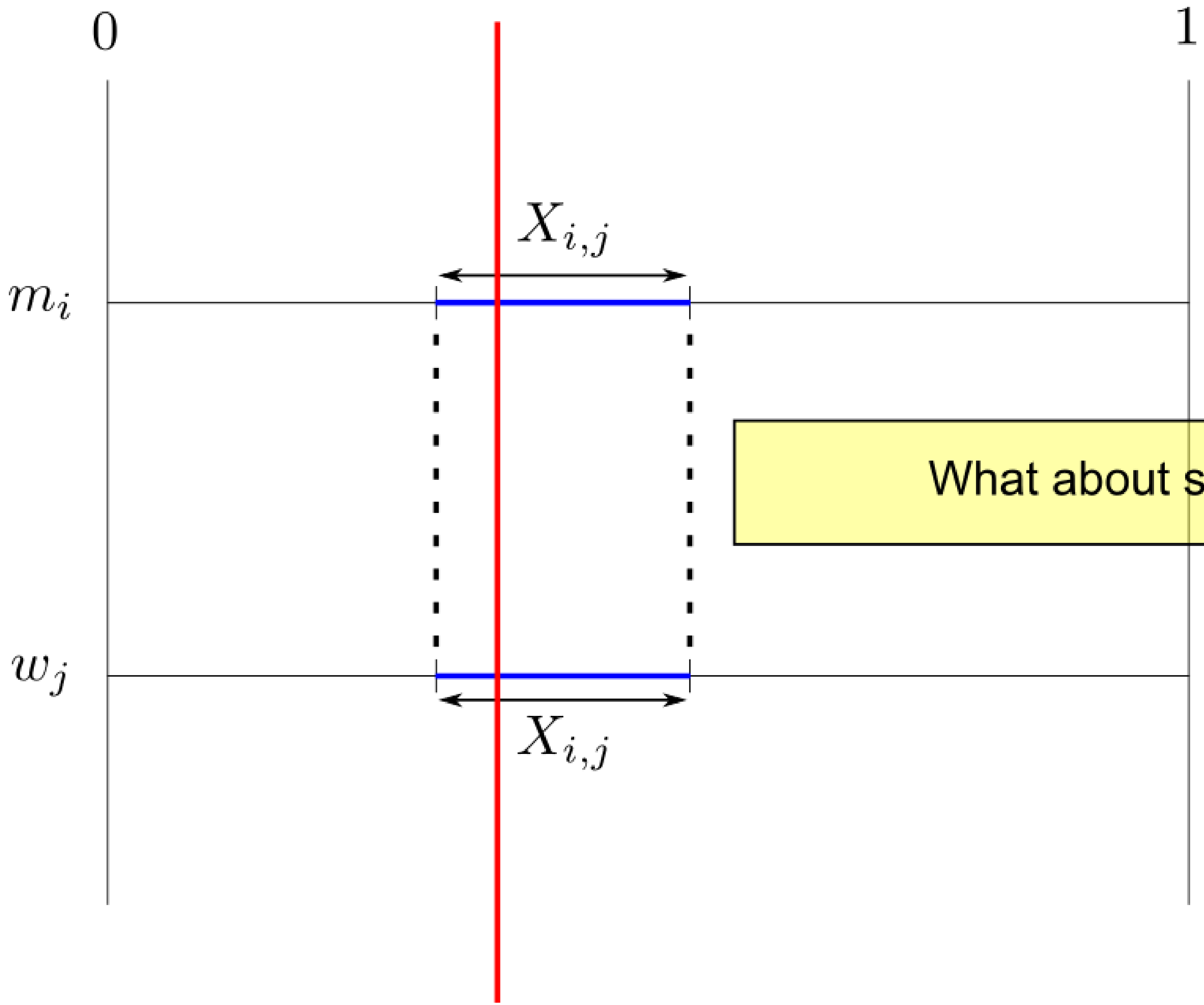
0
1
decreasing preference for m_i
$\sum_{k:w_k \succ_{m_i} w_j} X_{i,k}$
$X_{i,j}$
m_i
w_j
$X_{i,j}$
$\sum_{k:m_k \succ_{w_j} m_i} X_{k,j}$
decreasing preference for w_j



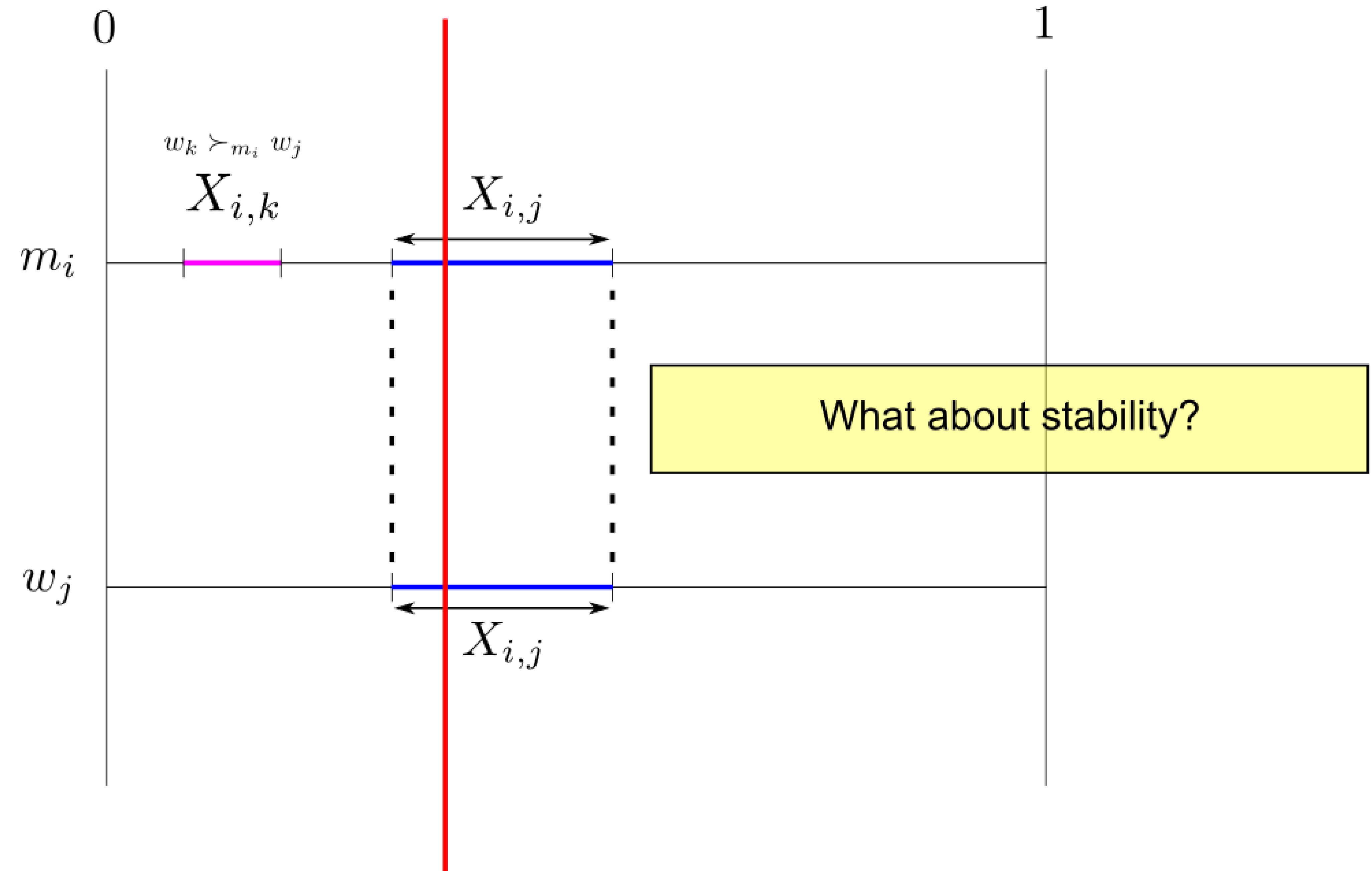
0
1
decreasing preference for m_i
$\sum_{k:w_k \succ_{m_i} w_j} X_{i,k}$
$X_{i,j}$
m_i
w_j
$X_{i,j}$
$\sum_{k:m_k \succ_{w_j} m_i} X_{k,j}$
decreasing preference for w_j
For any point in [0,1], a vertical line at that point induces an integral matching.

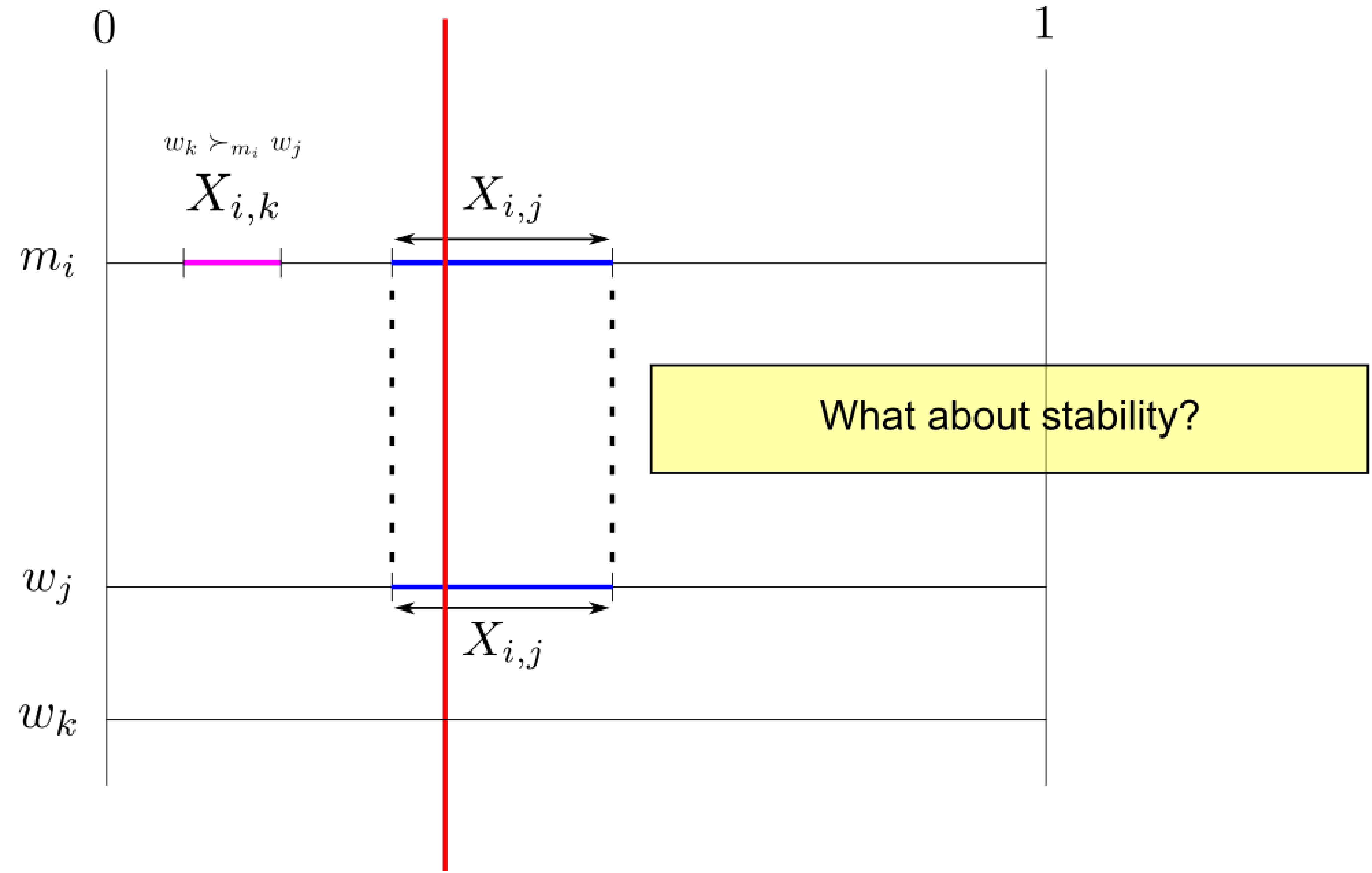

0
1
decreasing preference for m_i
$\sum_{k:w_k \succ_{m_i} w_j} X_{i,k}$
$X_{i,j}$
m_i
w_j
$X_{i,j}$
$\sum_{k:m_k \succ_{w_j} m_i} X_{k,j}$
decreasing preference for w_j
A left-to-right scan gives a convex decomposition over integral matchings.

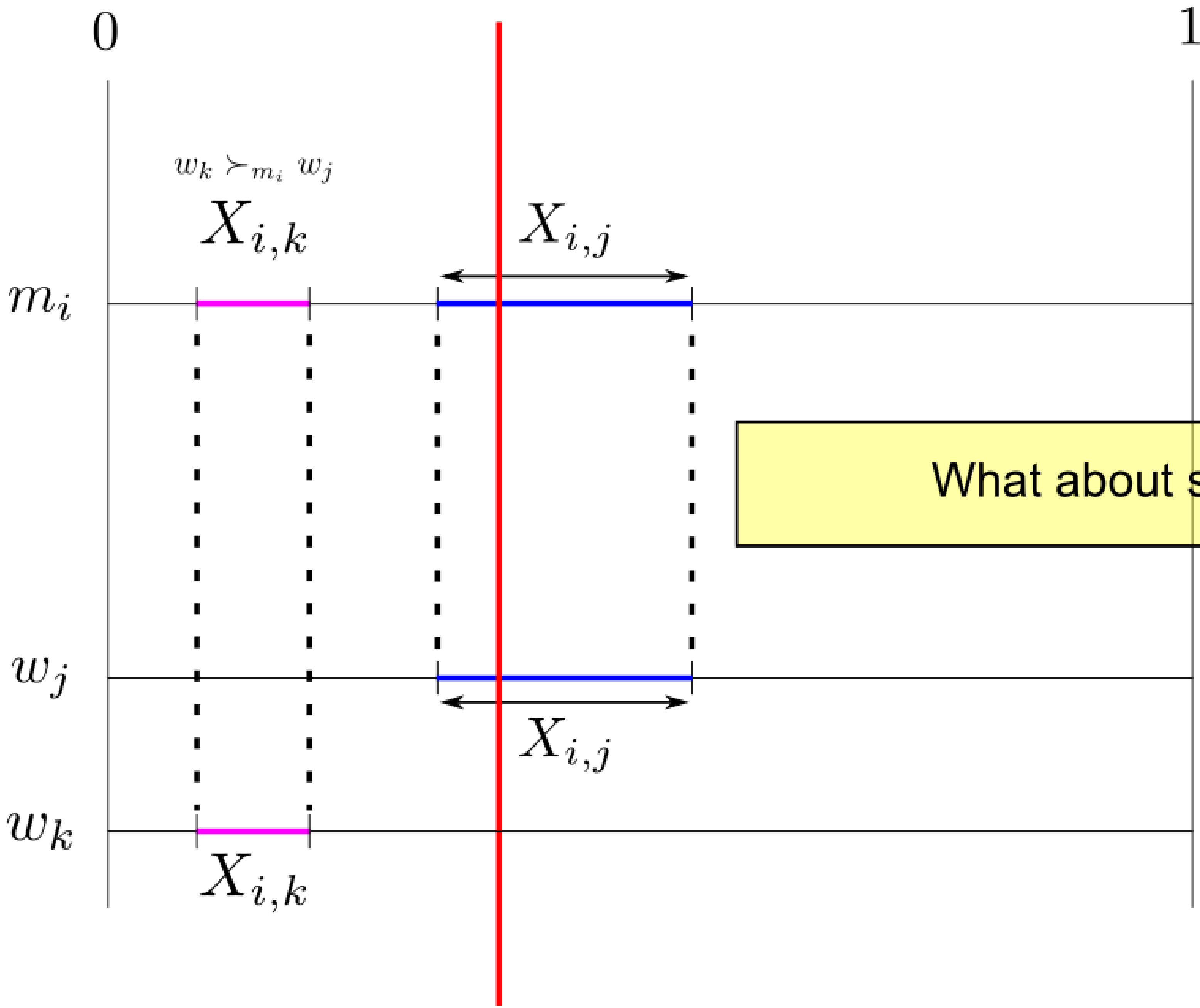

0
1
decreasing preference for m_i
$\sum_{k:w_k \succ_{m_i} w_j} X_{i,k}$
$X_{i,j}$
m_i
What about stability?
w_j
$X_{i,j}$
$\sum_{k:m_k \succ_{w_j} m_i} X_{k,j}$
decreasing preference for w_j



