

Lecture 3

Manipulation in Stable Matching Problem

Stable Matching Problem

$w_1 > w_2 > w_3$



$m_3 > m_2 > m_1$

$w_2 > w_1 > w_3$



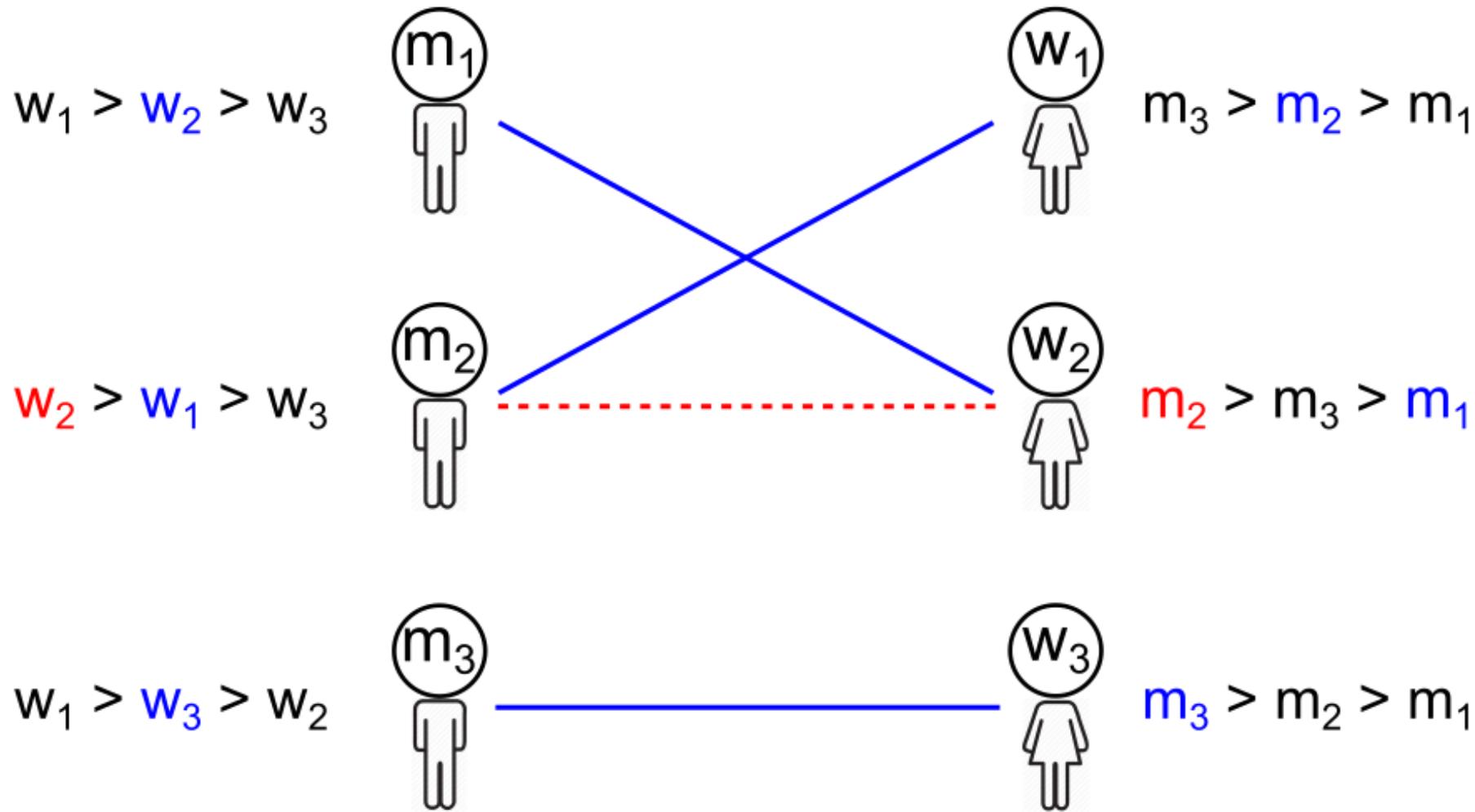
$m_2 > m_3 > m_1$

$w_1 > w_3 > w_2$

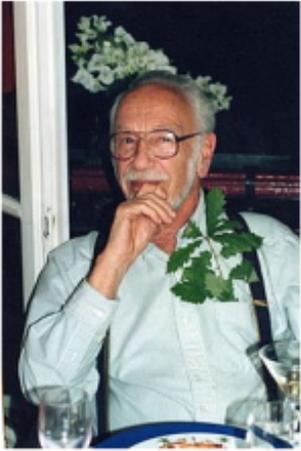


$m_3 > m_2 > m_1$

Stable Matching Problem



A matching is **stable** if there is no **blocking pair**.



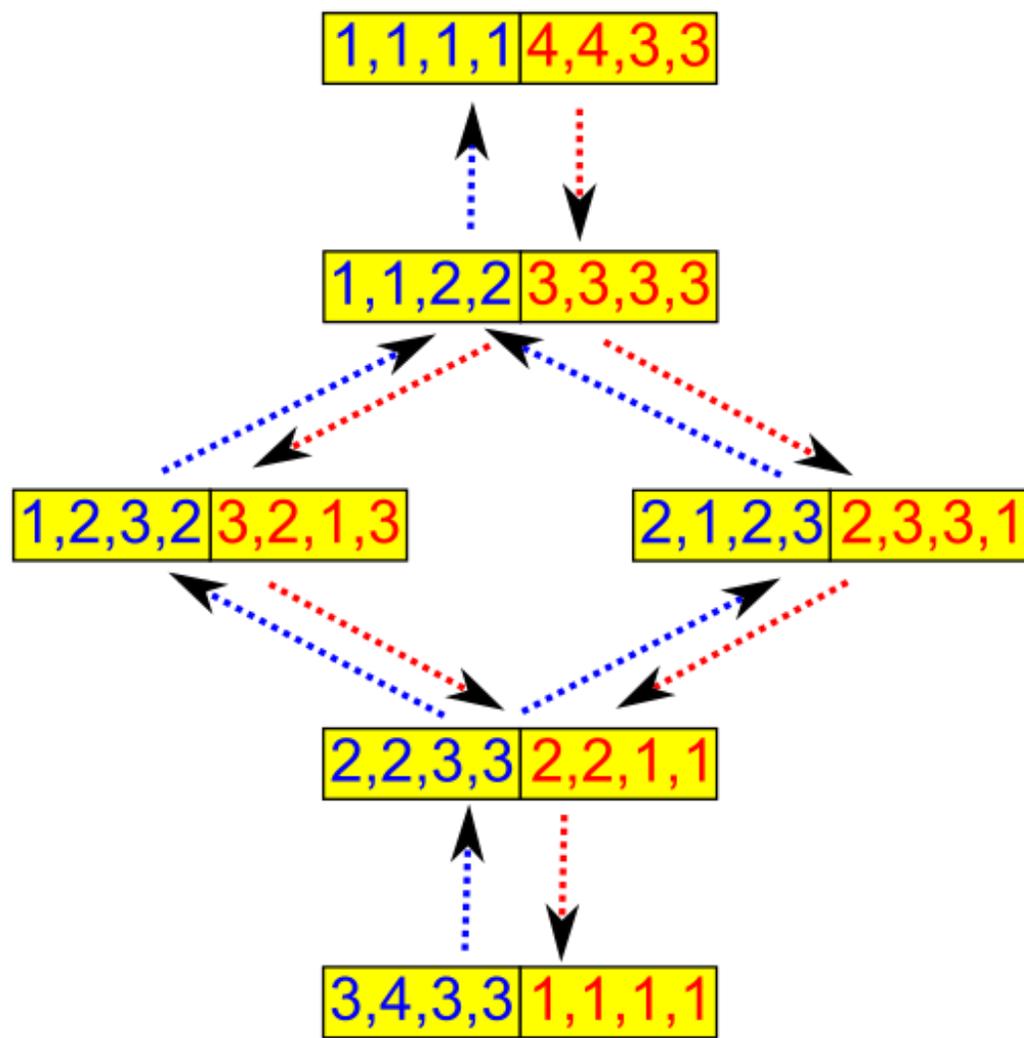
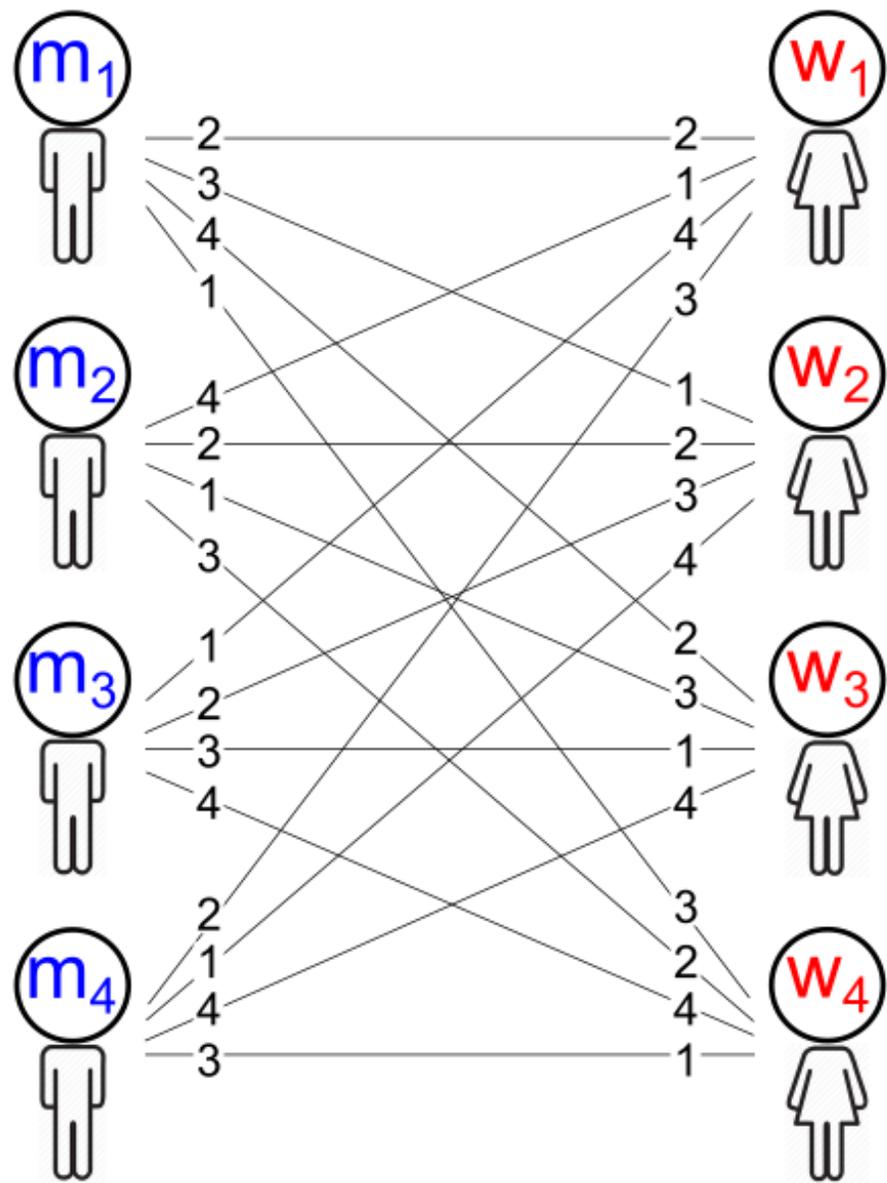
COLLEGE ADMISSIONS AND THE STABILITY OF MARRIAGE

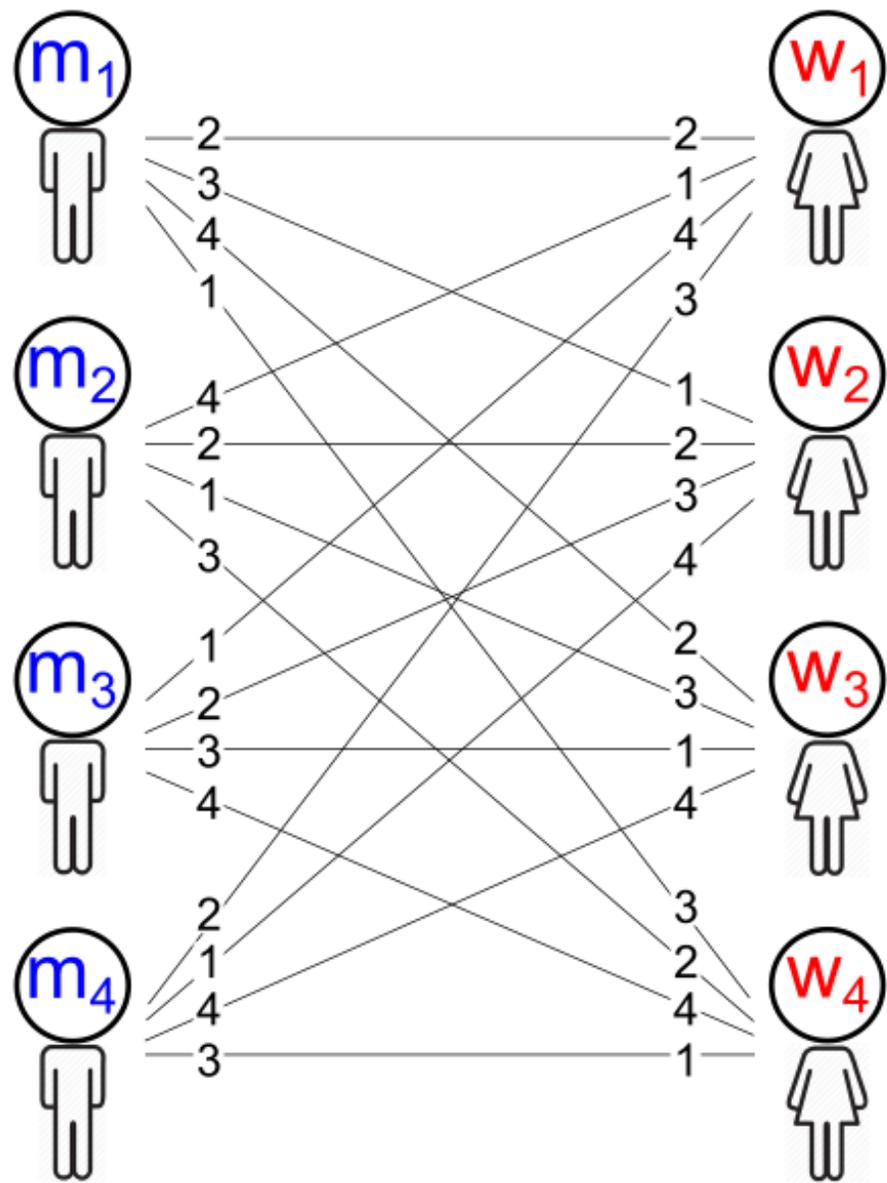
D. GALE* AND L. S. SHAPLEY, Brown University and the RAND Corporation



Source: *The American Mathematical Monthly*, Jan., 1962, Vol. 69, No. 1 (Jan., 1962), pp. 9-15

Given any preference profile, a stable matching for that profile always exists and can be computed in polynomial time.

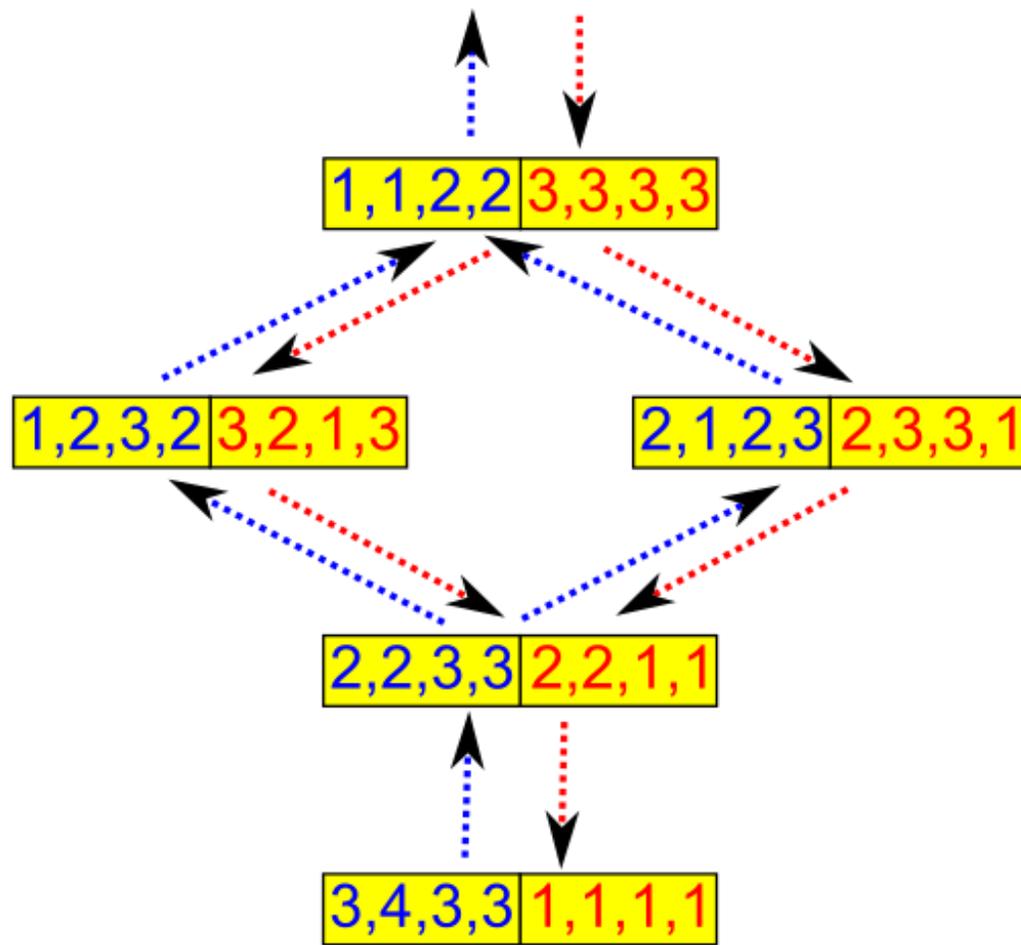


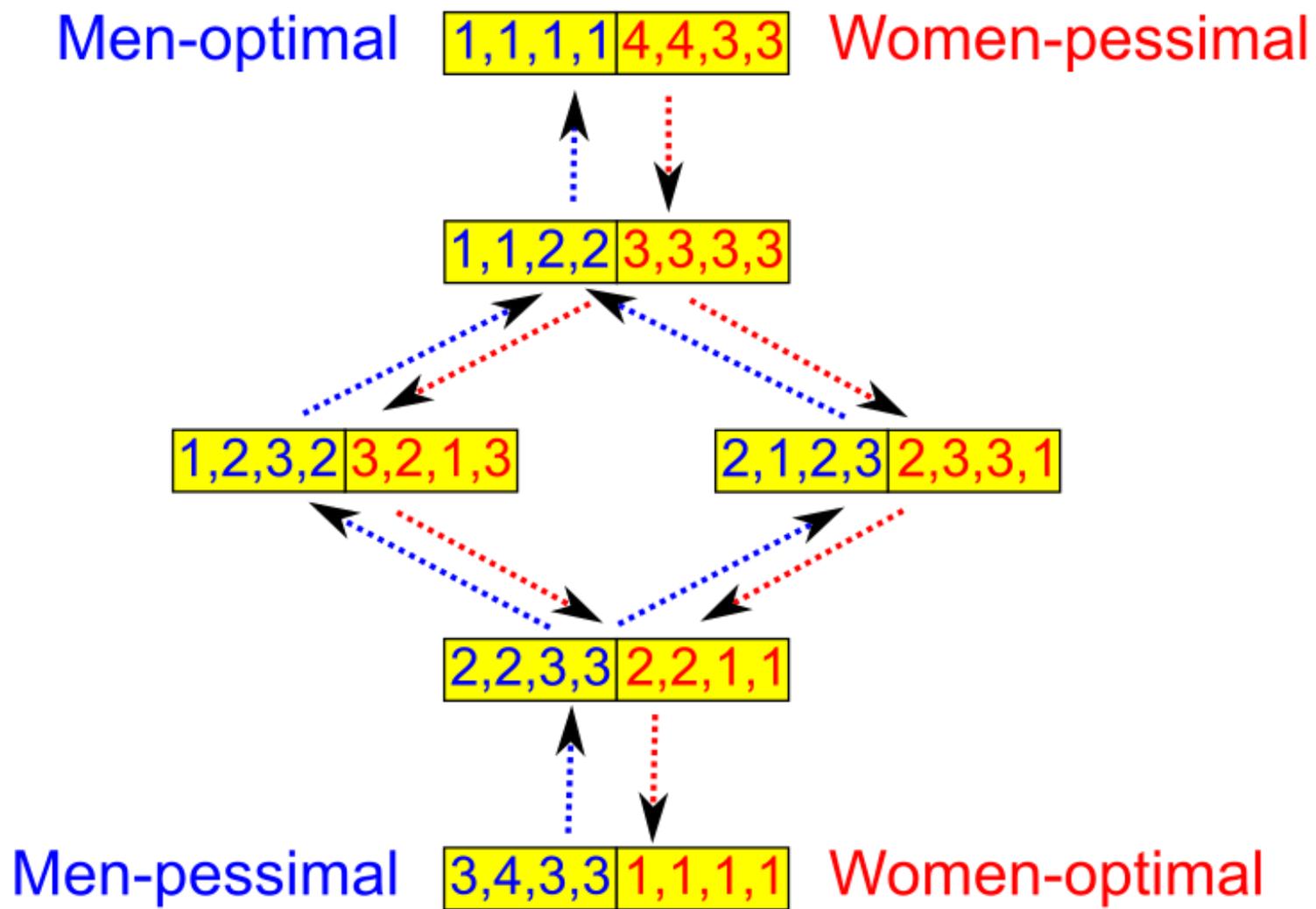
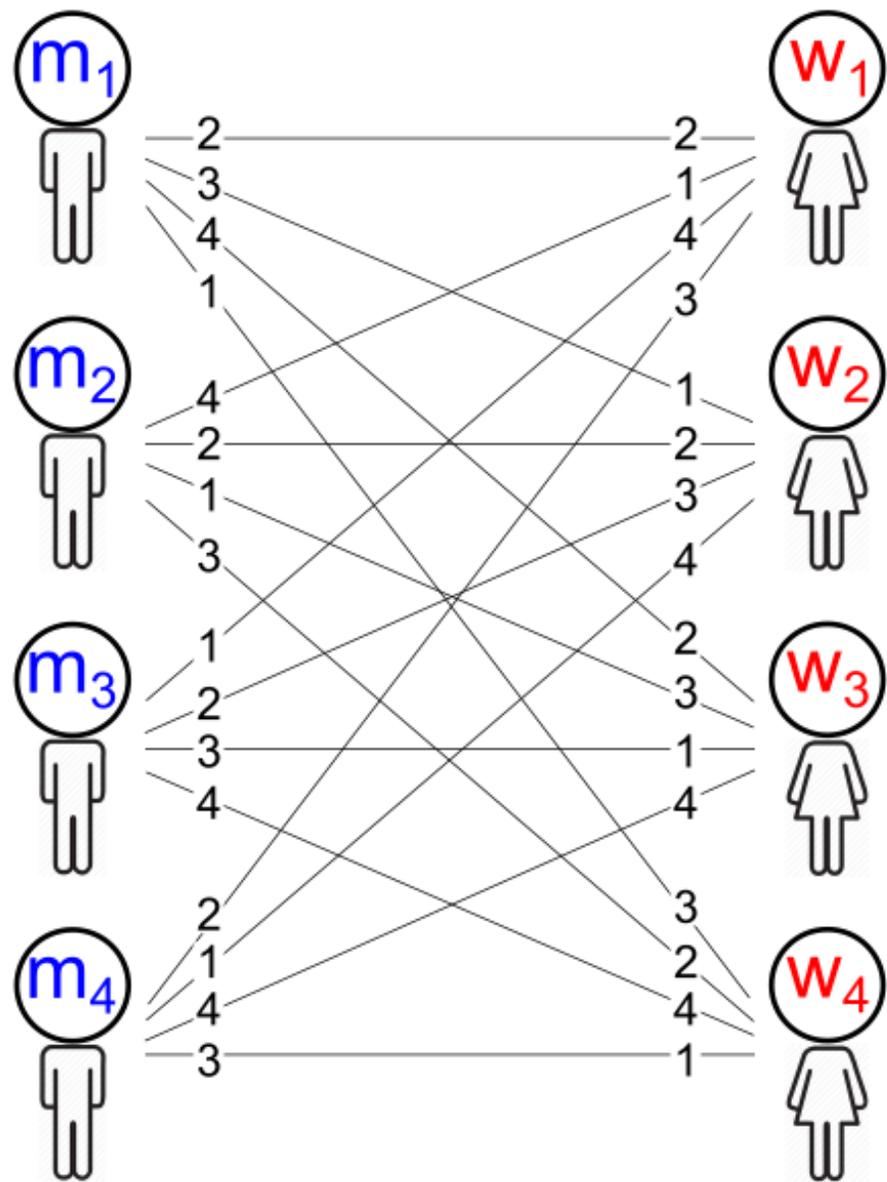


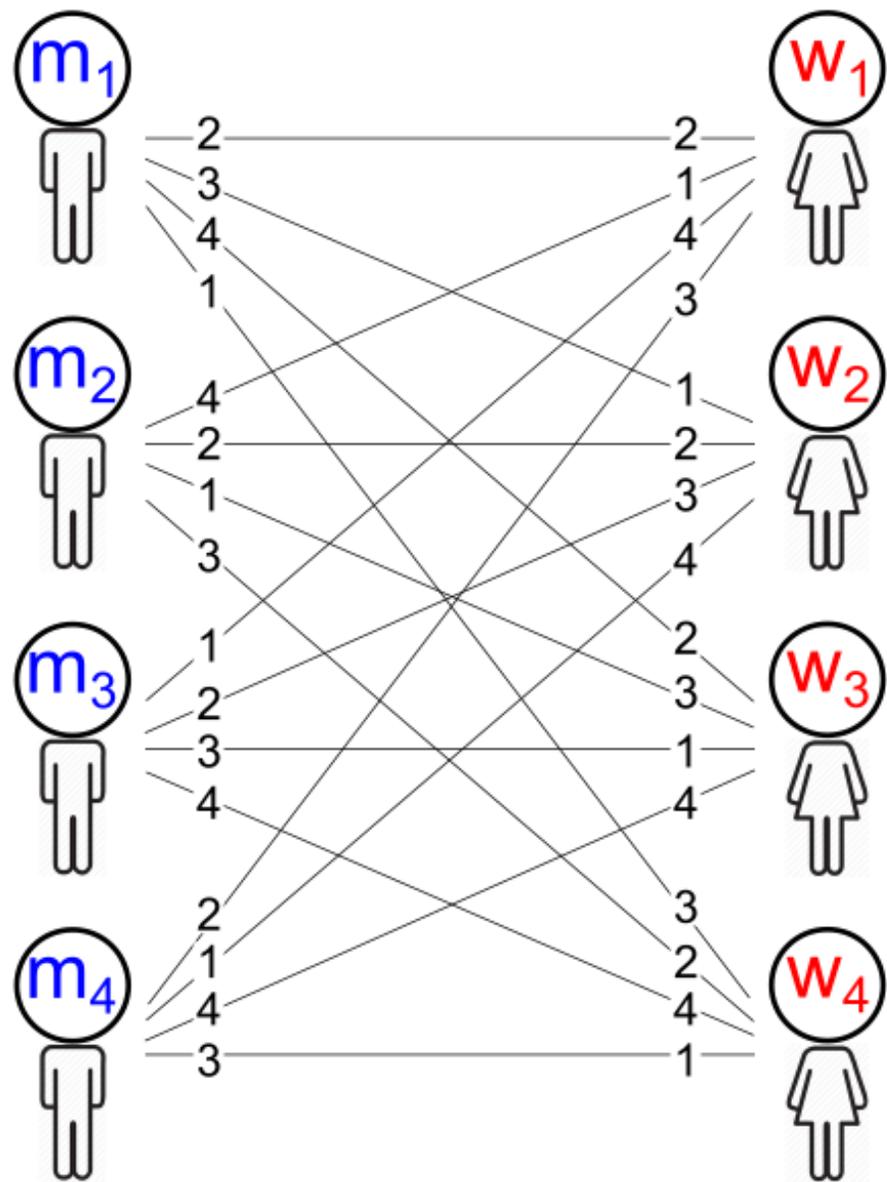
Men-optimal

1,1,1,1 | 4,4,3,3

Women-pessimal

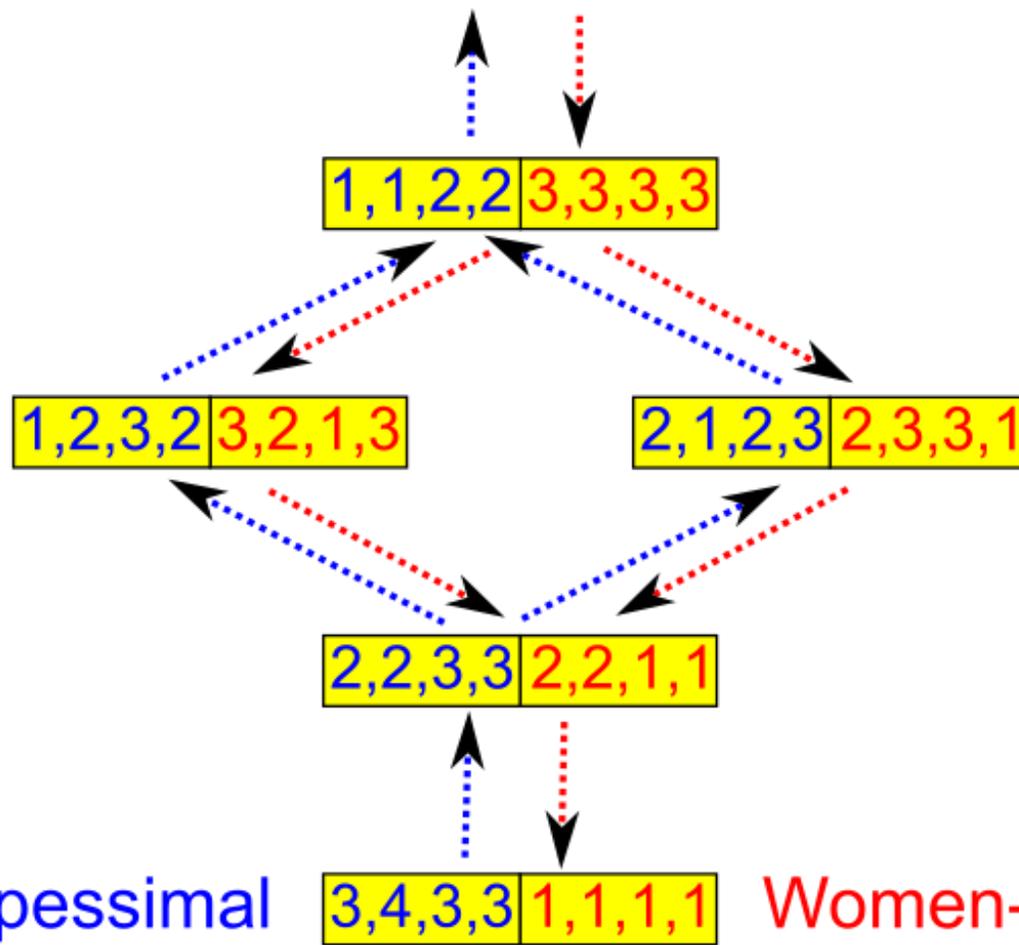


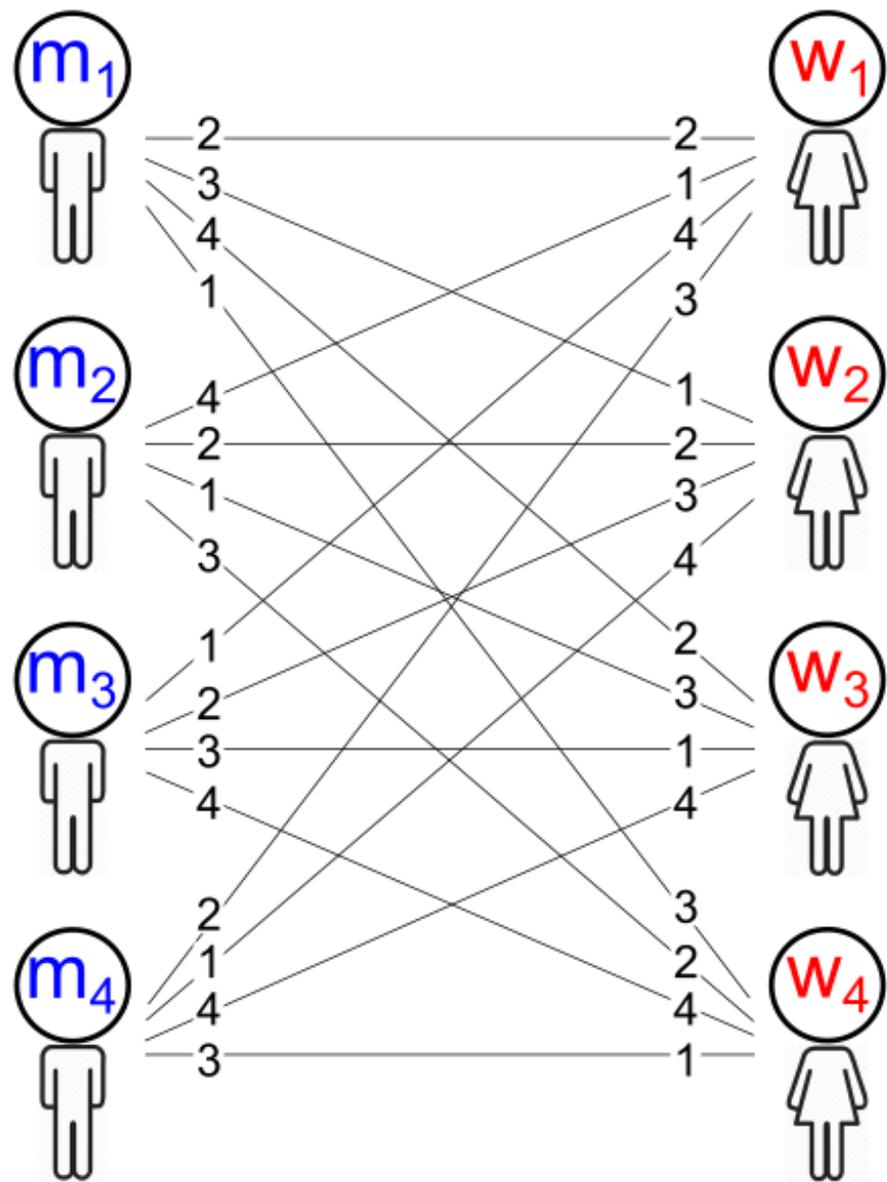




Men-proposing DA algorithm computes this

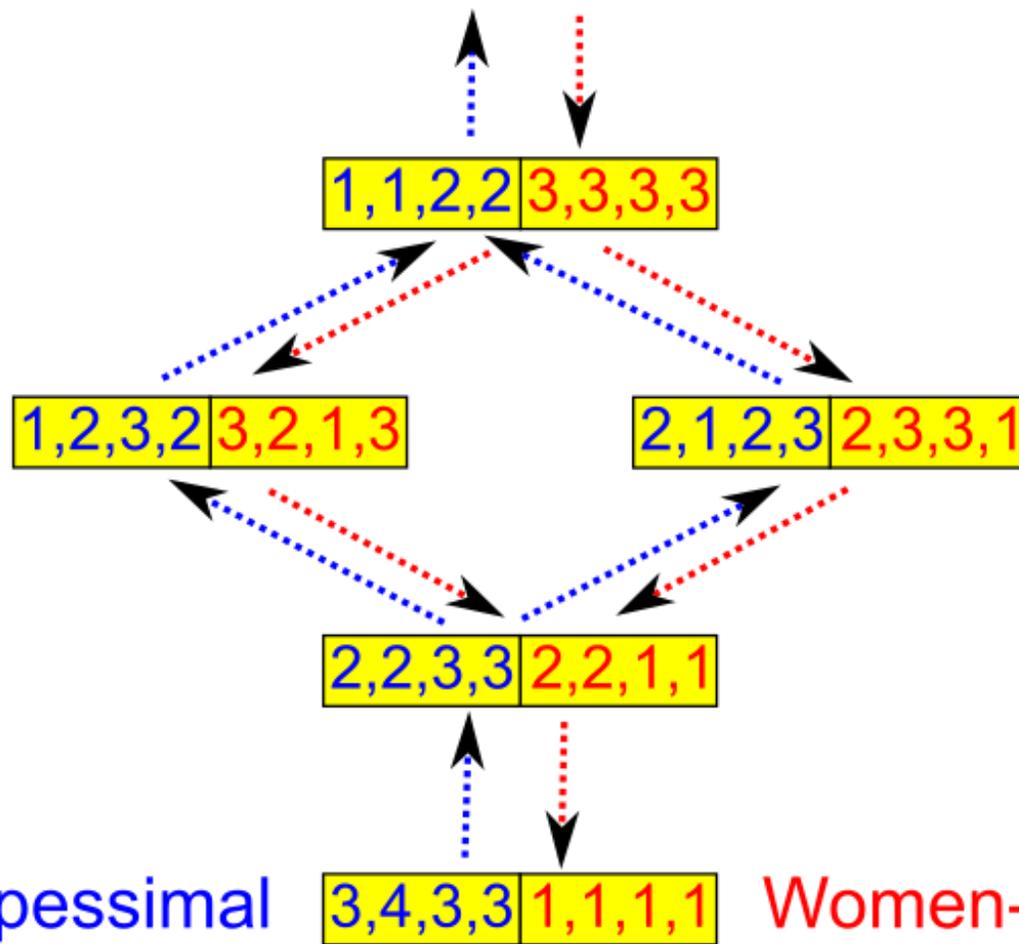
Men-optimal $1,1,1,1$ $4,4,3,3$ Women-pessimal





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Men-optimal $1, 1, 1, 1$ $4, 4, 3, 3$ Women-pessimal



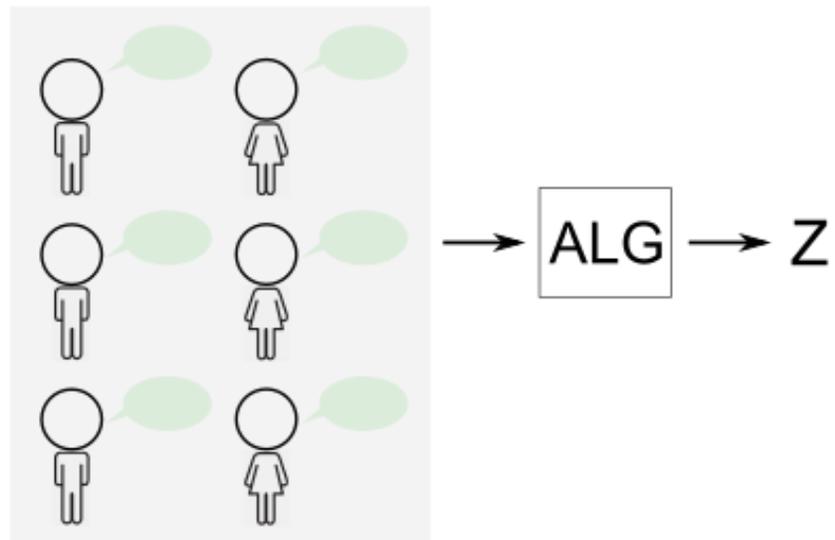
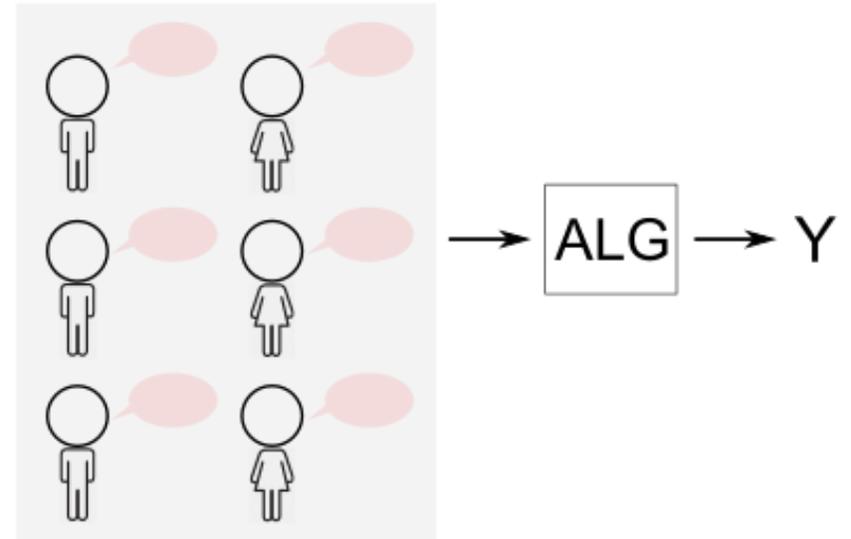
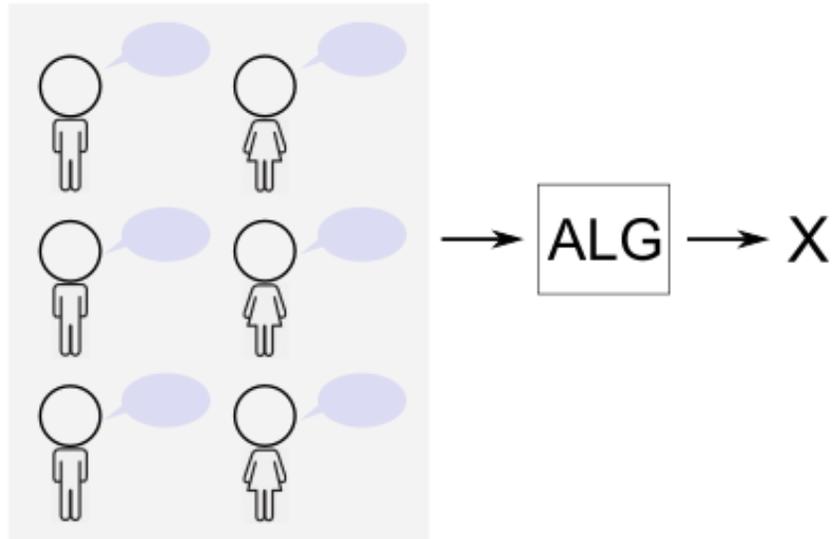
Men-pessimal $3, 4, 3, 3$ $1, 1, 1, 1$ Women-optimal

Women-proposing DA algorithm computes this

Can an agent get a better partner under DA algorithm
by misreporting his/her preferences?