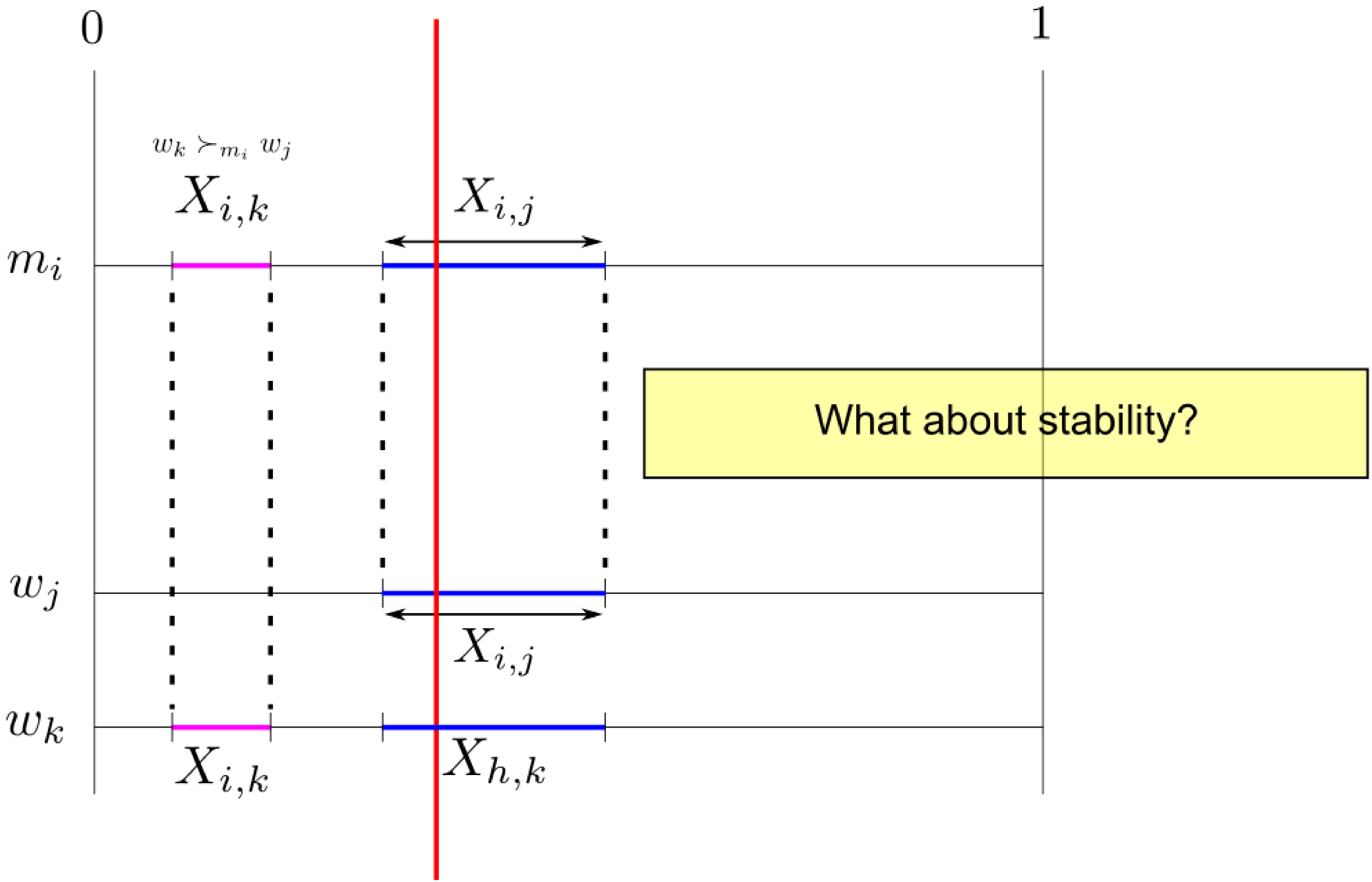
What about stability?

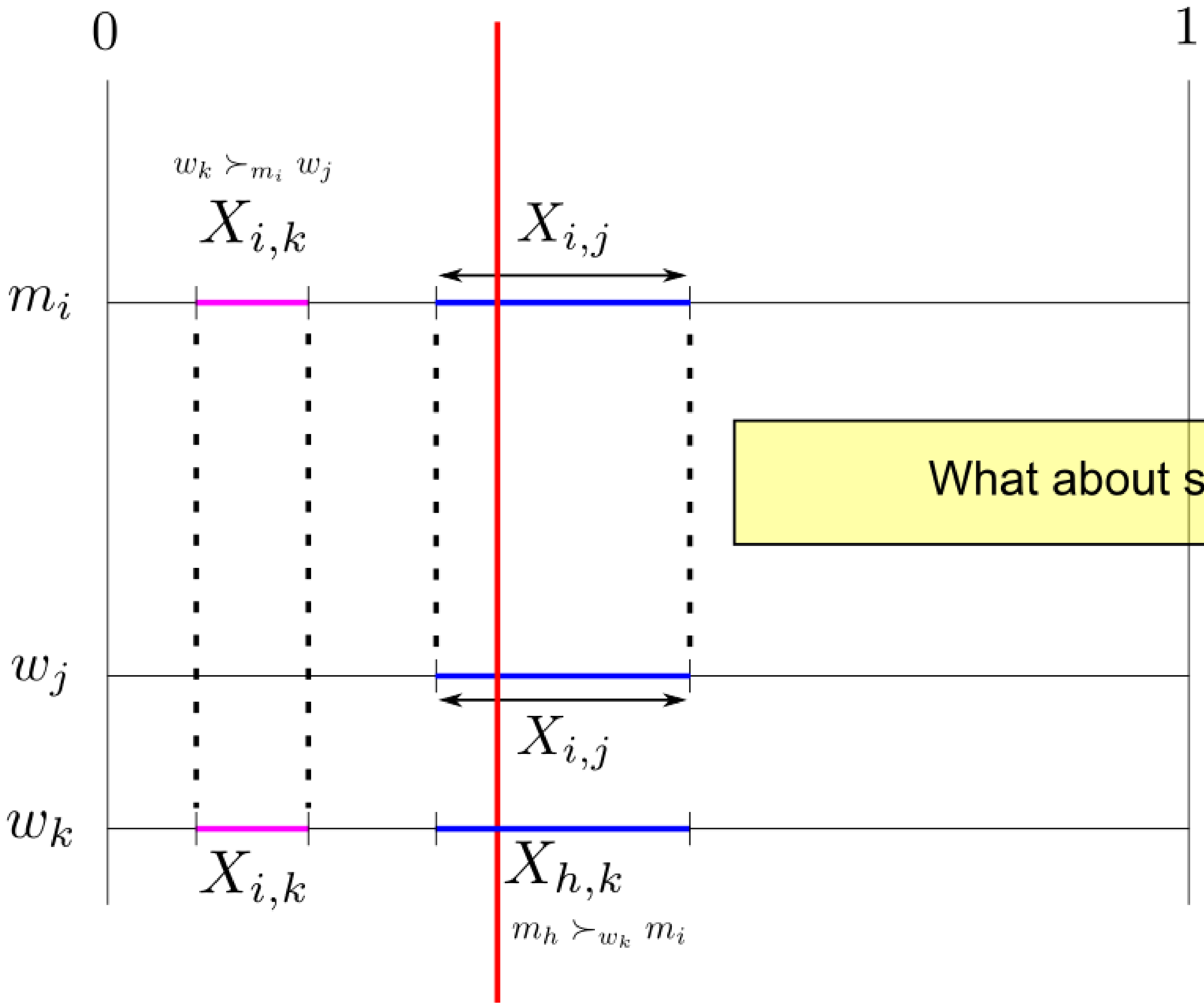

0
1
$w_k \succ_{m_i} w_j$
$X_{i,k}$
$X_{i,j}$
m_i
w_j
$X_{i,j}$
What about stability?


0
1
$w_k \succ_{m_i} w_j$
$X_{i,k}$
$X_{i,j}$
m_i
What about stability?
w_j
$X_{i,j}$
w_k

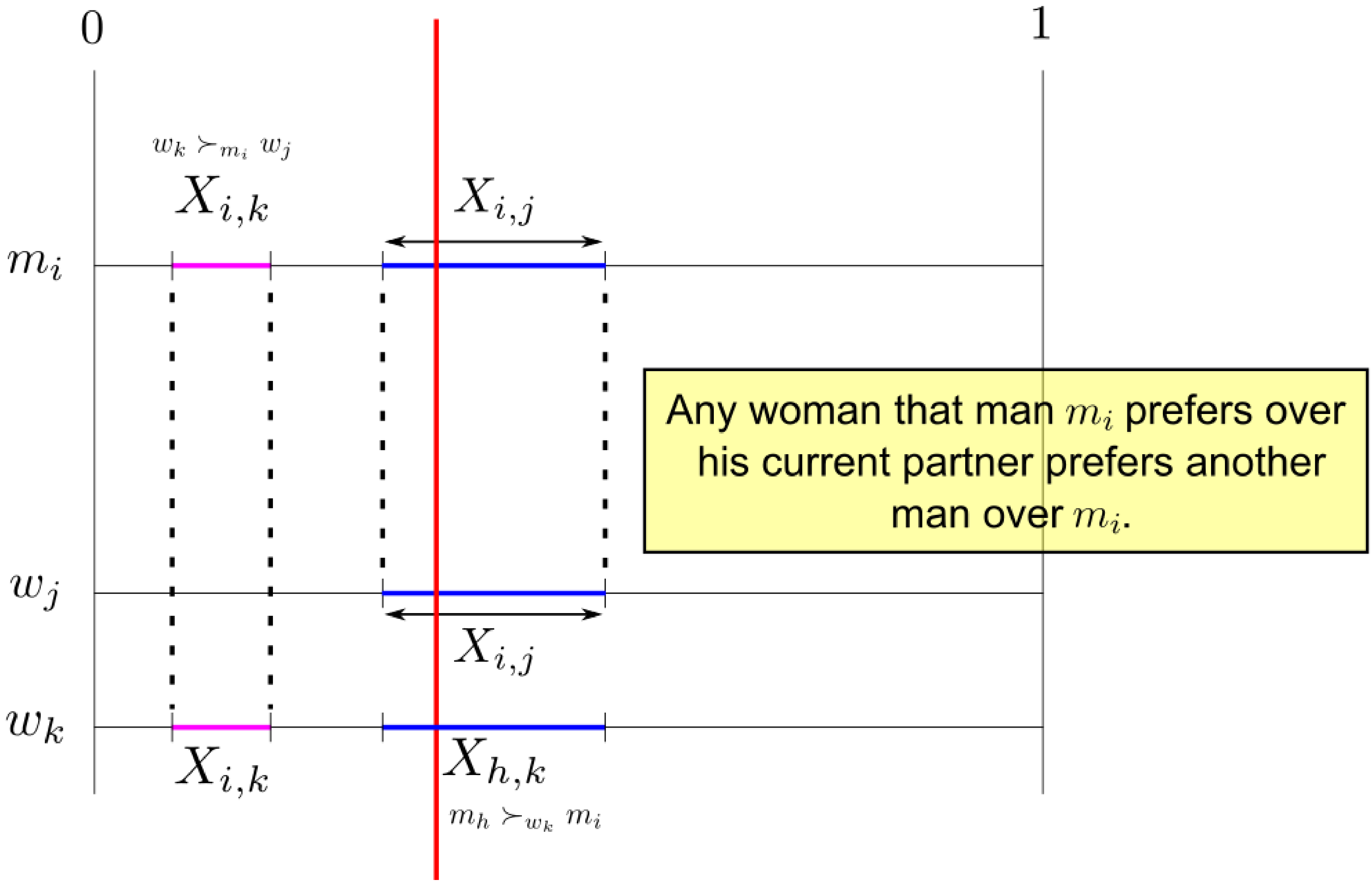


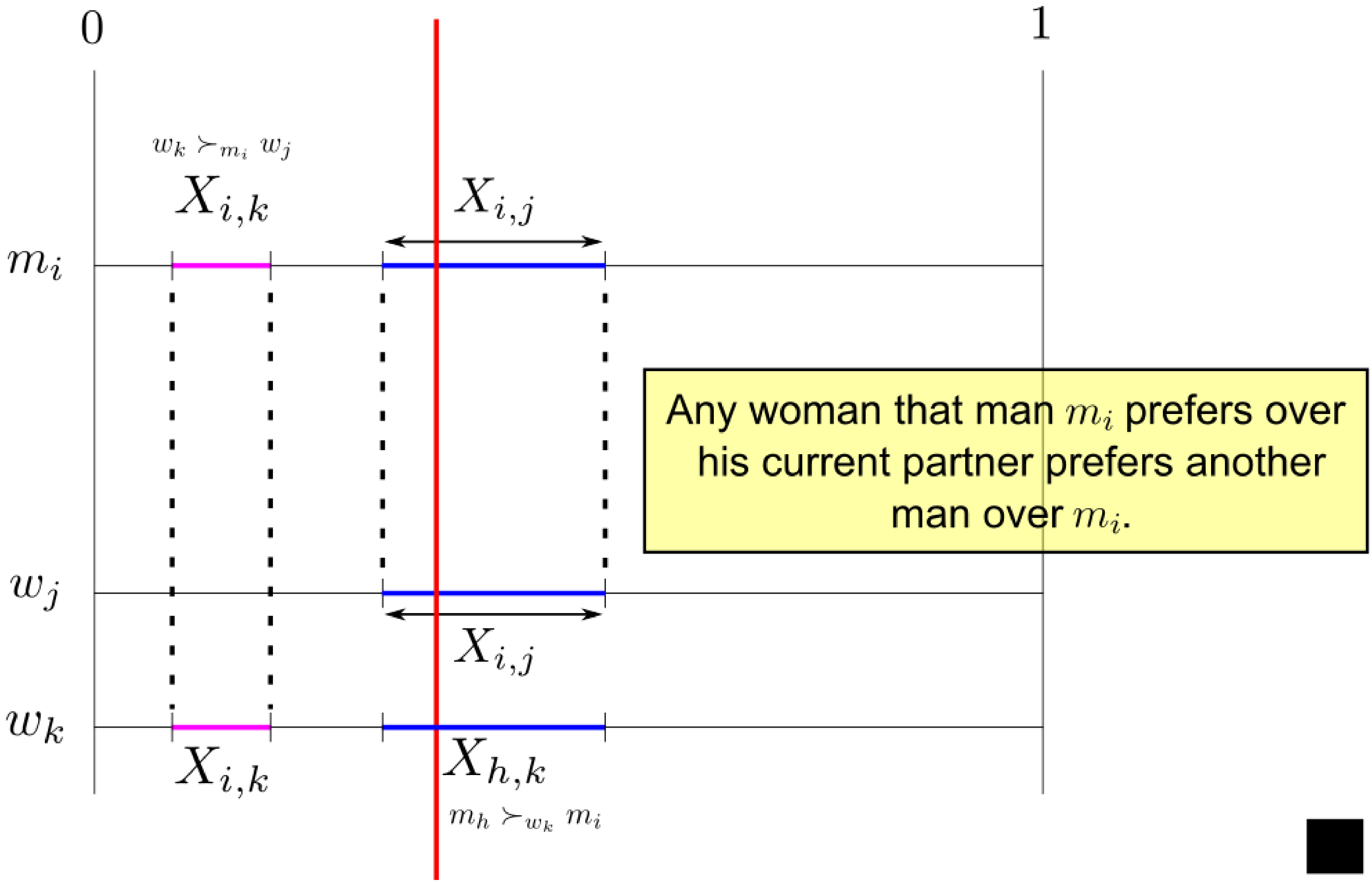
What about stability?


0
1
$w_k \succ_{m_i} w_j$
$X_{i,k}$
$X_{i,j}$
m_i
w_j
$X_{i,j}$
w_k
$X_{i,k}$
$X_{h,k}$
What about stability?



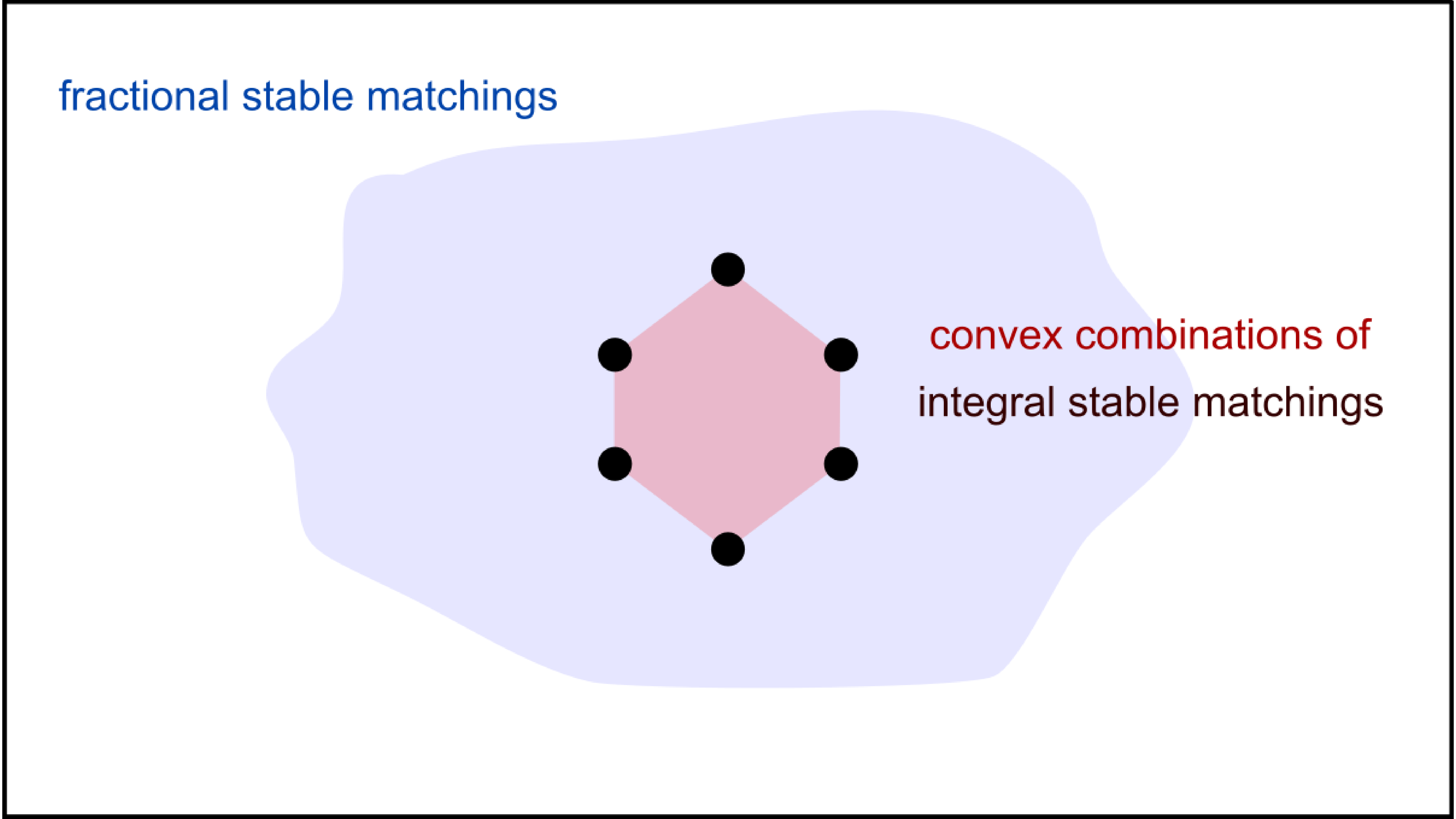
What about stability?

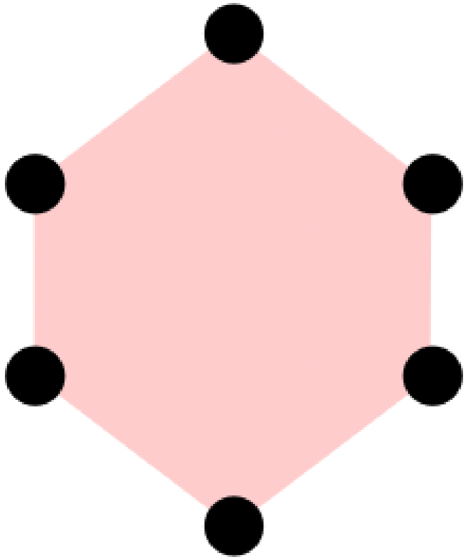

0
1
$w_k \succ_{m_i} w_j$
$X_{i,k}$
$X_{i,j}$
m_i
Any woman that man m_i prefers over his current partner prefers another man over m_i.
w_j
$X_{i,j}$
w_k
$X_{i,k}$
$X_{h,k}$
$m_h \succ_{w_k} m_i$


0
1
$w_k \succ_{m_i} w_j$
$X_{i,k}$
$X_{i,j}$
m_i
w_j
$X_{i,j}$
w_k
$X_{i,k}$
$X_{h,k}$
$m_h \succ_{w_k} m_i$
Any woman that man m_i prefers over his current partner prefers another man over m_i.

[Vande Vate, *Oper. Res. Let.* 1989]

Any fractional stable matching can be expressed as a convex combination of integral stable matchings.


fractional stable matchings
convex combinations of
integral stable matchings



convex combinations of
integral stable matchings
=
fractional stable matchings

Let us use the decomposition technique to show the existence of a "fair" matching.

A "Fair" Stable Matching

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Suppose there are L stable matchings $\mu_1, \ldots, \mu_L$ for a given instance.

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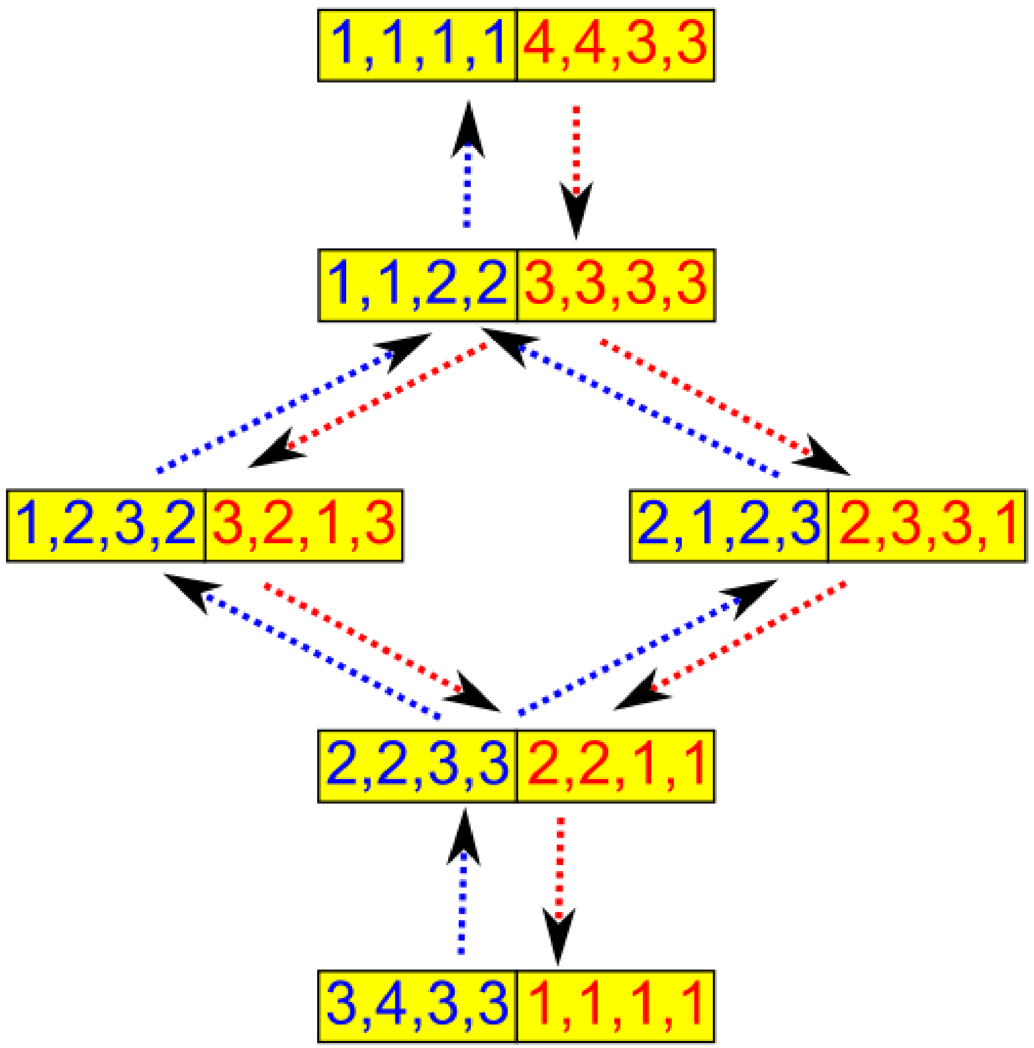
1,1,1,1 4,4,3,3
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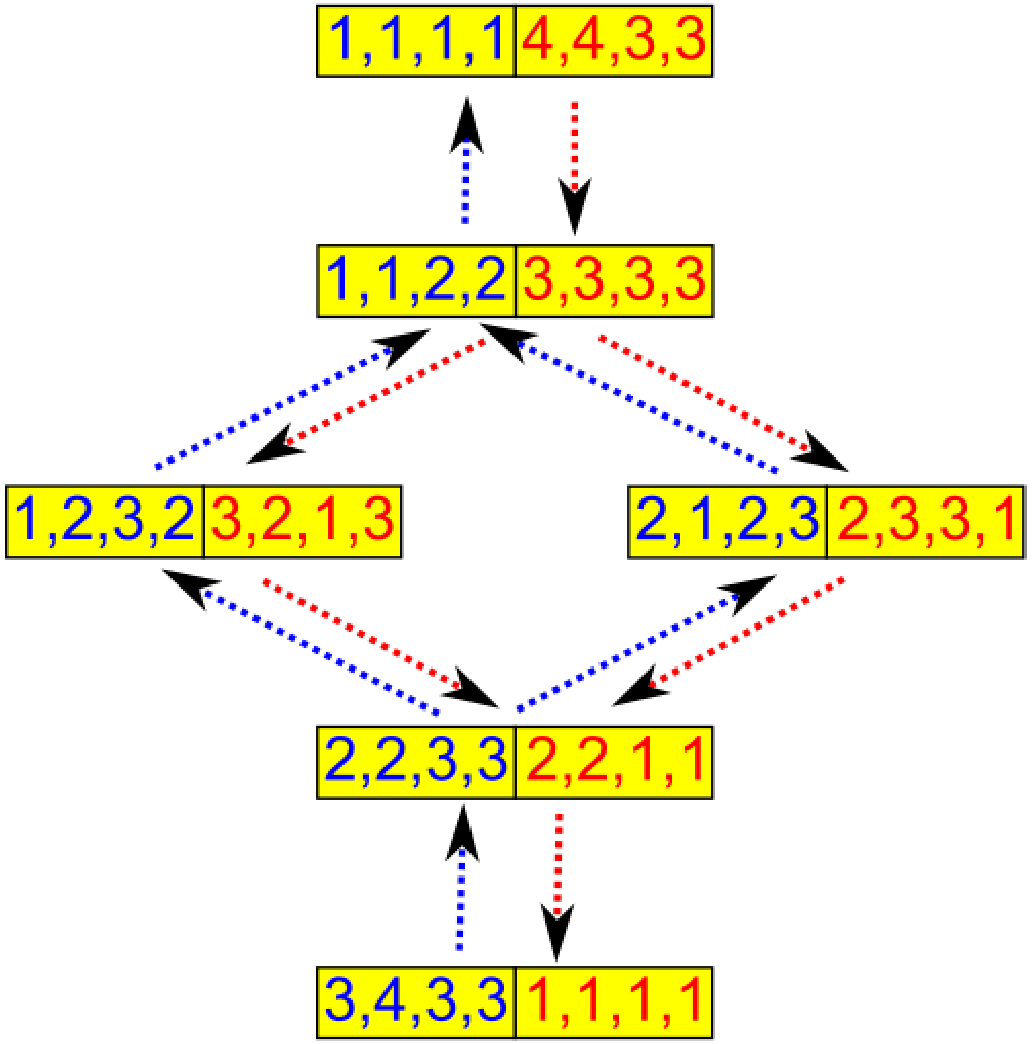
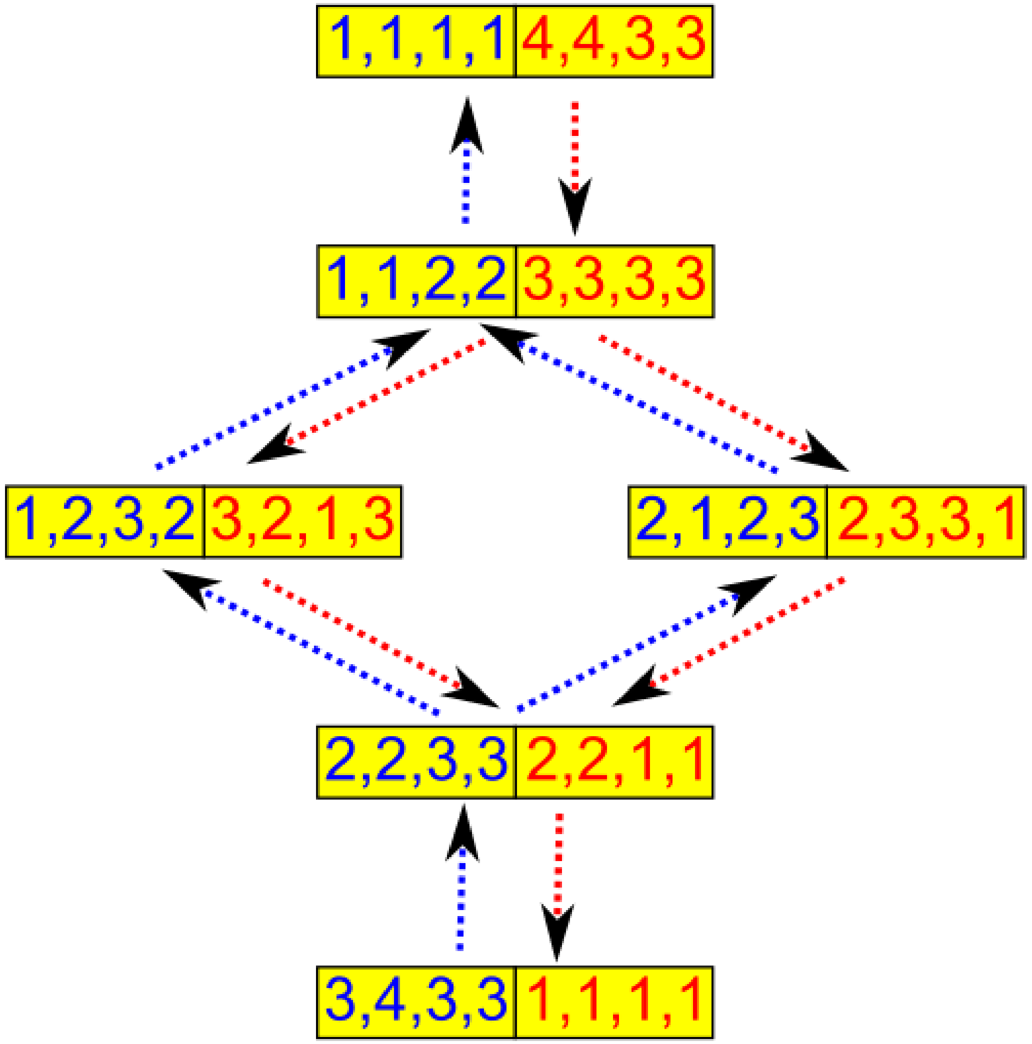
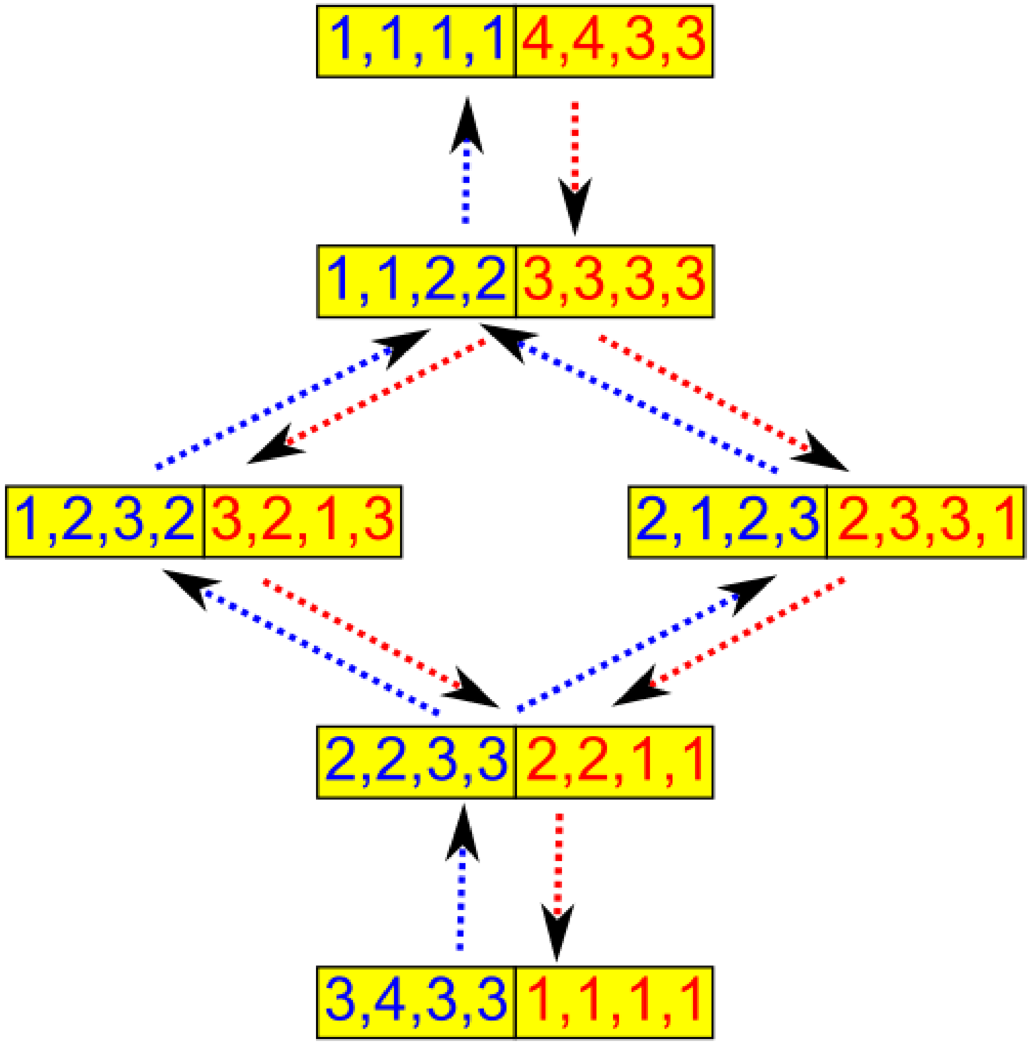