Strategyproofness

Strategyproofness



...

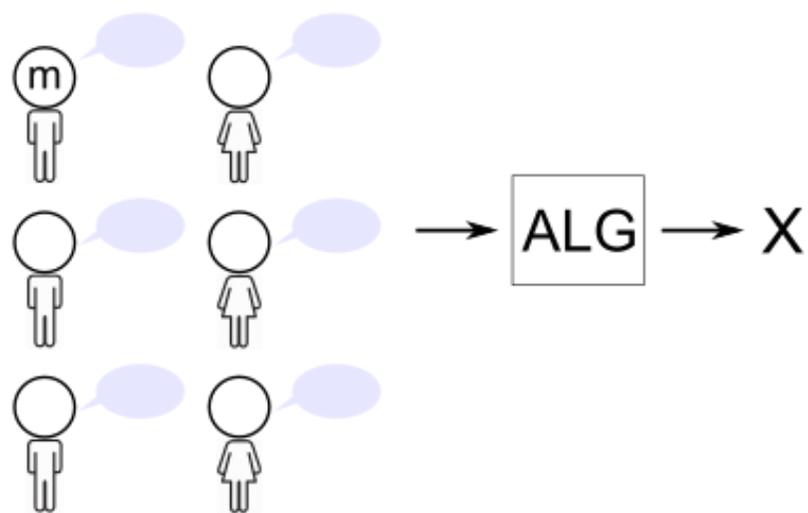
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An algorithm fails strategyproofness if, for some set of input preferences, some agent can unilaterally misreport and improve (w.r.t. true preference).

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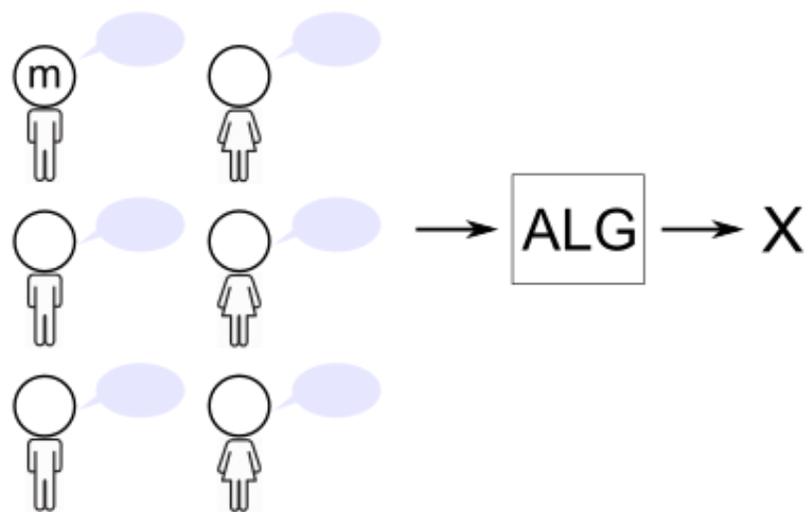
True profile $P = (P_{-m}, P_m)$



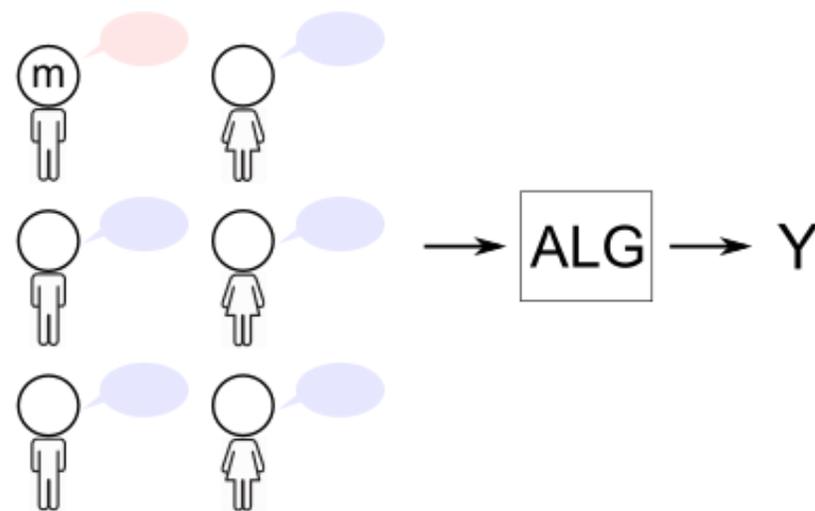
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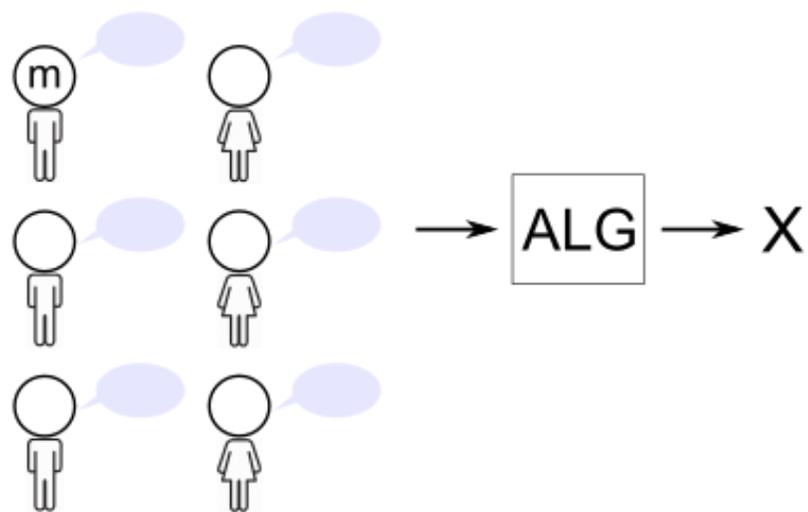
Manipulated profile $P' = (P_{-m}, P'_m)$



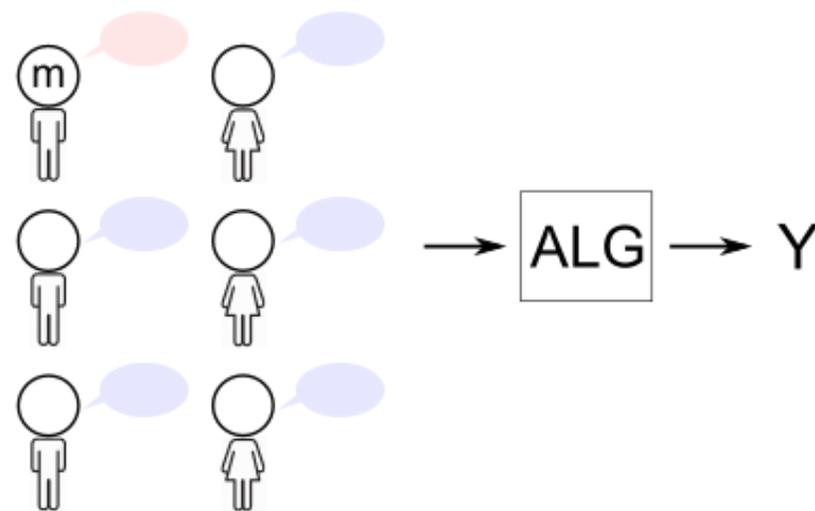
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True preference list P_m ... $> Y(m) > \dots > X(m) > \dots$

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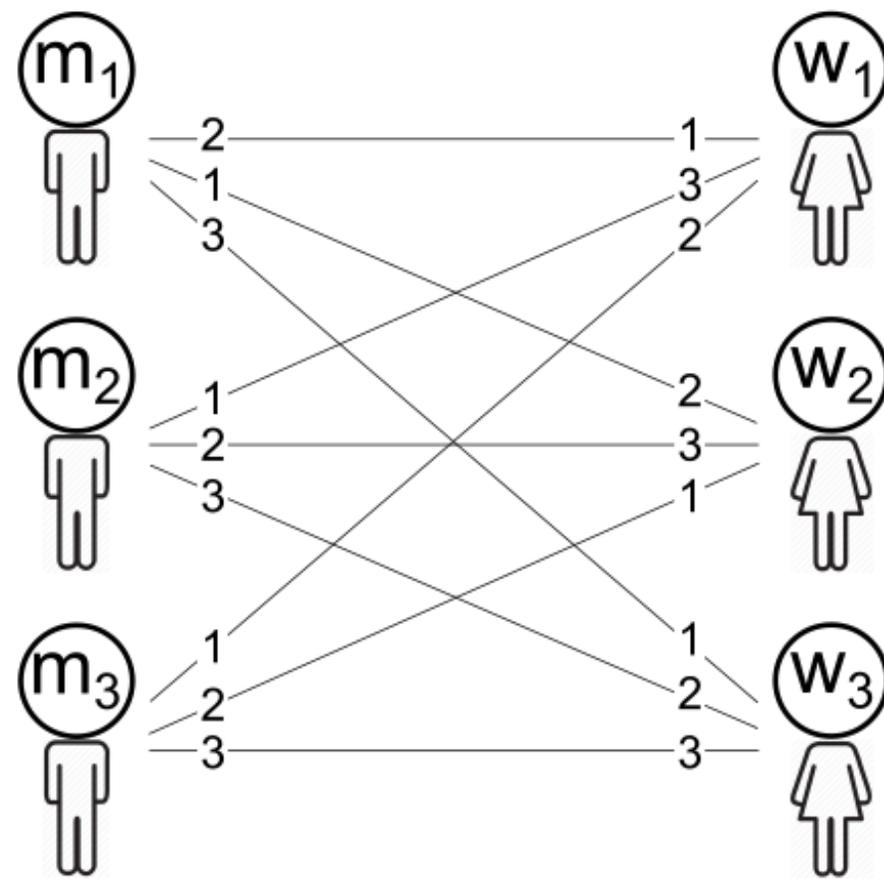
An algorithm is **strategyproof** if it does not fail strategyproofness on any input.

[Dubins and Freedman, *Amer. Math. Mon.* 1981; Roth, *MOR* 1982]

DA algorithm is not strategyproof.

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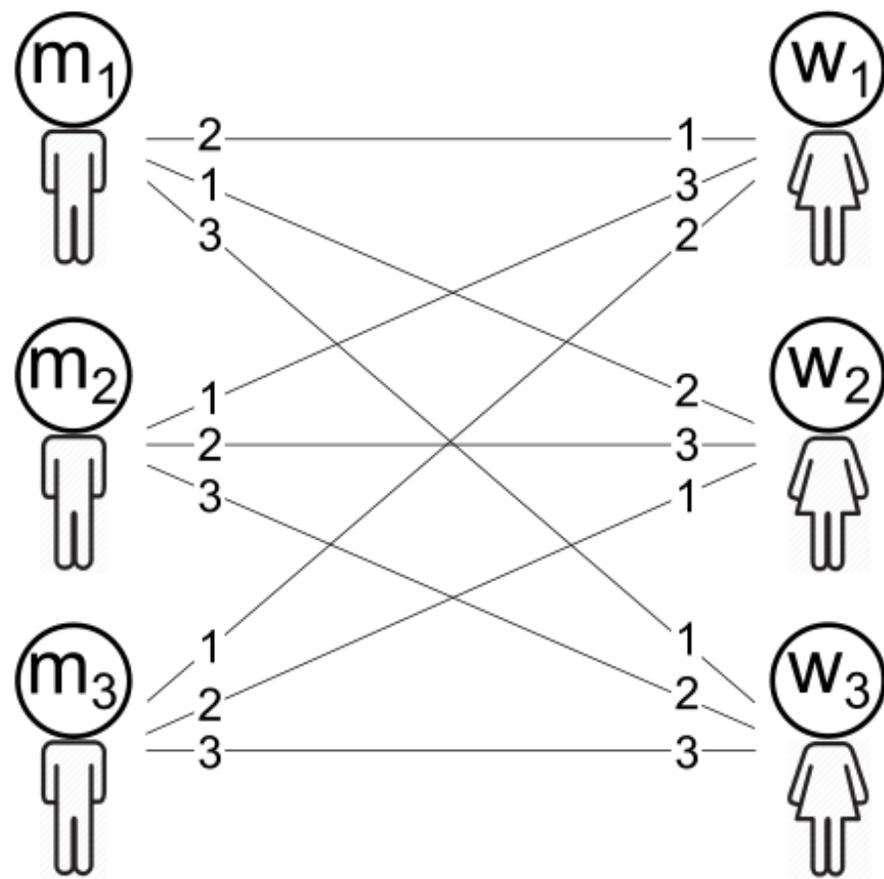
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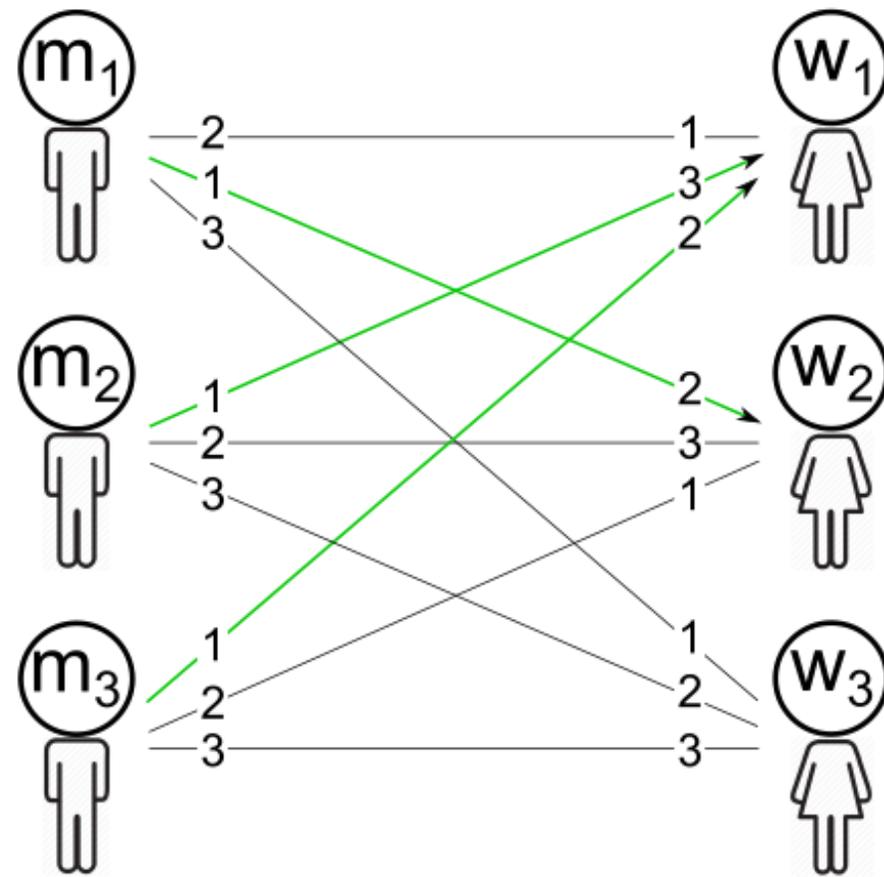
Round 1



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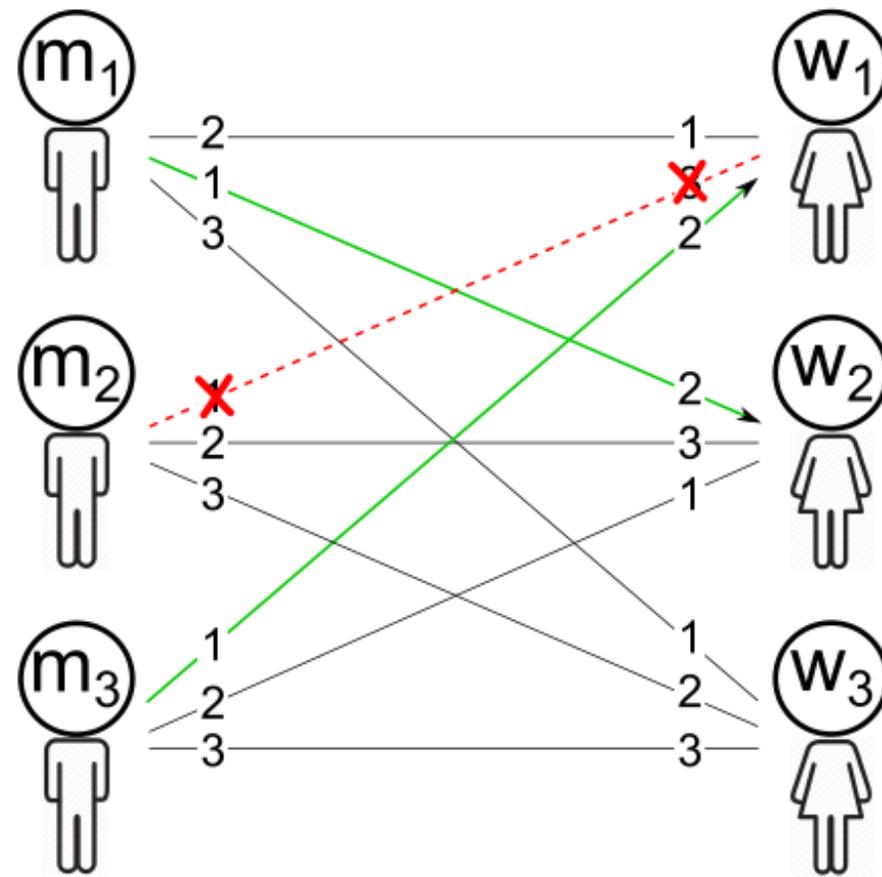
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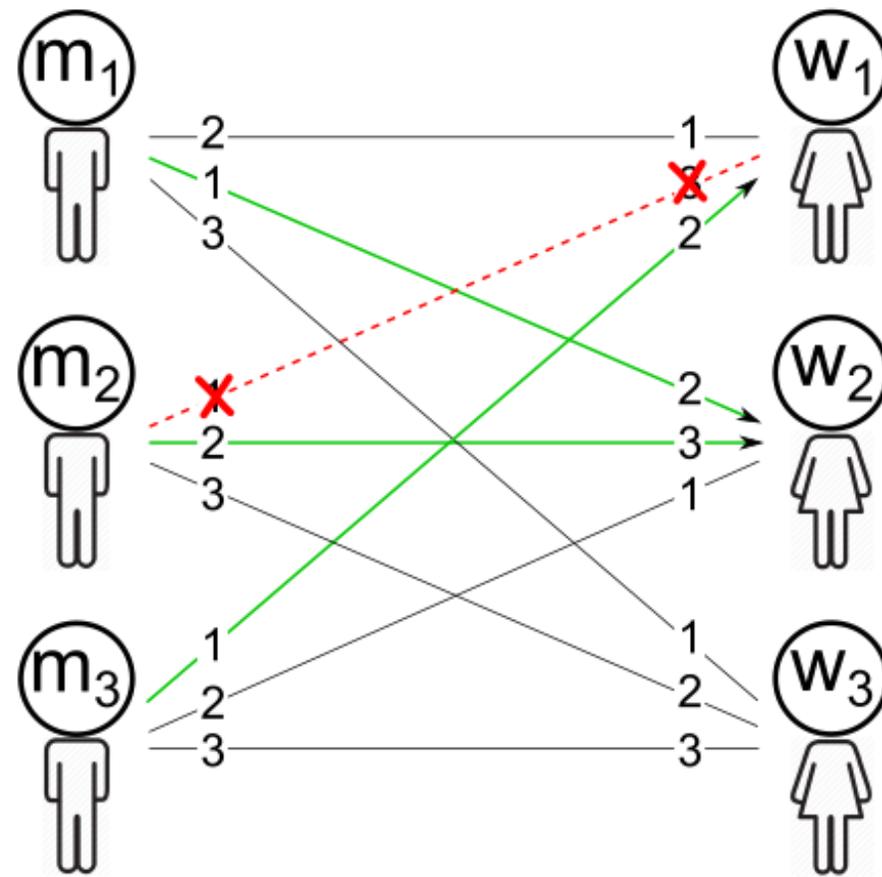
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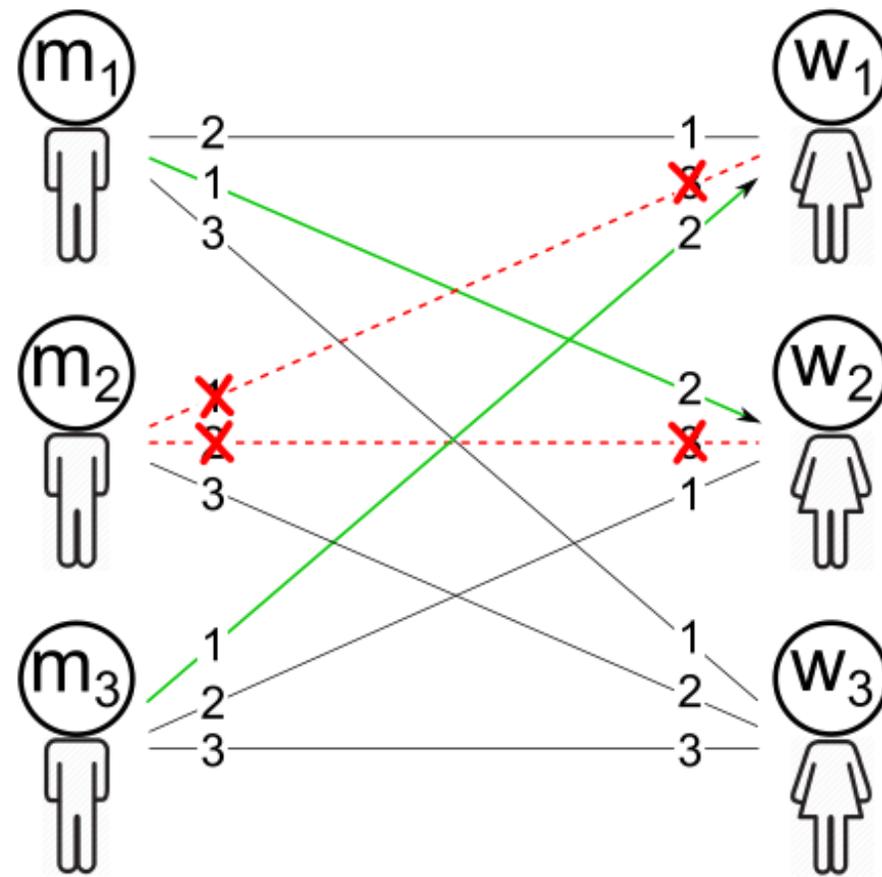
Round 2



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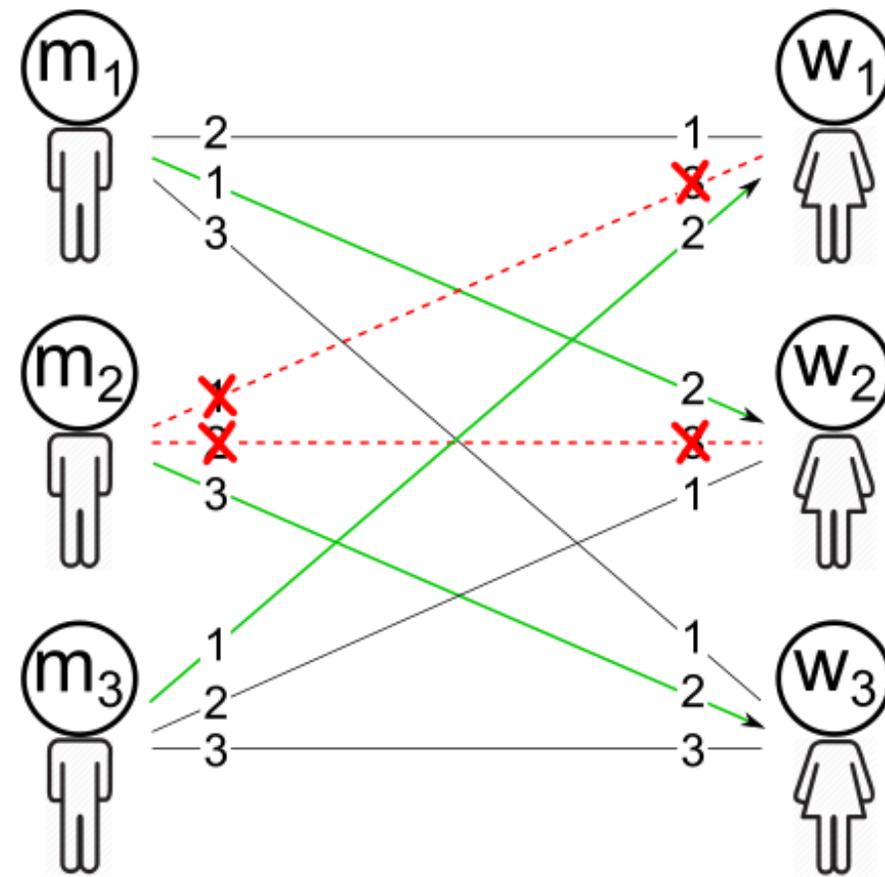
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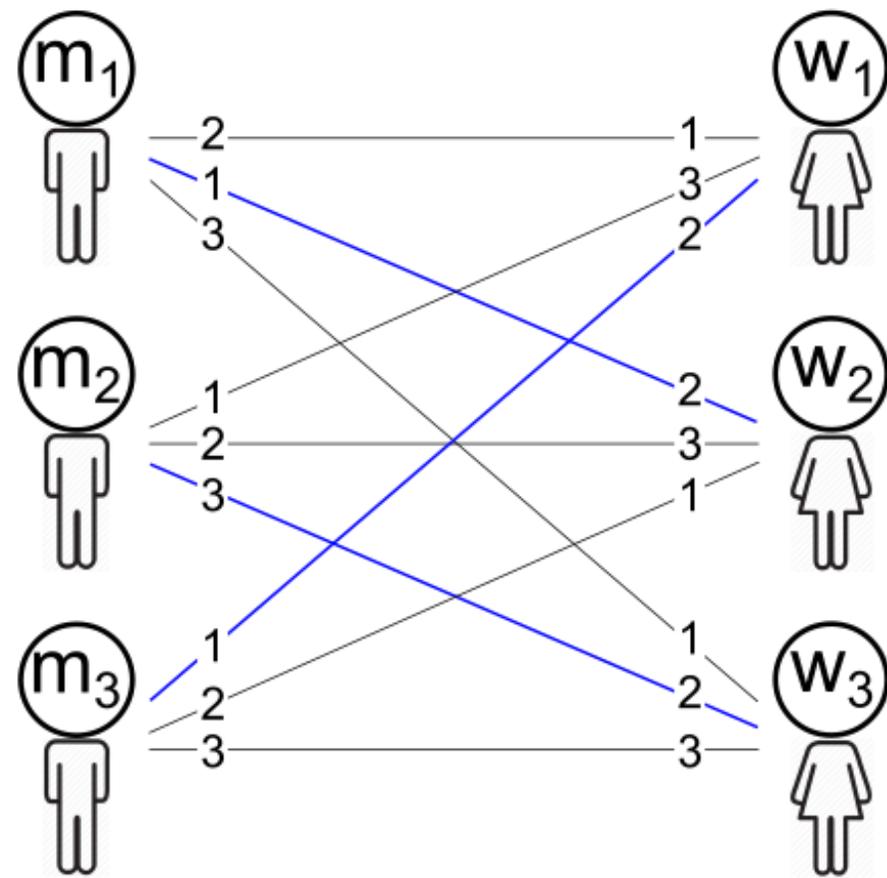
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Round 3



[Dubins and Freedman, *Amer. Math. Mon.* 1981; Roth, *MOR* 1982]

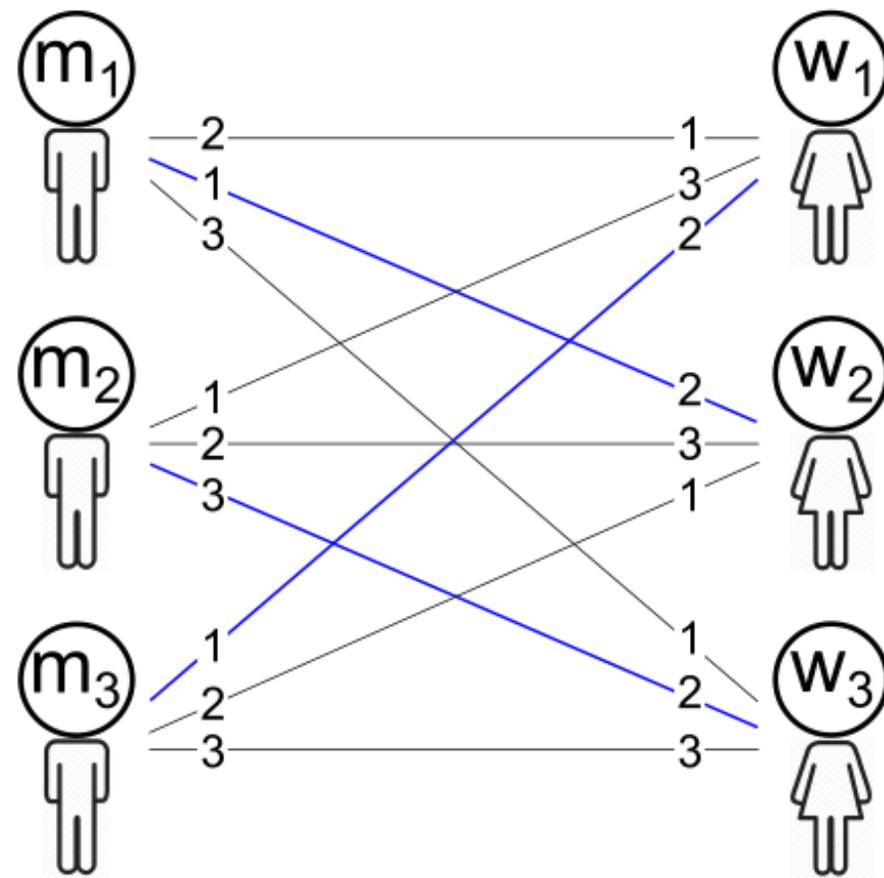
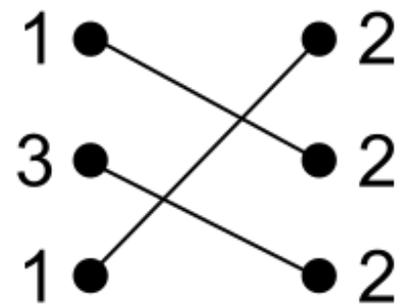
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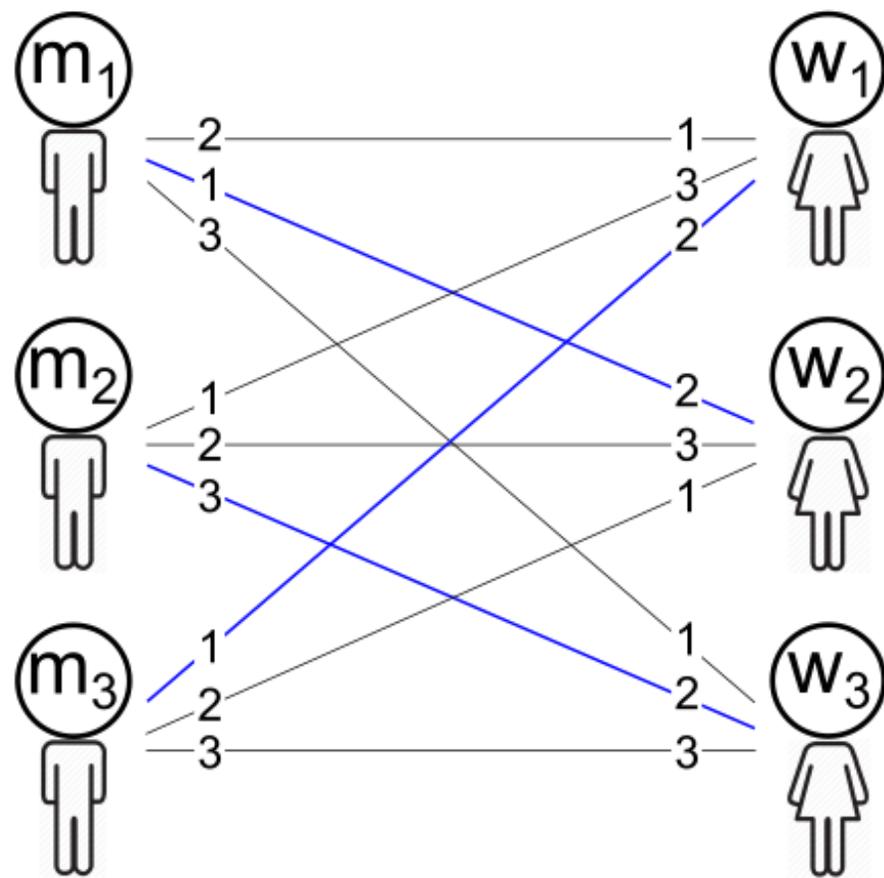
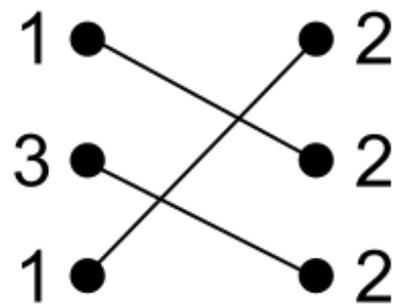
Outcome for true prefs



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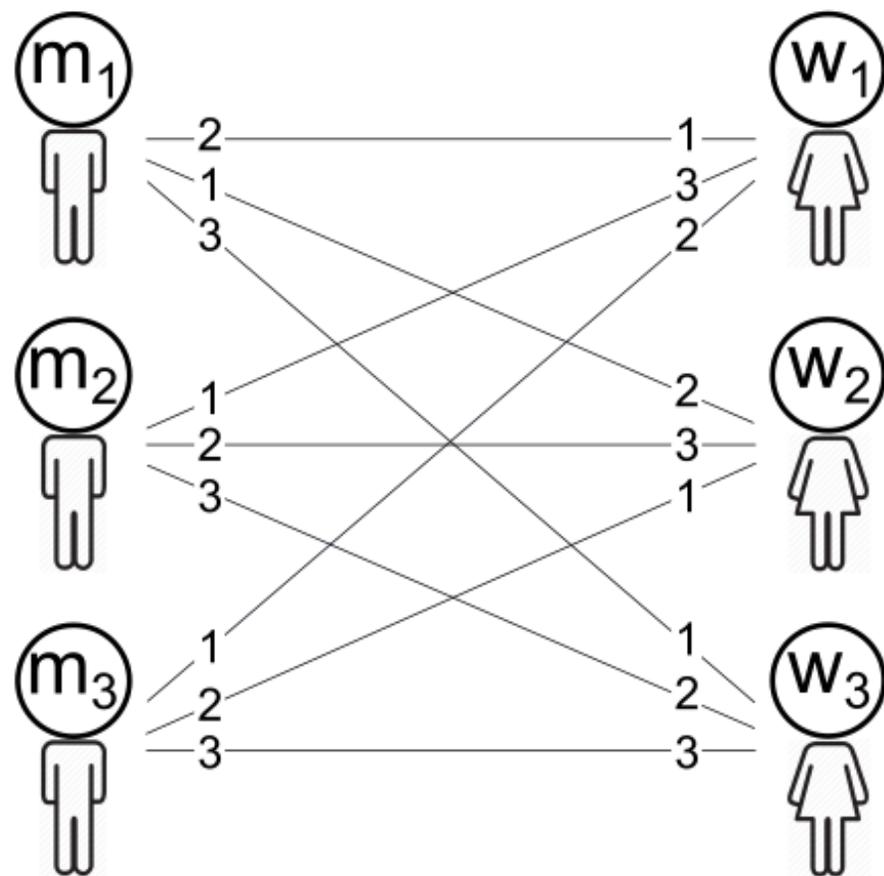
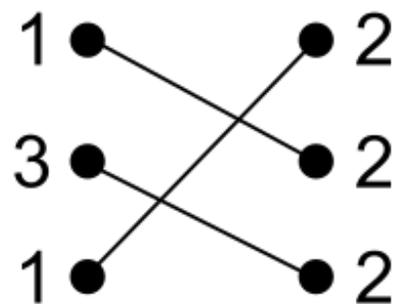


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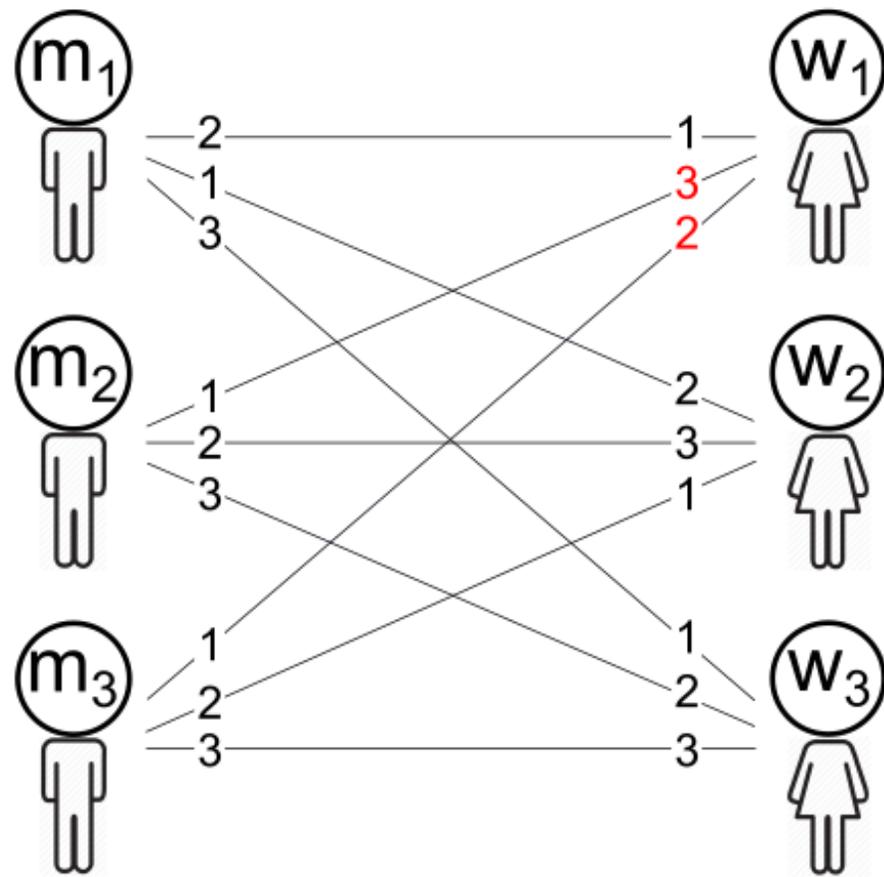
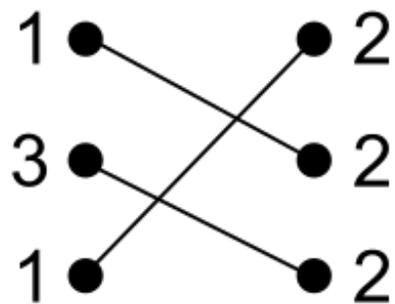


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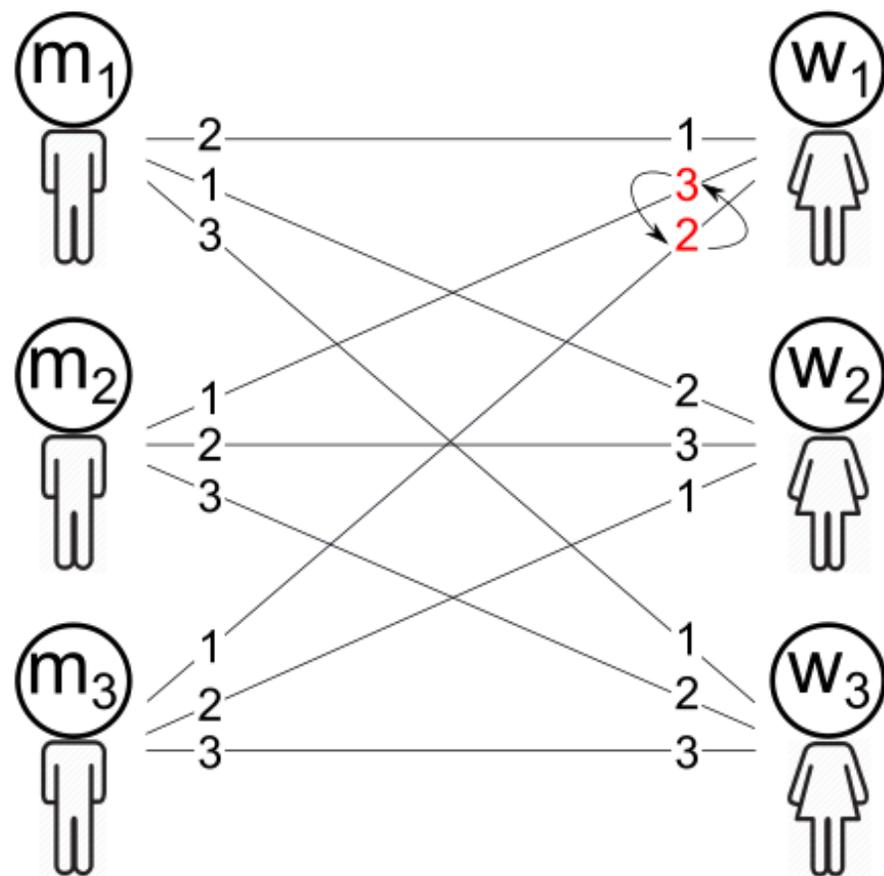
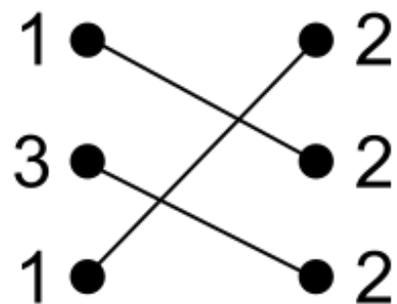


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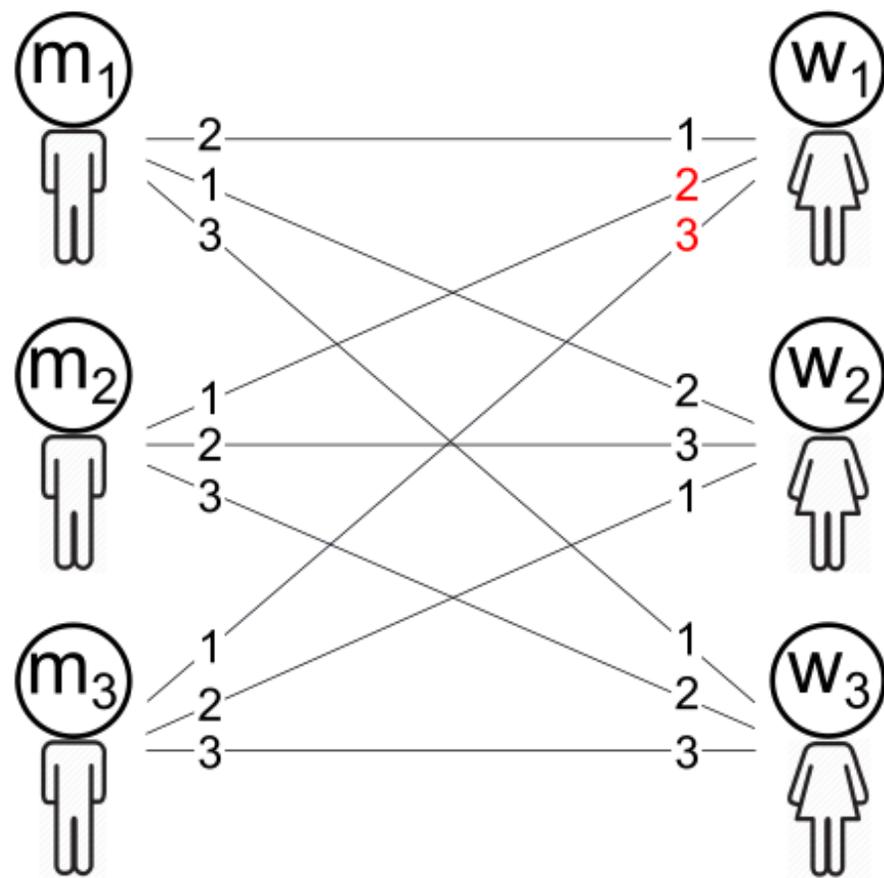
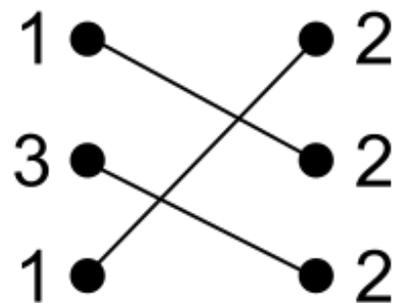


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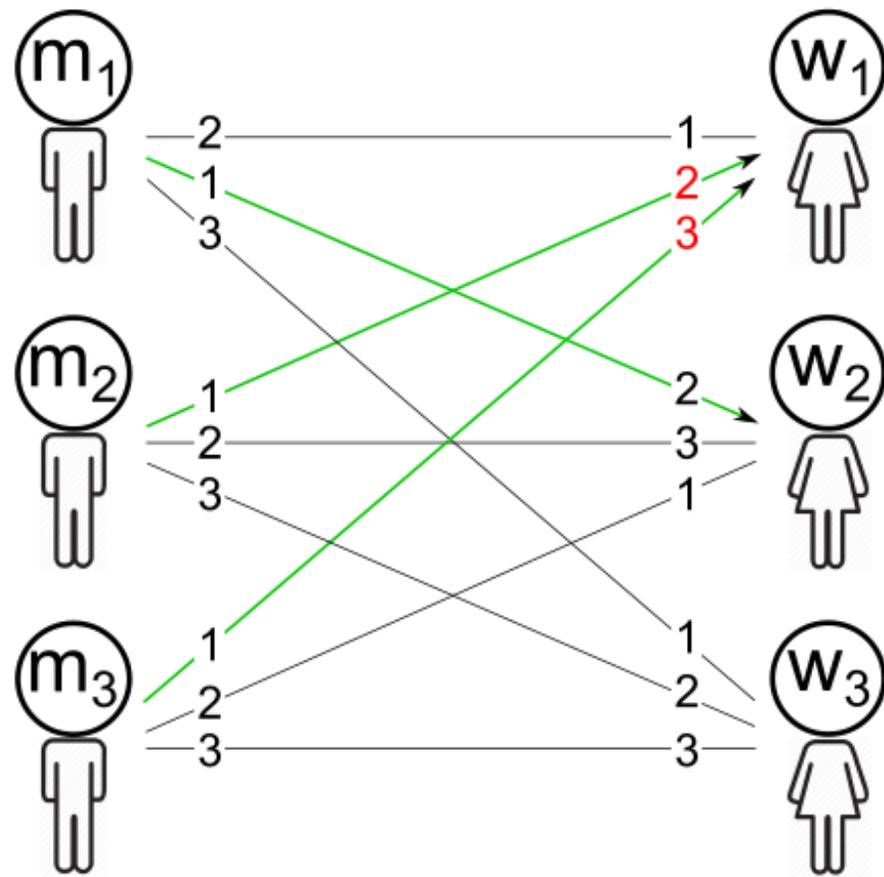
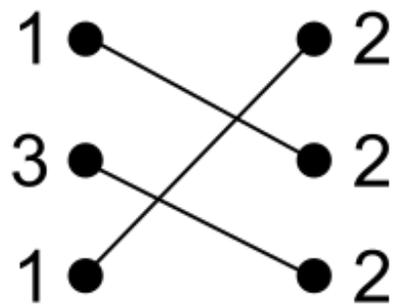


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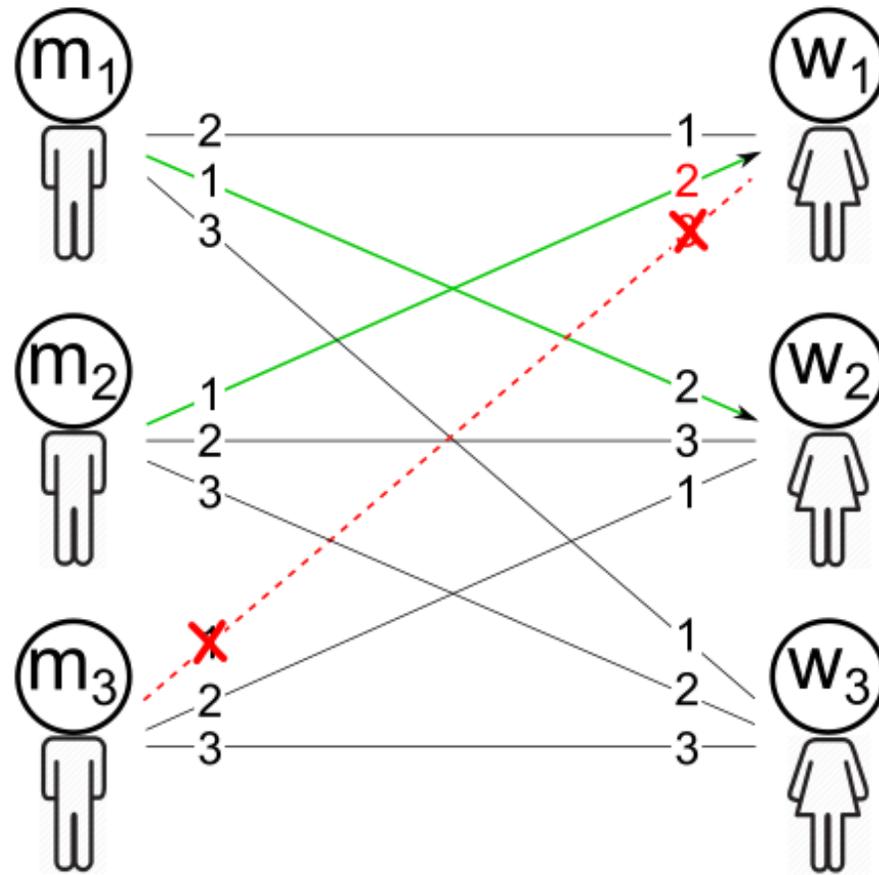
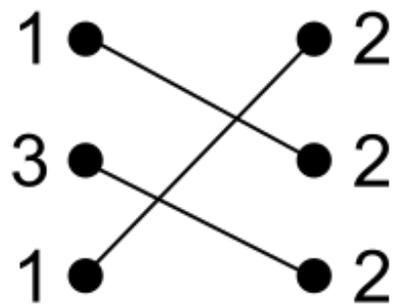


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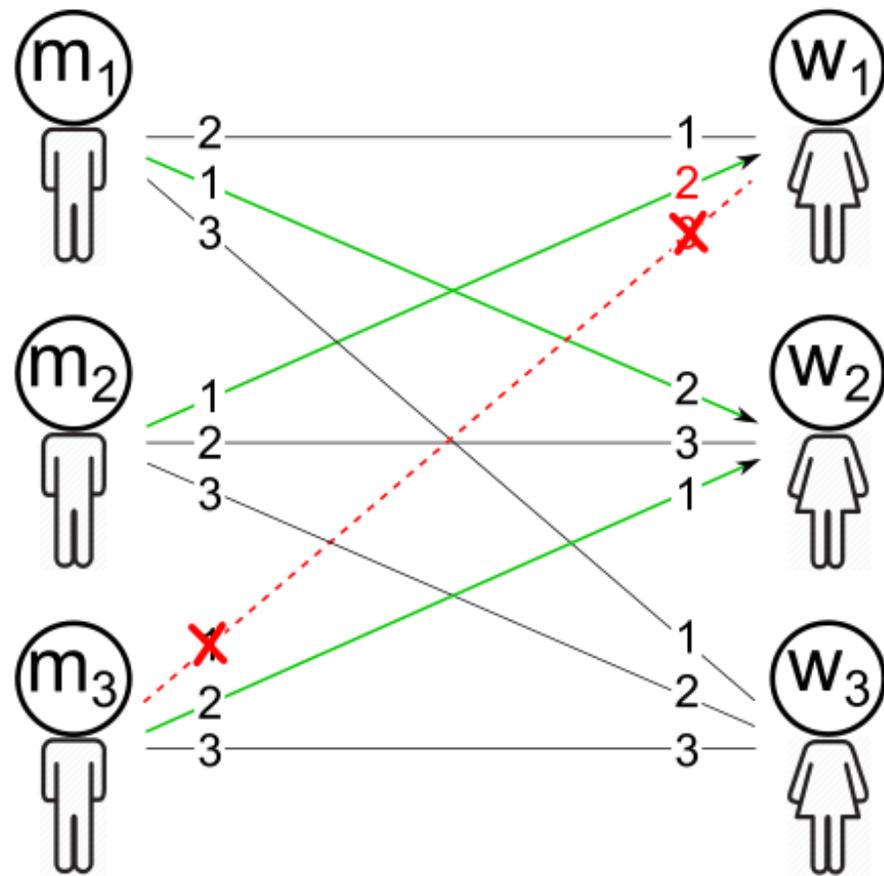
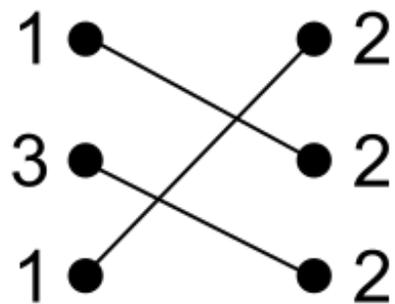


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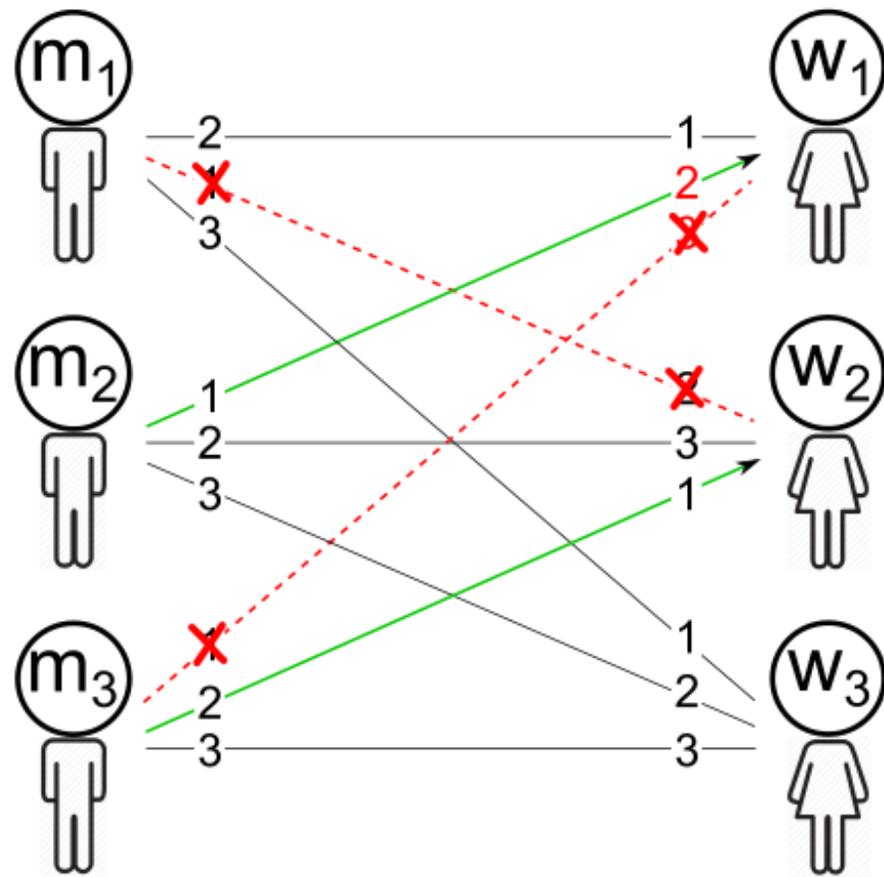
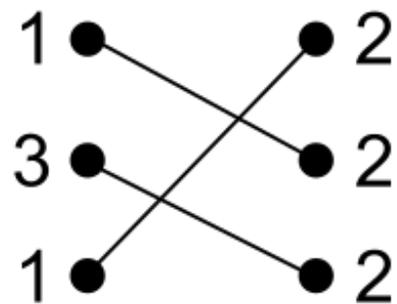


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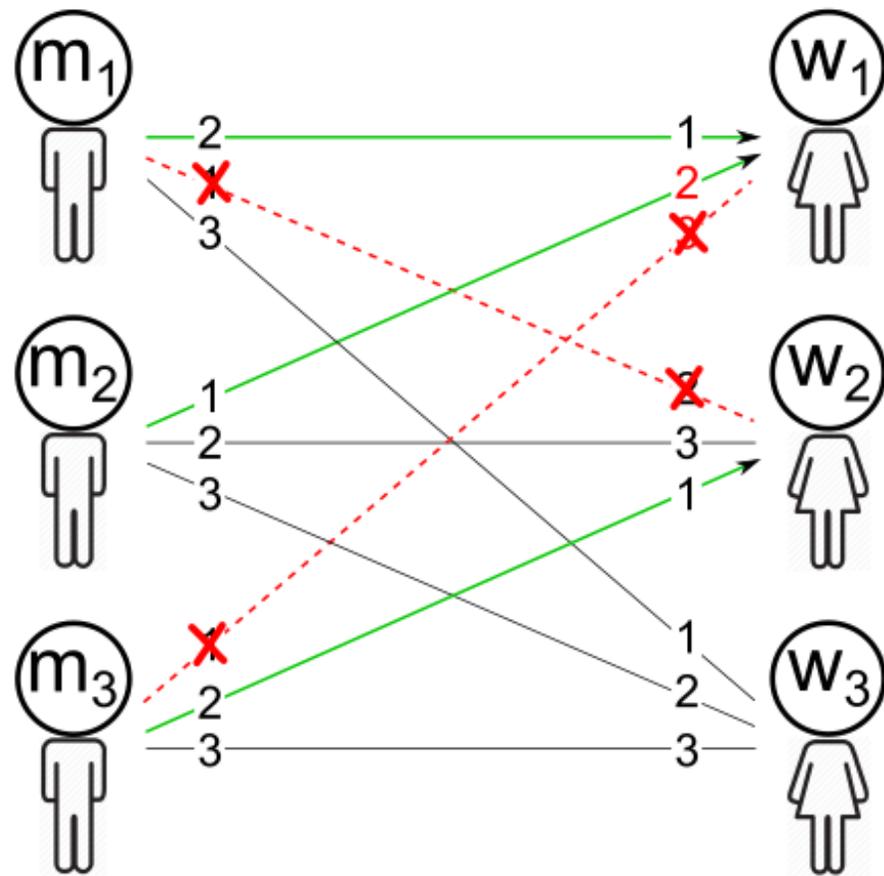
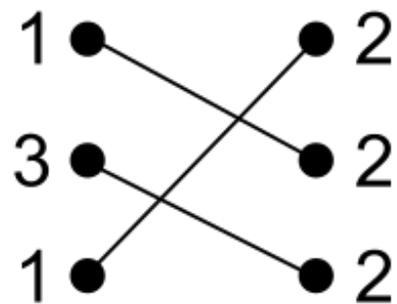


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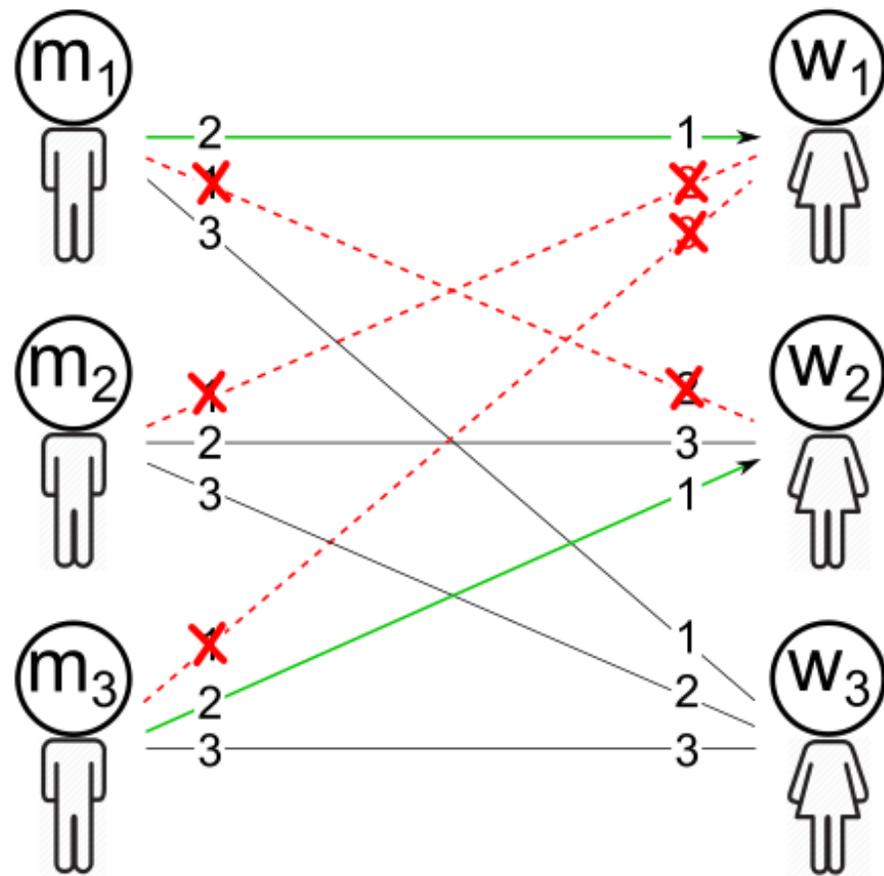
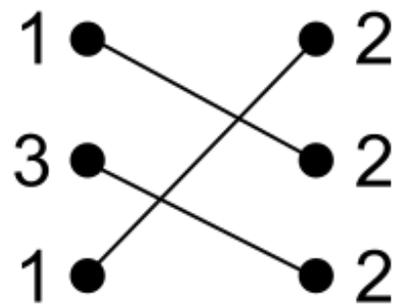


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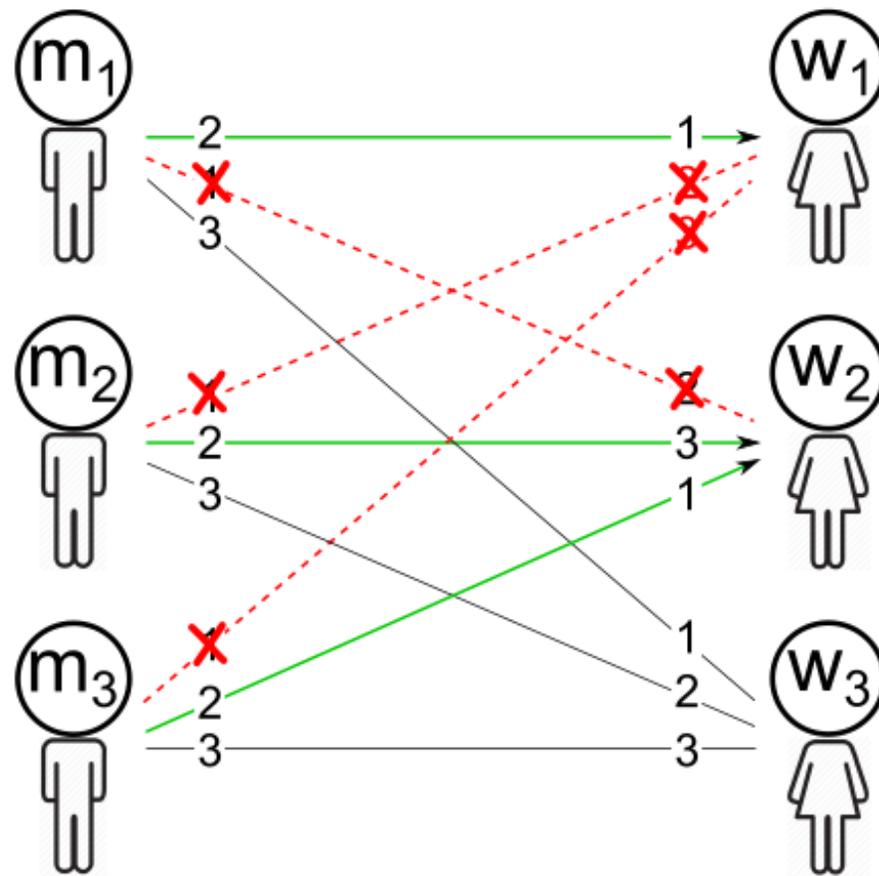
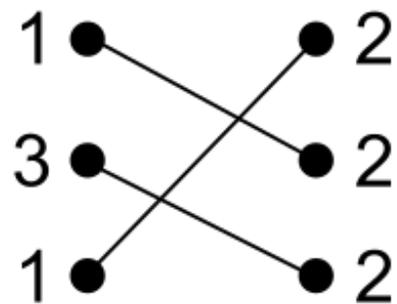


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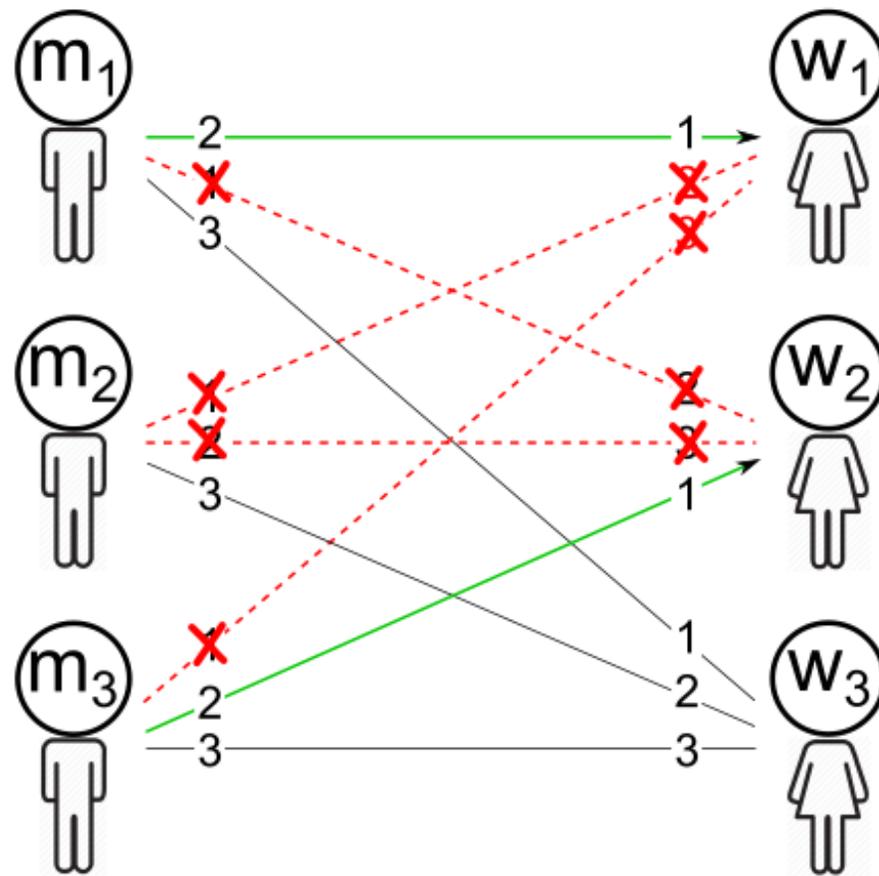
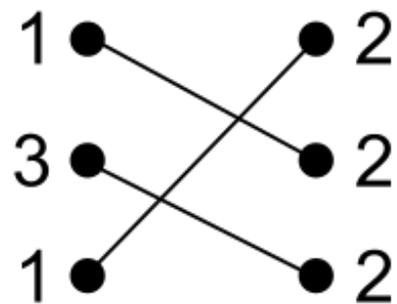


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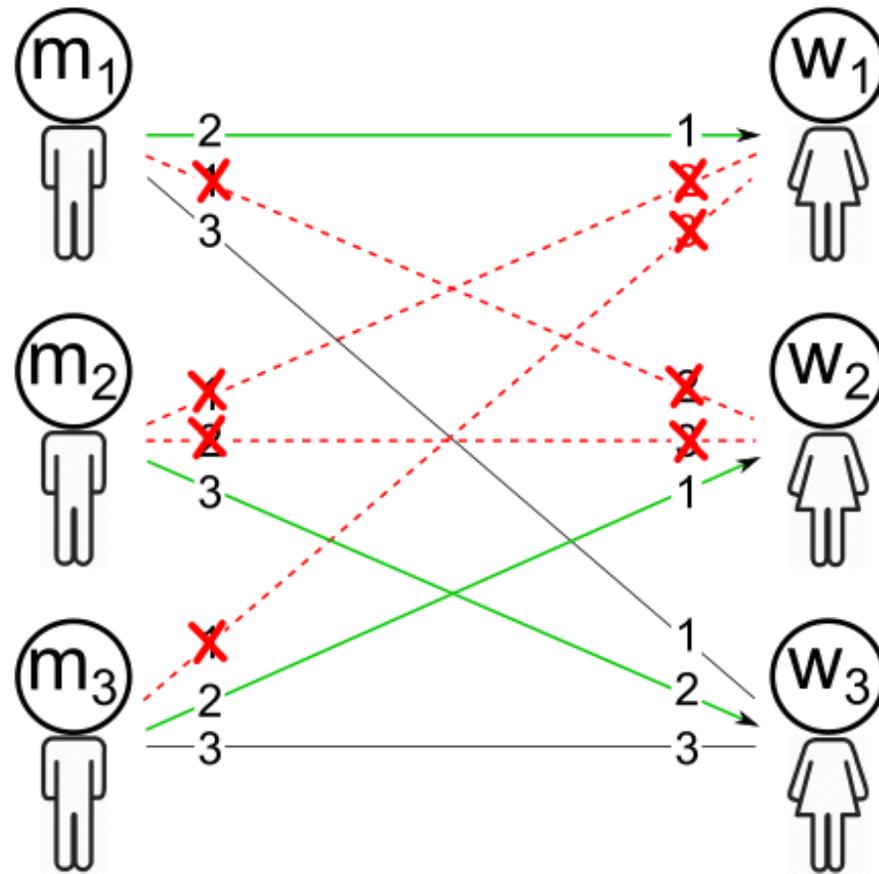
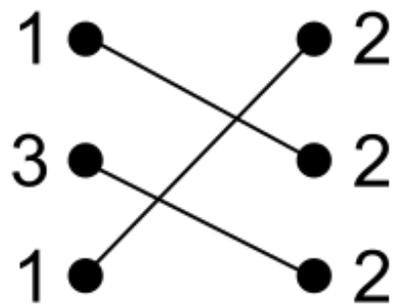


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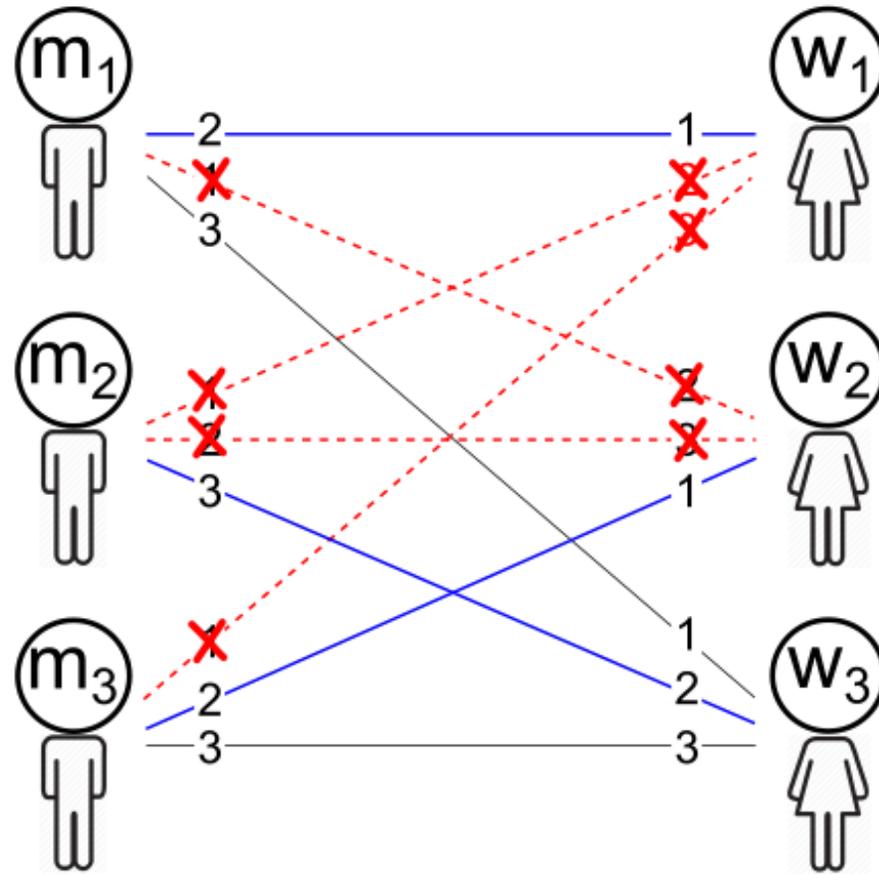
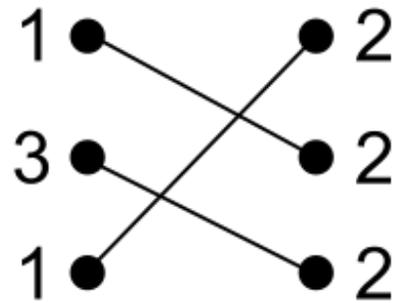


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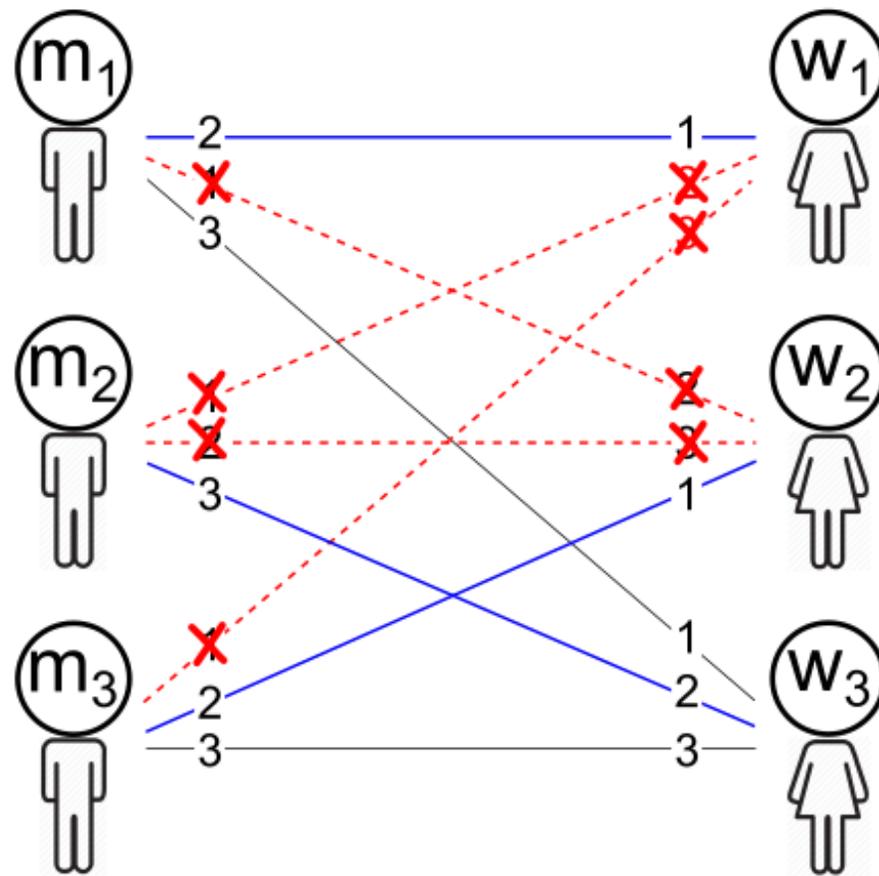
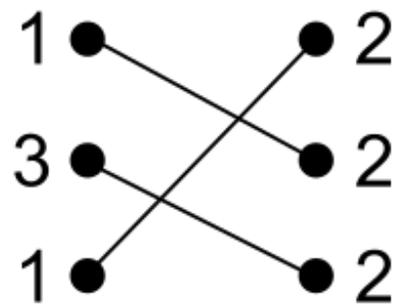