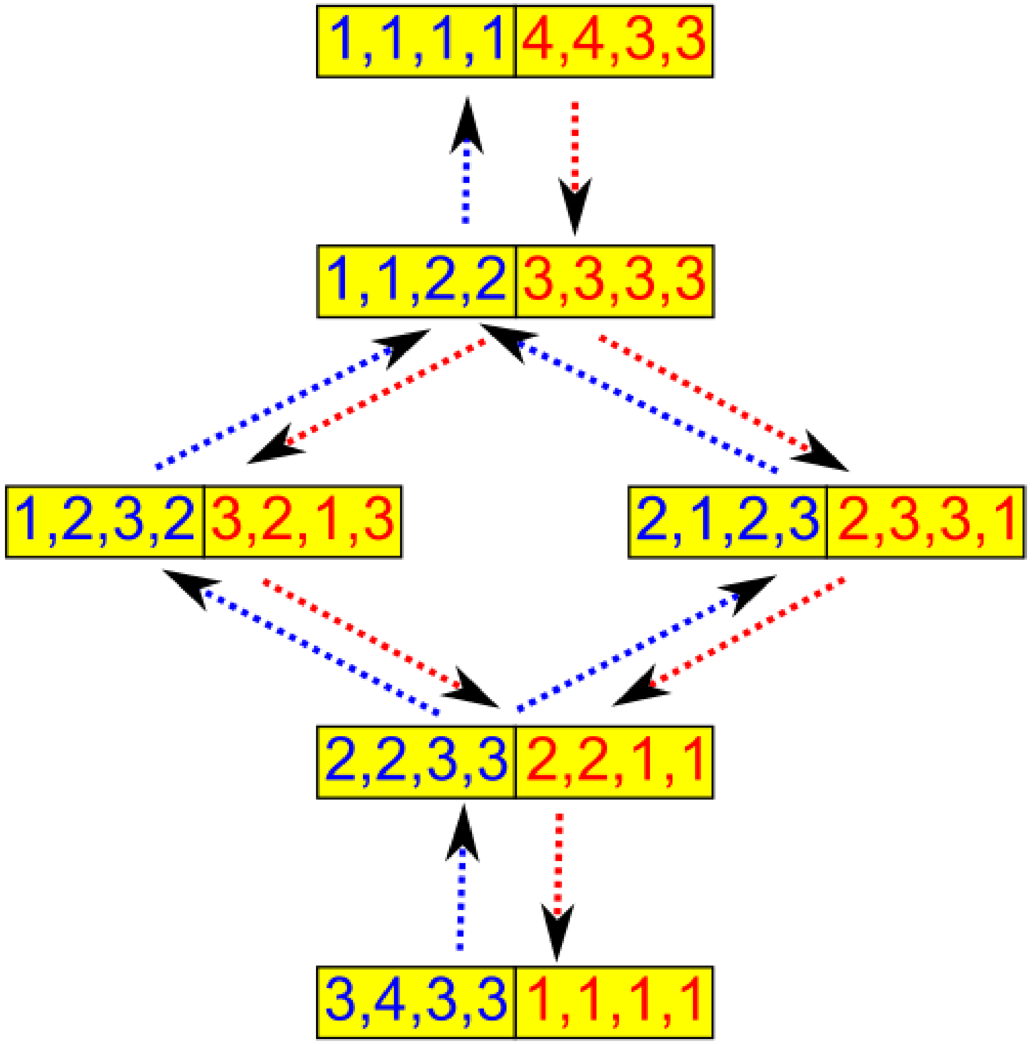
m_1	1	1	1	2	2	3



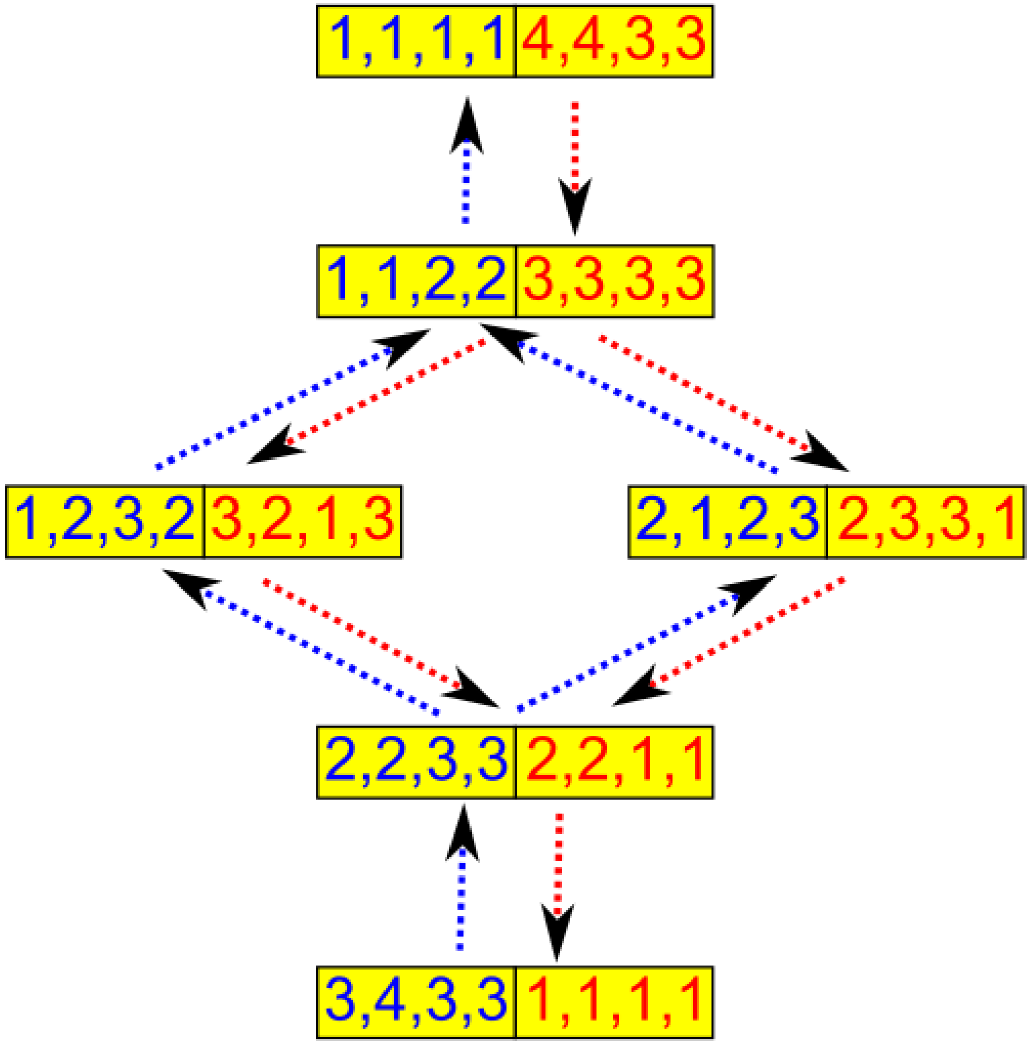
m_1	1	1	1	2	2	3
m_2	1	1	1	2	2	4



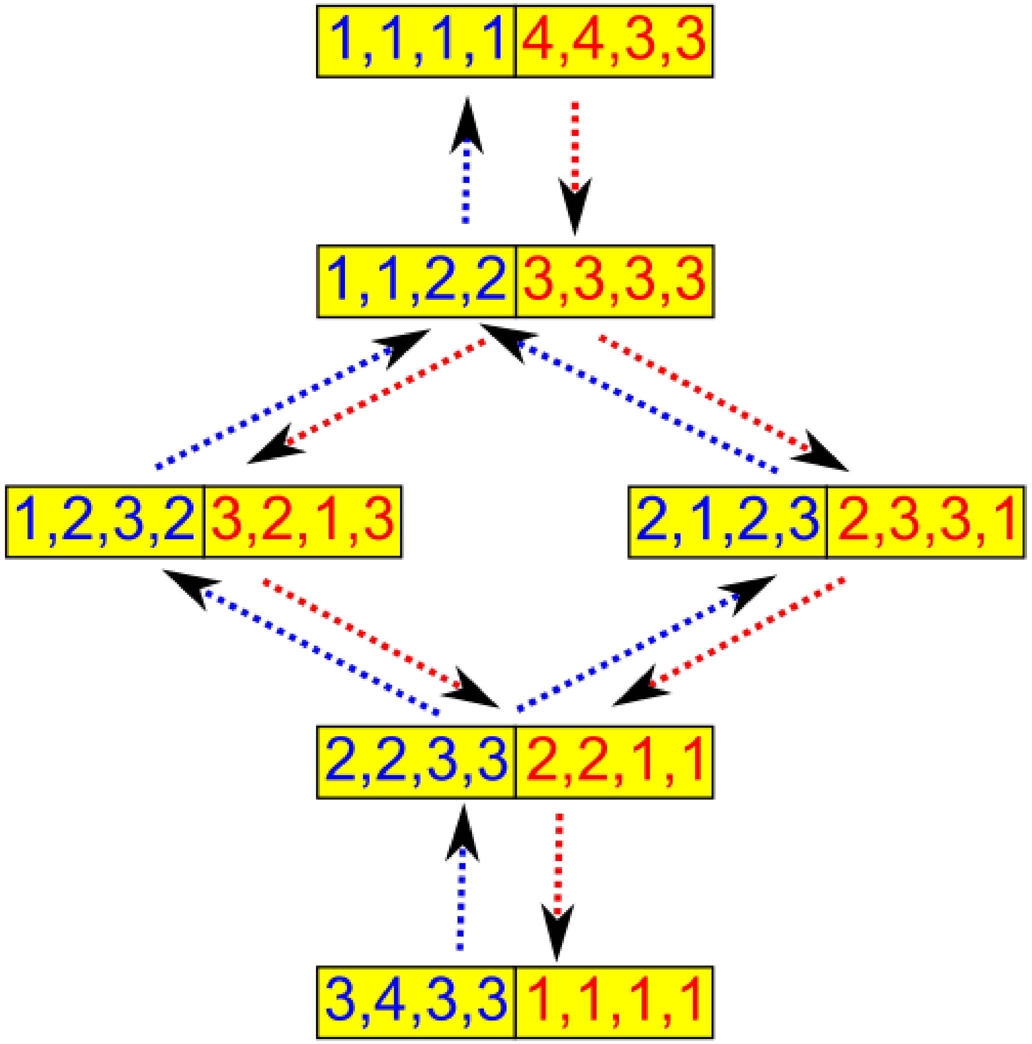
m_1	1	1	1	2	2	3
m_2	1	1	1	2	2	4
m_3	1	2	2	3	3	3
m_4	1	2	2	3	3	3



m_1	1	1	1	2	2	3
m_2	1	1	1	2	2	4
m_3	1	2	2	3	3	3
m_4	1	2	2	3	3	3
w_1	4	3	3	2	2	1
w_2	4	3	3	2	2	1
w_3	3	3	3	1	1	1
w_4	3	3	3	1	1	1



m_1	1	1	1	2	2	3
m_2	1	1	1	2	2	4
m_3	1	2	2	3	3	3
m_4	1	2	2	3	3	3
w_1	4	3	3	2	2	1
w_2	4	3	3	2	2	1
w_3	3	3	3	1	1	1
w_4	3	3	3	1	1	1



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w_3	3	3	3	1	1	1
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[Teo and Sethuraman, *MOR* 1998]

The median mapping induces a stable matching.

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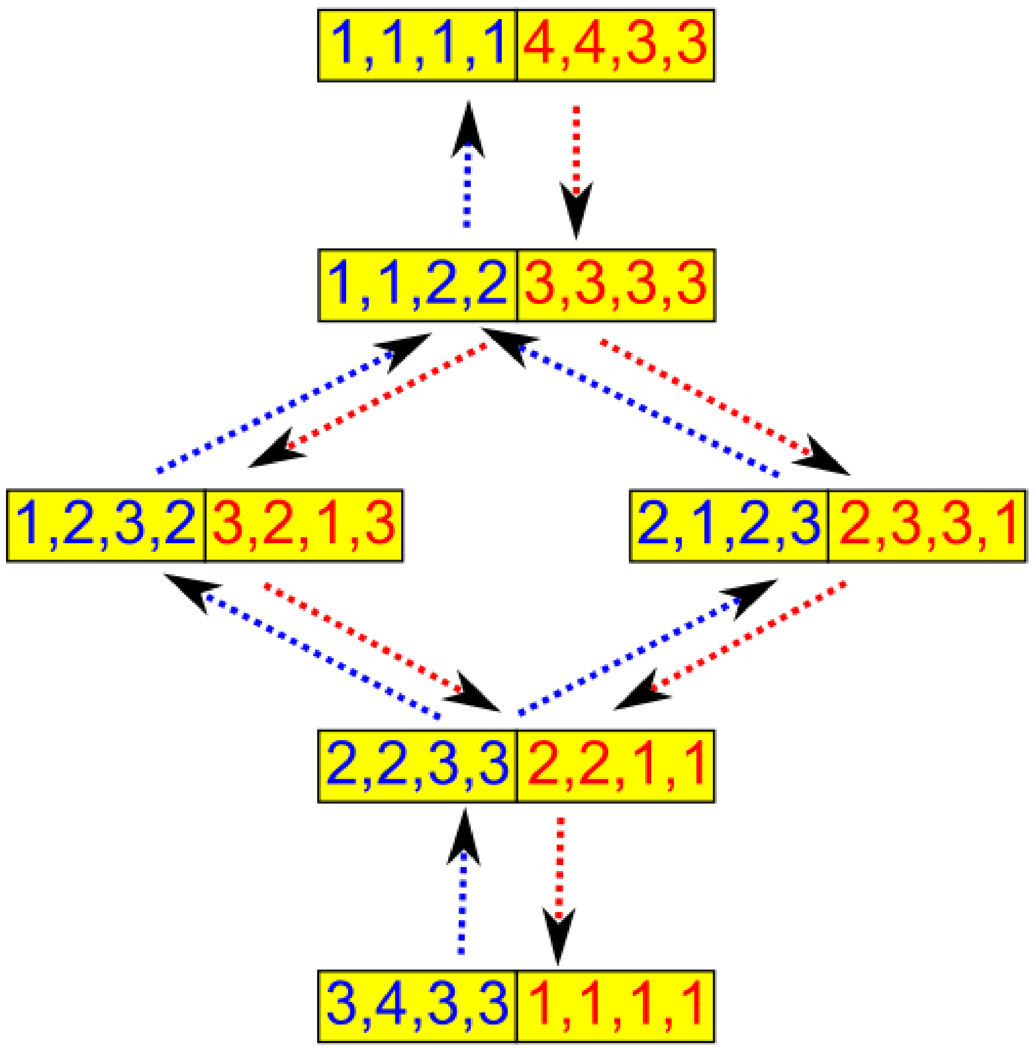
Proof by example.



1,1,1,1 4,4,3,3
1,1,2,2 3,3,3,3
1,2,3,2 3,2,1,3
2,1,2,3 2,3,3,1
2,2,3,3 2,2,1,1
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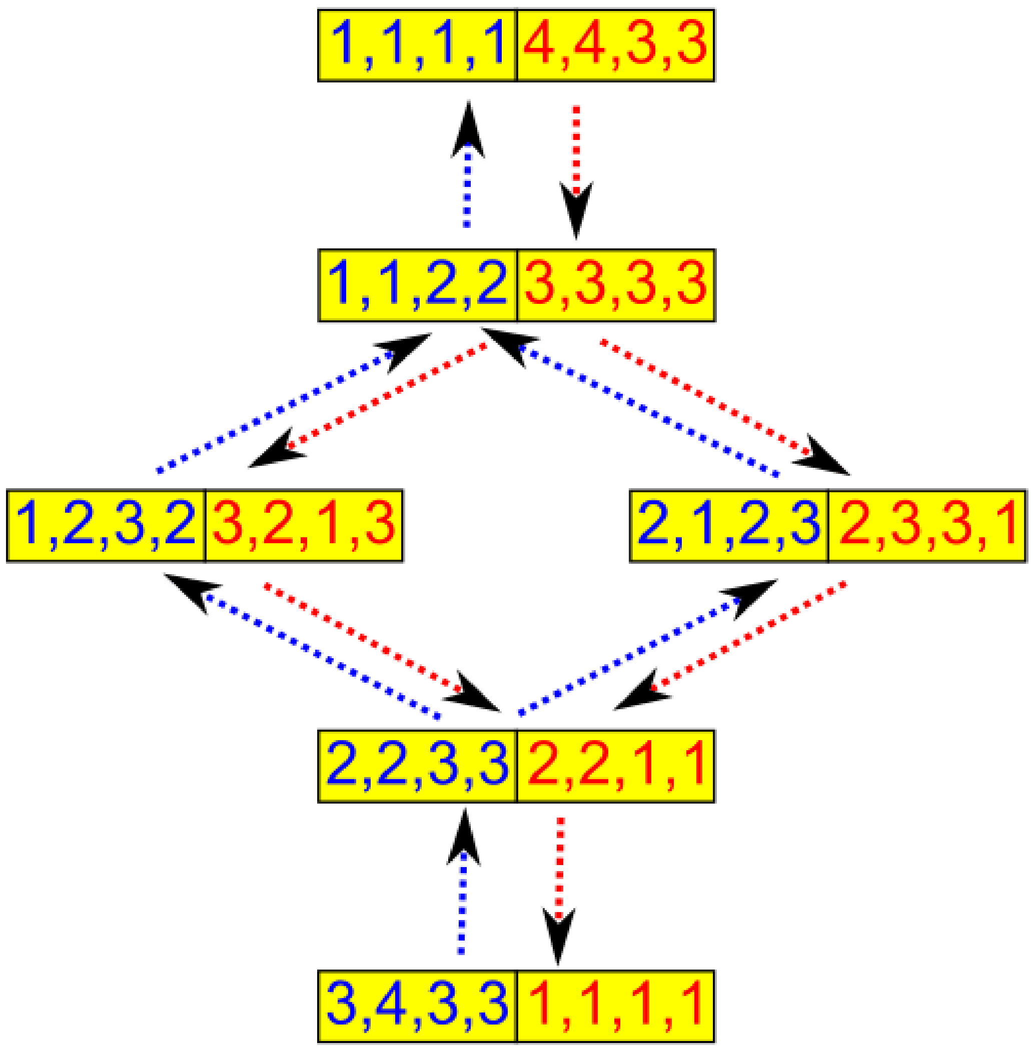
Consider a uniform combination of all integral stable matchings.



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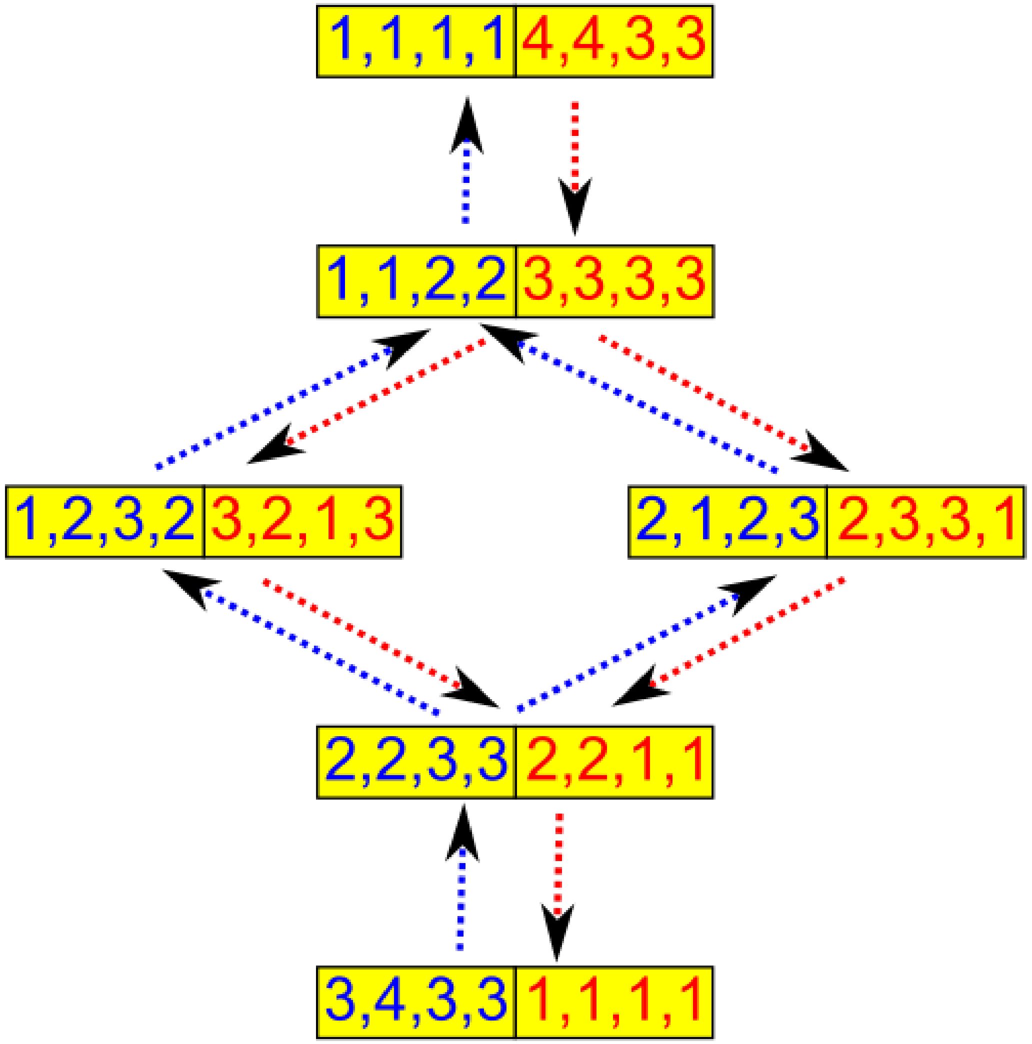

m_1	1	1	1	2	2	3

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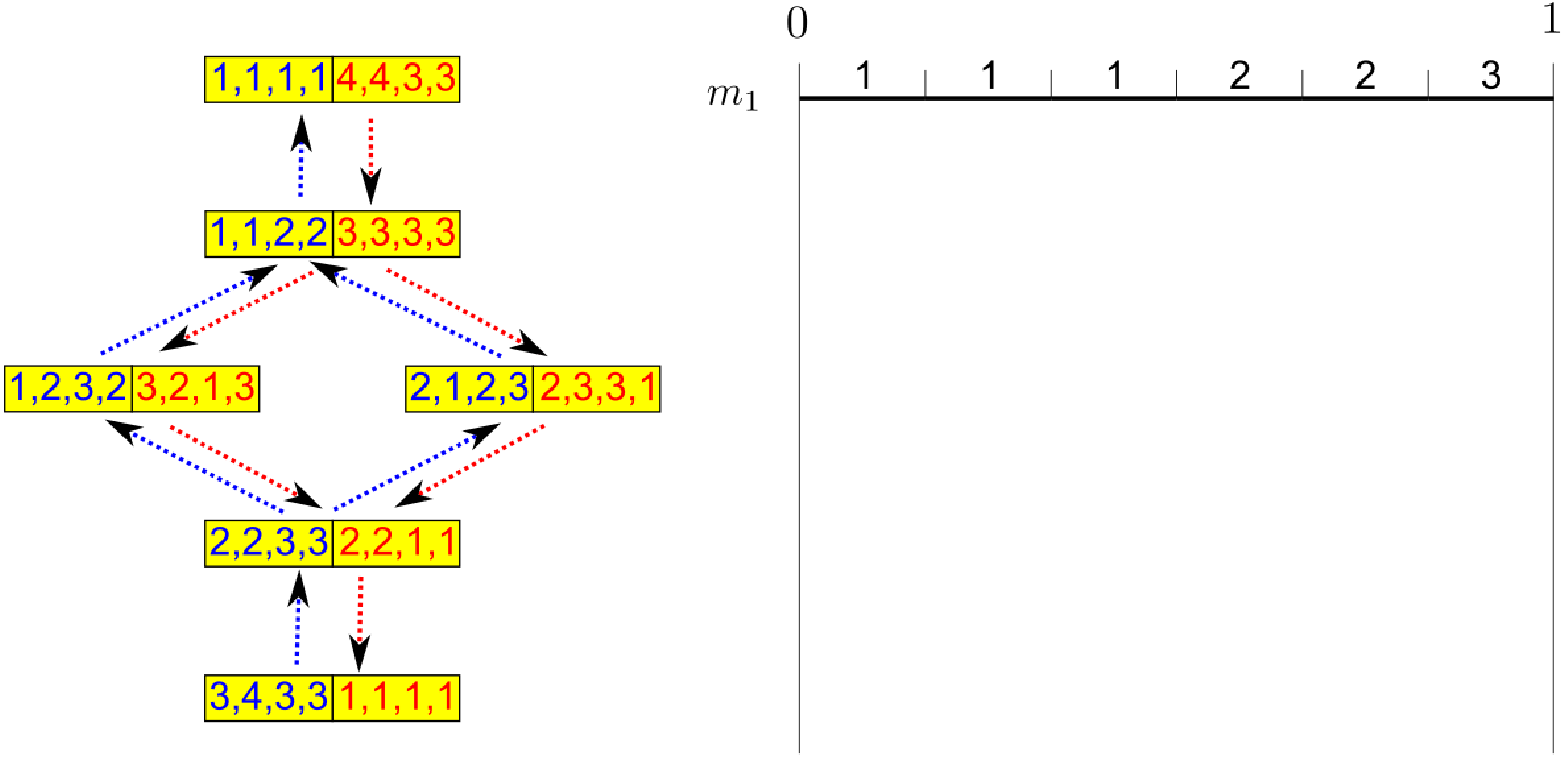

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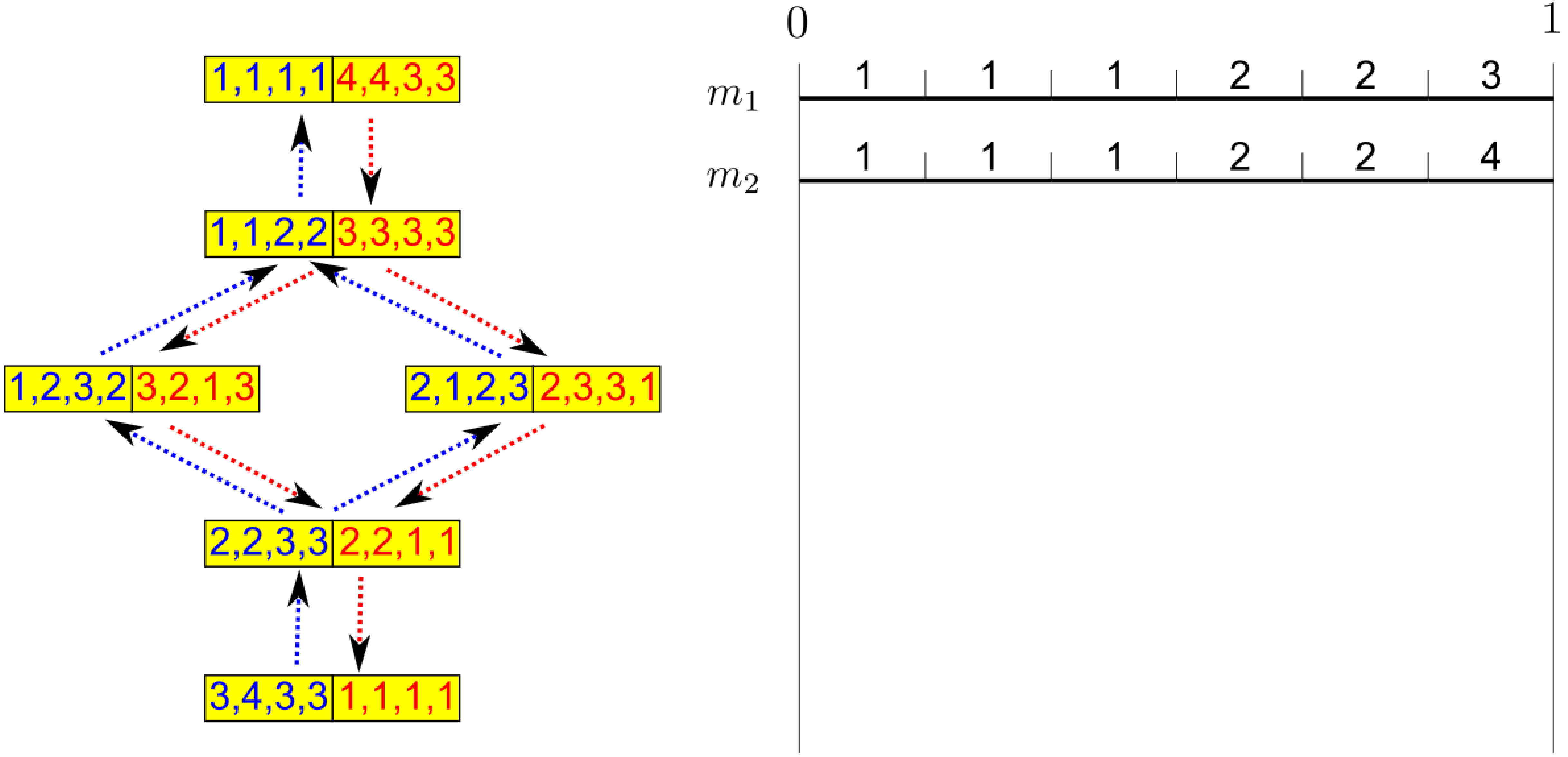

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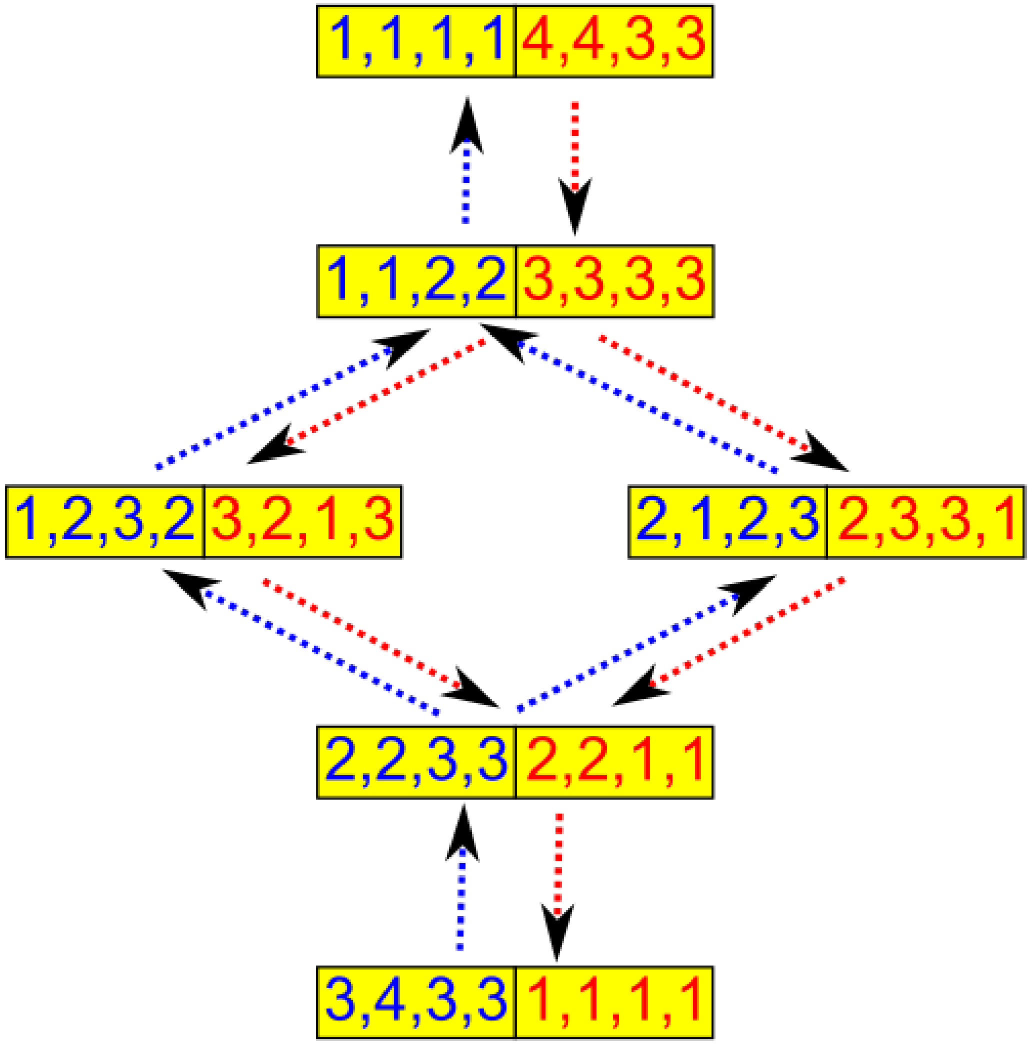