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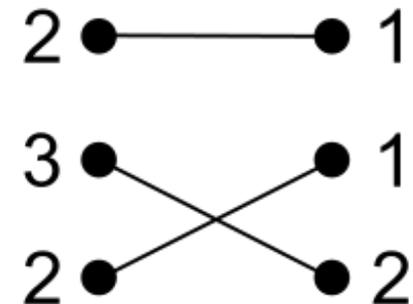
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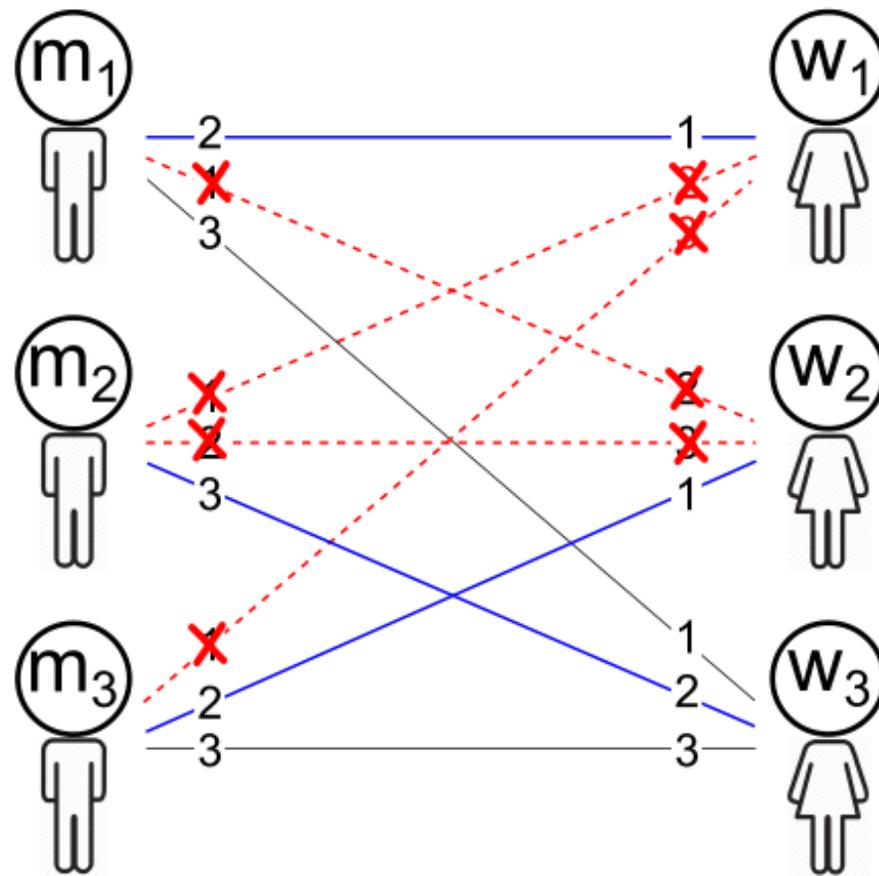
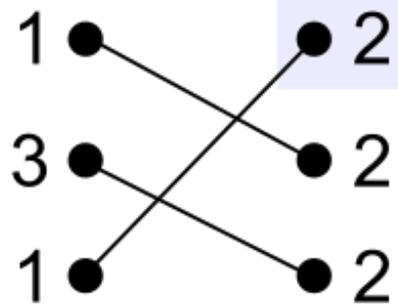
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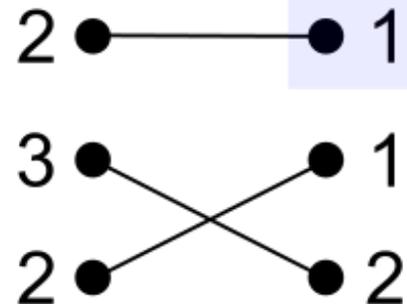
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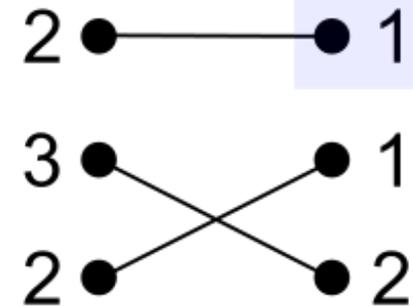
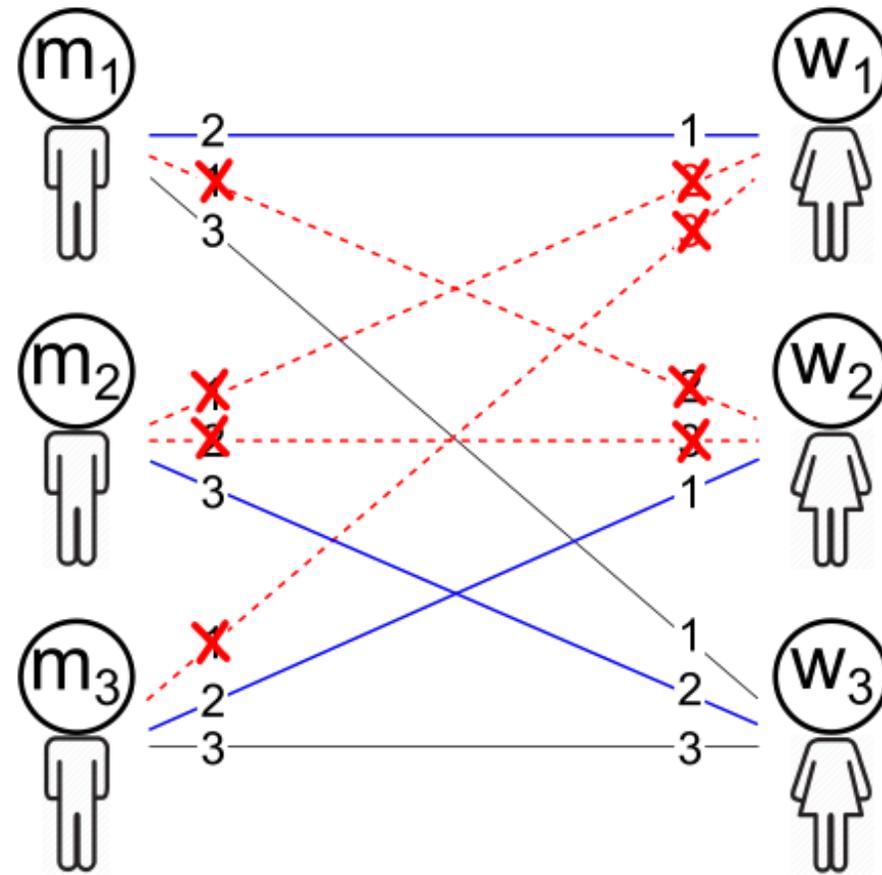
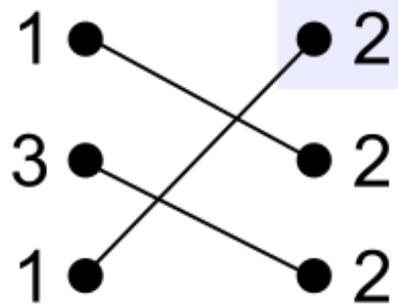
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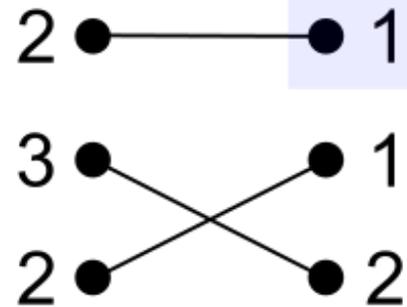
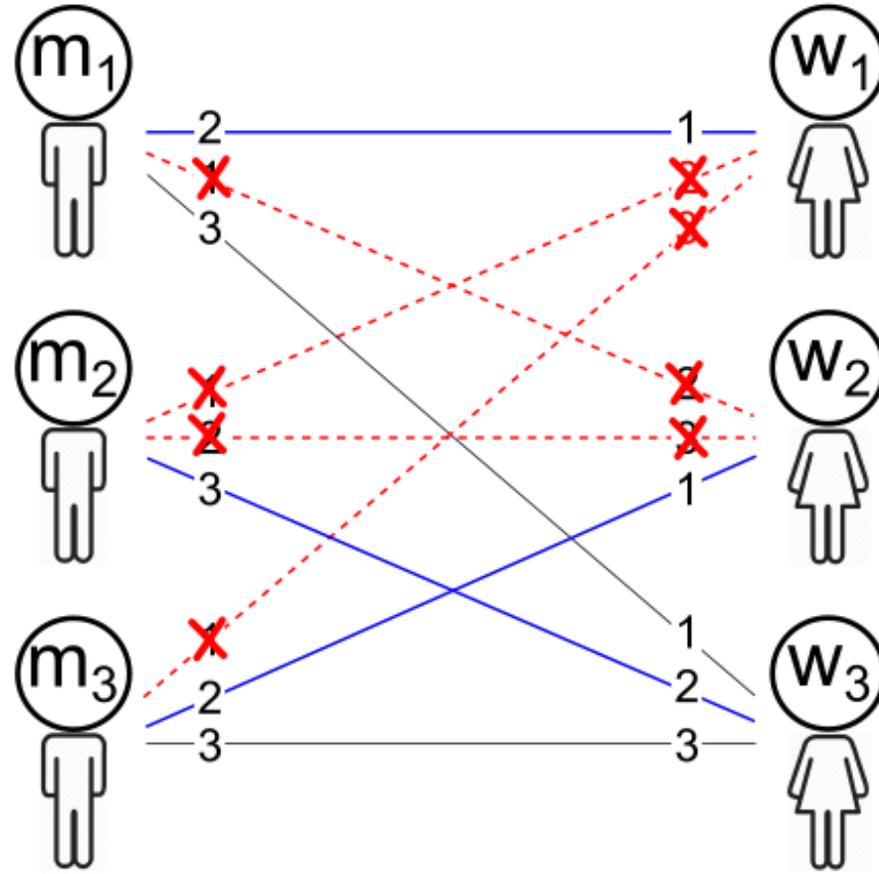
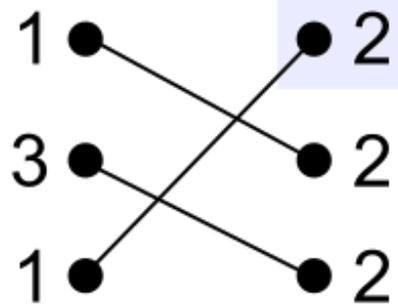
DA algorithm is not strategyproof.

Outcome for true prefs



DA algorithm is not strategyproof.

Outcome for true prefs



Any luck for the men?

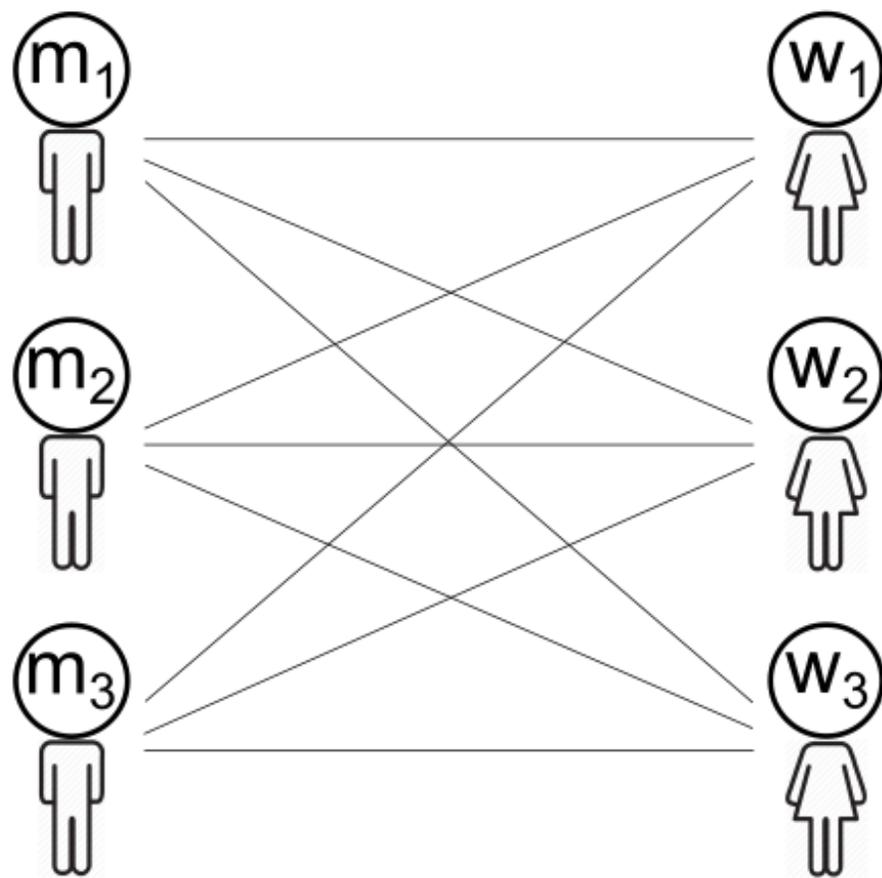
[Dubins and Freedman, *Amer. Math. Mon.* 1981; Roth, *MOR* 1982]

DA algorithm is strategyproof for the men.

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DA algorithm is strategyproof for the men.

Can I get a better partner by misreporting my preferences?



[Dubins and Freedman, *Amer. Math. Mon.* 1981; Roth, *MOR* 1982]

DA algorithm is strategyproof for the men.

Can I get a better partner by misreporting my preferences?

Nope.

Nope.

Nope.

Nope.

Nope.

m_1

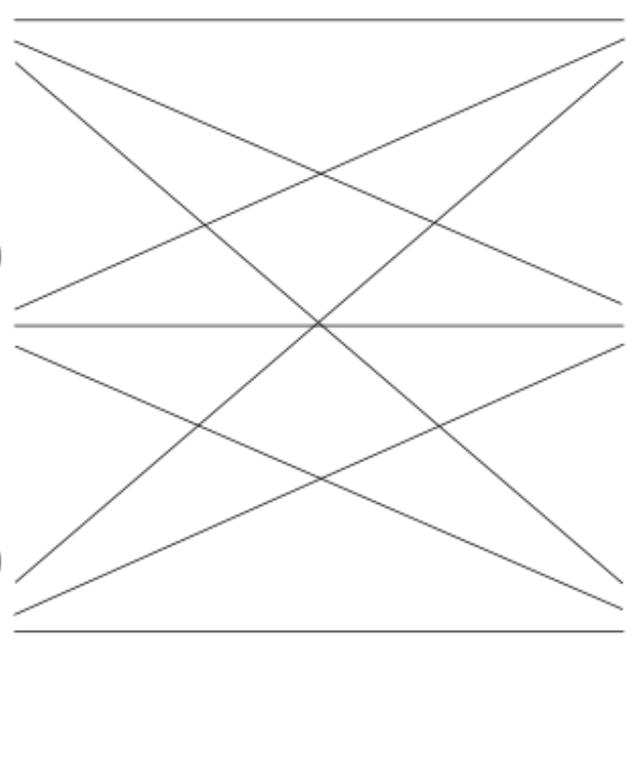
m_2

m_3

w_1

w_2

w_3



[Dubins and Freedman, *Amer. Math. Mon.* 1981; Roth, *MOR* 1982]

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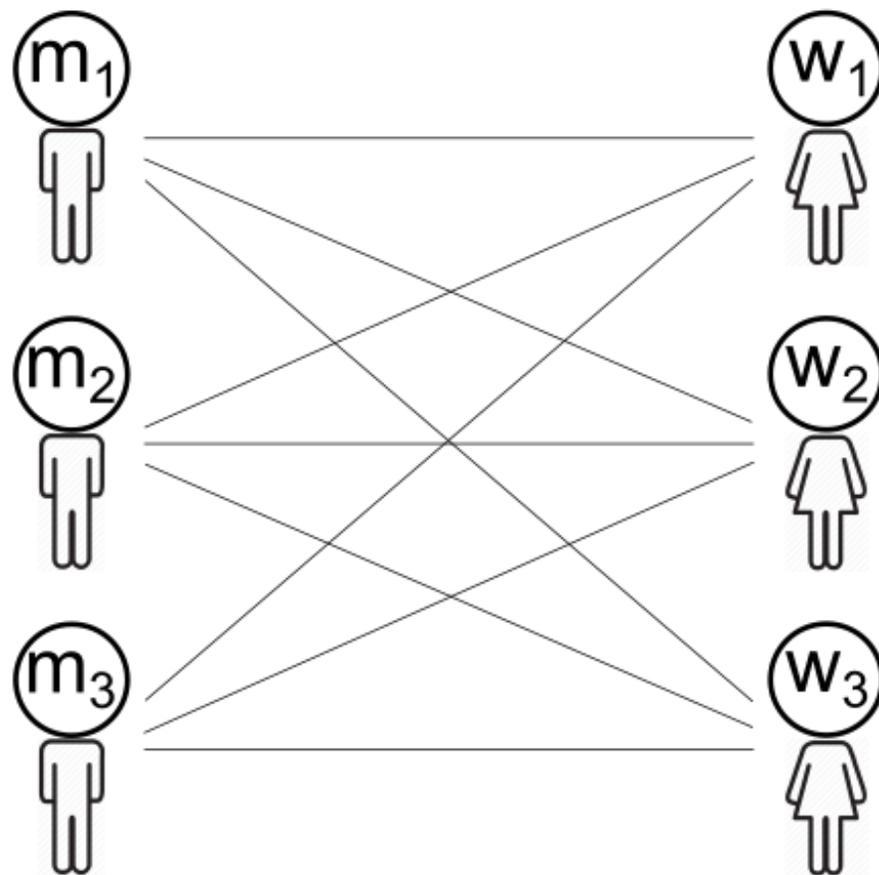
Nope.

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Nope.



Proof ~~later in today's lecture~~ in the lecture slides.

So, men can't cheat in the men-proposing DA algorithm but there **exists** an opportunity for a woman to manipulate.

So, men can't cheat in the men-proposing DA algorithm but there **exists** an opportunity for a woman to manipulate.

Is it possible to efficiently **compute** a beneficial misreport whenever one exists?

[Teo, Sethuraman and Tan, *Manag. Sci.* 2001; Vaish and Garg, *IJCAI* 2017]

An optimal manipulation for a woman can be computed
in polynomial time.

[Teo, Sethuraman and Tan, *Manag. Sci.* 2001; Vaish and Garg, *IJCAI* 2017]

An optimal manipulation for a woman can be computed
in polynomial time.

We will use a structural result about optimal manipulation strategies.

Any optimal manipulation for a woman can also be achieved by an "inconspicuous" misreport.

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True list of woman w : $m_1 > m_2 > m_3 > m_4 > \boxed{m_5} > m_6 > m_7 > m_8$

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An optimal manipulation for a woman can be computed in polynomial time.

But what about stability (w.r.t. true preferences)?

The DA matching after optimal manipulation by a woman is stable with respect to the true preferences.

The DA matching after optimal manipulation by a woman is stable with respect to the true preferences.

We will use the following observation:

Suppose a woman promotes a man m in her list and no other changes are made.

If m proposed to her during DA on the old profile, then he proposes to her during DA on the new profile.

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If m proposed to her during DA on the old profile, then he proposes to her during DA on the new profile.

Idea: Any deviation between old and new runs of the DA must involve tentative acceptance/rejection of m , but that can happen only after m proposes.

The DA matching after optimal manipulation by a woman is stable with respect to the true preferences.

The DA matching after optimal manipulation by a woman is stable with respect to the true preferences.

P = profile with **true** preferences

P' = profile with **manipulated** preferences

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$$P = (P_{-w}, P_w)$$

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If $w' \neq w$, then m and w' are both truthful and will block X' w.r.t. P' ---contradicting the stability of DA.

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So, (m, w) blocks X' w.r.t. P .

P_m	P_w
•	•
•	•
w	m
•	•
•	•
$X'(m)$	$X'(w)$
•	•
•	•

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P_m	P_w	P'_m	P'_w
•	•	•	•
•	•	•	•
w	m	w	$X'(w)$
•	•	•	•
•	•	•	•
$X'(m)$	$X'(w)$	$X'(m)$	m
•	•	•	•
•	•	•	•

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•	•	•	•
w	m	w	$X'(w)$
•	•	•	•
•	•	•	•
$X'(m)$	$X'(w)$	$X'(m)$	m
•	•	•	•
•	•	•	•

m must propose to w during $DA(P')$

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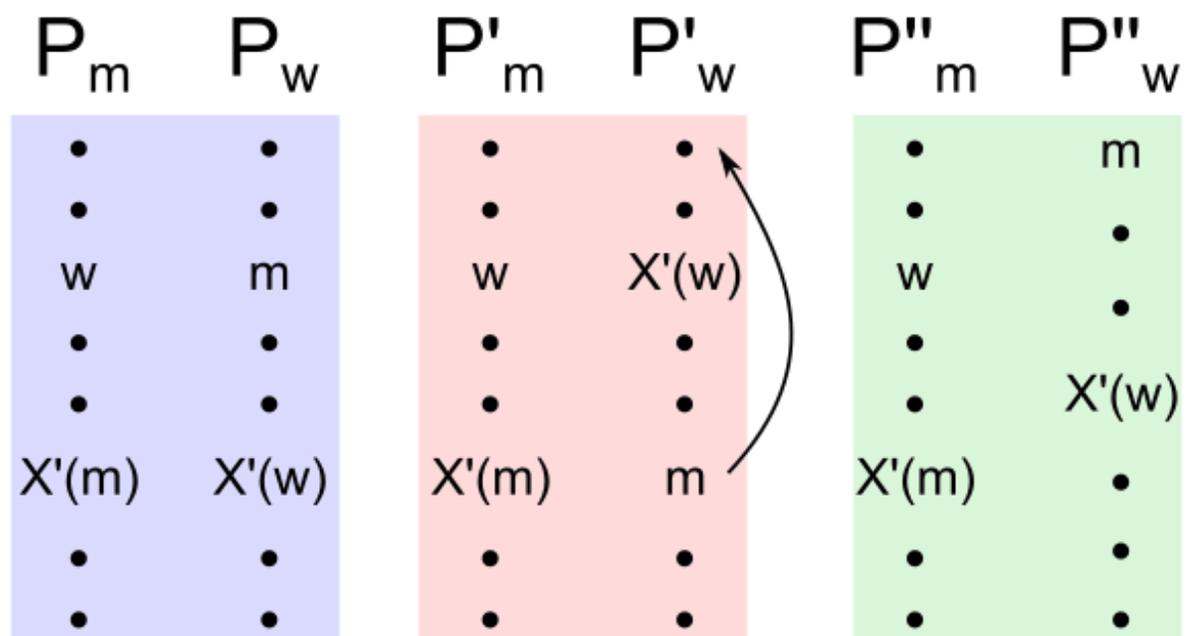
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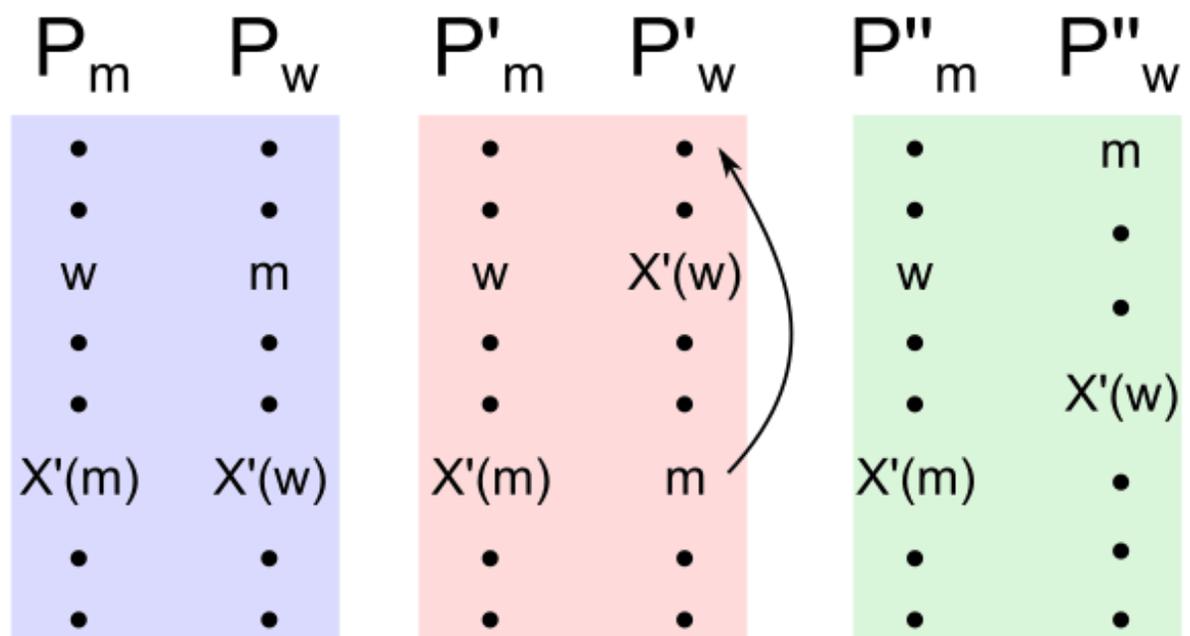
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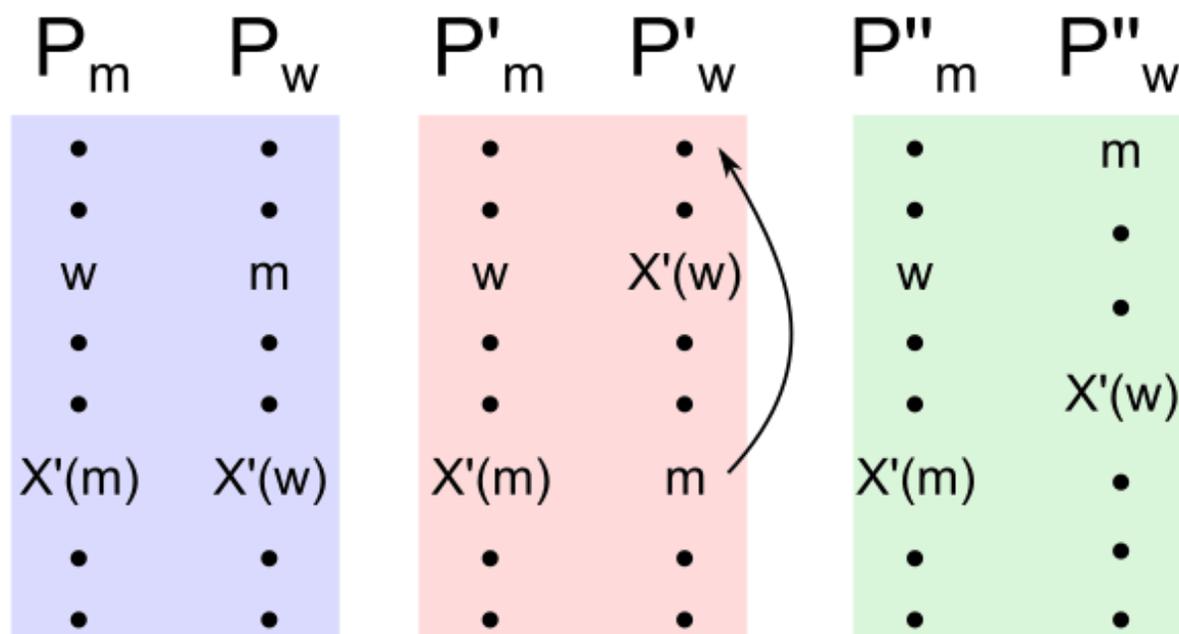
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m must propose to w during $DA(P')$
 $\Rightarrow m$ must propose to w during $DA(P'')$

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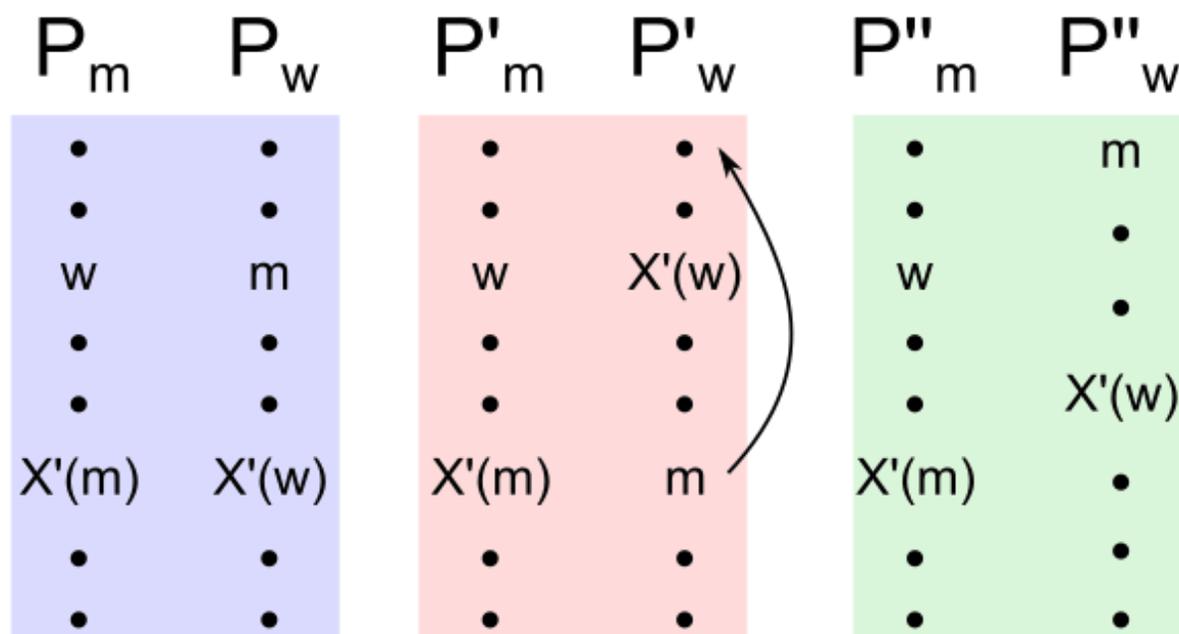
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m must propose to w during $DA(P')$
 $\Rightarrow m$ must propose to w during $DA(P'')$
 $\Rightarrow X''(w) = m$, where $X'' = DA(P'')$

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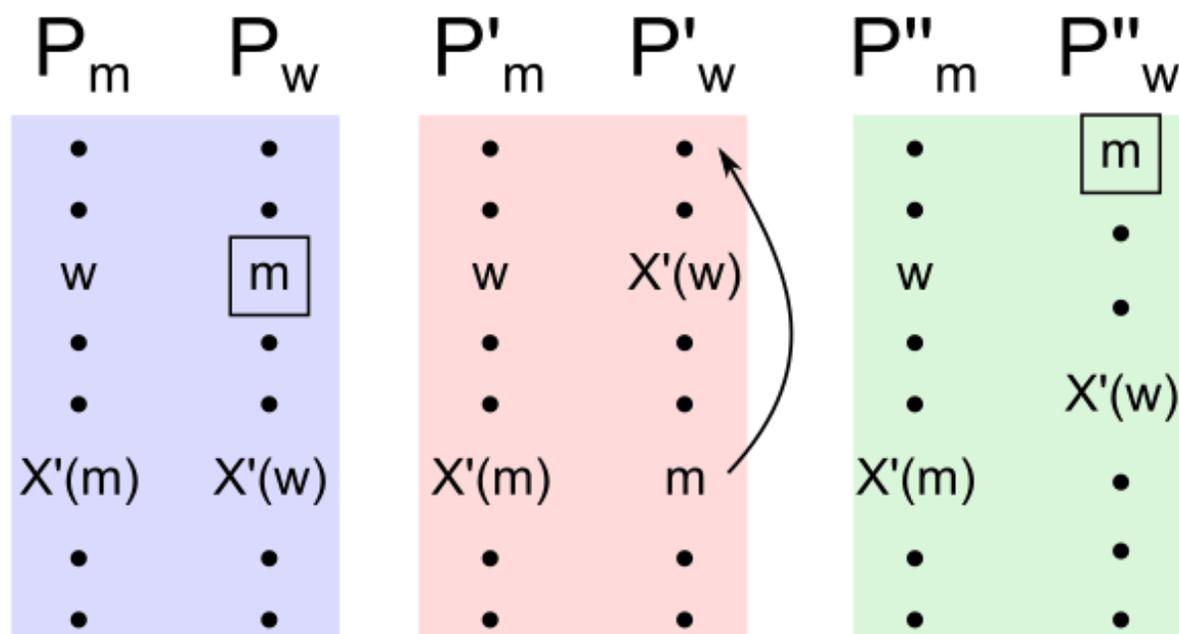
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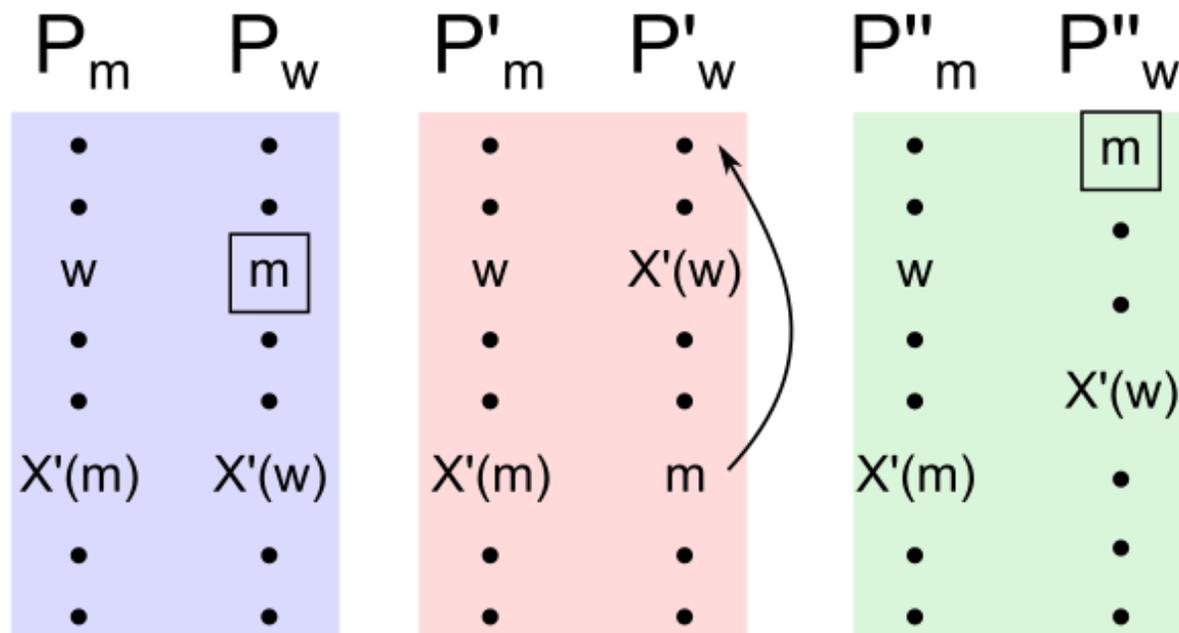
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 But then, P''_w gives w a better partner than under her optimal strategy!

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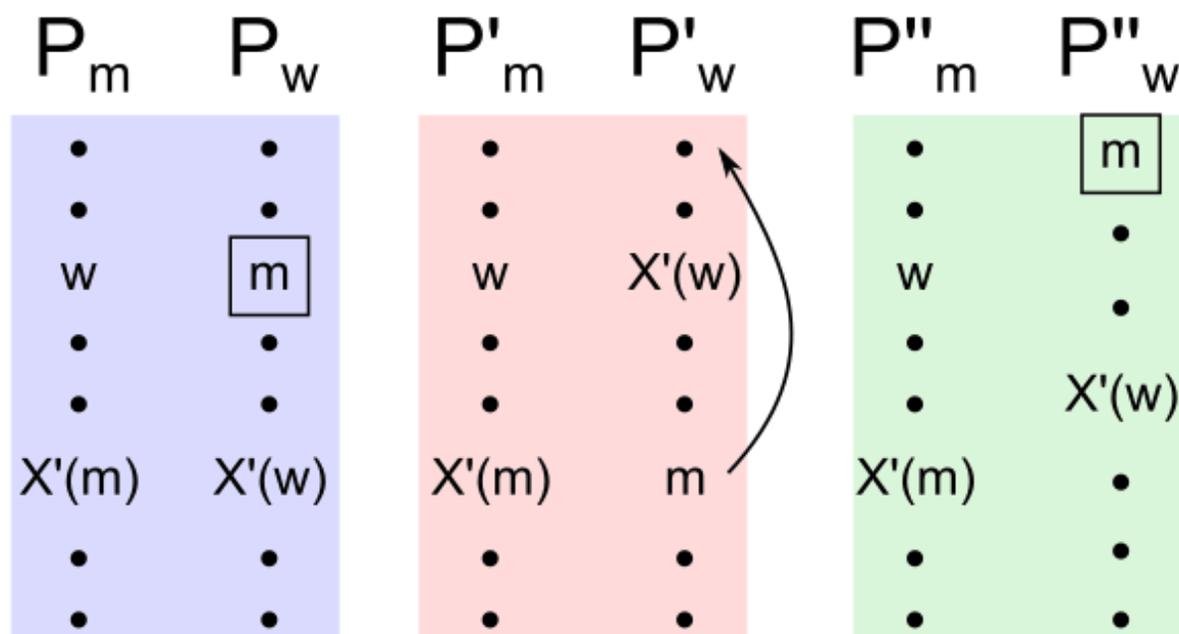
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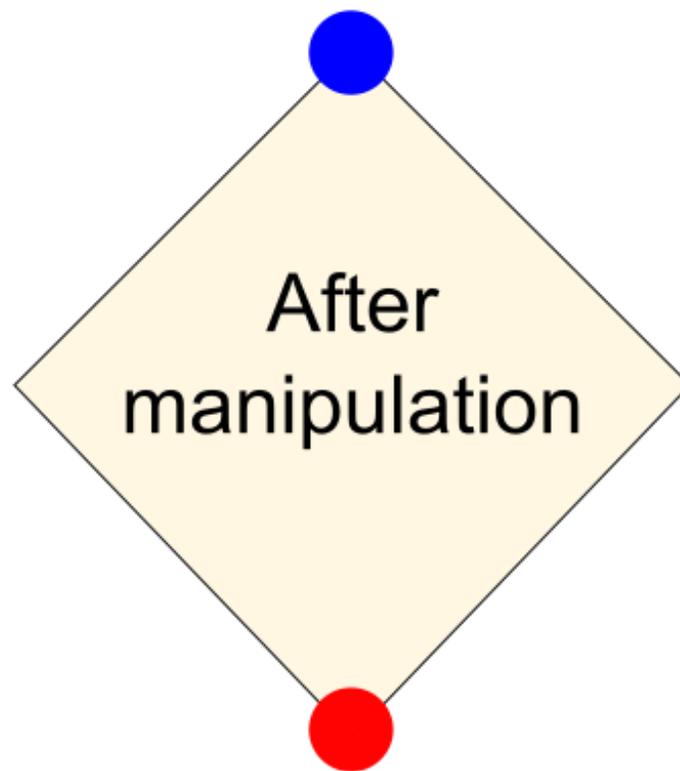
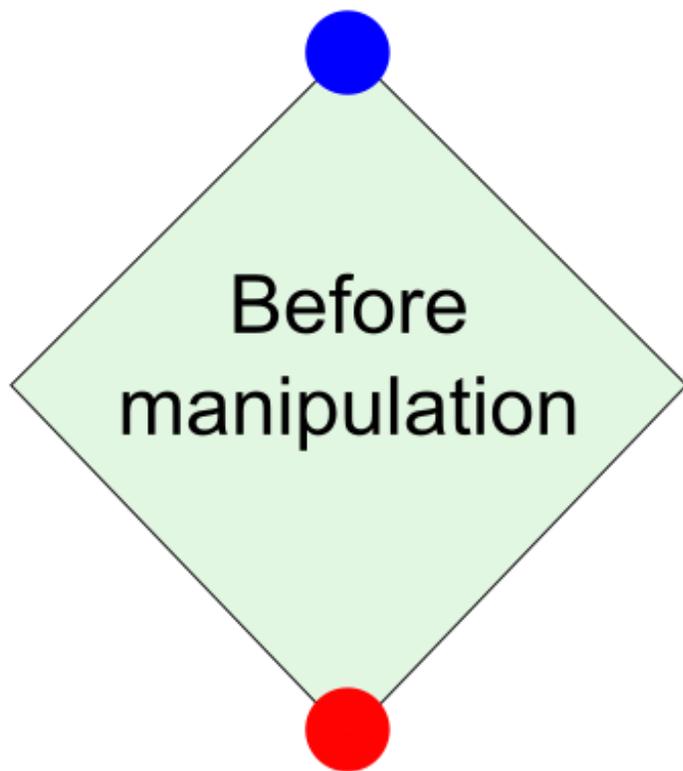
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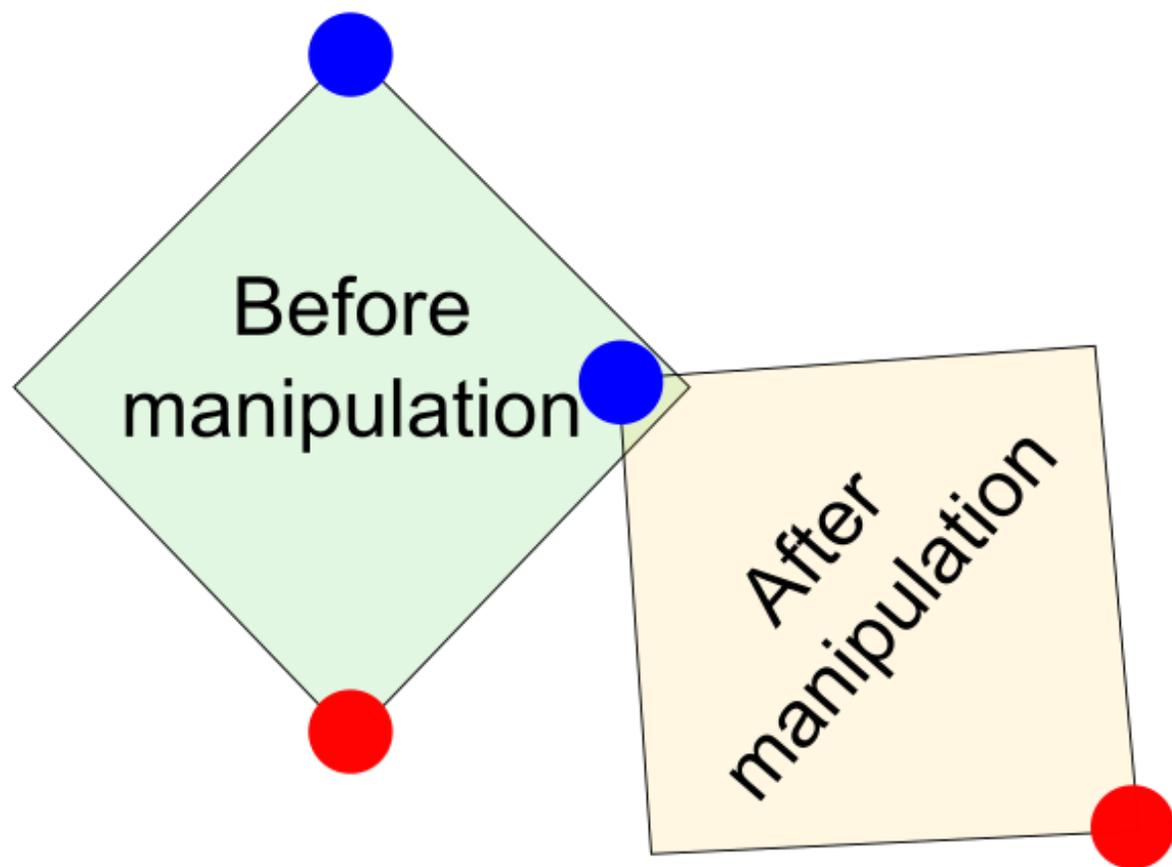
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*Stable marriages are manipulable,
but optimally manipulated marriages are stable.*

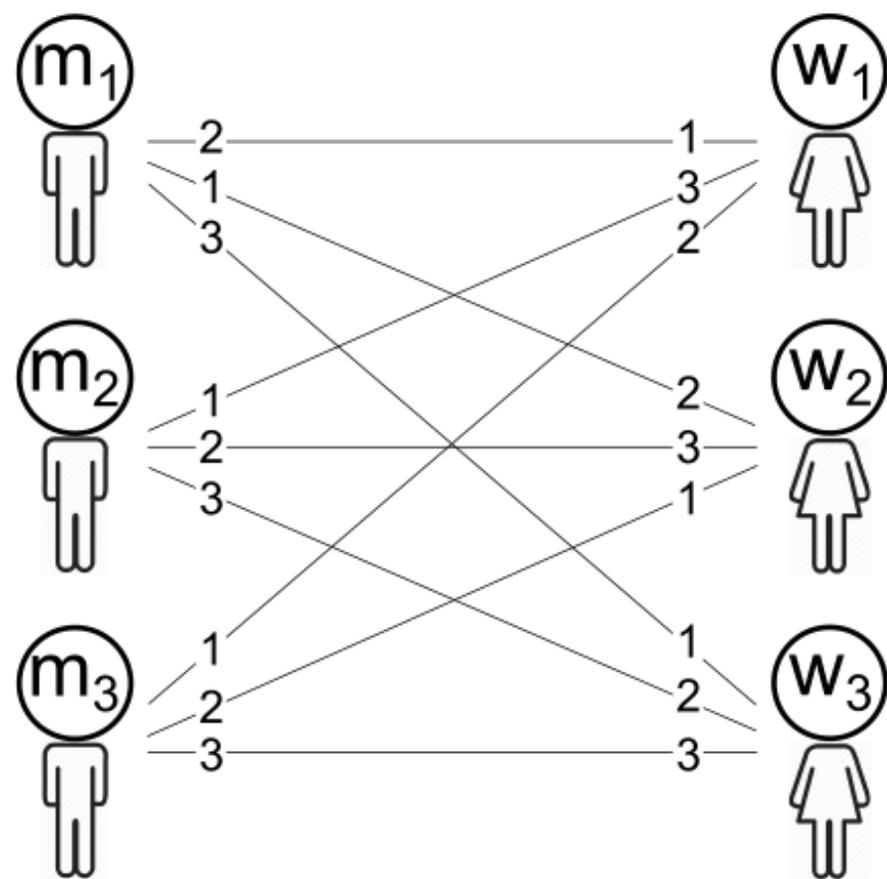
DA fails strategyproofness---too bad!

Let's think of a different stable matching algorithm that is truthful.

[Roth, *MOR* 1982]

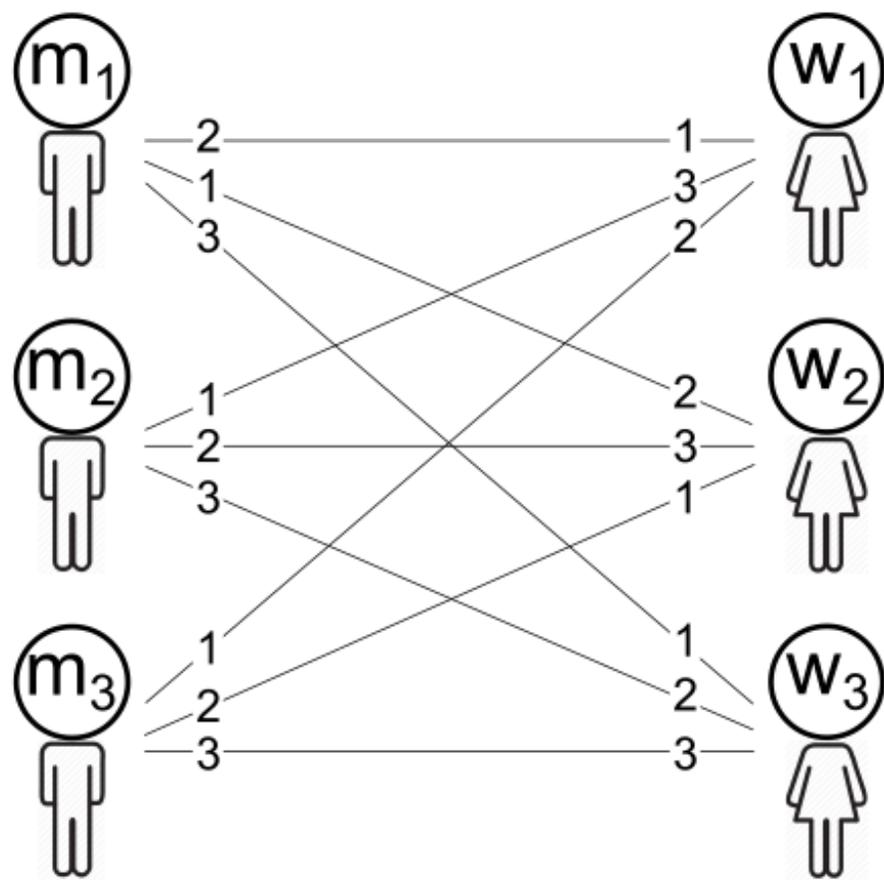
No stable matching procedure can be strategyproof.

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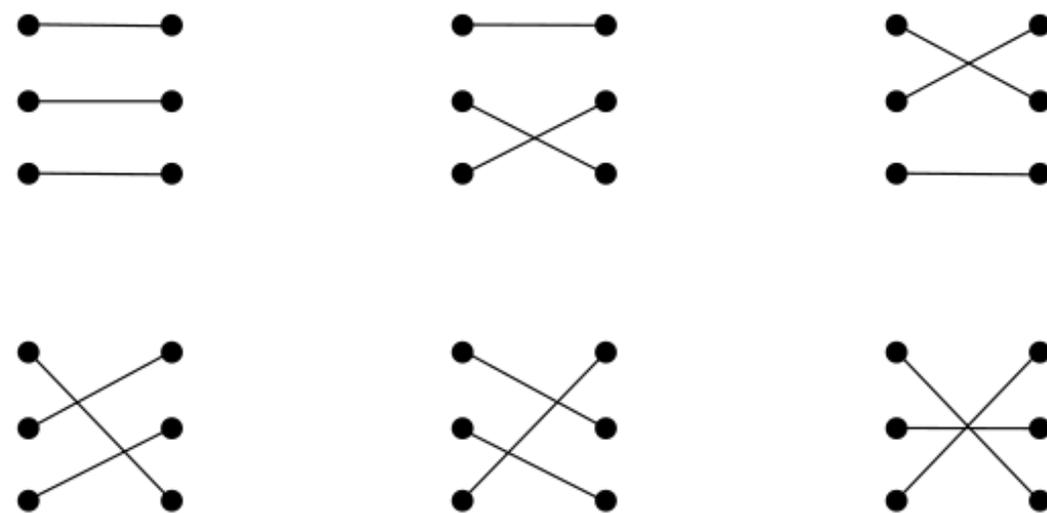


Instance I_0

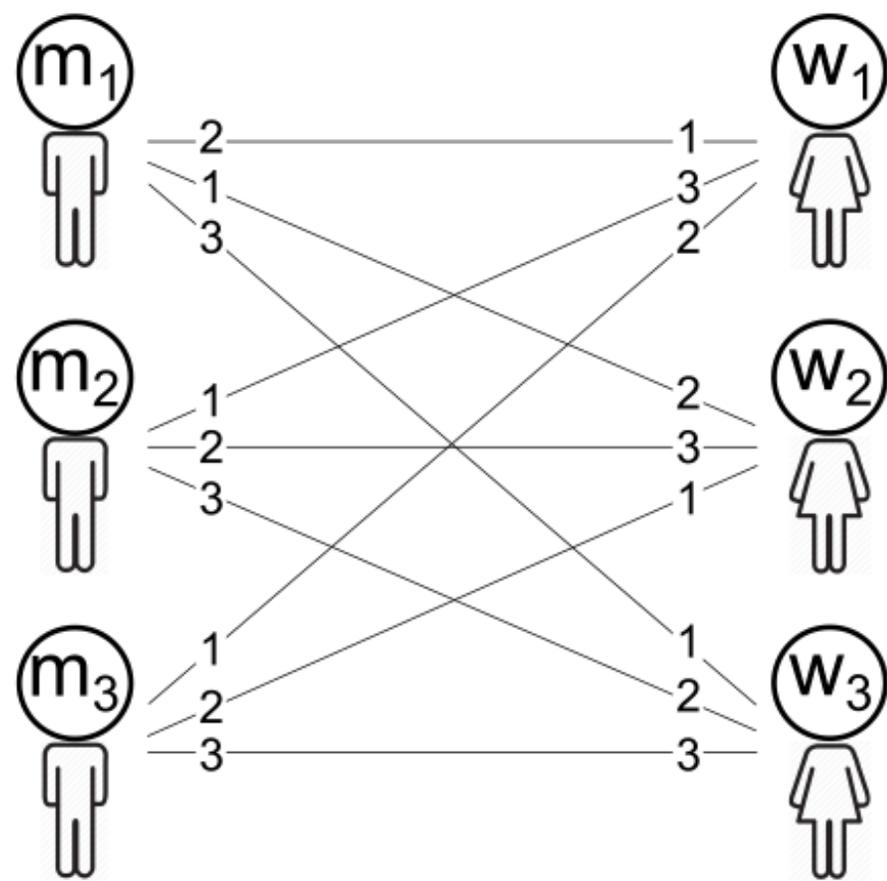
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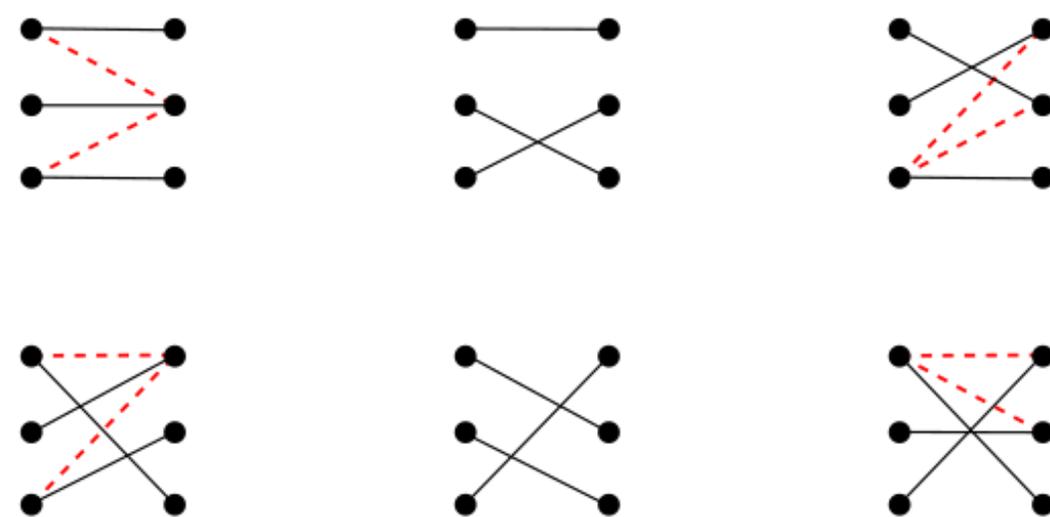
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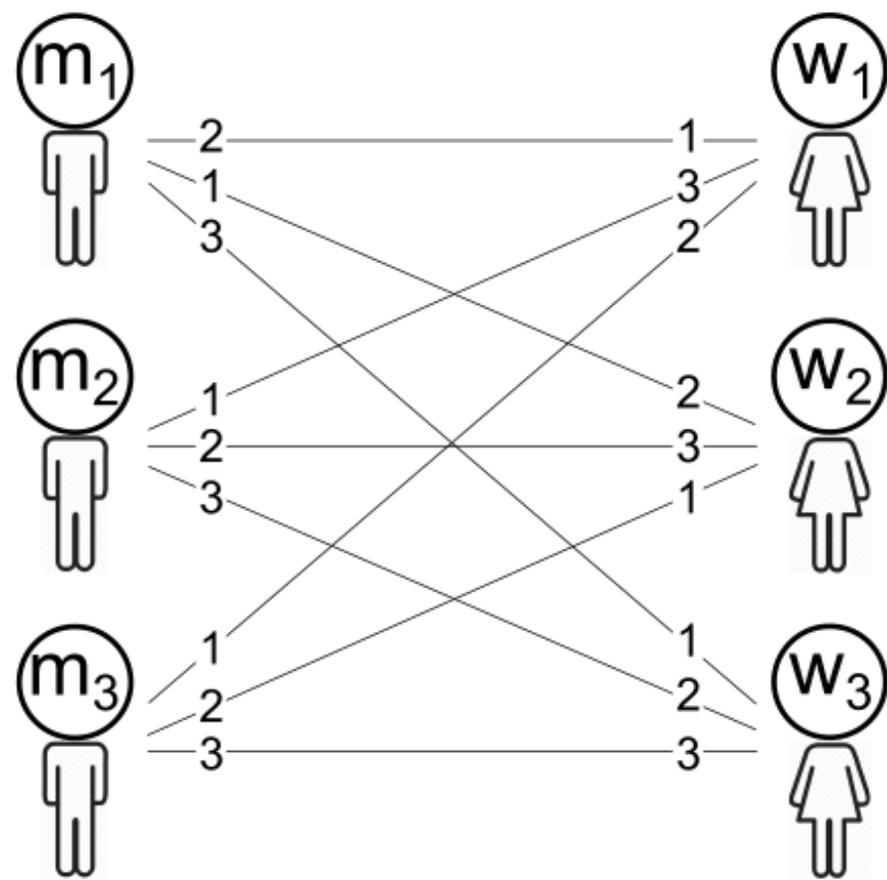
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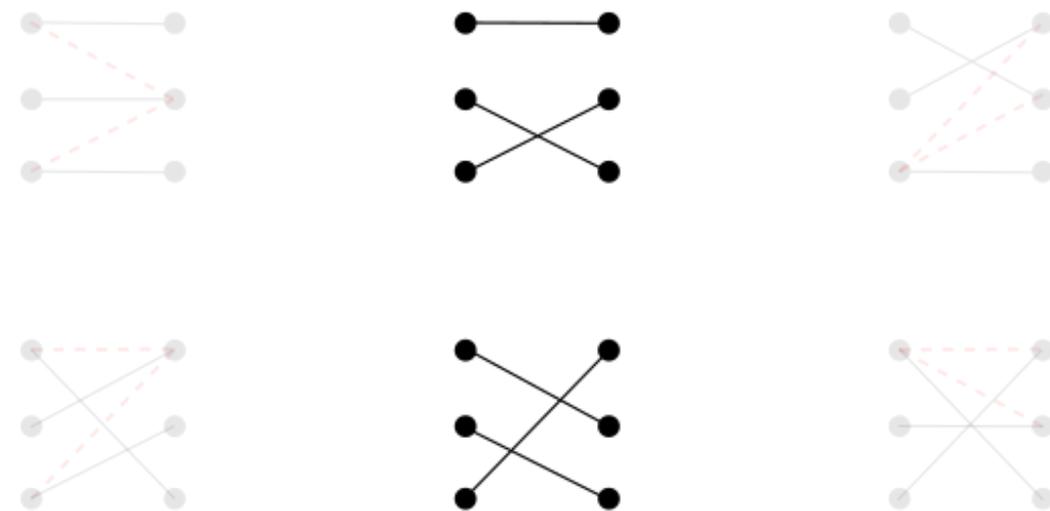
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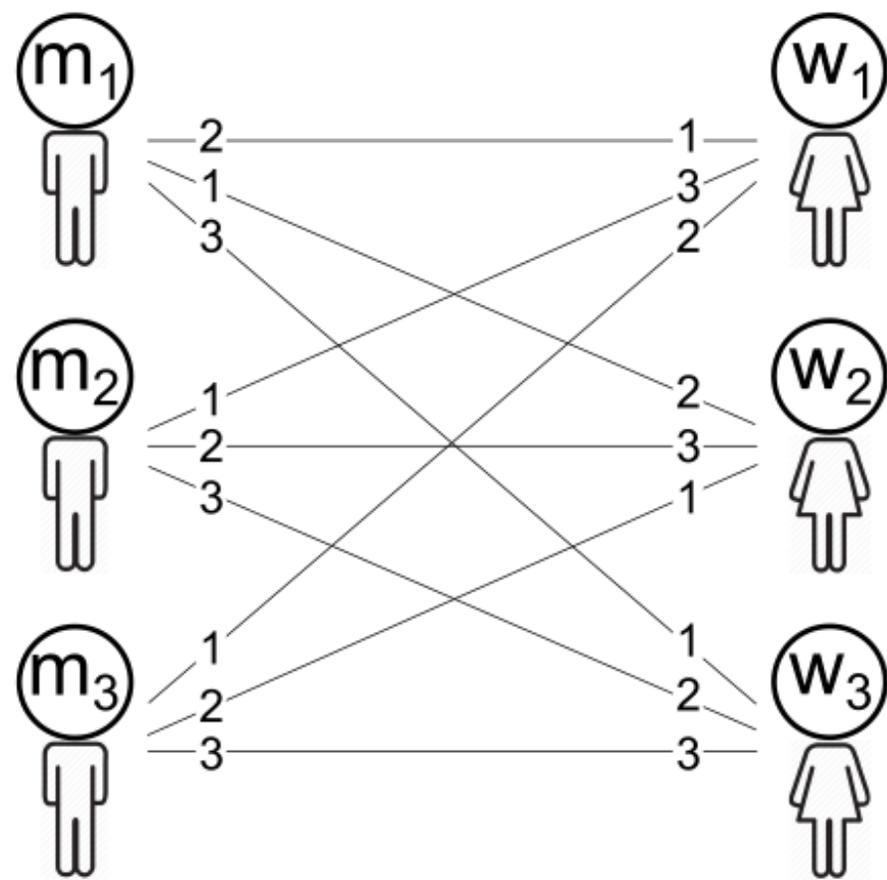
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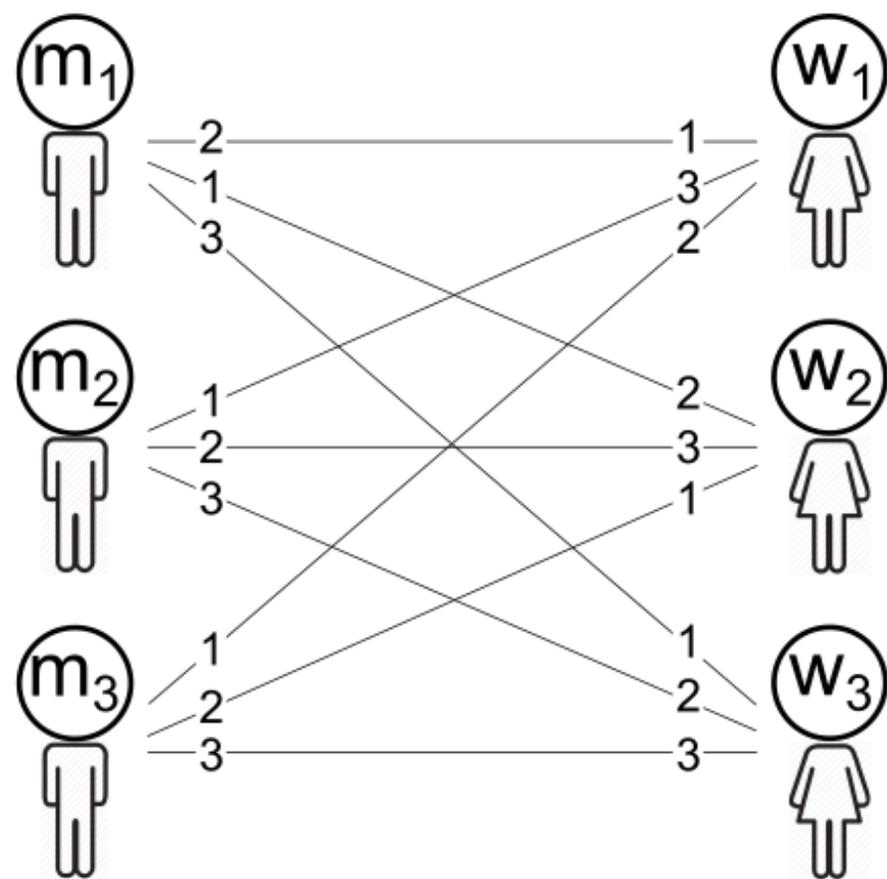


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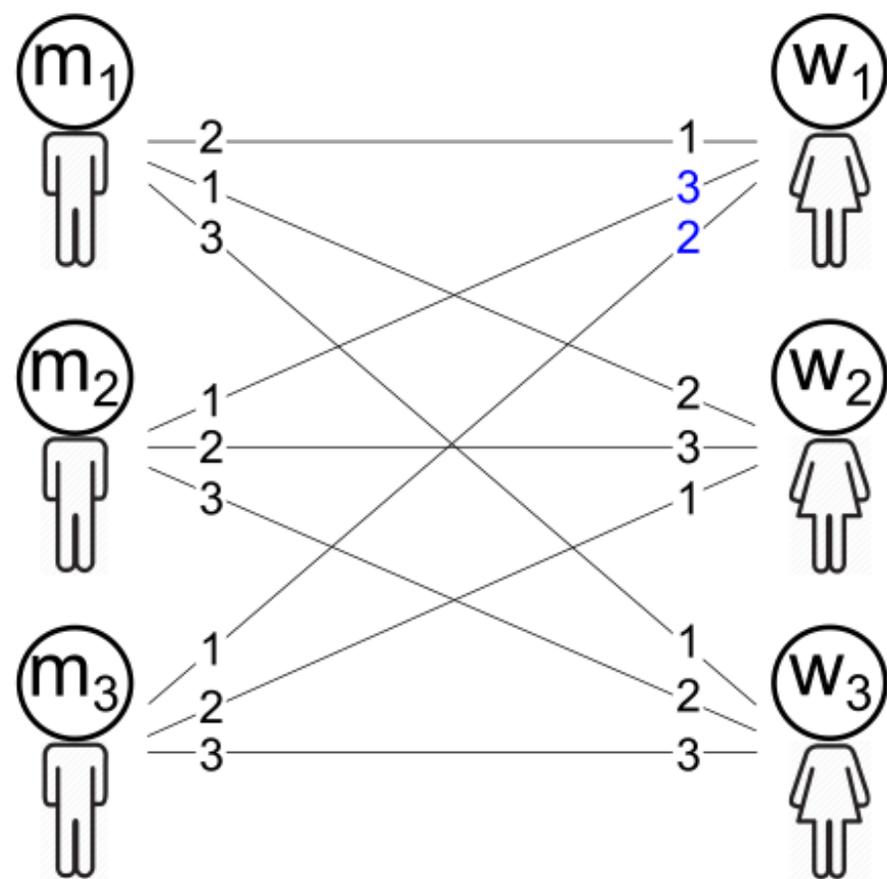
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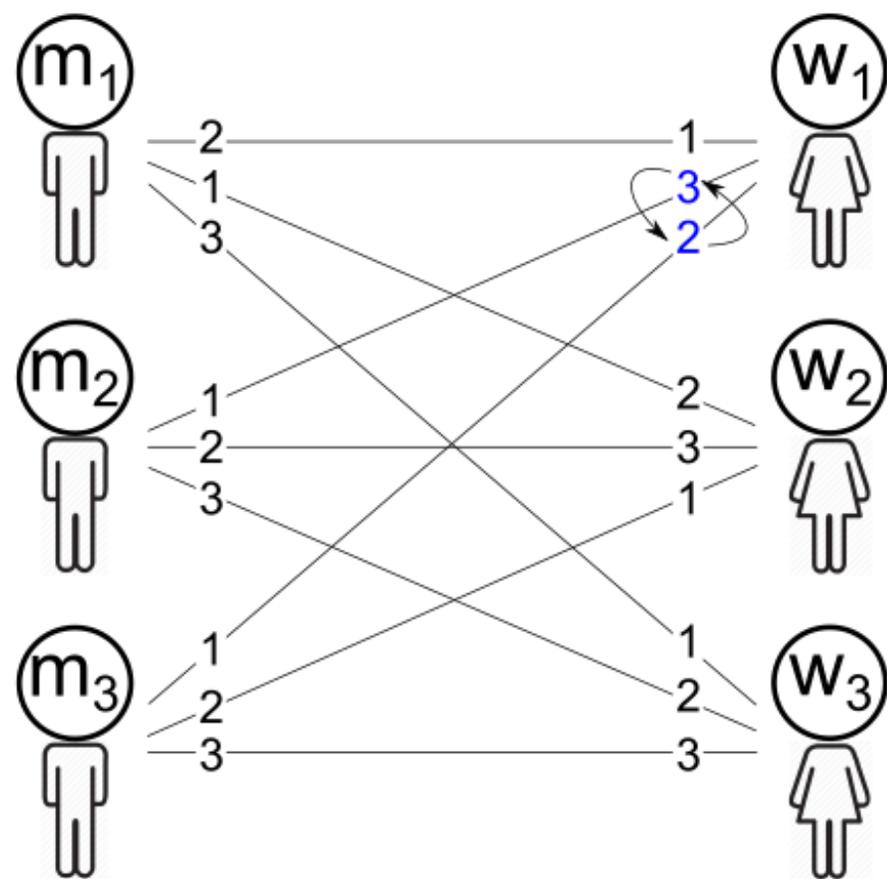
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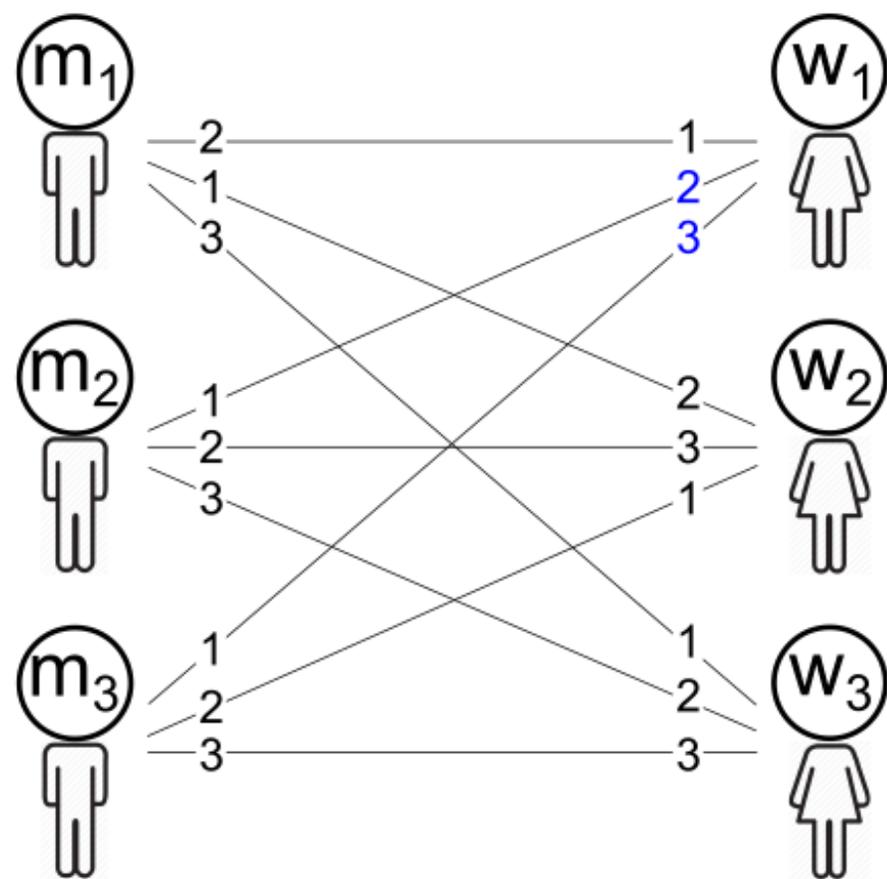
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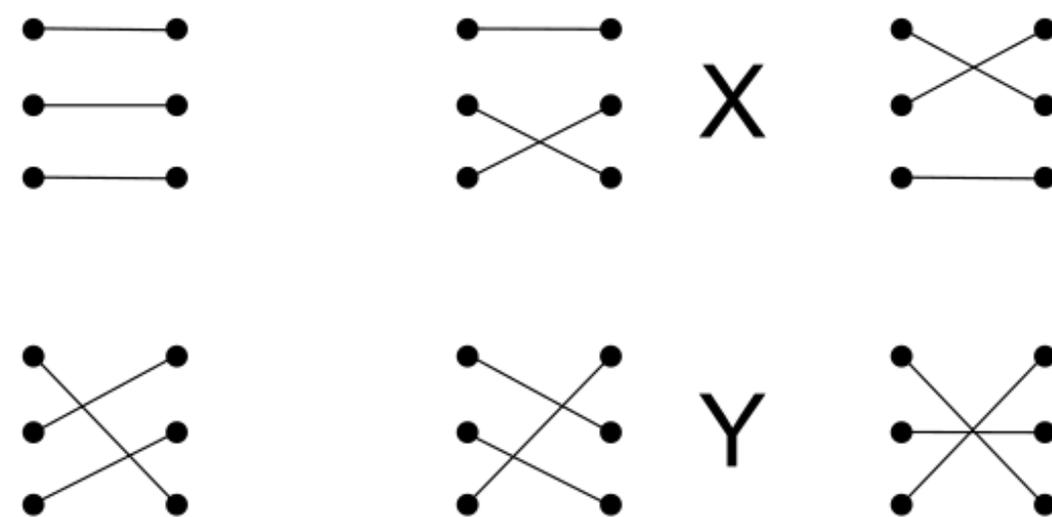
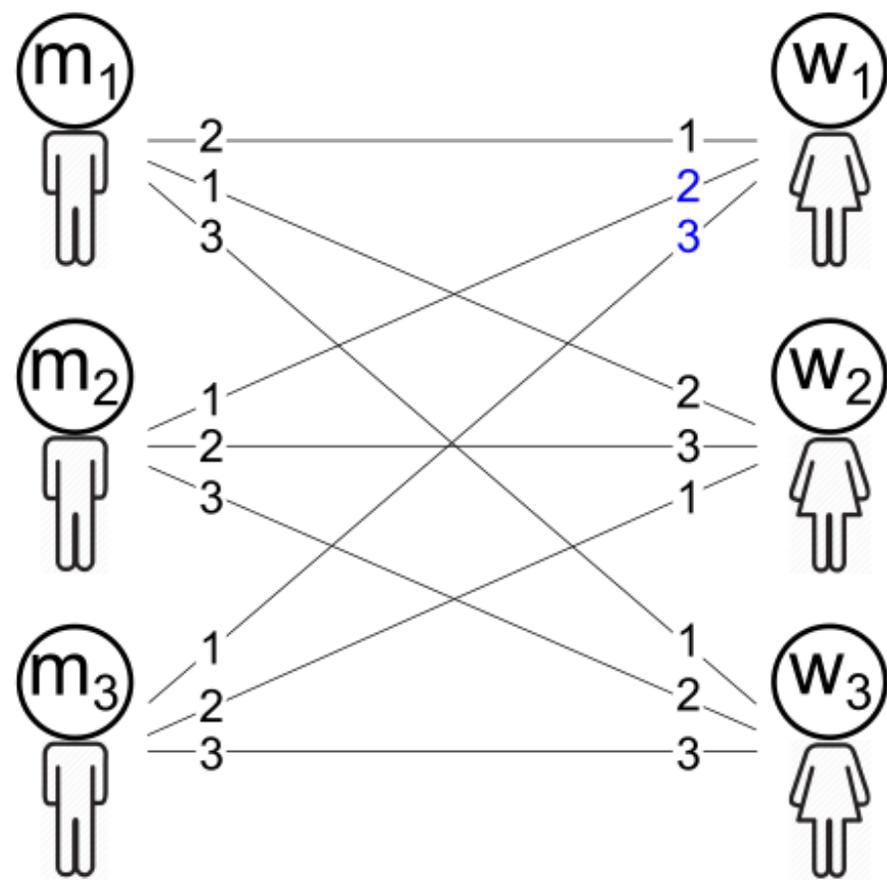
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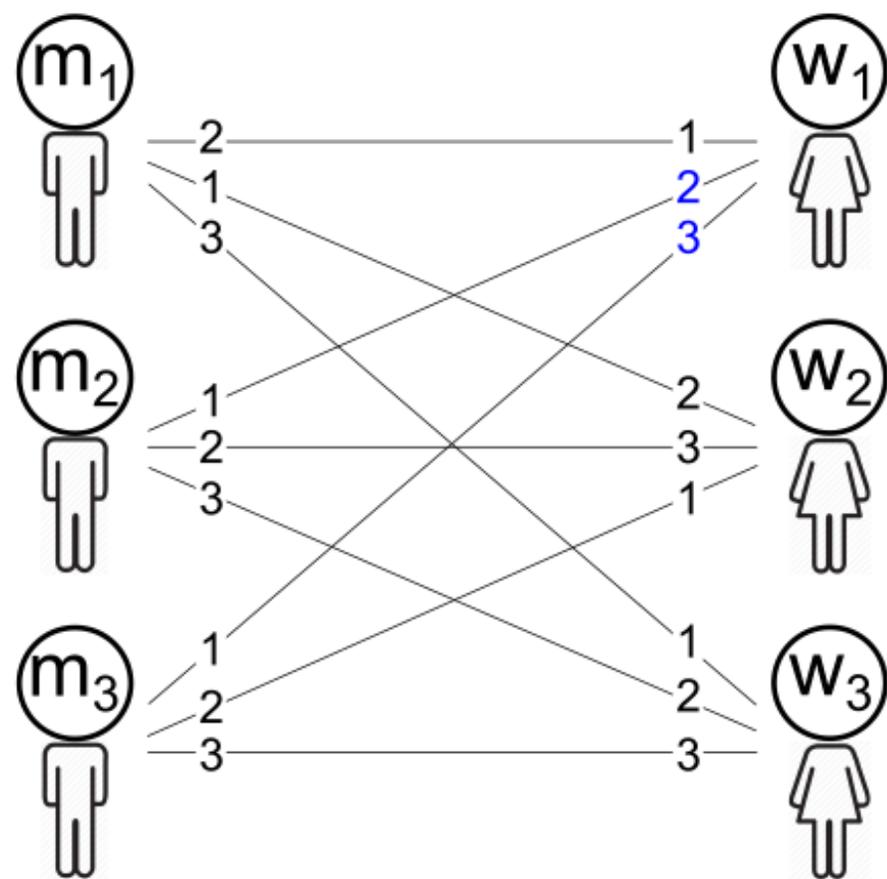


Instance I_1

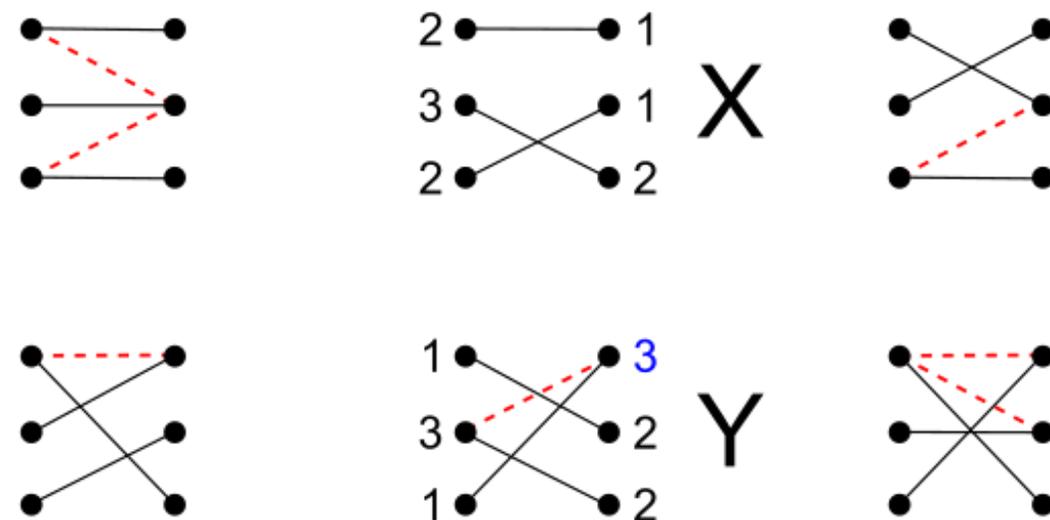
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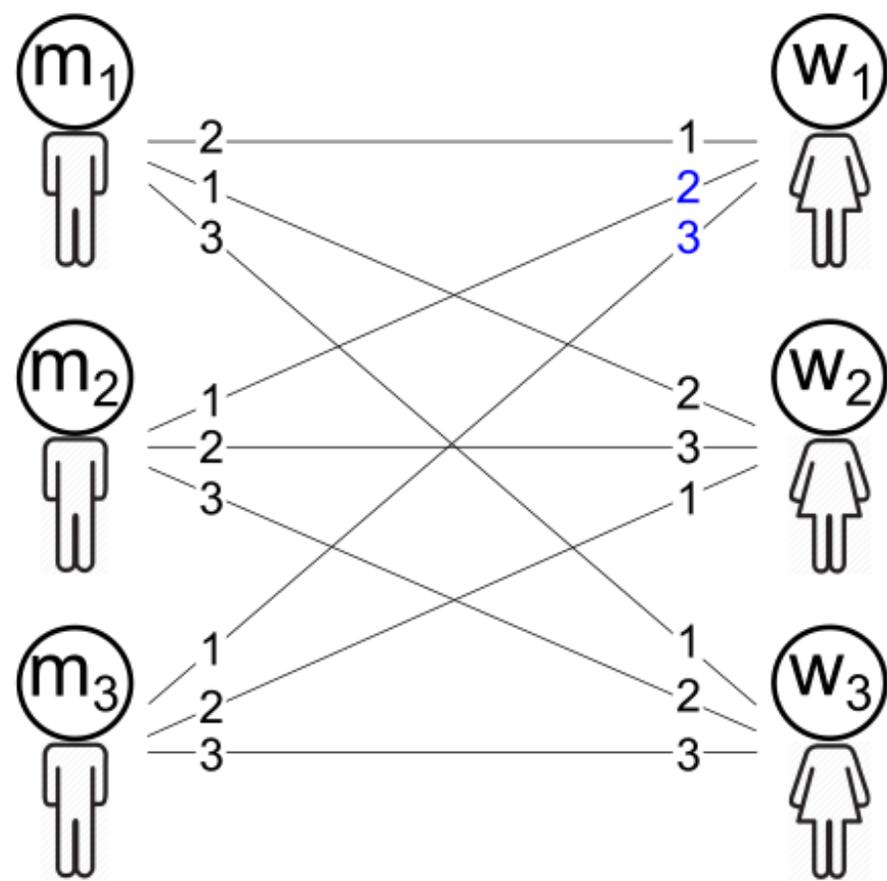
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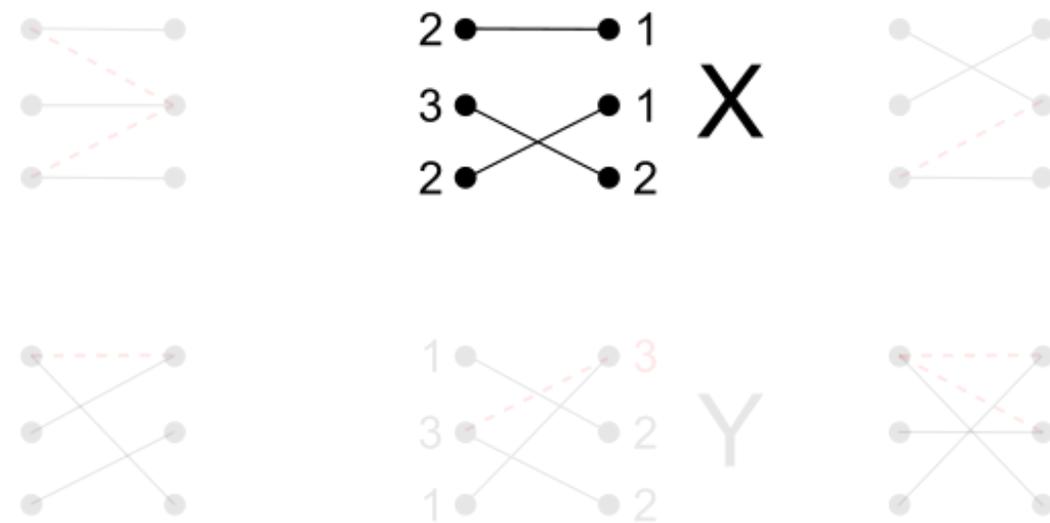
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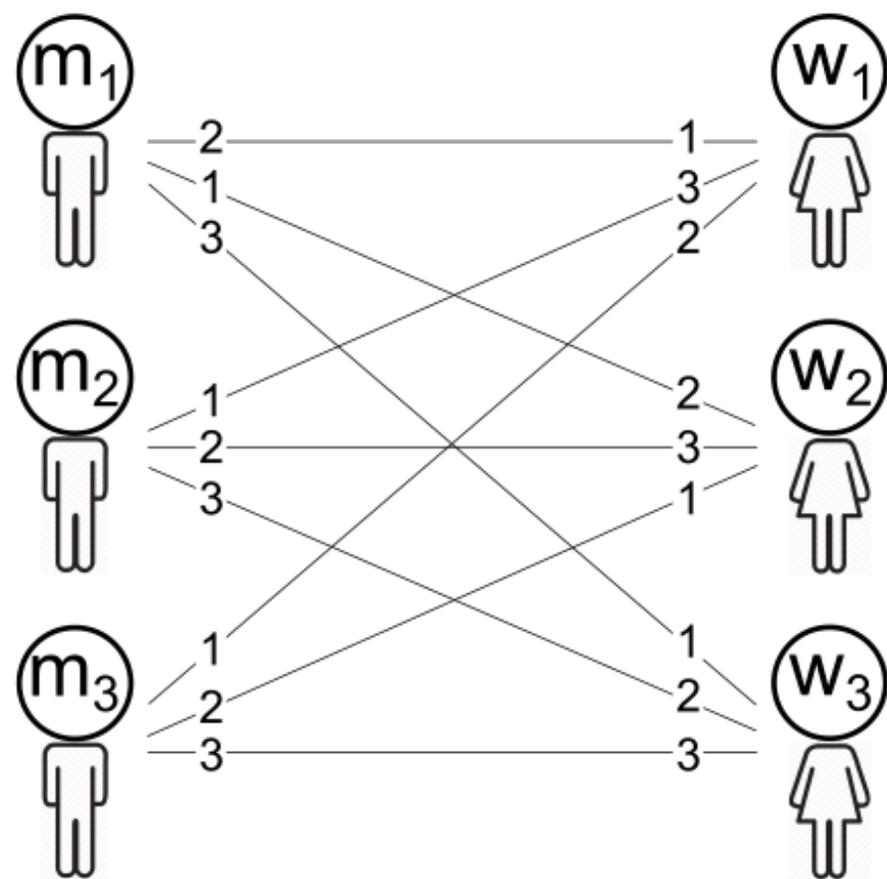
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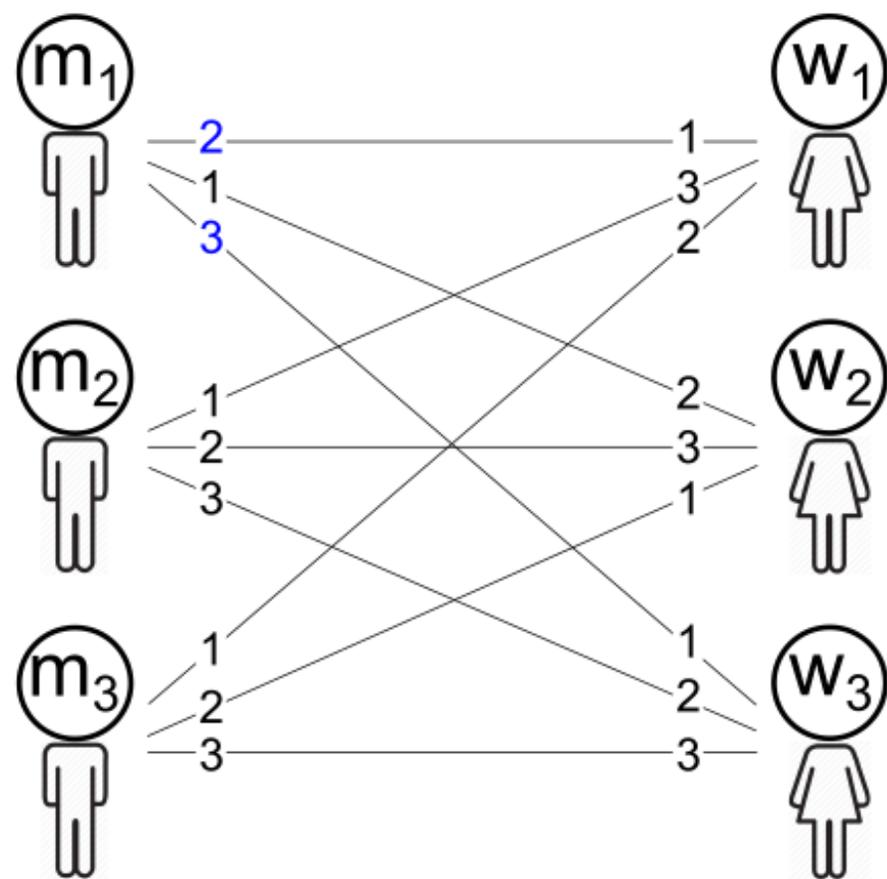