0 — 1

m_1	1	1	1	2	2	3
m_2	1	1	1	2	2	4
m_3	1	2	2	3	3	3
m_4	1	2	2	3	3	3
w_1	4	3	3	2	2	1
w_2	4	3	3	2	2	1
w_3	3	3	3	1	1	1
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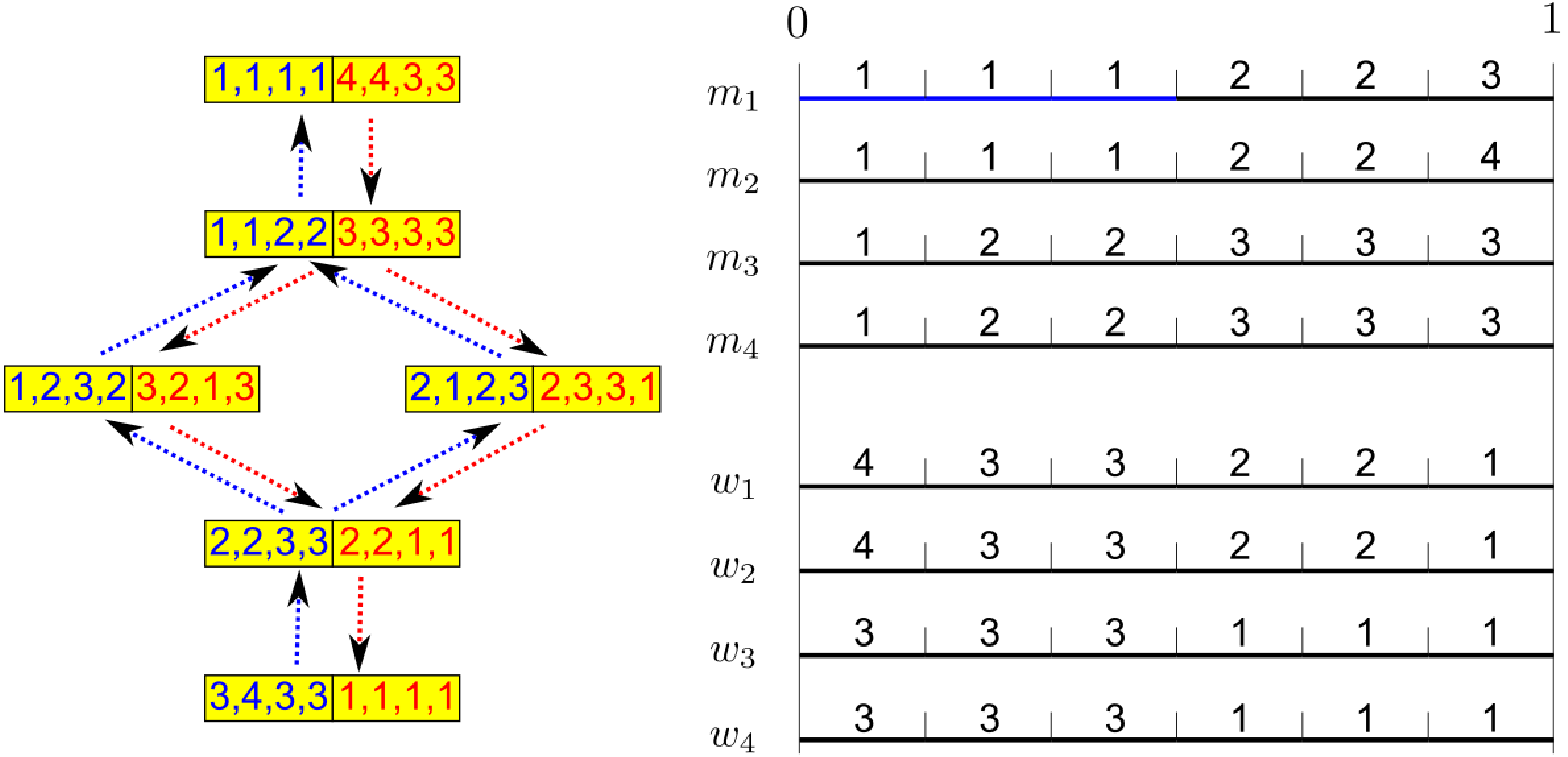

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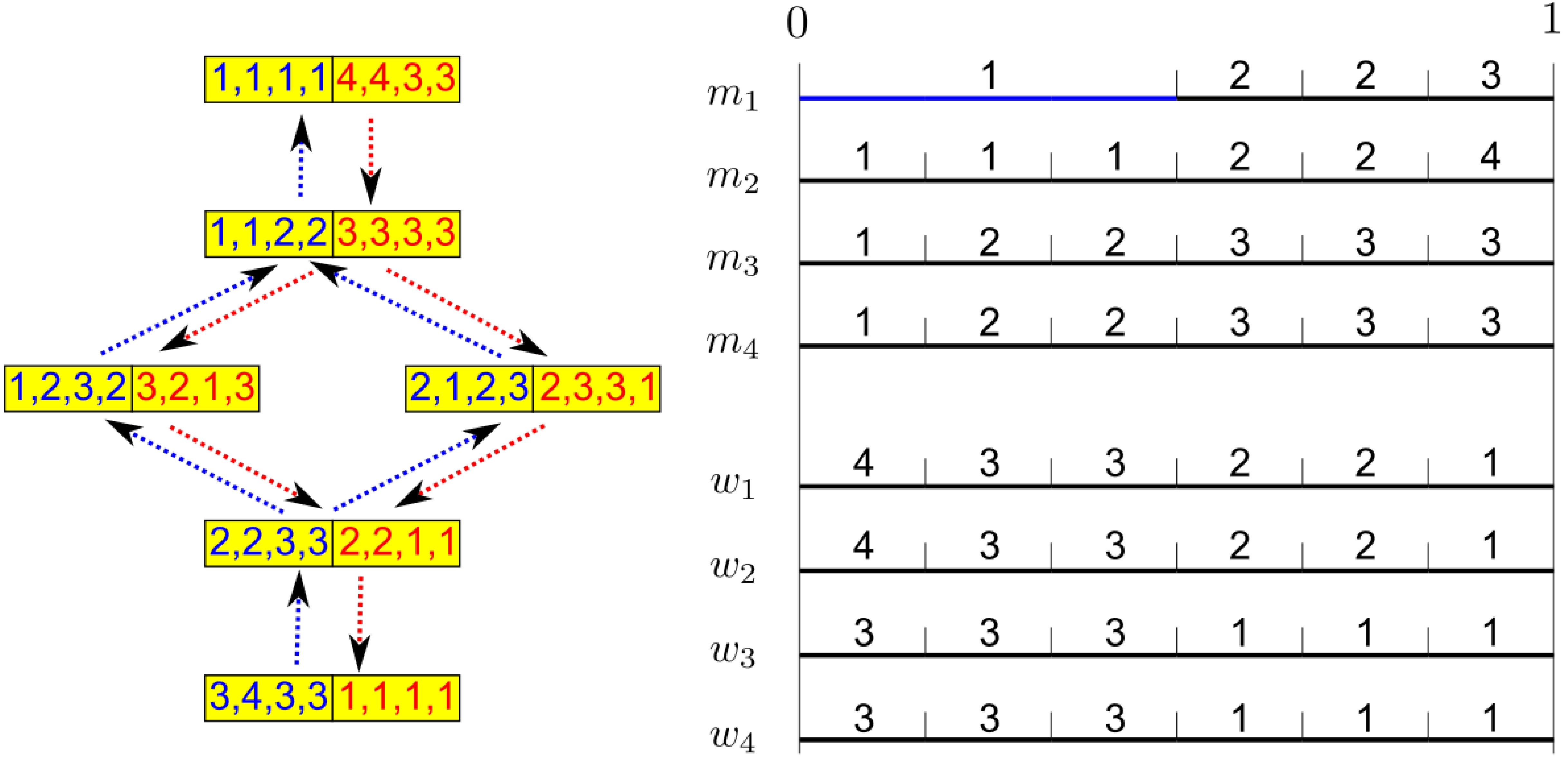

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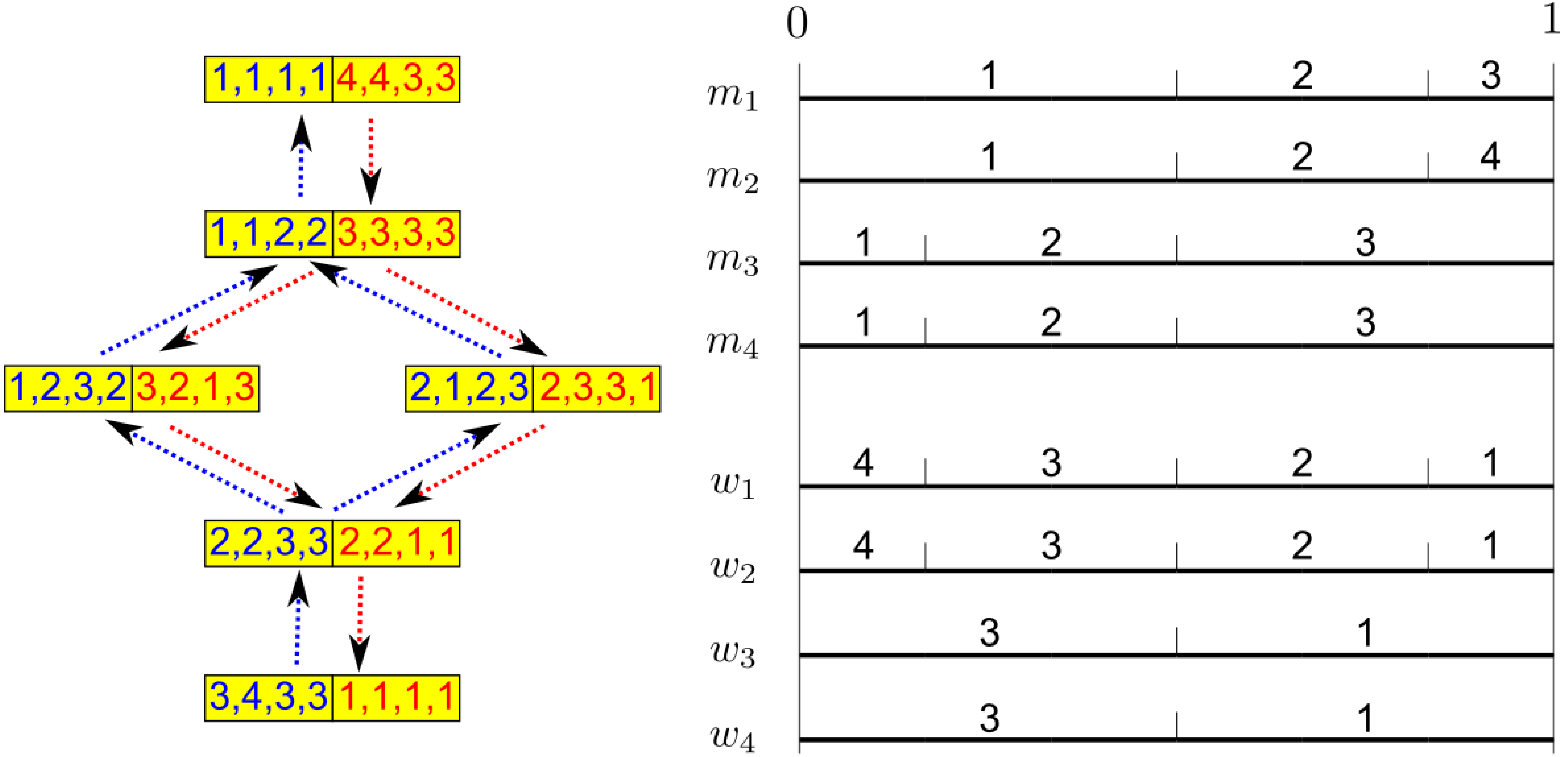

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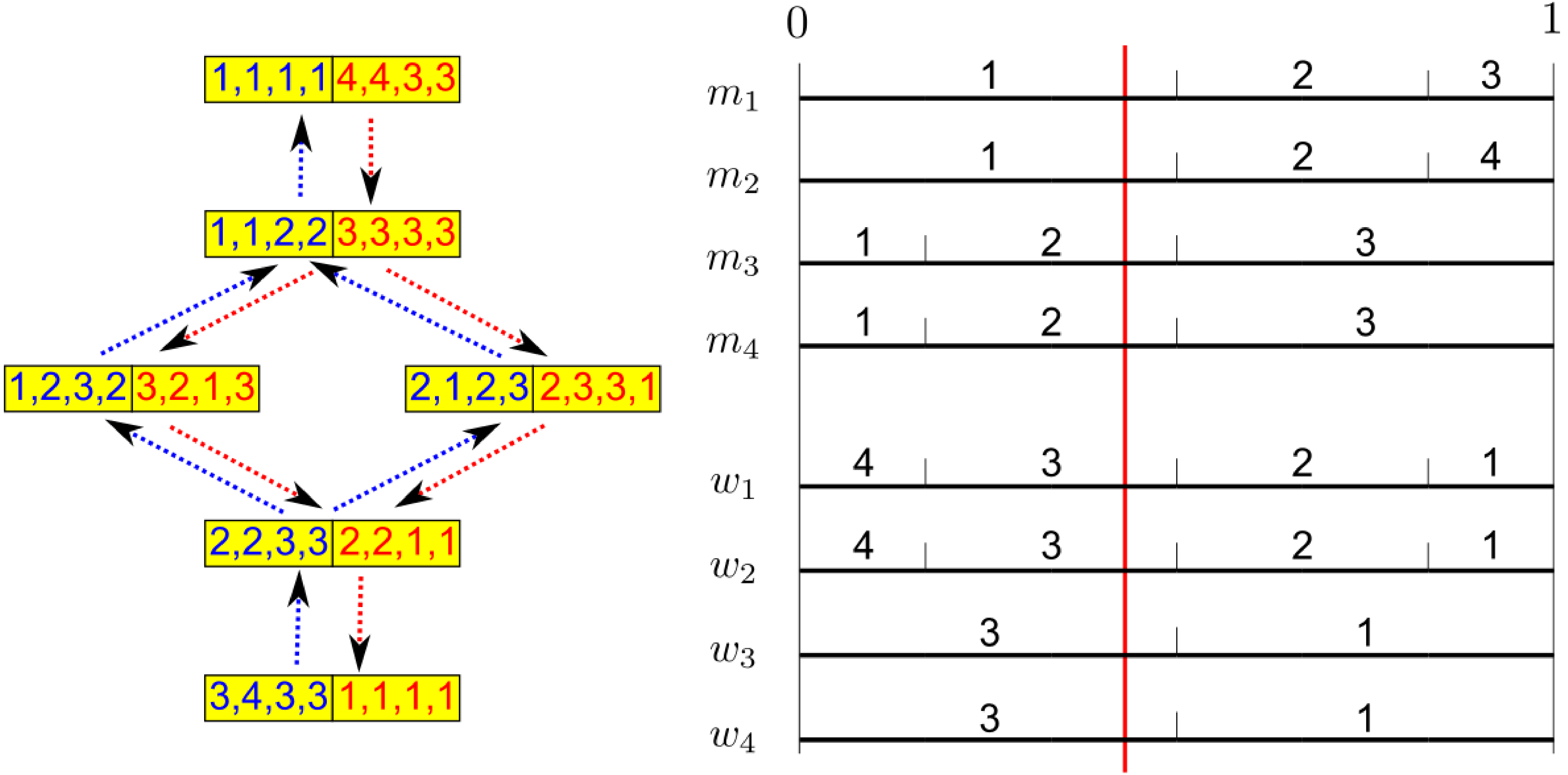
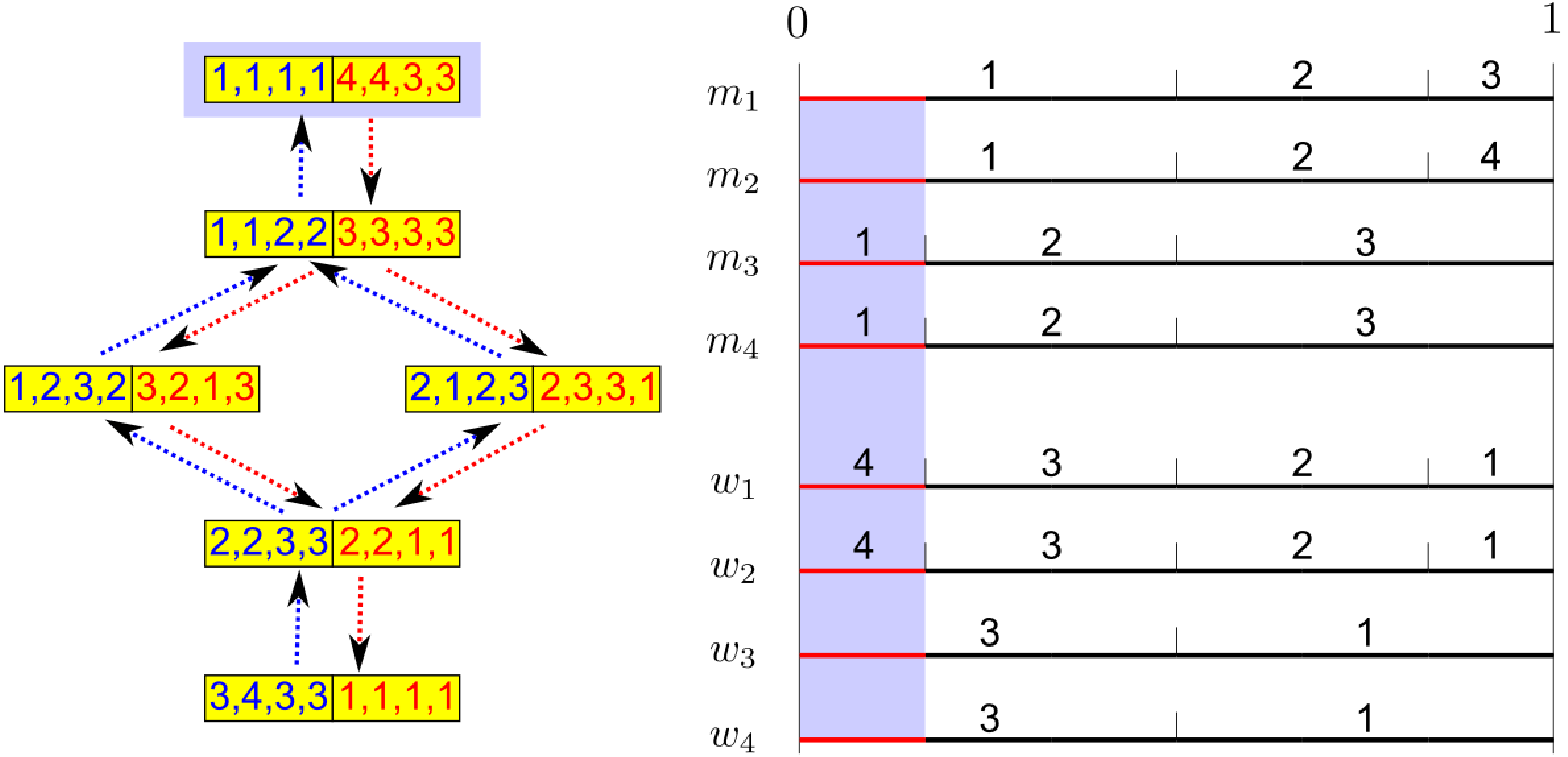


A vertical cut at any point in [0,1] induces a stable matching.