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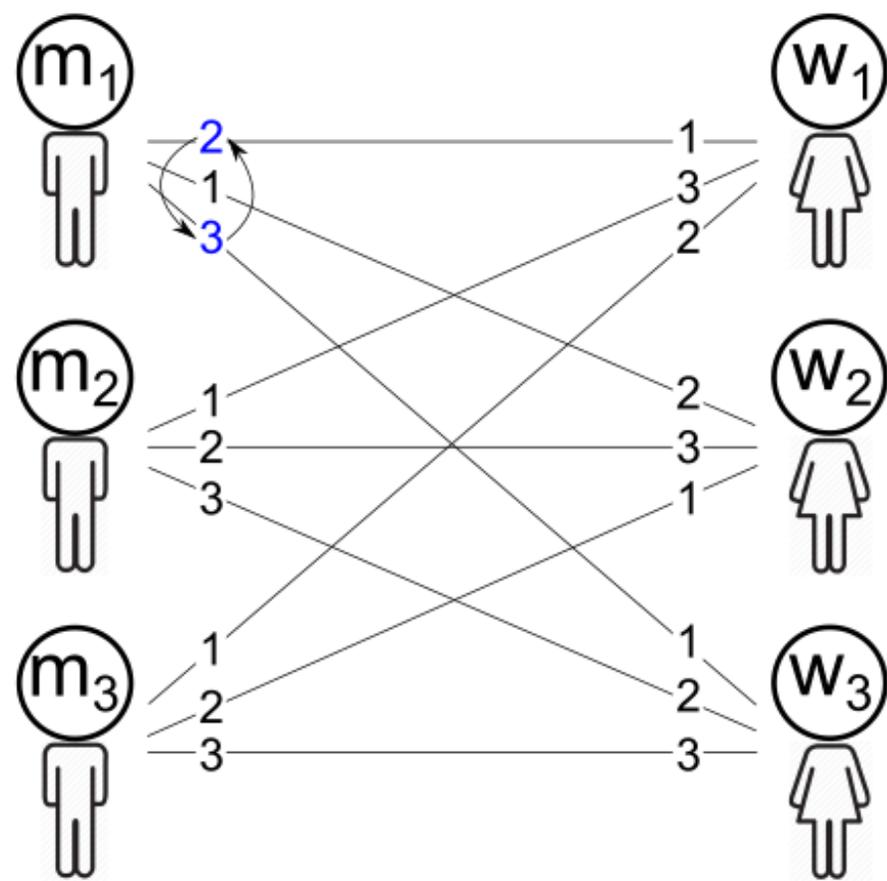
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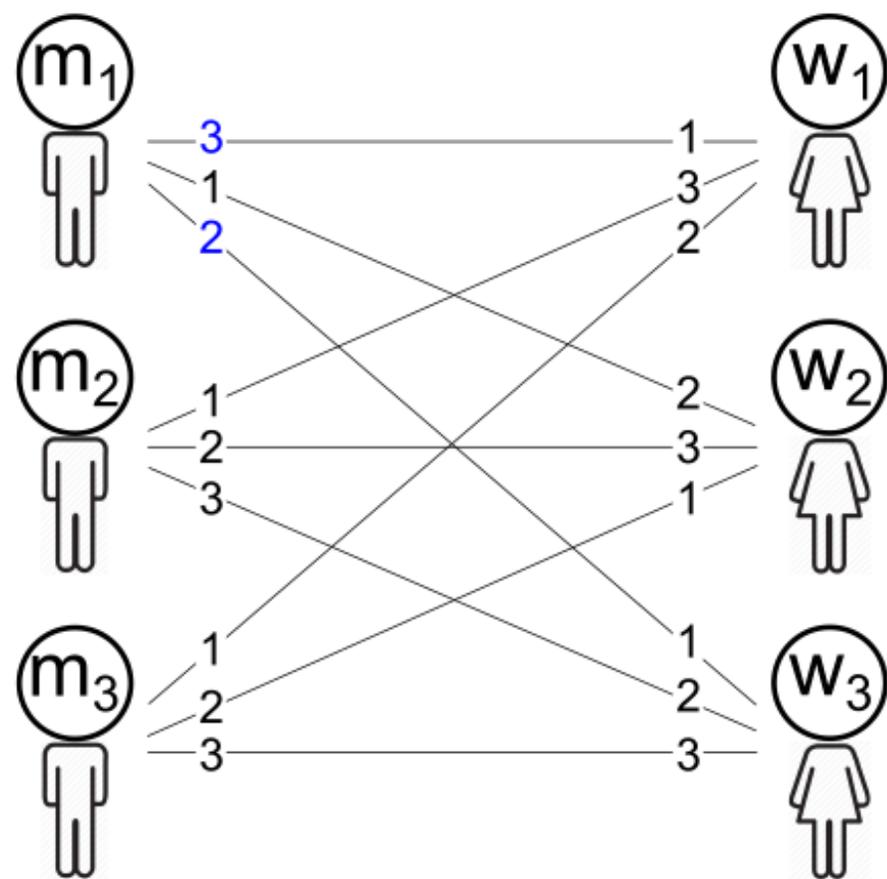
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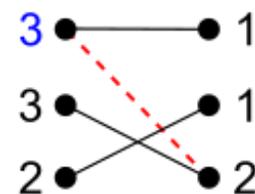
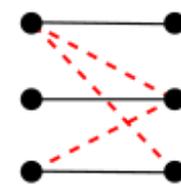
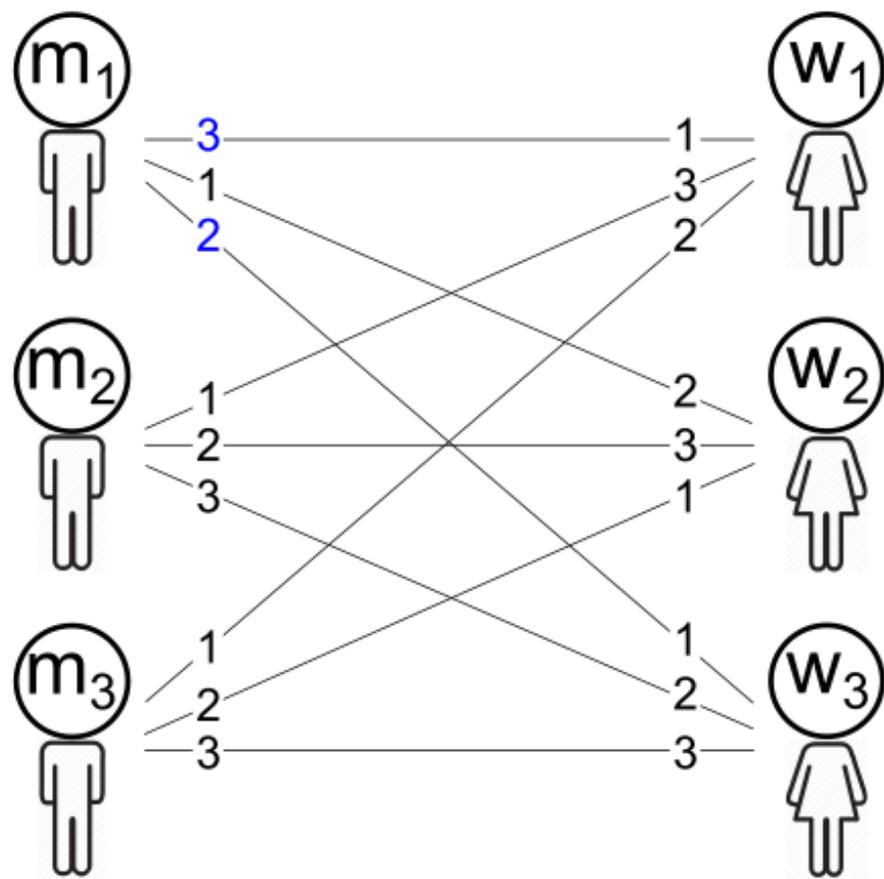
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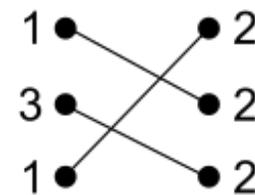
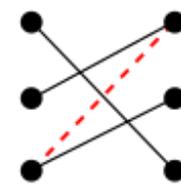
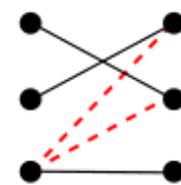


Instance I_2

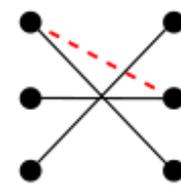
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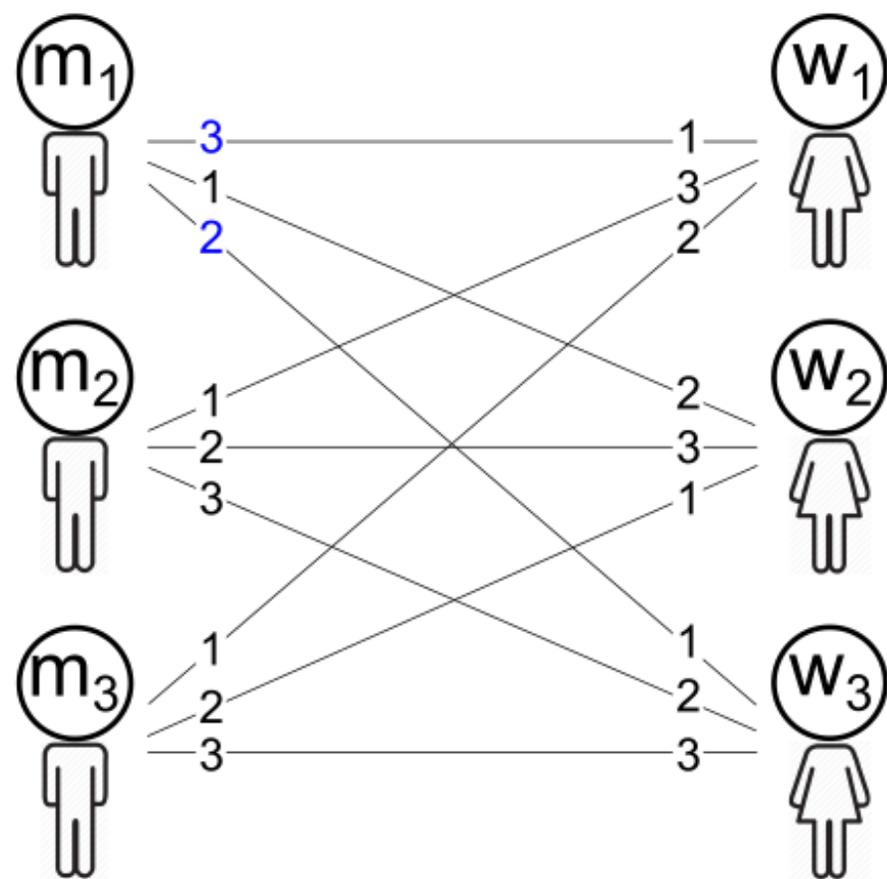
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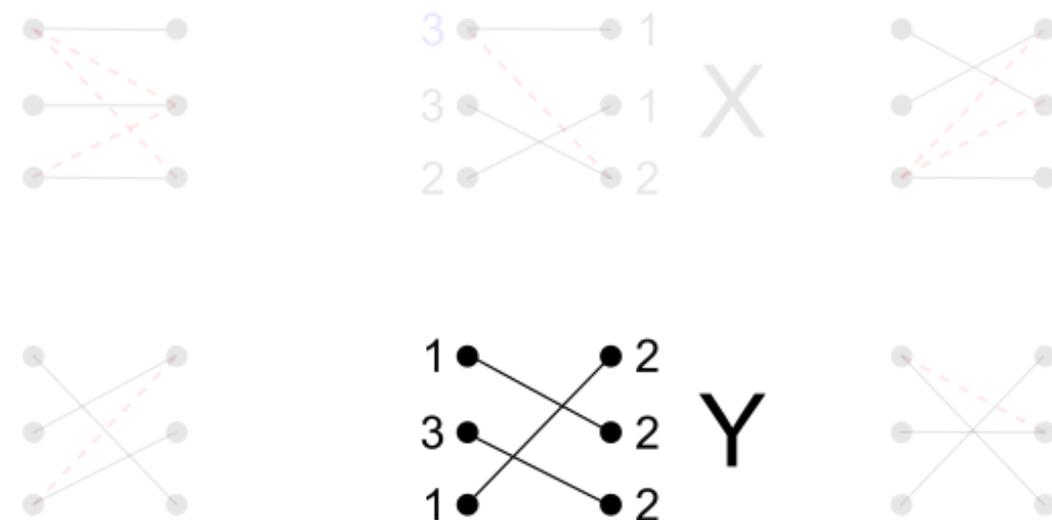
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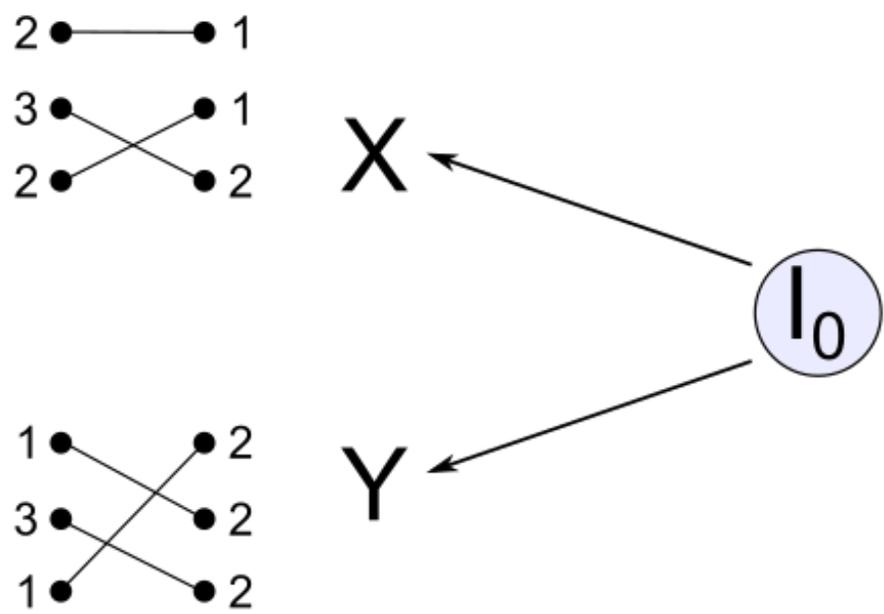
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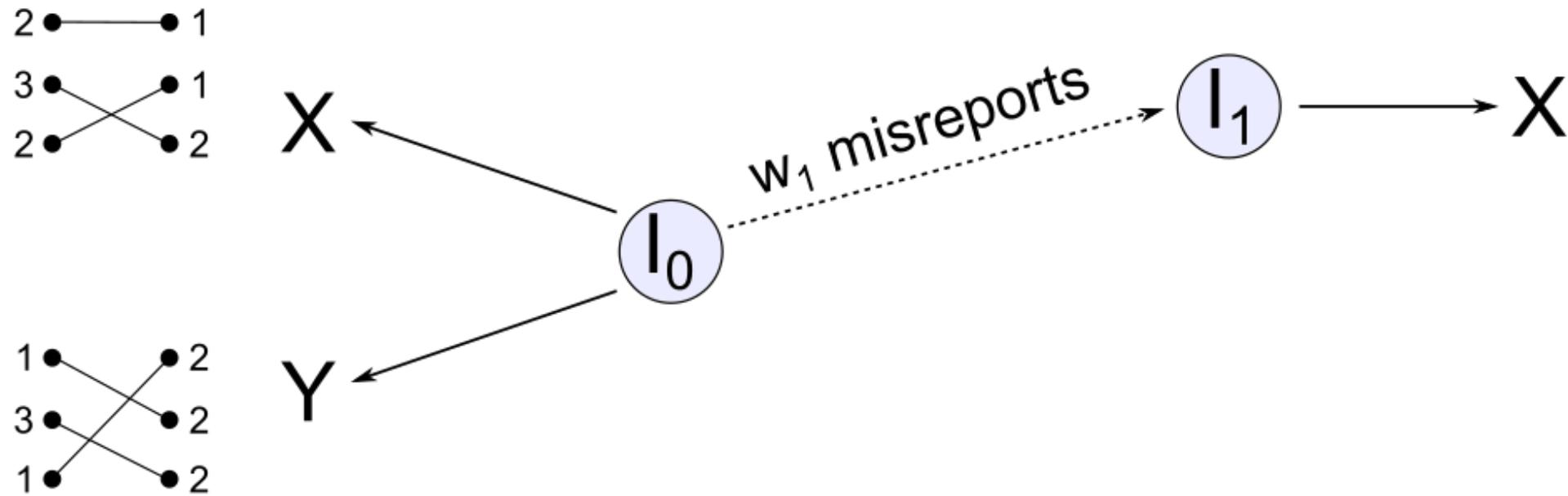
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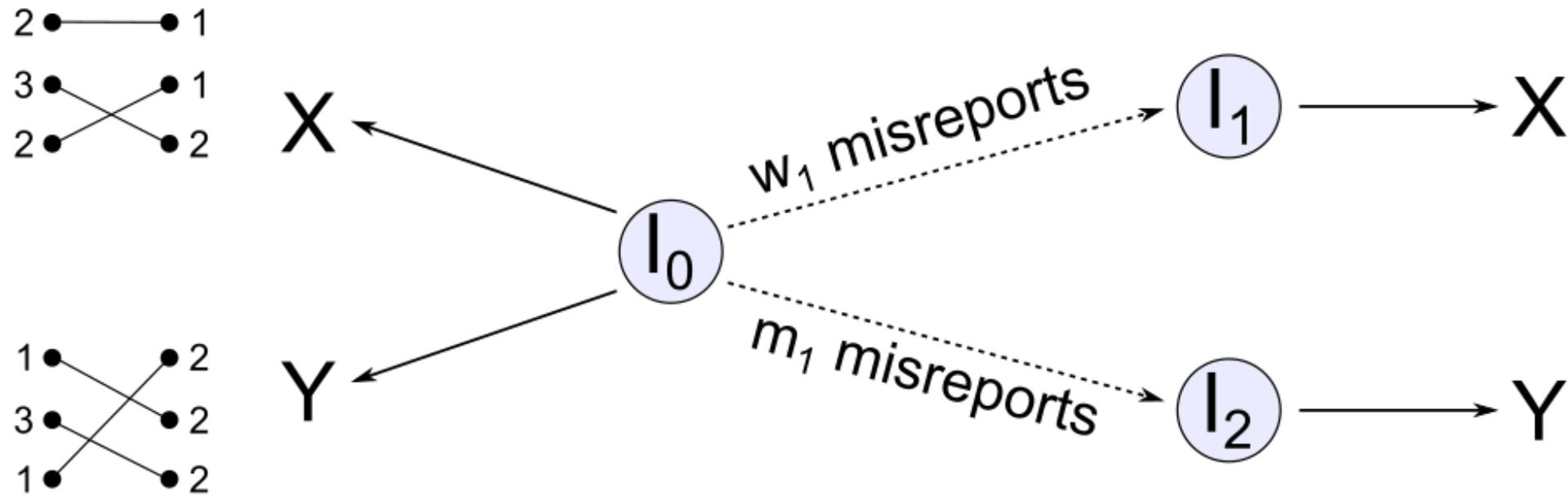
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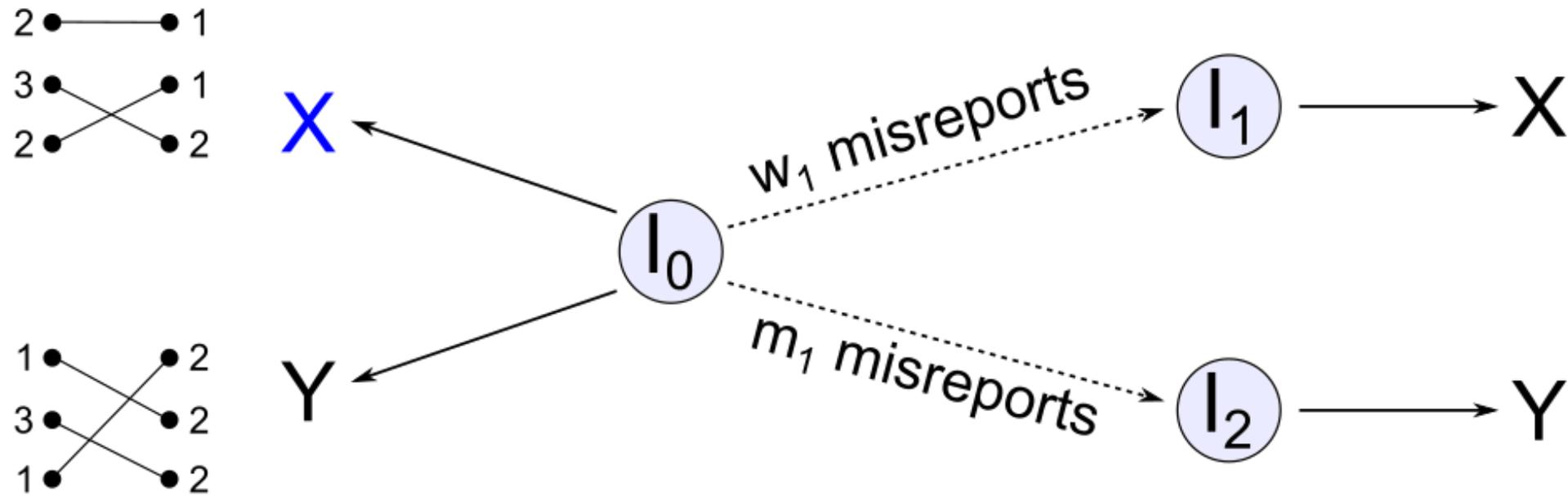
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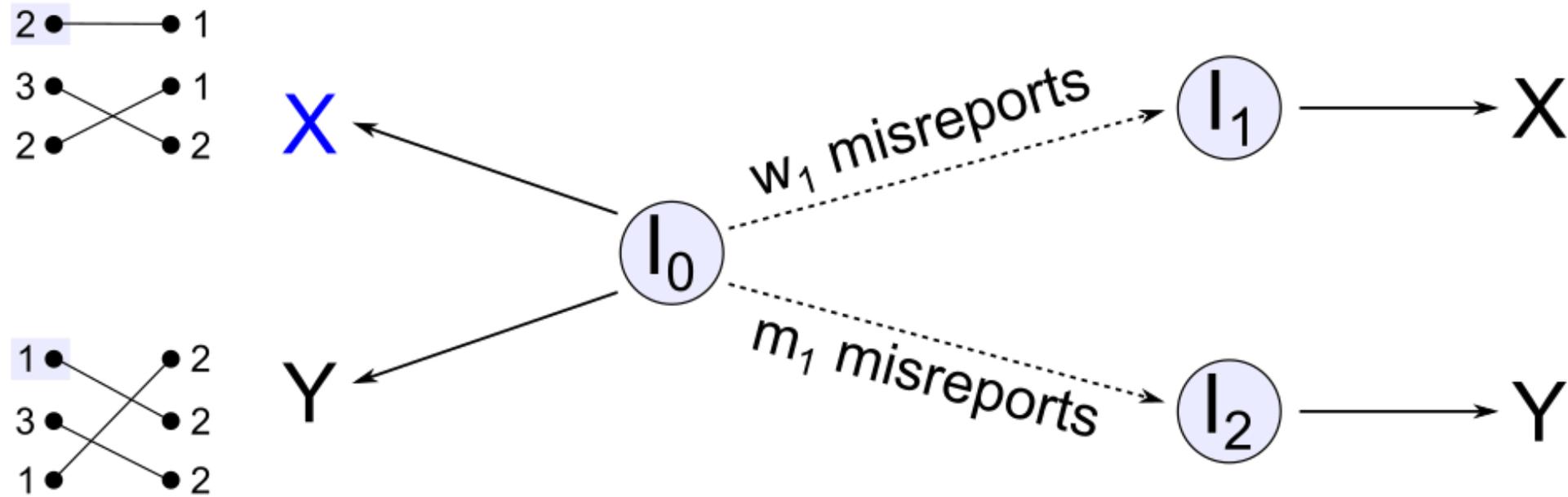
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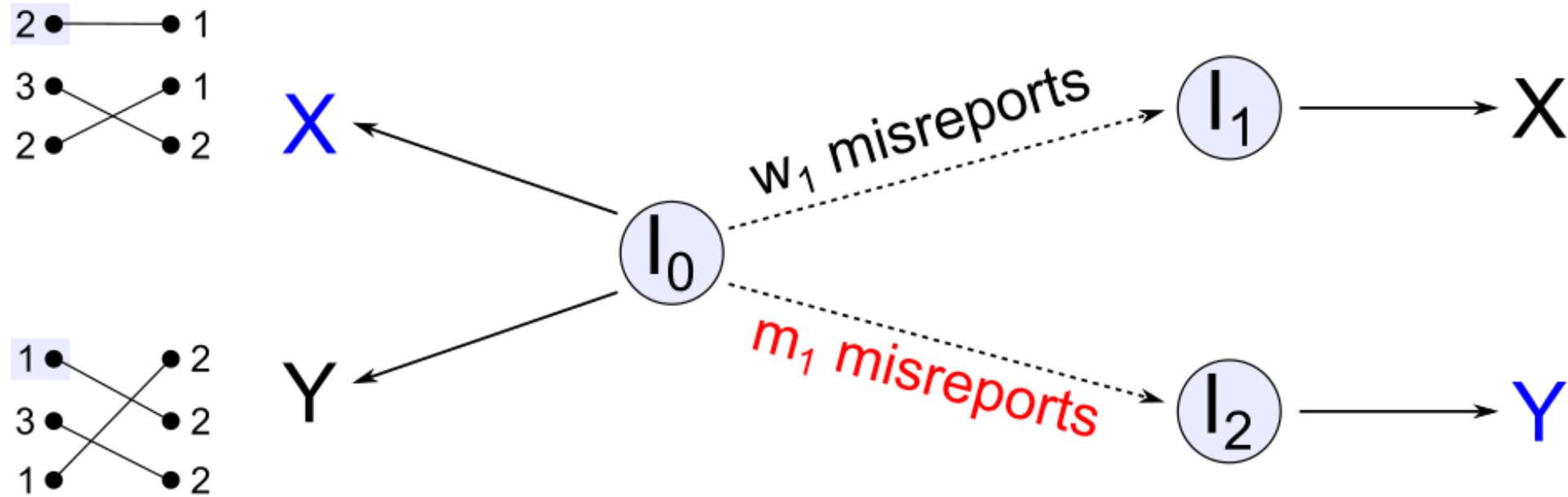
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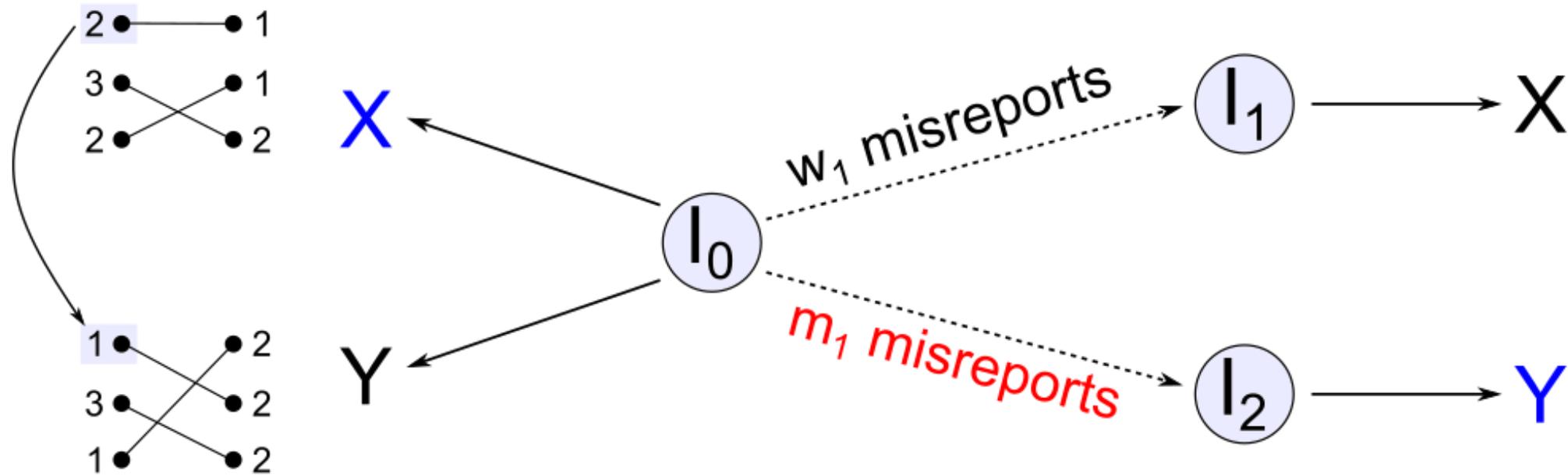
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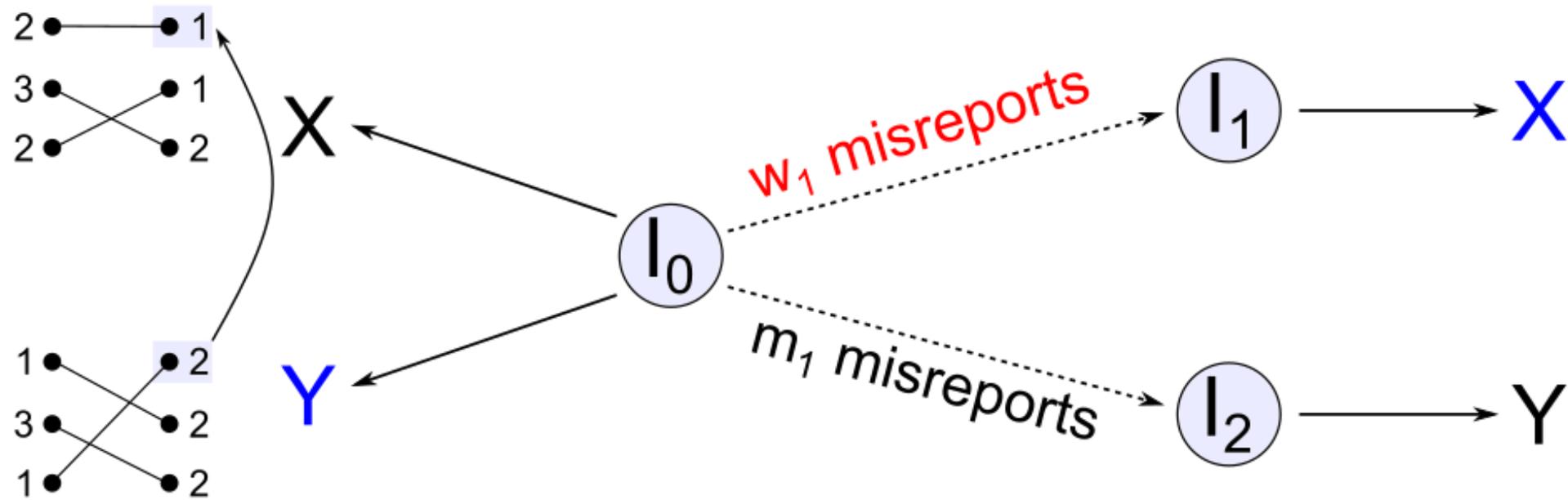
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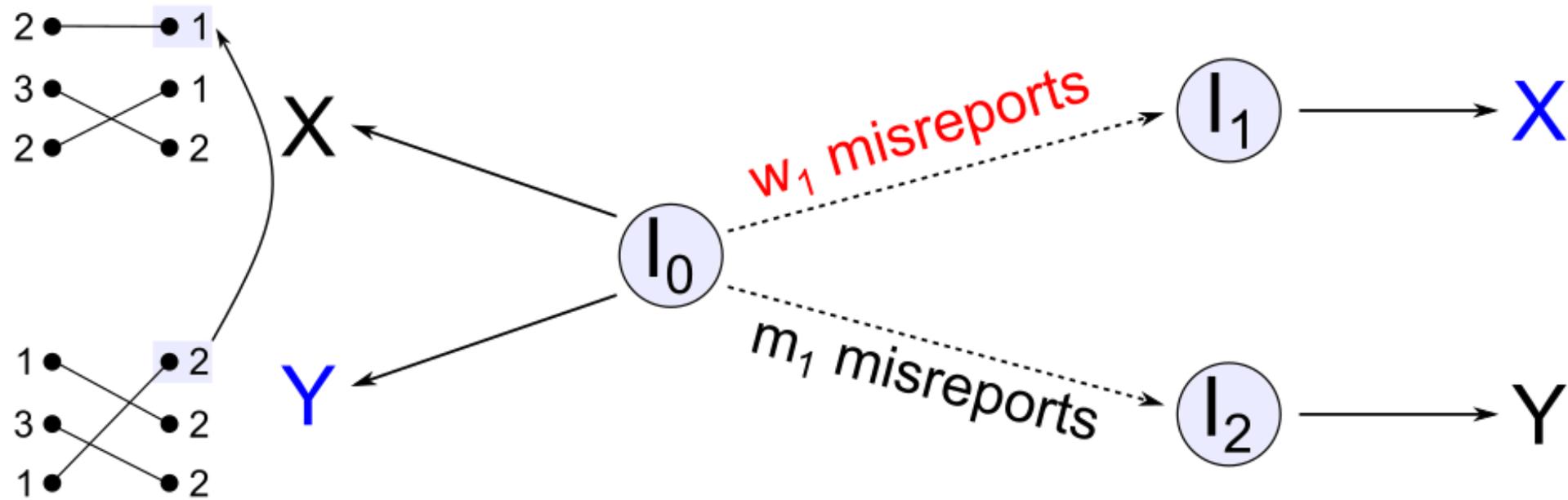
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No stable matching procedure is strategyproof for all agents.

Next Time

Finding Fair Stable Matchings



Quiz

Quiz

Identify the optimal manipulation strategy for each woman.

$$m_1: w_2 > w_1 > w_3 > w_4 > w_5$$

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References

- DA algorithm fails to be strategyproof.

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- Optimally manipulated marriages are stable.