Consider a uniform combination of all integral stable matchings.



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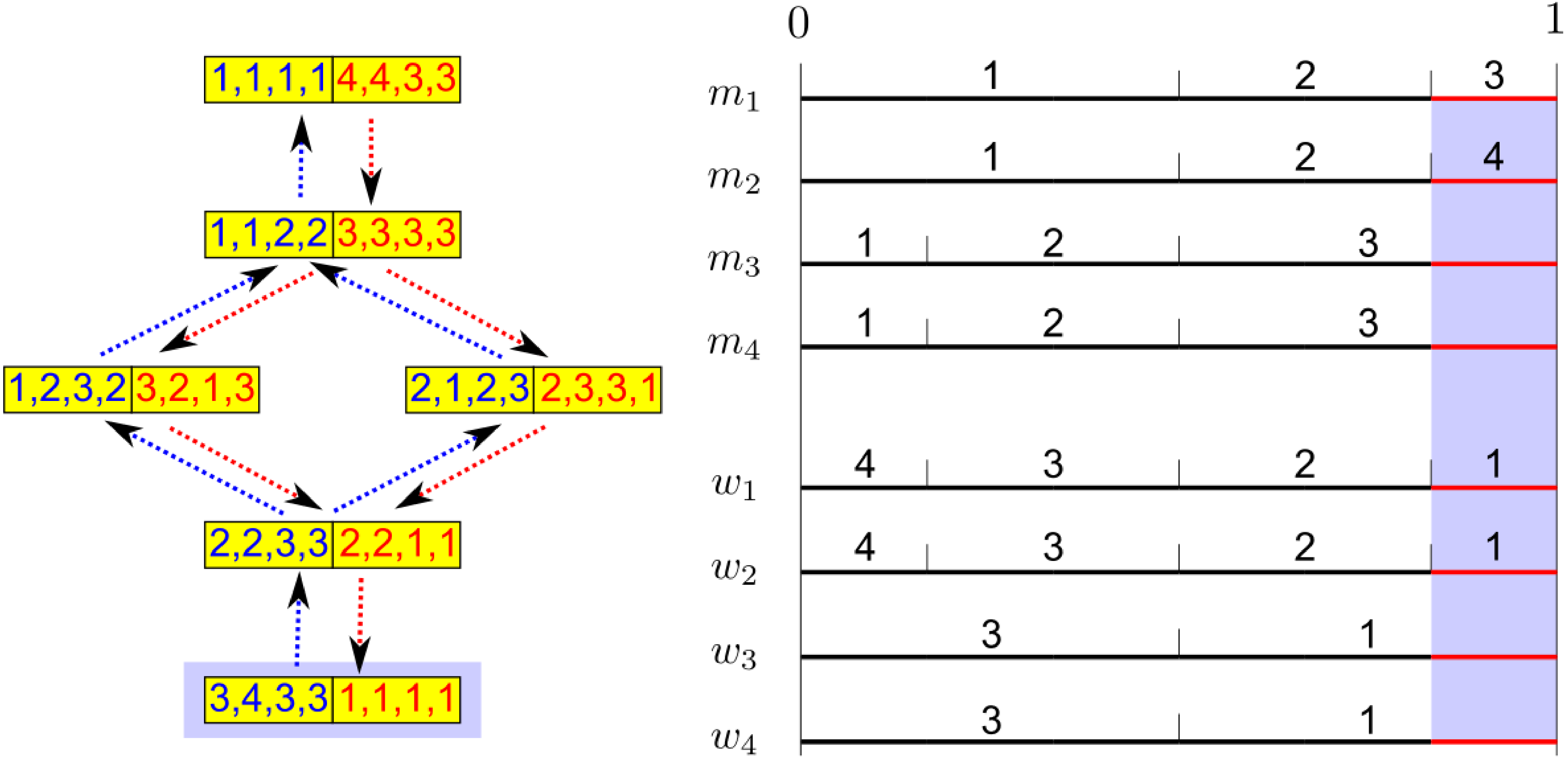

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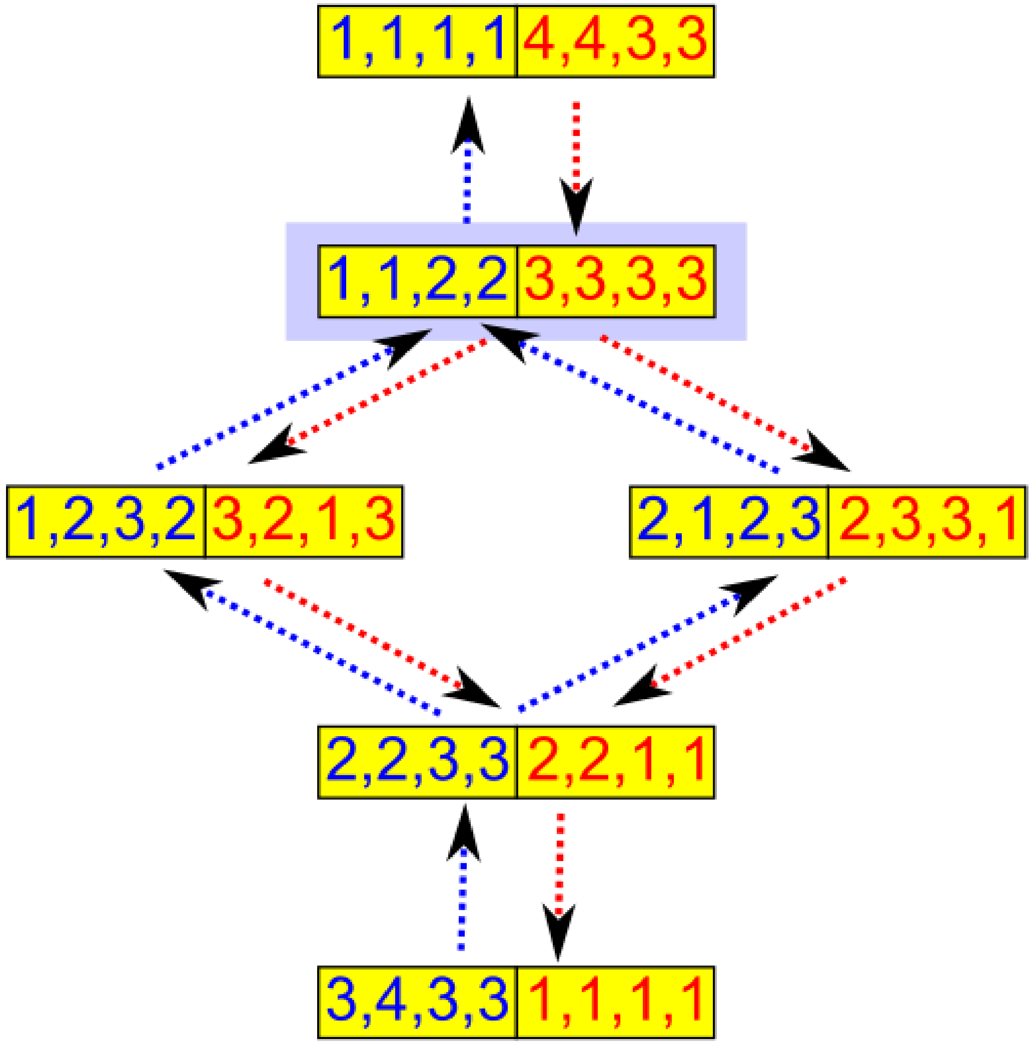

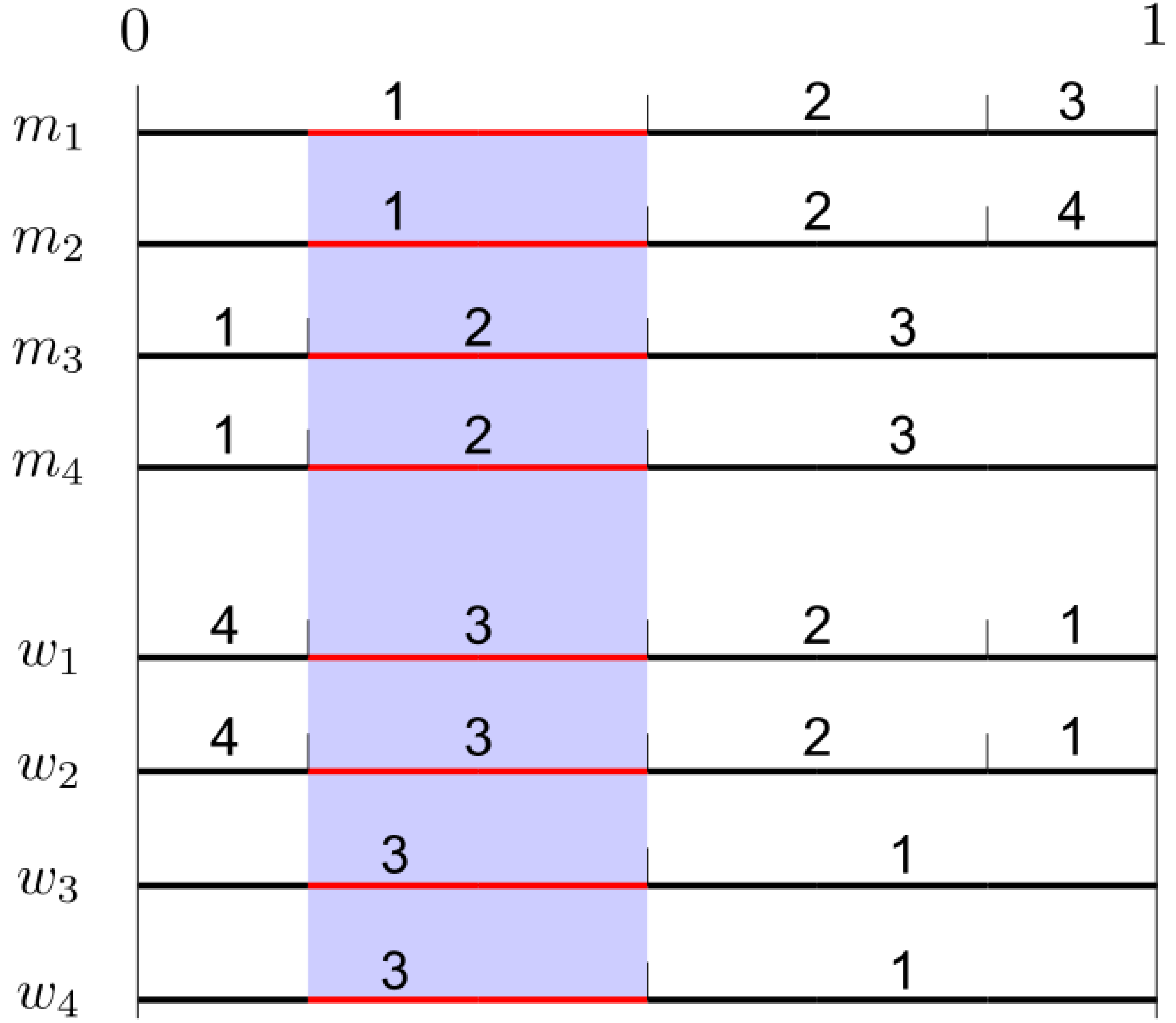

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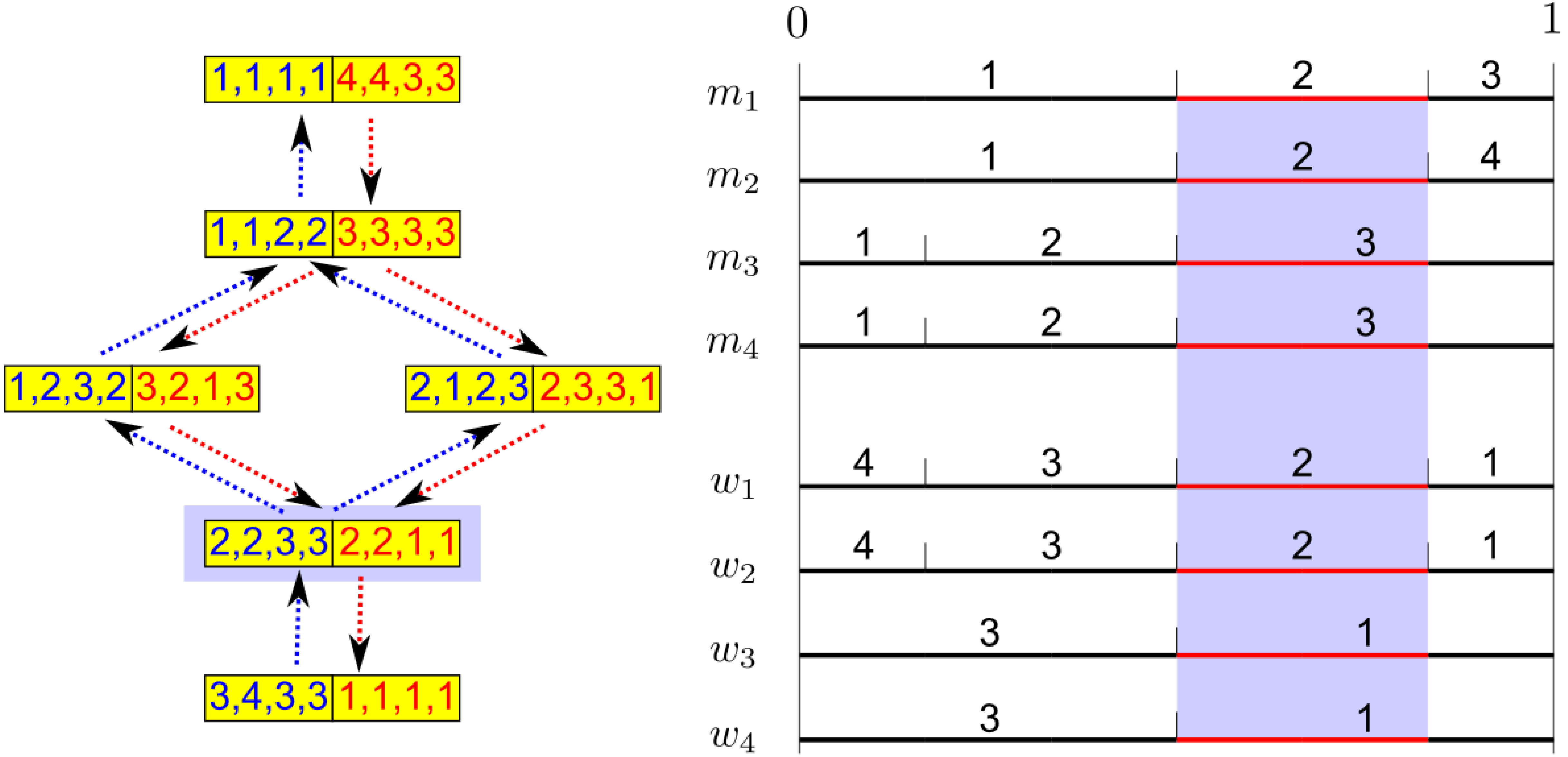

Bad News

Bad News

[Cheng, *Algorithmica* 2010]

Computing a median stable matching is #P-hard.

Bad News

[Cheng, *Algorithmica* 2010]

Computing a median stable matching is #P-hard.

as hard as...

- computing the permanent of a 0-1 matrix
- counting the number of perfect matchings in a bipartite graph
- counting the number of stable matchings for a given instance
- and many others...

Other Notions of Fairness

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Egalitarian

[Irving, Leather, and Gusfield, 1987]

a stable matching that minimizes the *average rank* of matched partners across all men and women

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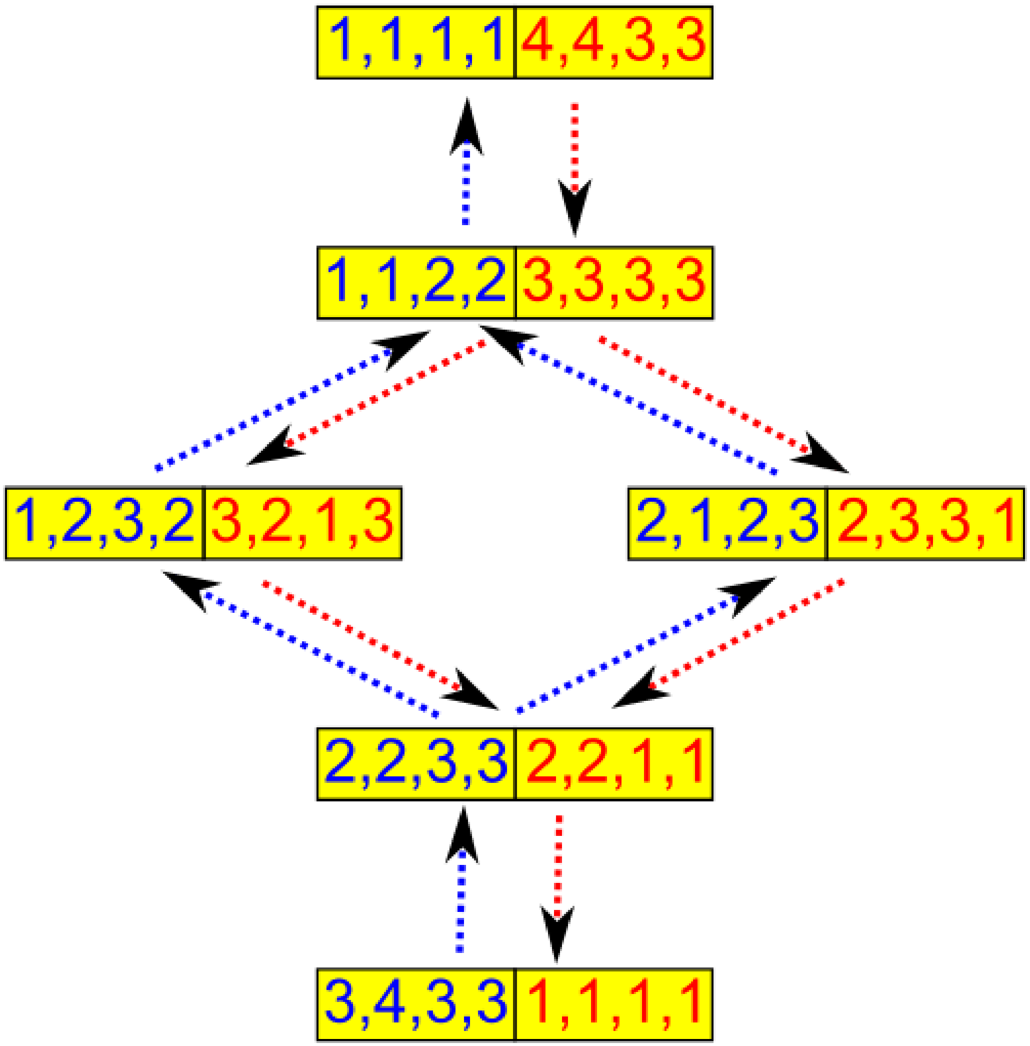
1,1,1,1 4,4,3,3
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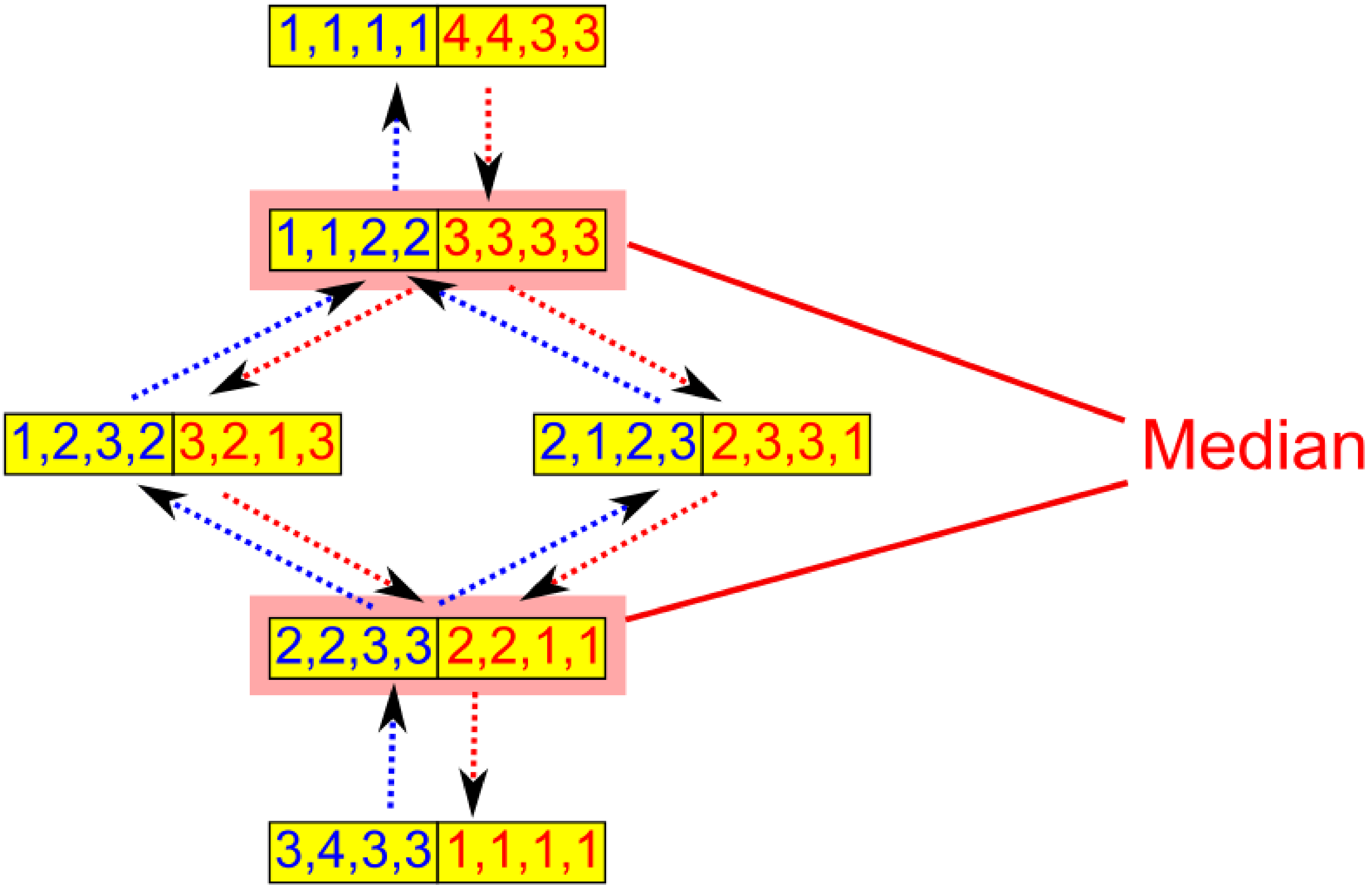

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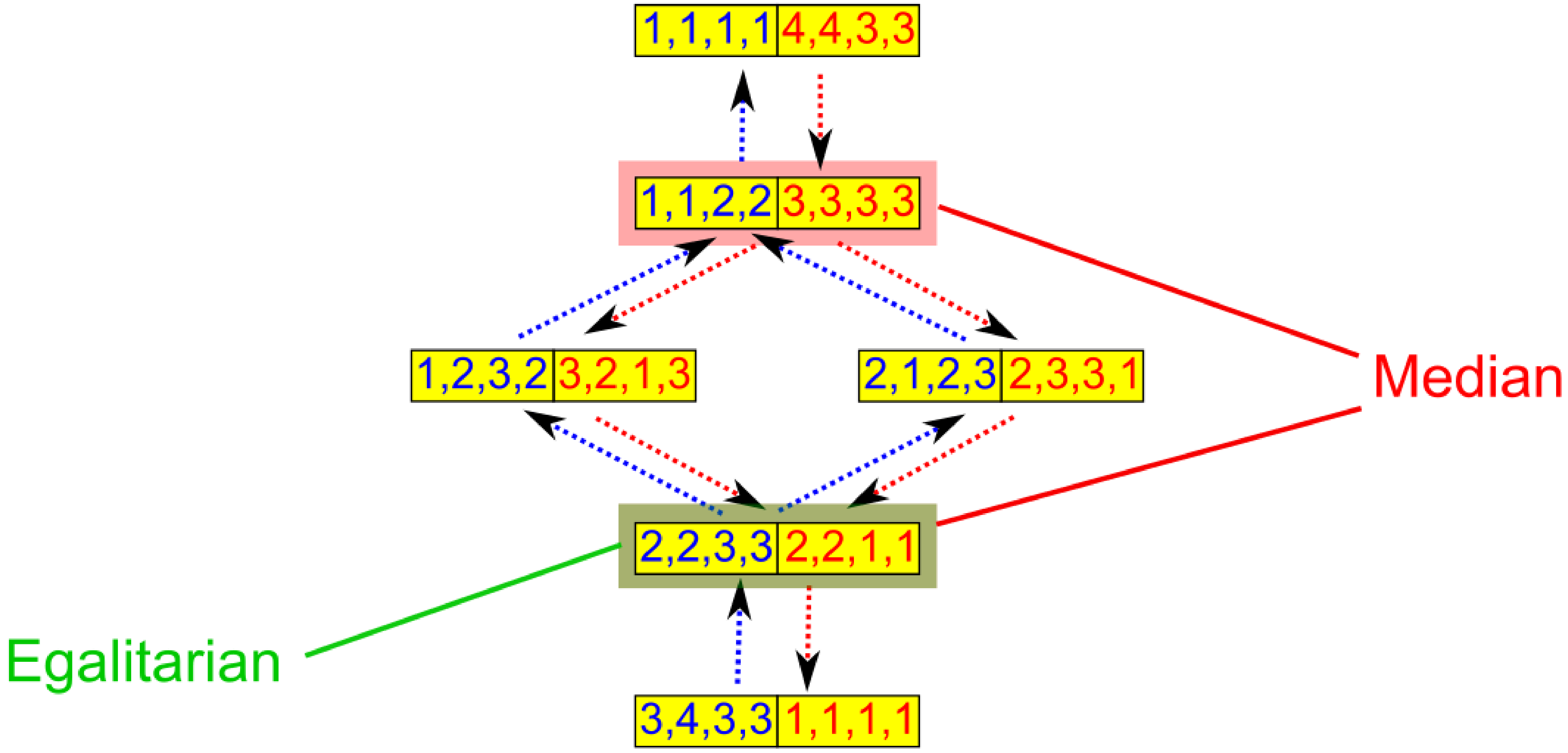
2,2,3,3 2,2,1,1
3,4,3,3 1,1,1,1
Median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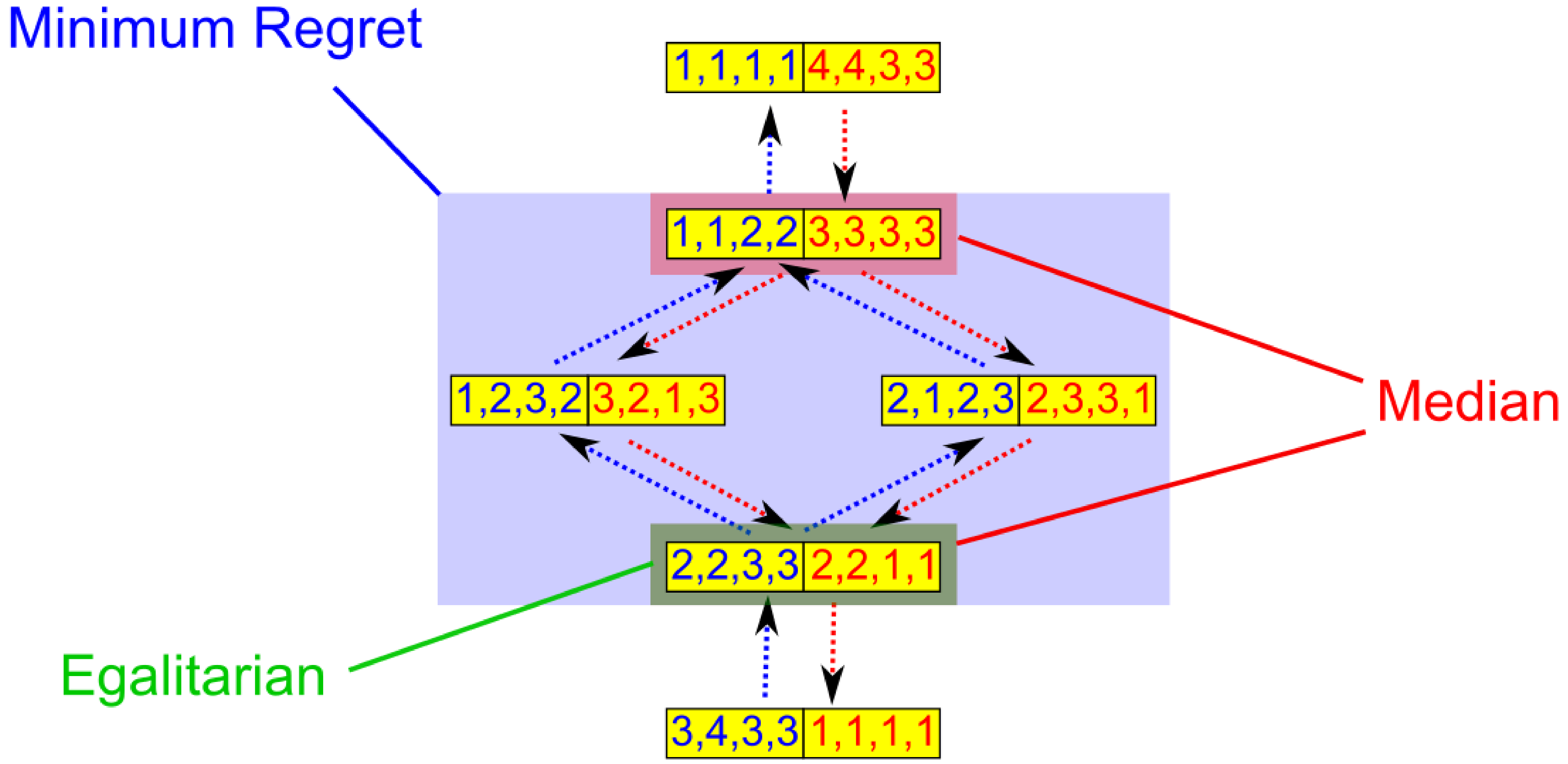
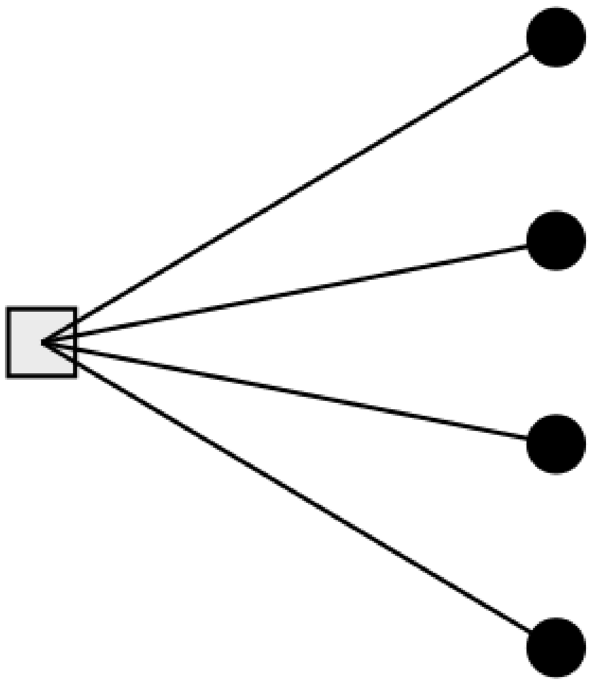
3,4,3,3 1,1,1,1
Median
Egalitarian


Minimum Regret
1,1,1,1 4,4,3,3
1,1,2,2 3,3,3,3
1,2,3,2 3,2,1,3
2,1,2,3 2,3,3,1
Median
2,2,3,3 2,2,1,1
Egalitarian
3,4,3,3 1,1,1,1

Next Time

Many-To-One Matchings



Quiz

Quiz

Use the geometric method to write the following fractional stable matching as a convex combination of integral stable matchings:

m_1: $w_1 > w_2 > w_3$

m_2: $w_2 > w_1 > w_3$

m_3: $w_3 > w_1 > w_2$

w_1: $m_3 > m_2 > m_1$

w_2: $m_3 > m_1 > m_2$

w_3: $m_1 > m_2 > m_3$

	w_1	w_2	w_3
m_1	1/2	1/2	0
m_2	1/6	1/2	1/3
m_3	1/3	0	2/3

References

- Slides by Christine Cheng on “Fair Stable Matchings”.

https://www.optimalmatching.com/MATCHUP2015/slides/ChristineCheng.pdf

- Linear programming-based formulation of the stable matching problem

John Vande Vate
“Linear Programming Brings Marital Bliss”
Operations Research Letters, 8(3), 1989 pg 147-153

References

- A median stable matching always exists.

Chung-Piaw Teo and Jay Sethuraman
"The Geometry of Fractional Stable Matchings and Its Applications"
Mathematics of Operations Research, 23(4), 1998 pg 874-891

- Computing a median stable matching is #P-hard.

Christine Cheng
"Understanding the Generalized Median Stable Matchings"
Algorithmica, 58, 2010 pg 34-51

References

- An algorithm for computing an egalitarian stable matching.

Robert Irving, Paul Leather, and Dan Gusfield
"An Efficient Algorithm for the Optimal Stable Marriage Problem"
Journal of the ACM, 34, 1987 pg 532-543