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[Dubins and Freedman, *Amer. Math. Mon.* 1981; Roth, *MOR* 1982]

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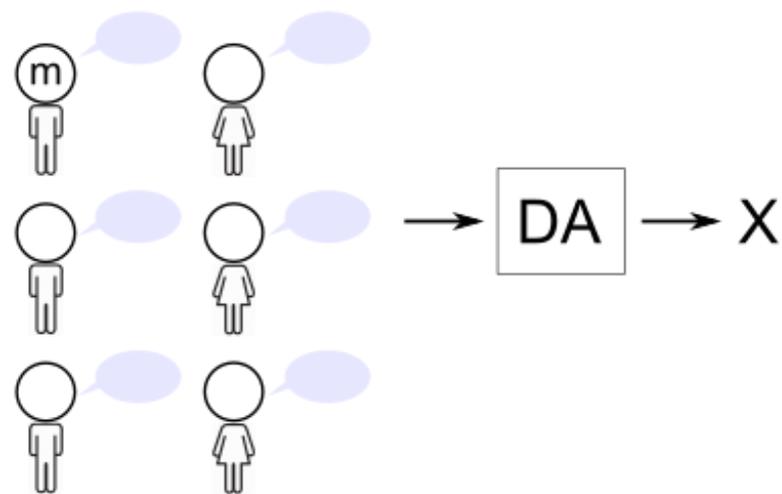
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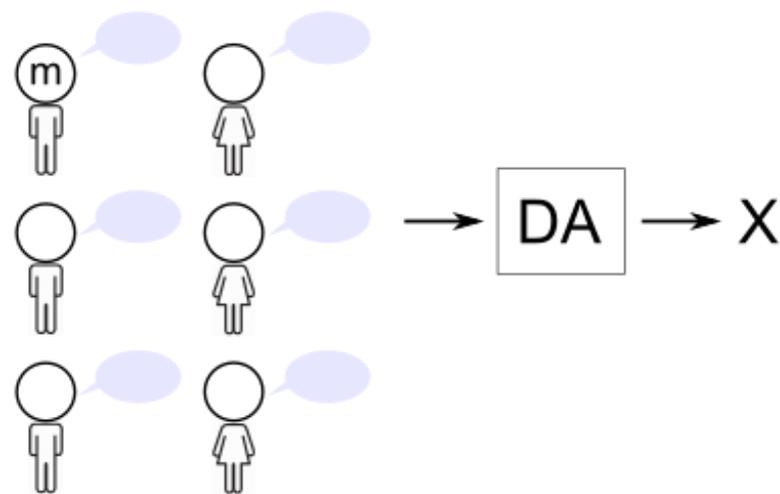


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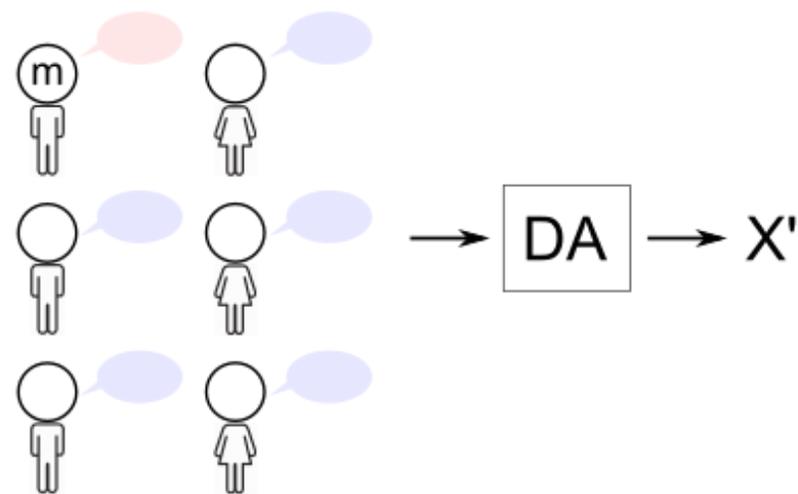
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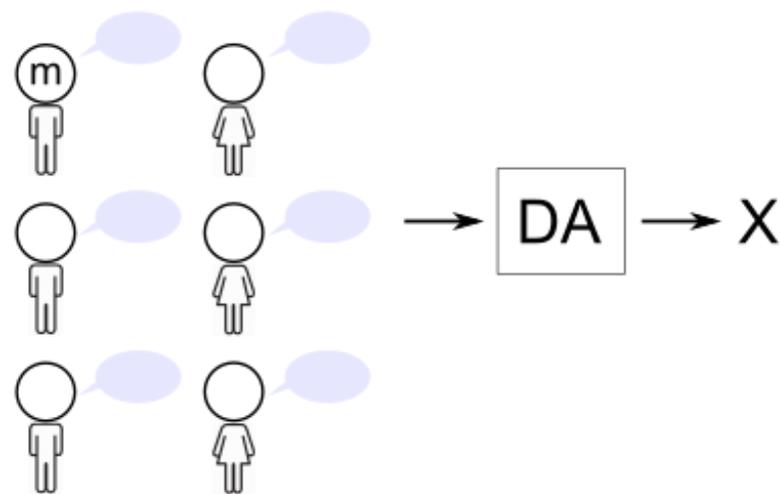


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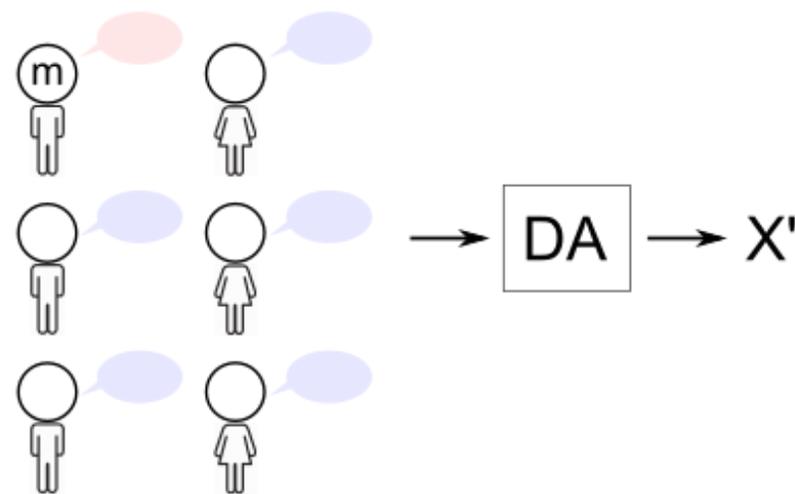
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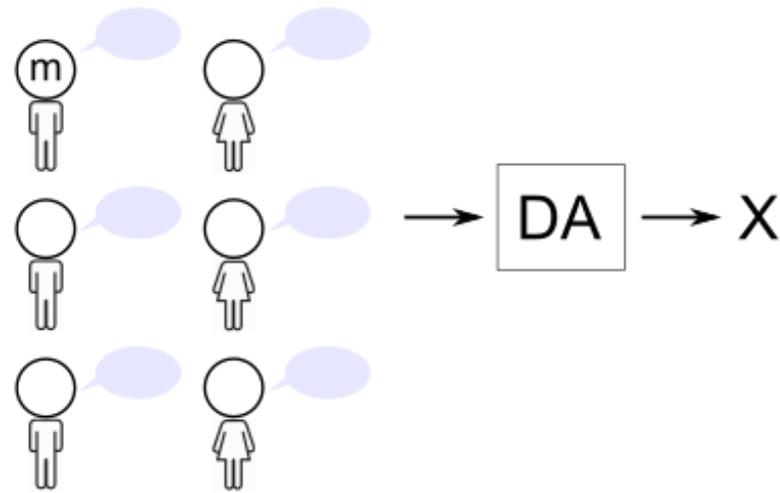
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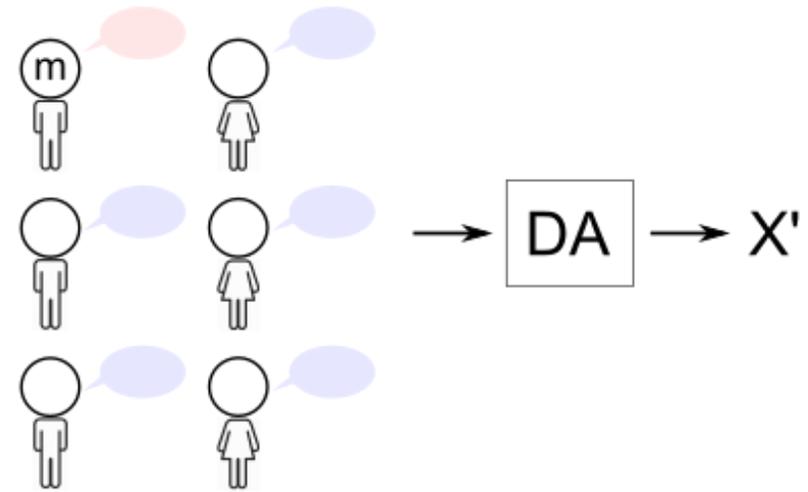
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We will use three lemmas to derive a contradiction.

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If there is a feasible strategy for manipulation, then there is a "simple" feasible strategy that achieves the same outcome.

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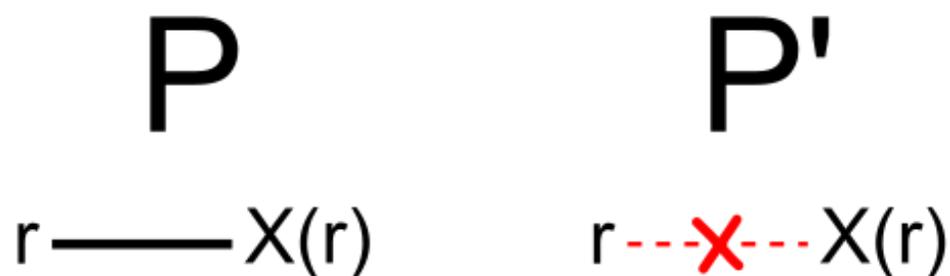
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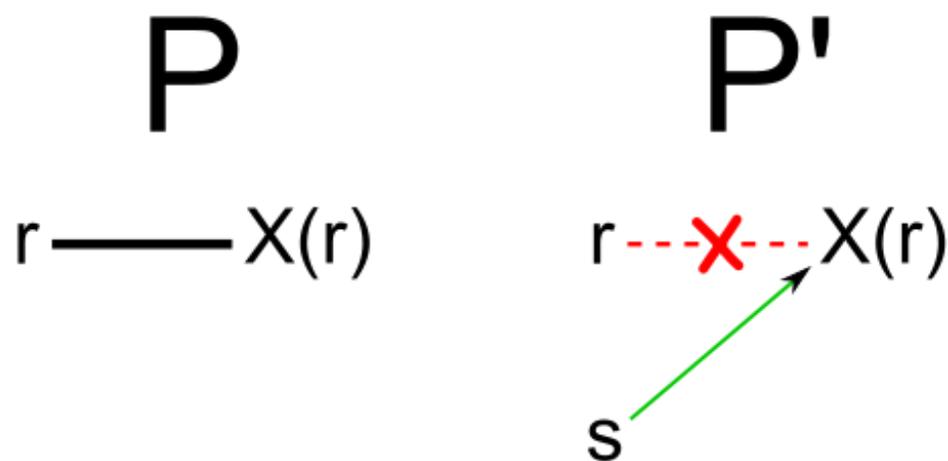
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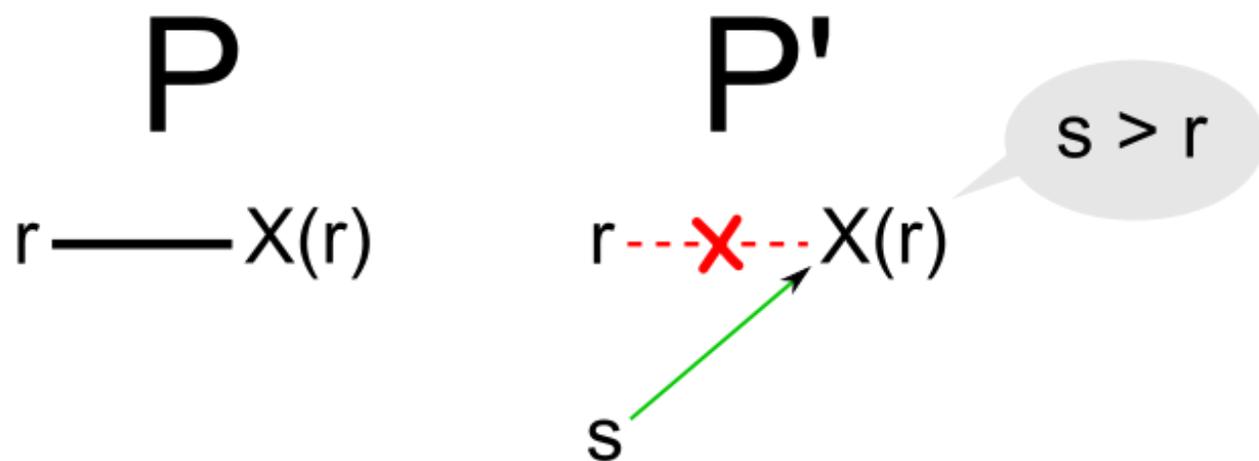
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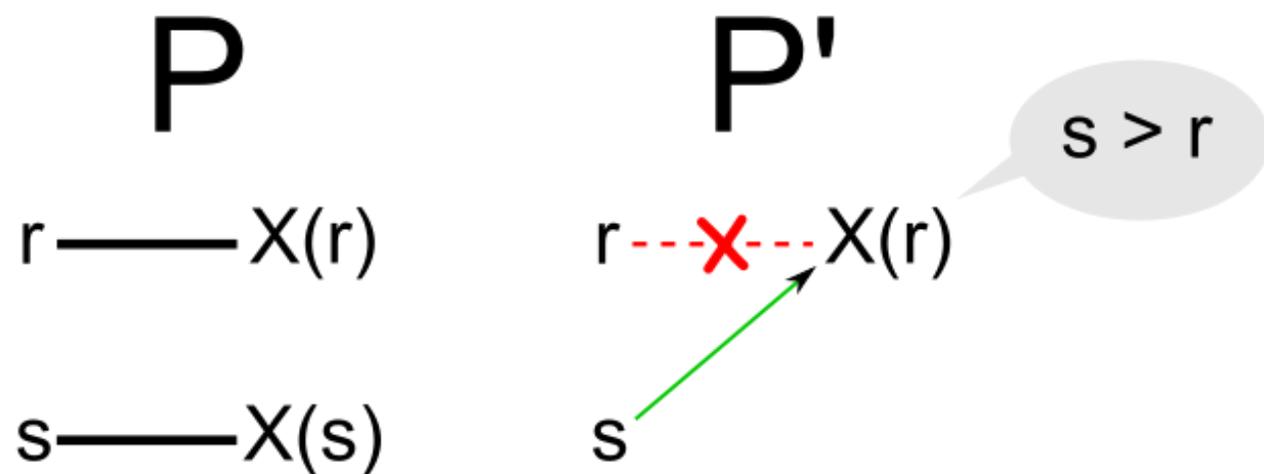
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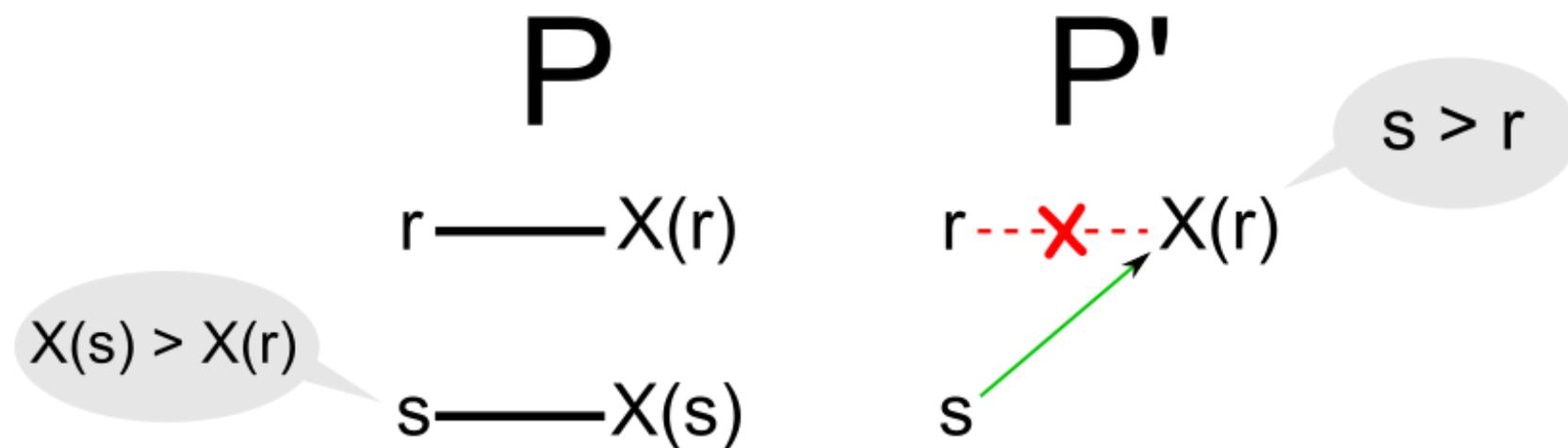
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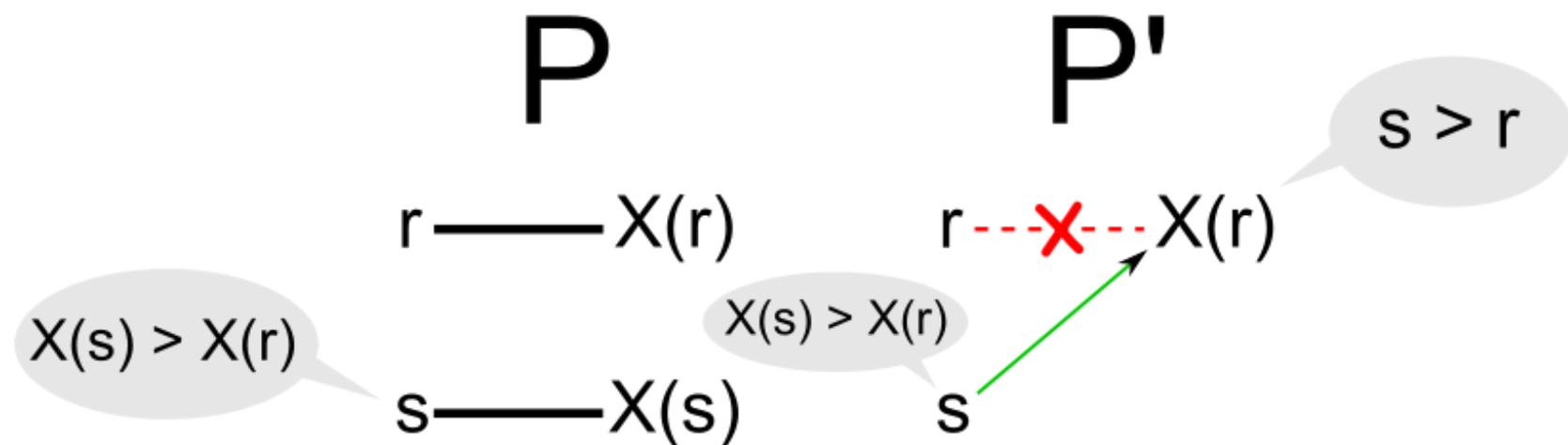
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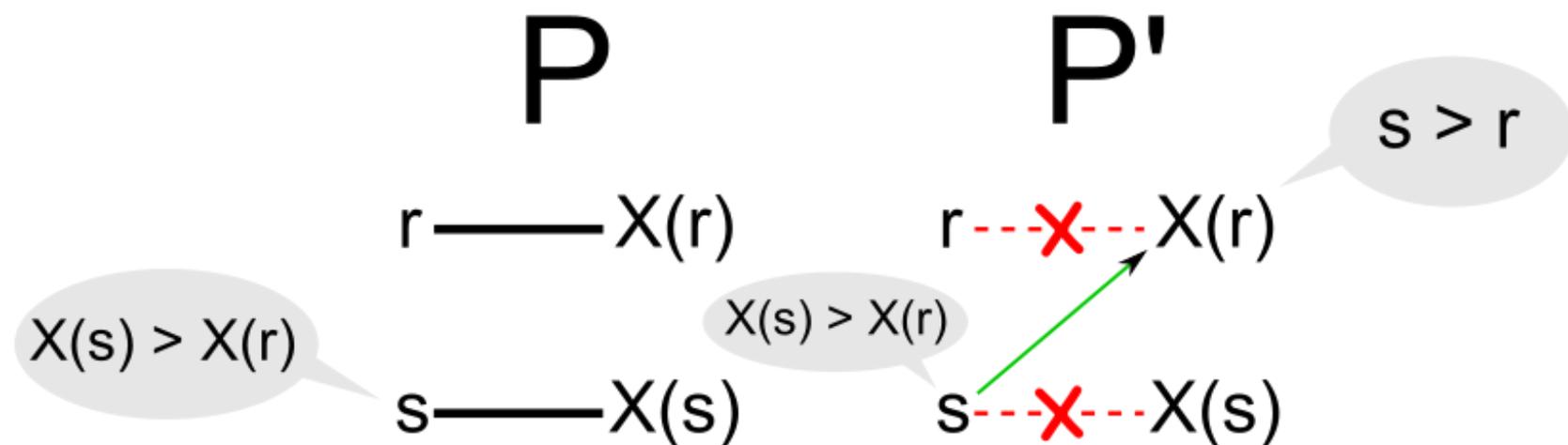
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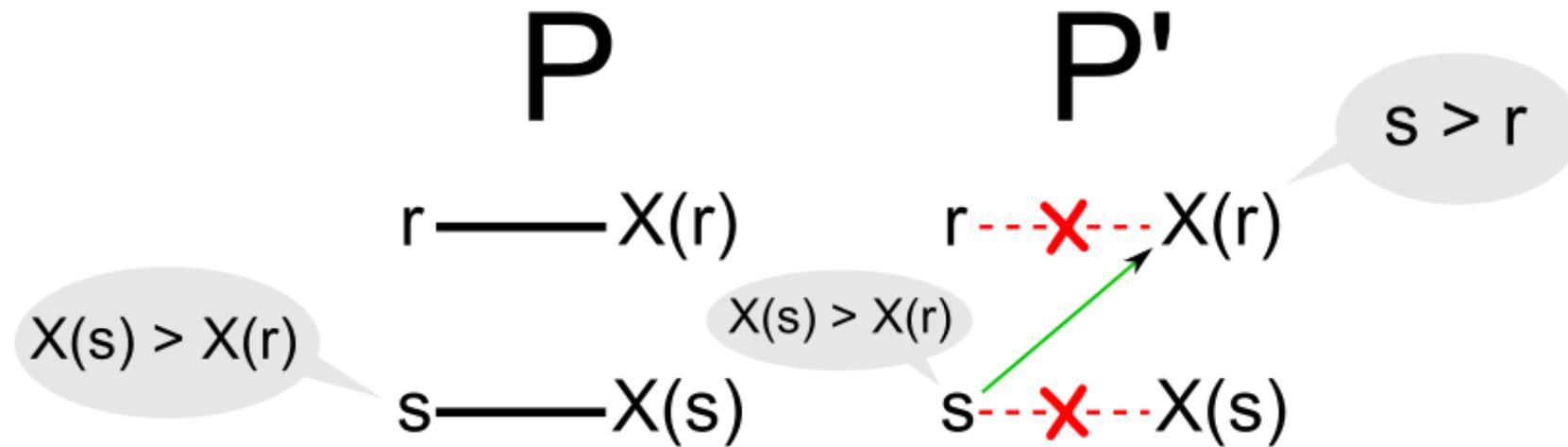
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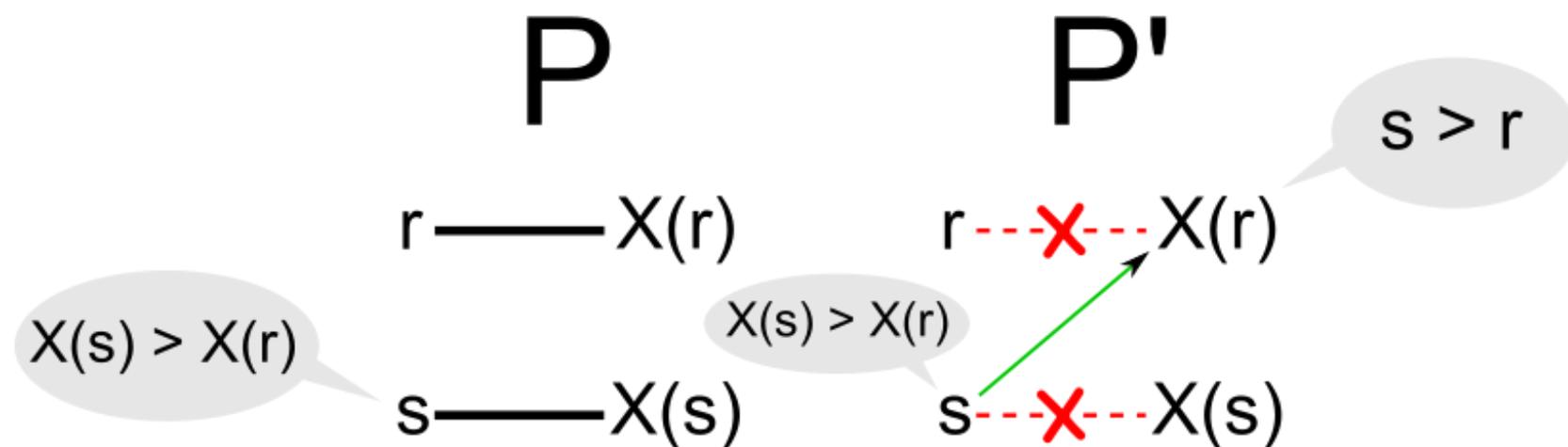
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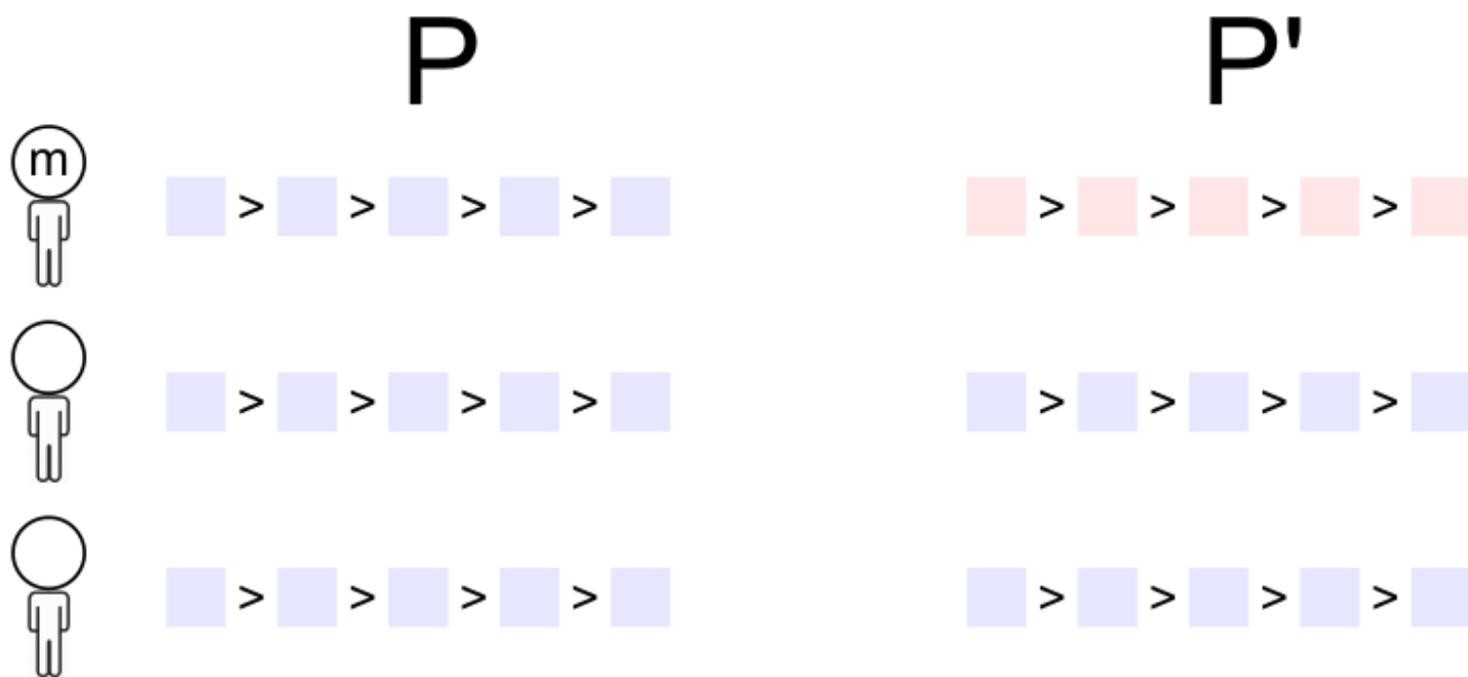
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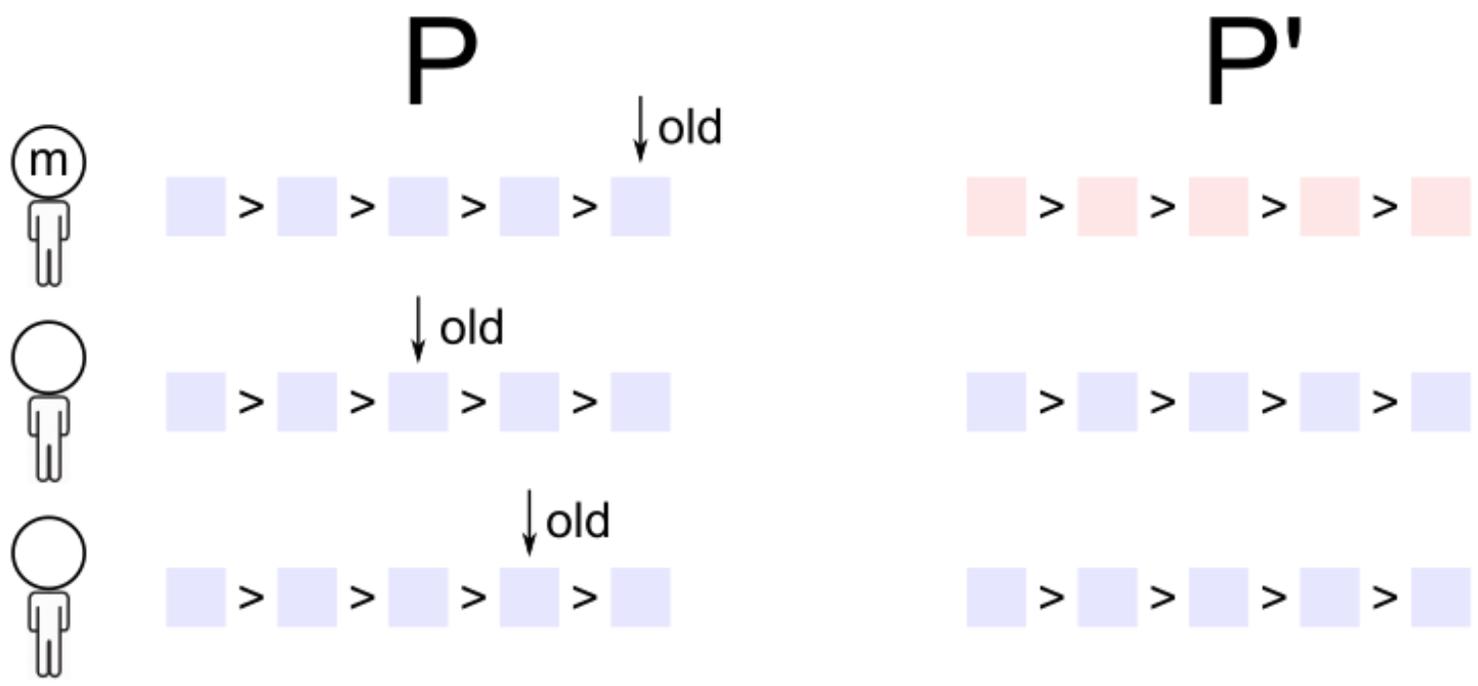
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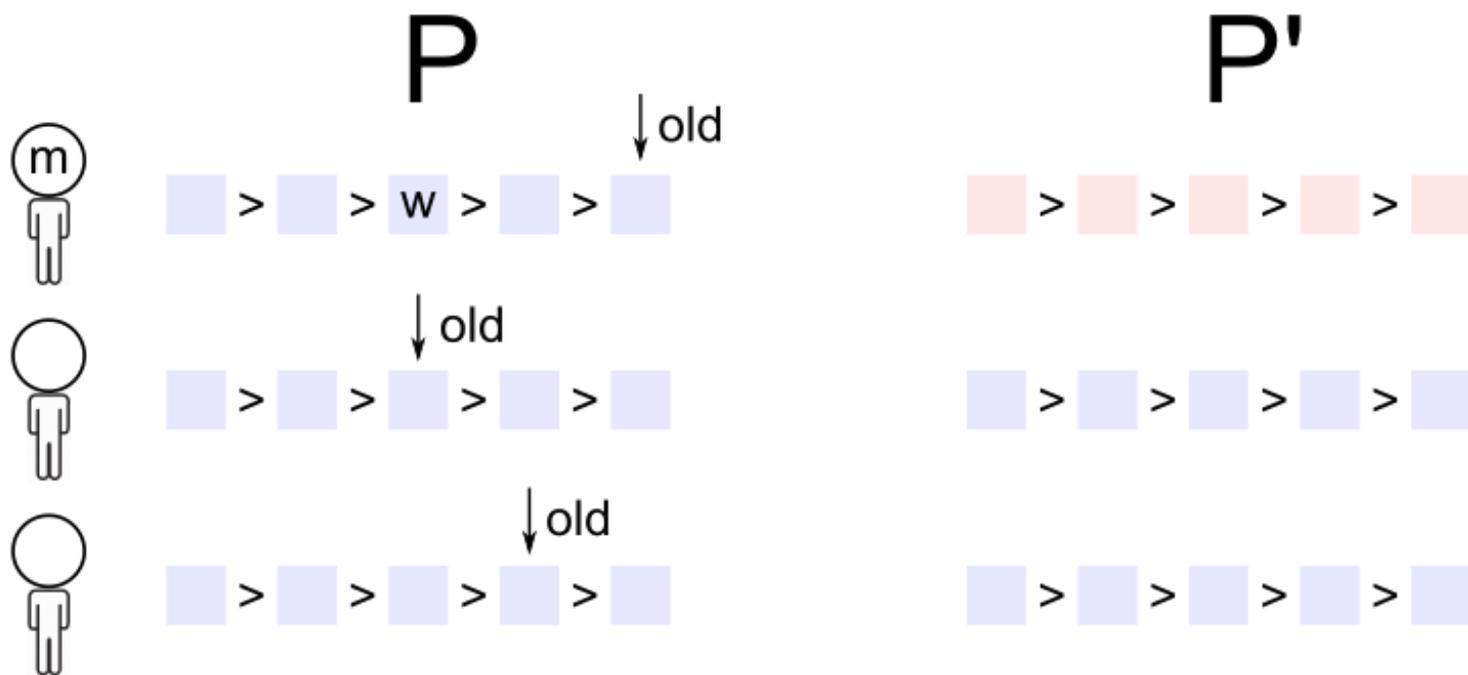
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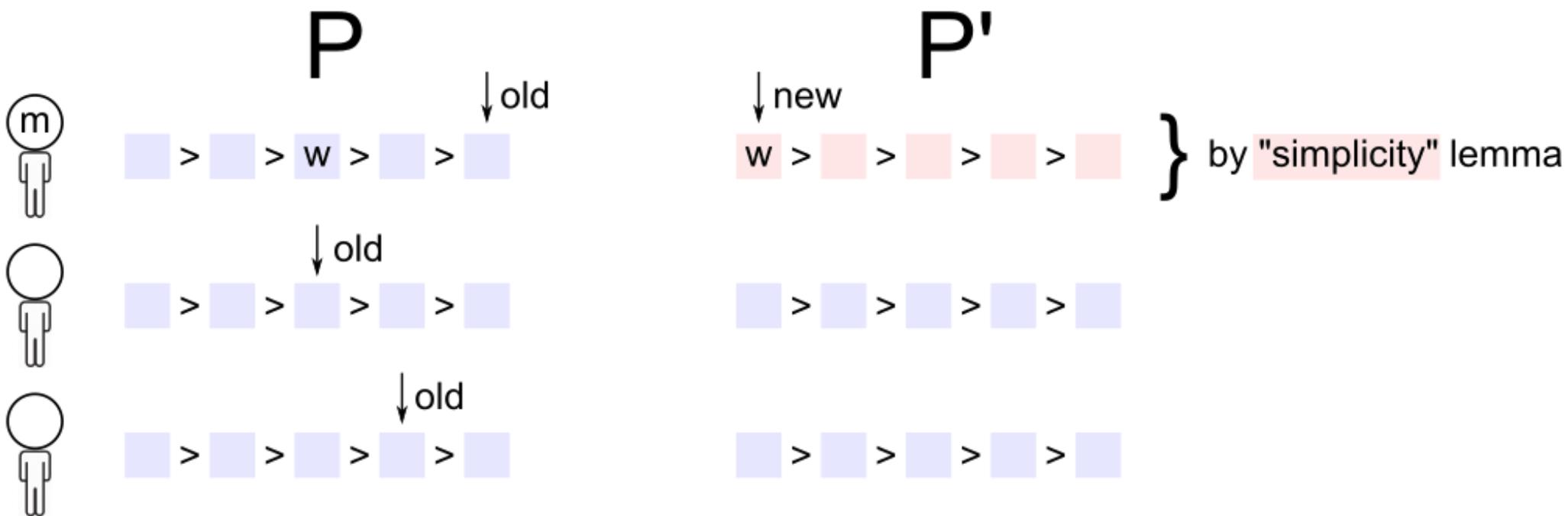
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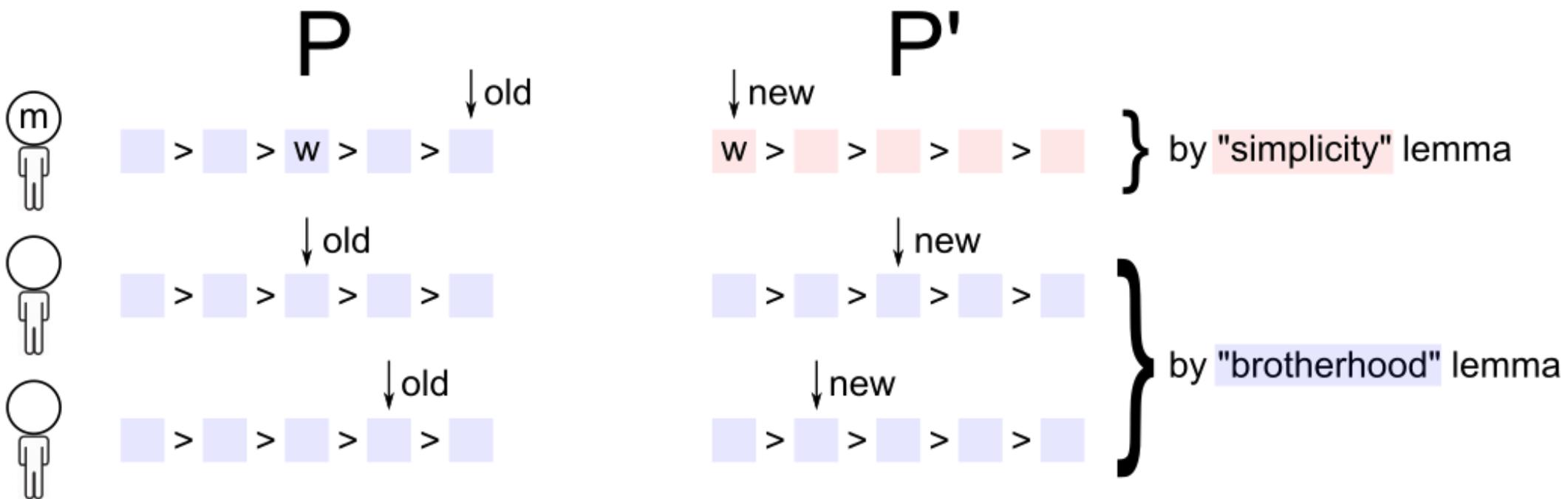
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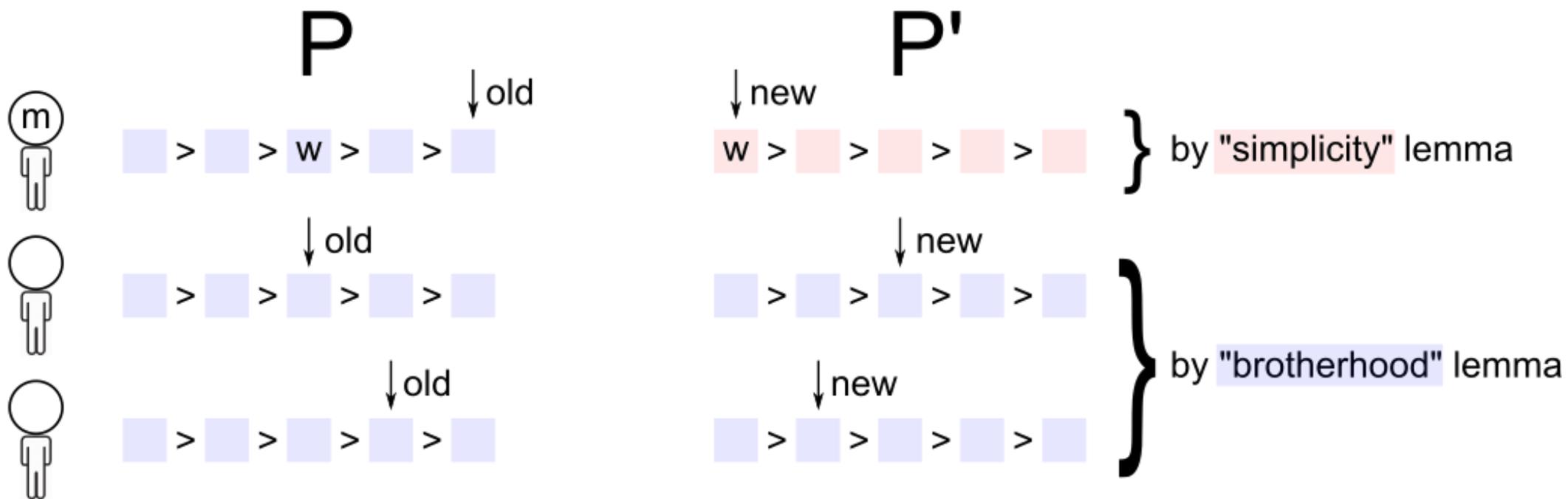
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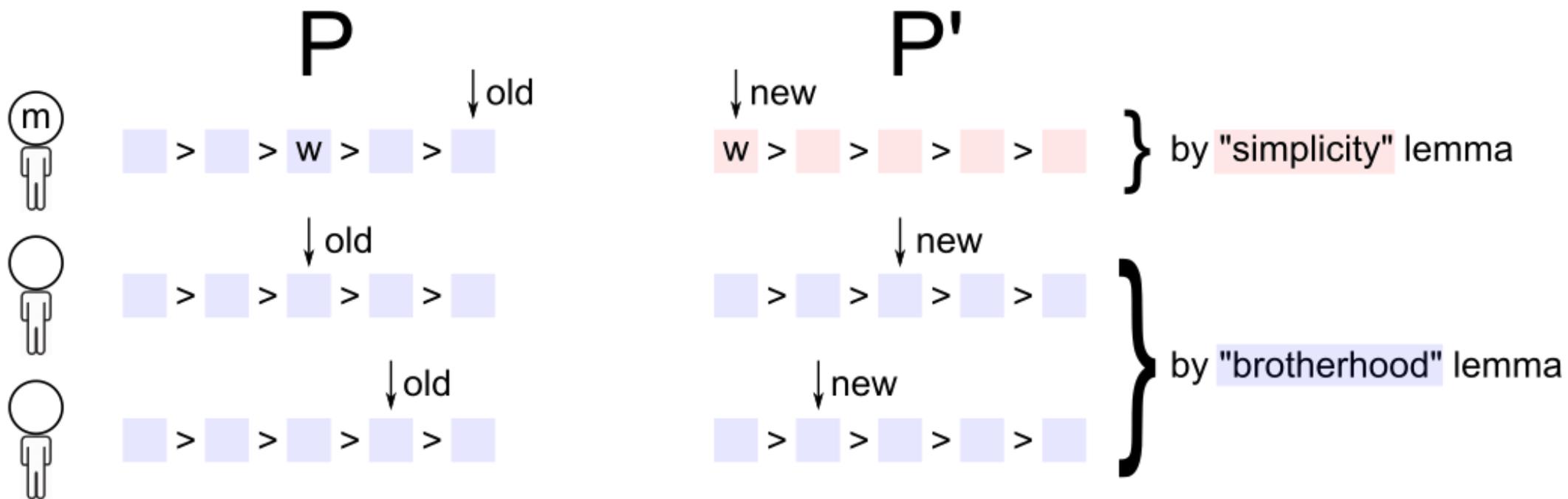
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Consider the execution of DA on the true profile P . Recall that $X = DA(P)$.

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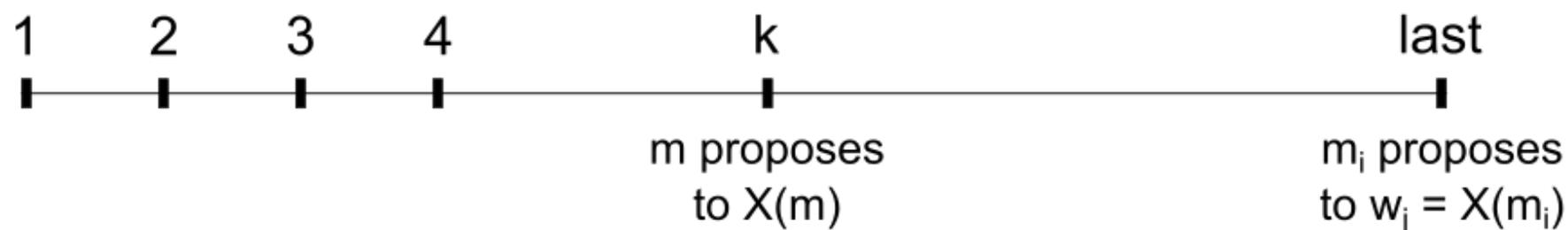
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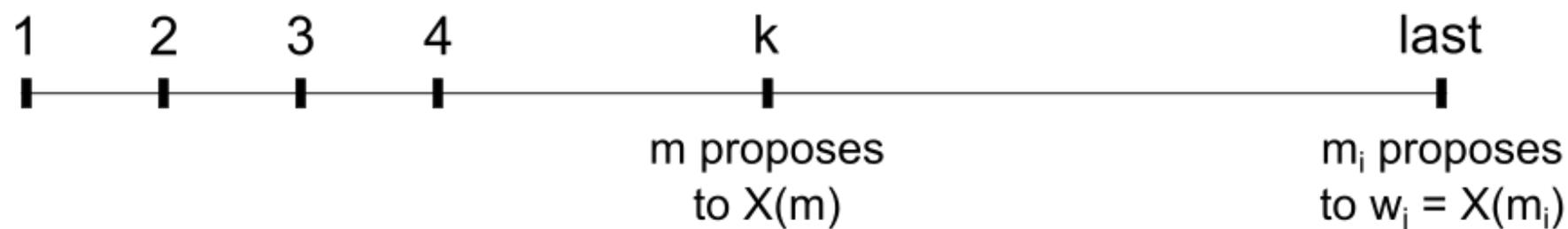


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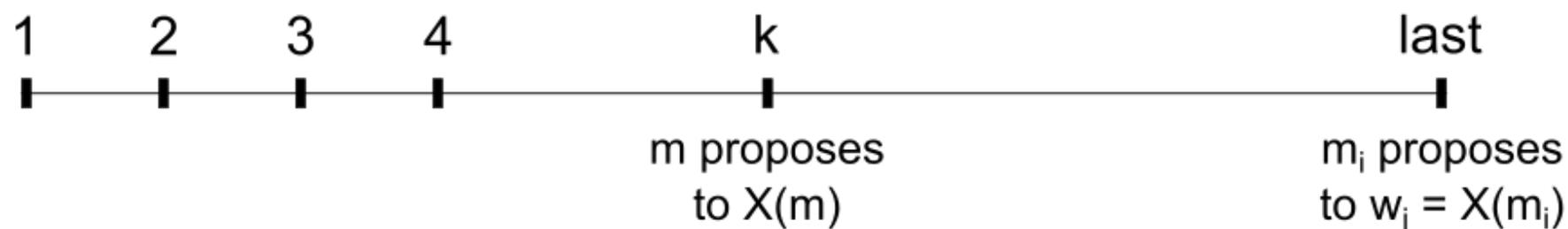
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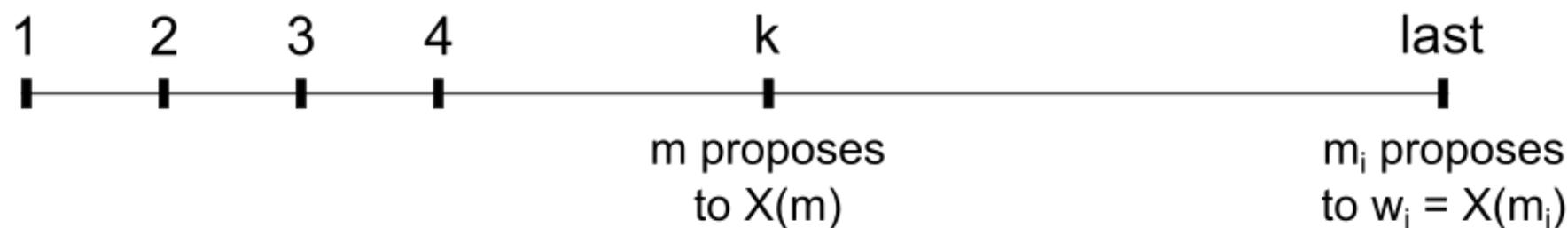
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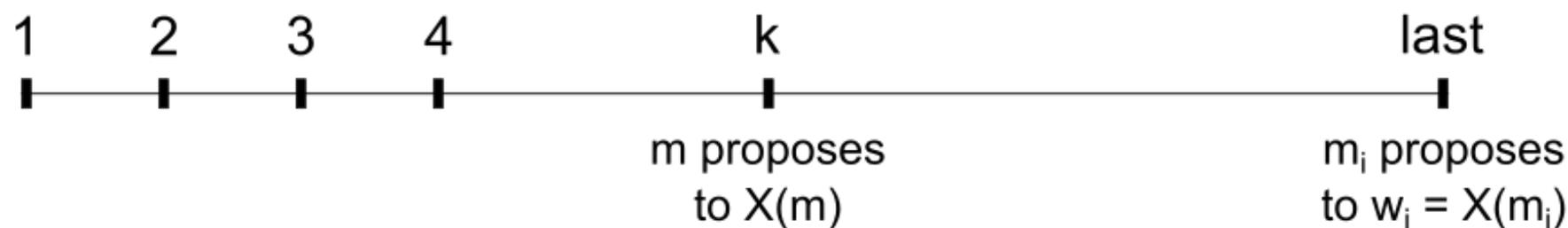
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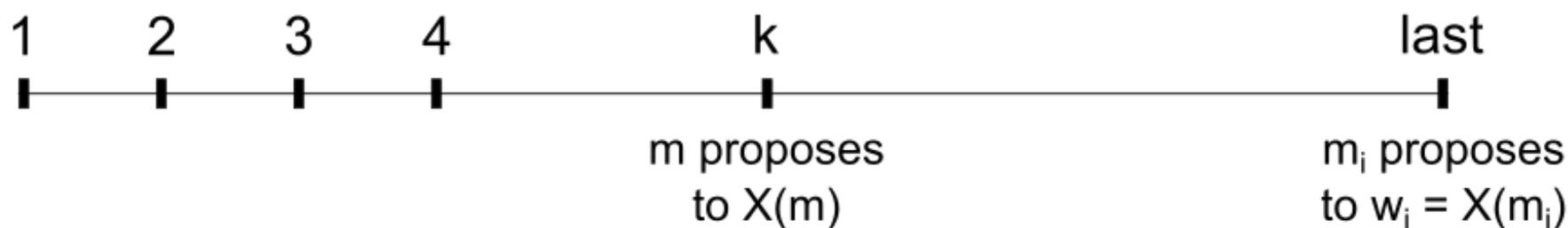
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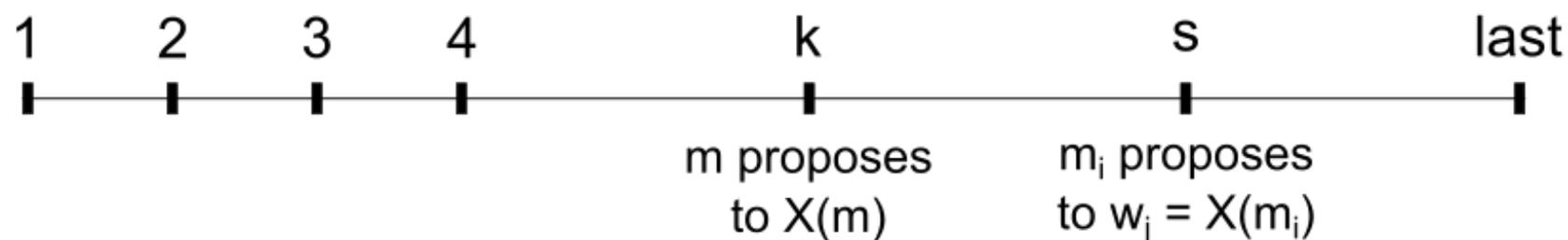
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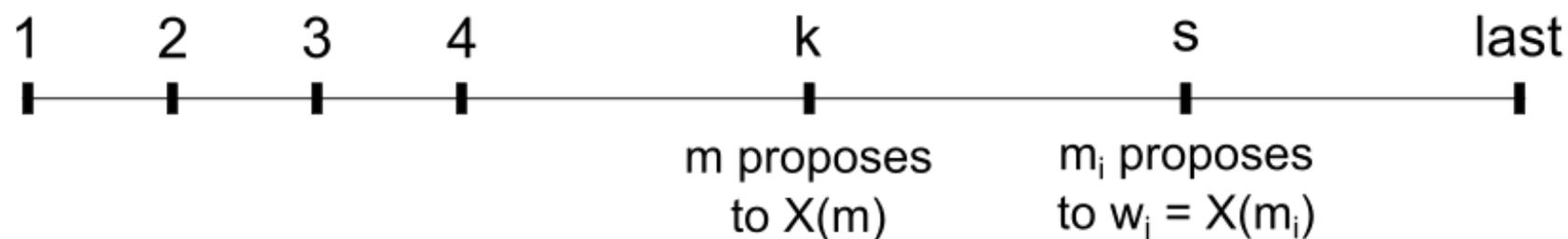


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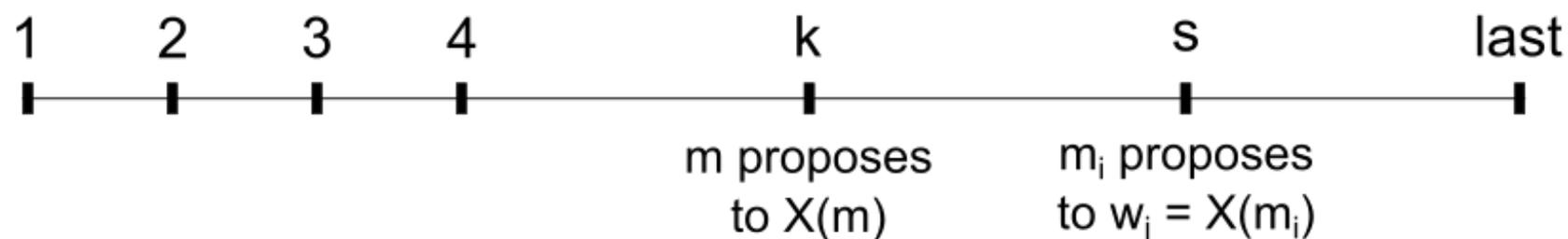
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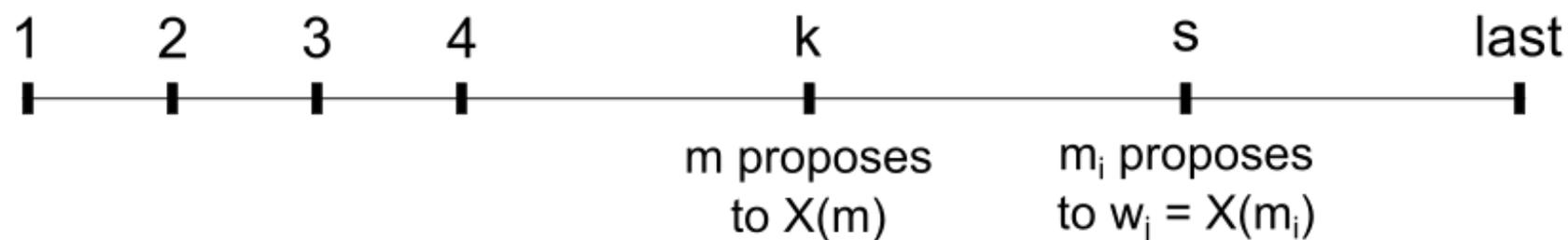
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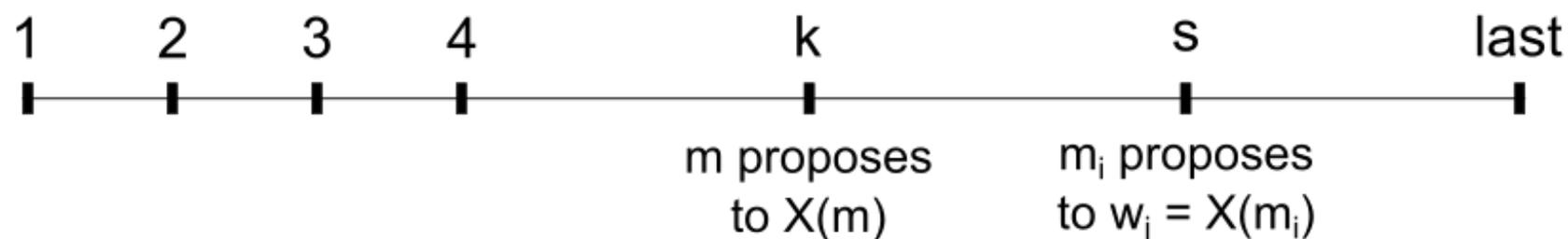


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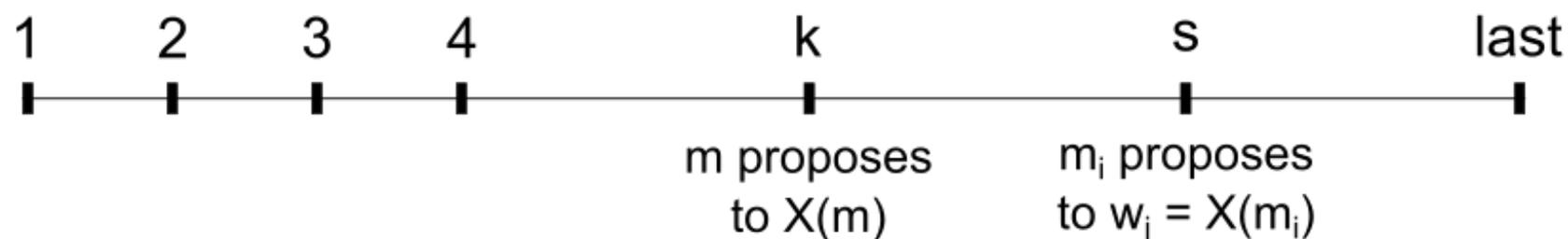
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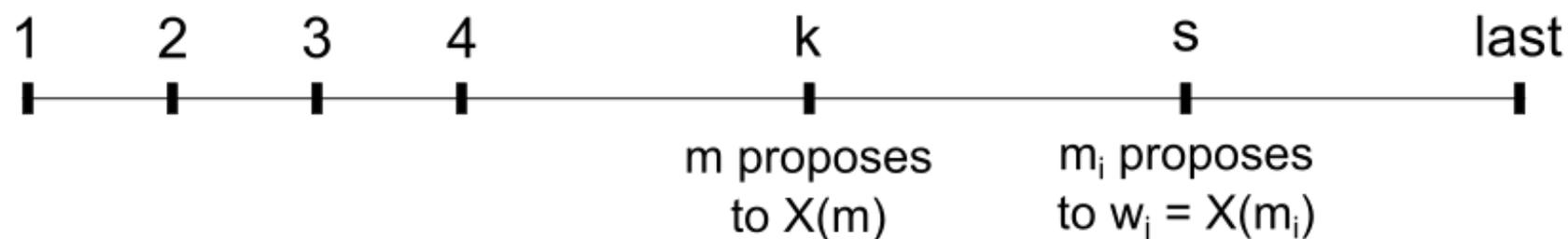
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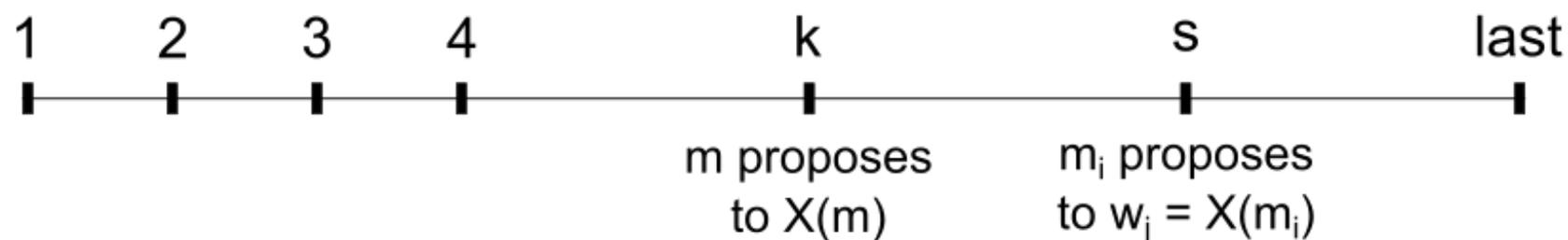
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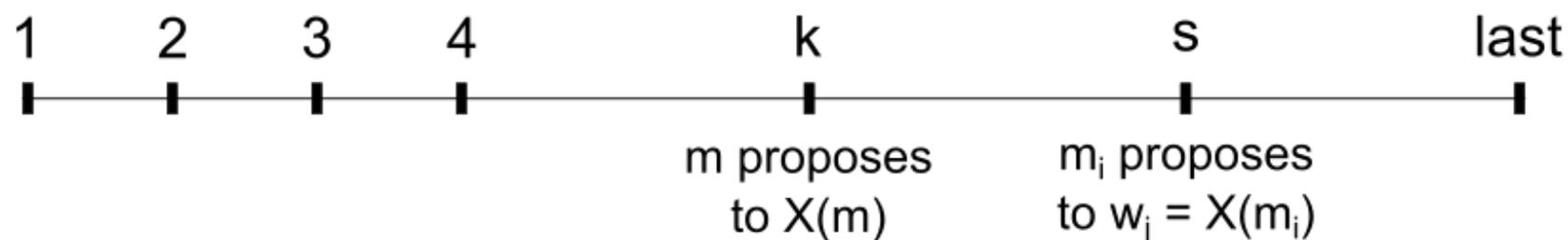
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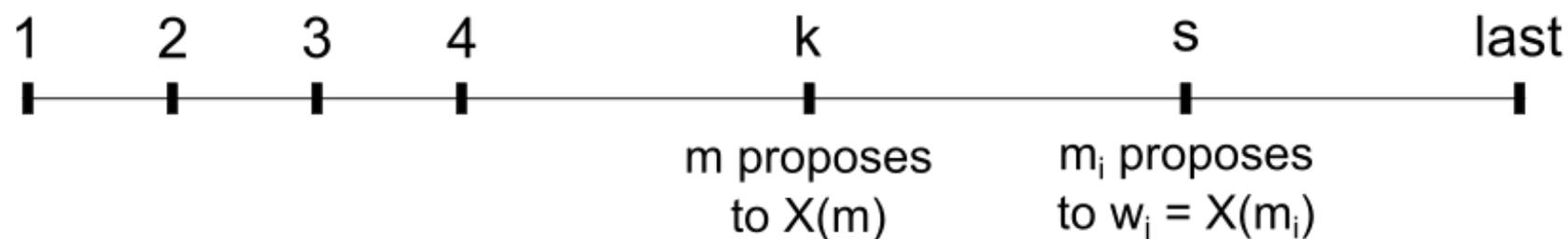
If R is non-empty, then let m_F be w_j 's favorite man in R .

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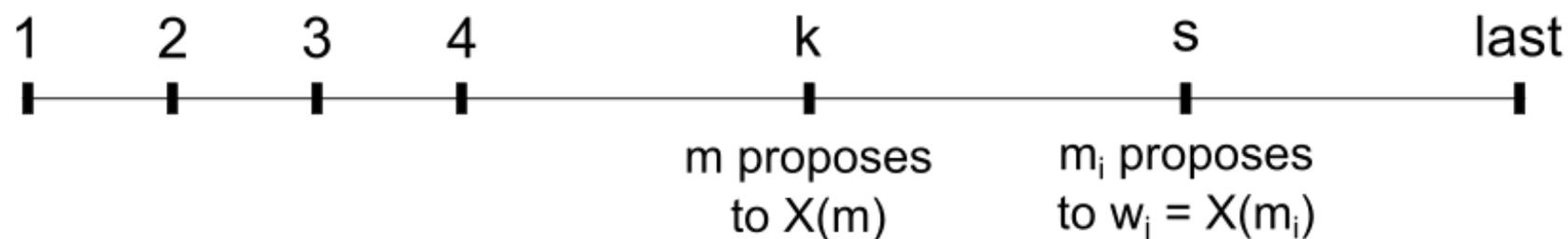
Then, m_F proposes to his X-partner in round $s+1$ or later. Thus, $X(m_F) = X'(m_F)$.

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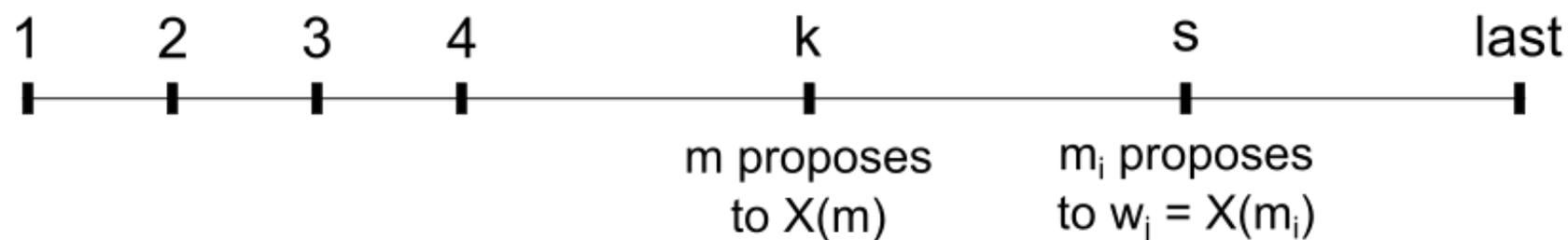
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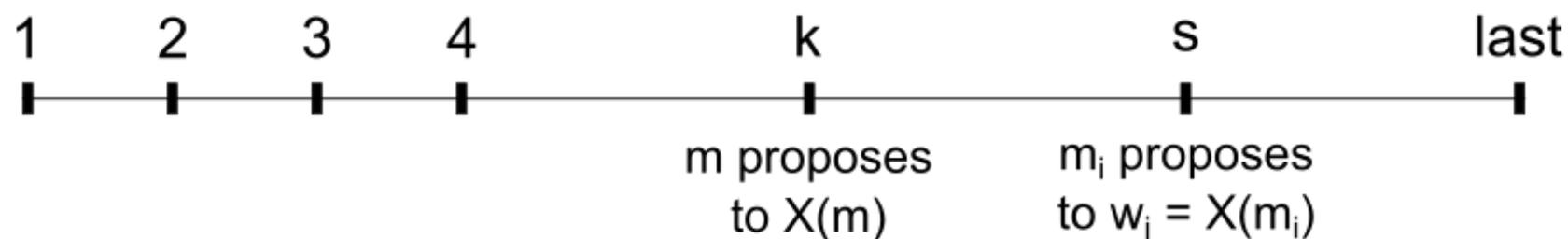


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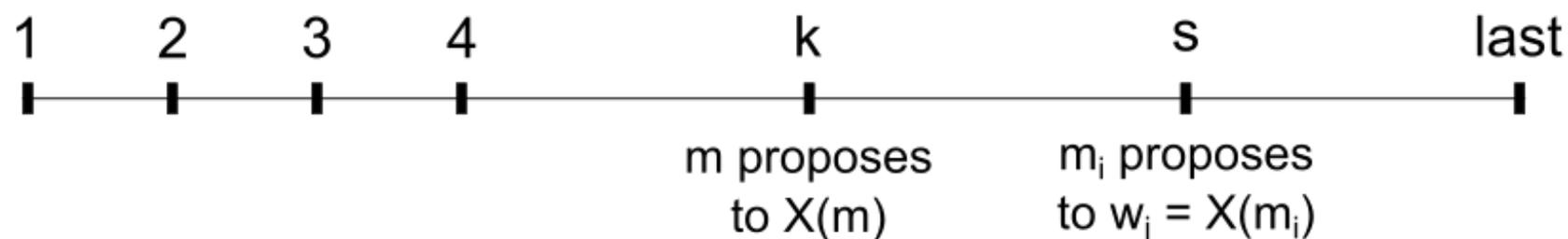
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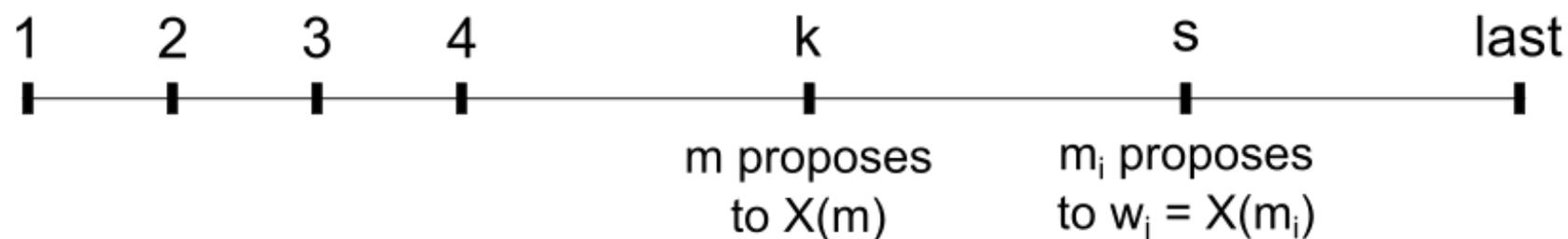
So, m_F must propose to w_j during DA on P' , and is again rejected since $X(m_F) = X'(m_F)$.

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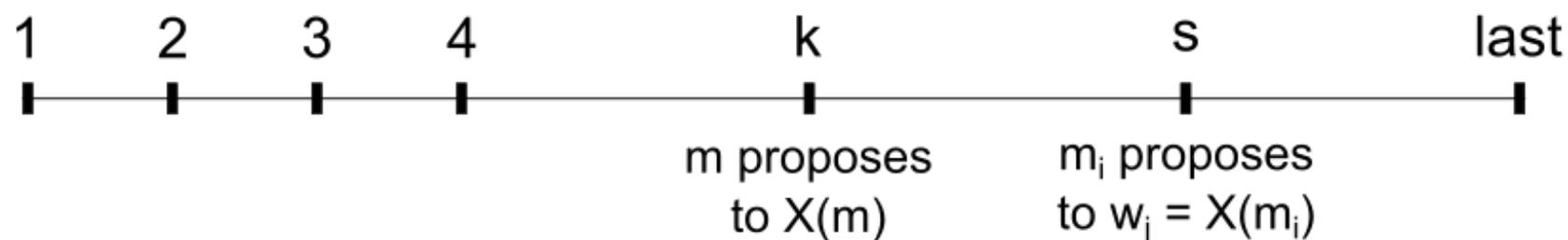
Thus, w_j receives at least one more proposal (besides m_F) during DA(P').

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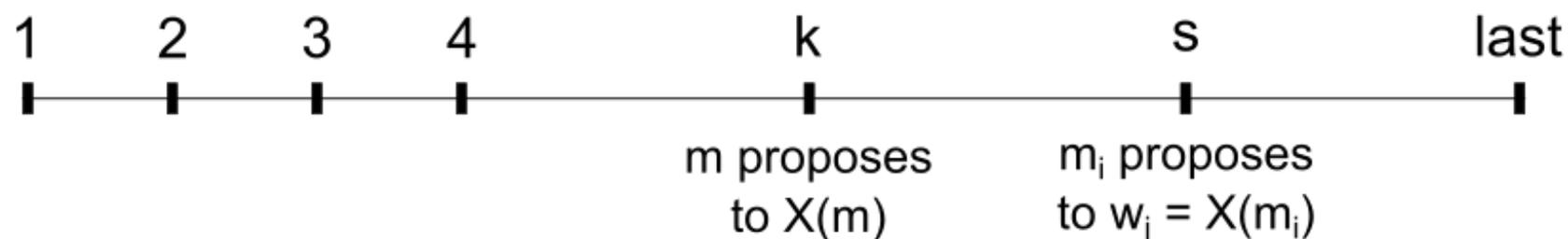


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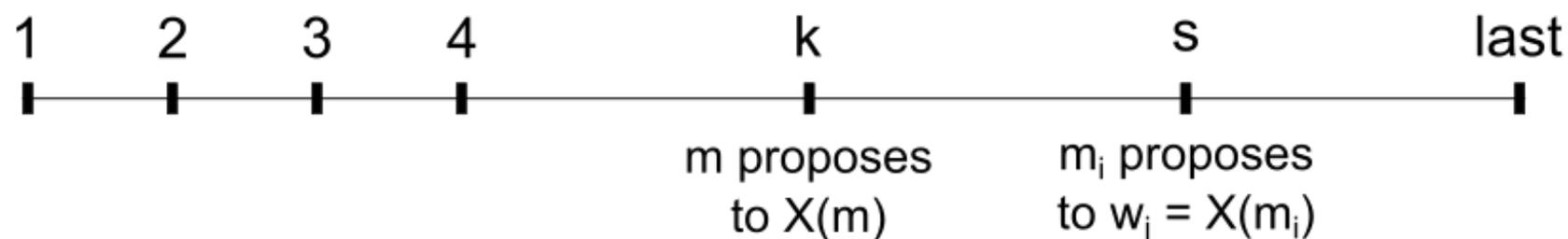
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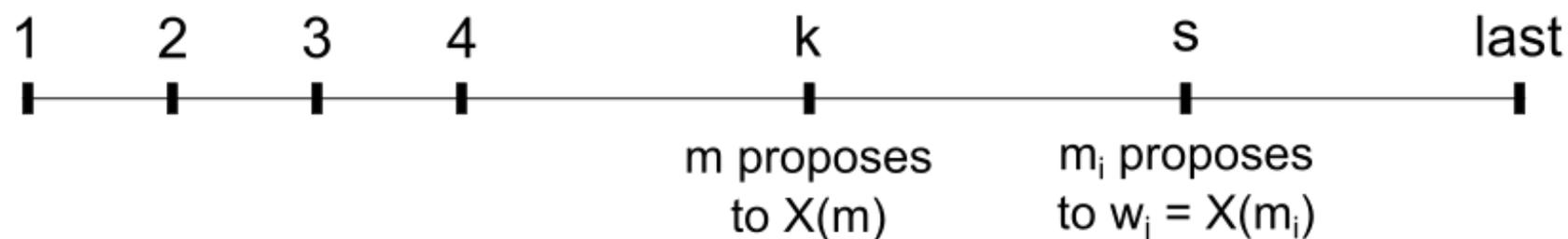
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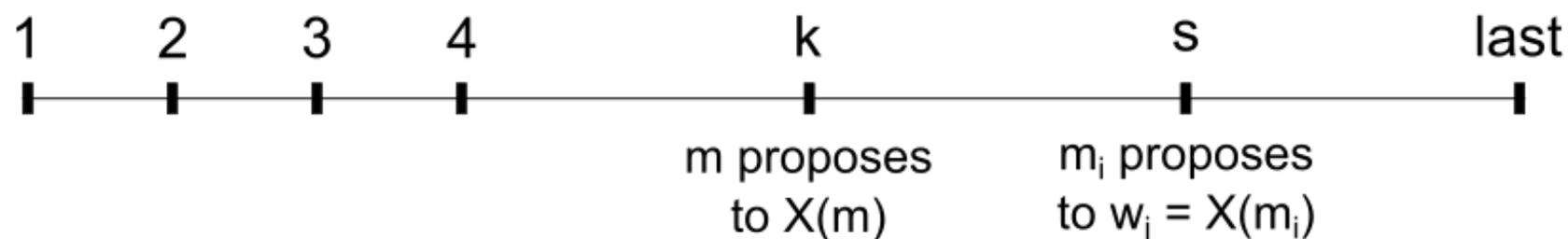
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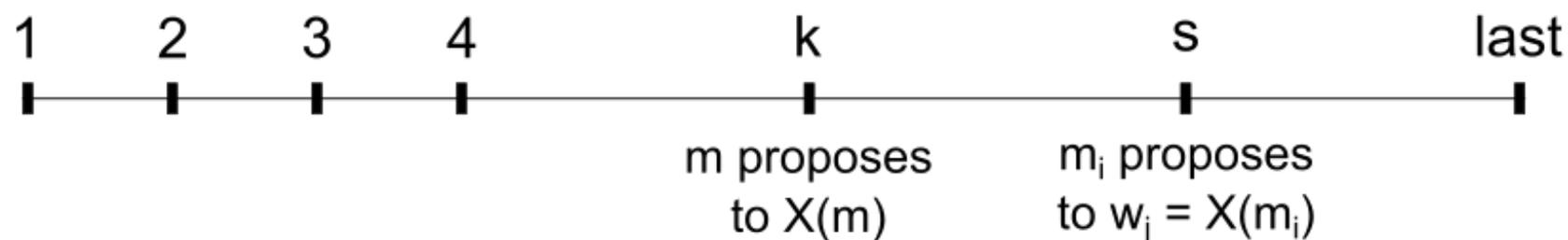
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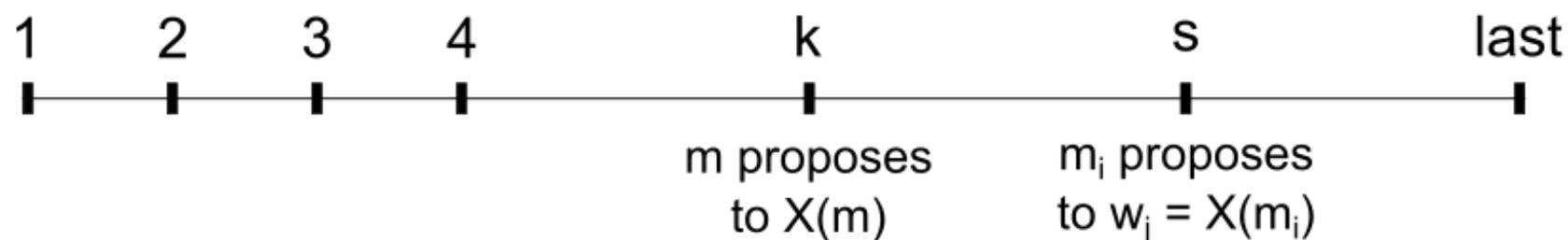


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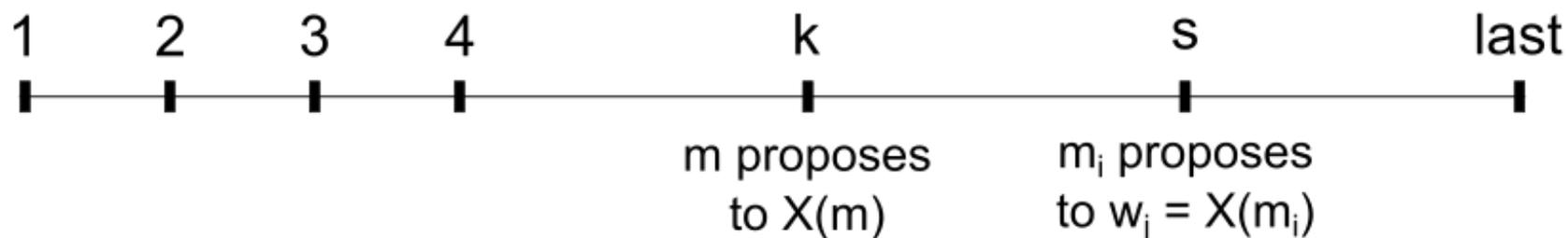
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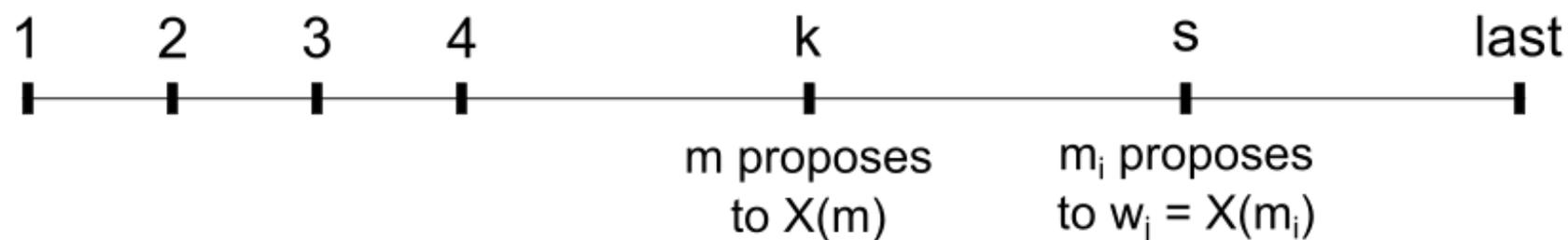


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