

COL749 : COMPUTATIONAL SOCIAL CHOICE

LECTURE 23

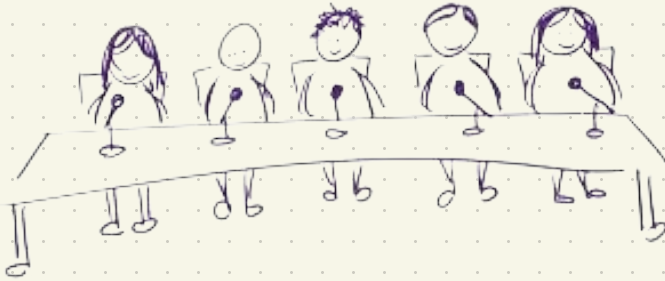
MULTIWINNER VOTING (CONTD.) & WRAP UP

APRIL 12, 2025

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ROHIT VAISH

ELECTING A COMMITTEE

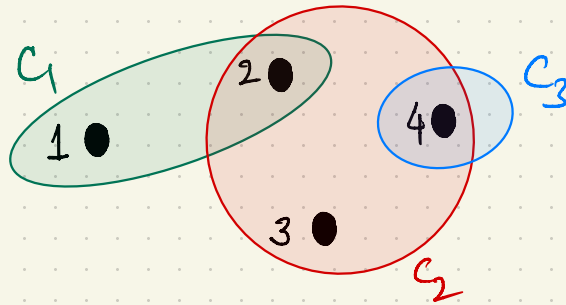


VOTING WITH APPROVAL PREFERENCES

- * Set of candidates C
- * Set of n voters $\{1, 2, \dots, n\}$
- * Each voter i *approves* a subset $A_i \subseteq C$

VOTING WITH APPROVAL PREFERENCES

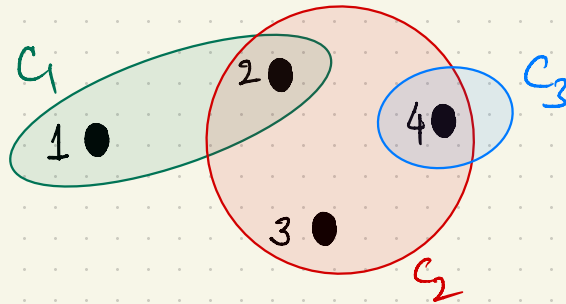
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1 : C_1
2 : C_1, C_2
3 : C_2
4 : C_2, C_3

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2 : c_1, c_2
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- * Goal : Select k candidates (k is given)

VOTING RULES

Thiele methods : Given weights $w_1 \geq w_2 \geq w_3 \geq \dots \geq 0$
find a size k committee W that maximizes

$$\text{Score}(W) = \sum_{\text{voter } i} w_i + w_2 + \dots + w_{|W \cap A_i|}$$

Approval Voting (AV) : 1, 1, 1, ...

Chamberlin Courant (CC) : 1, 0, 0, ...

Proportional Approval Voting (PAV) : $1, \frac{1}{2}, \frac{1}{3}, \dots$

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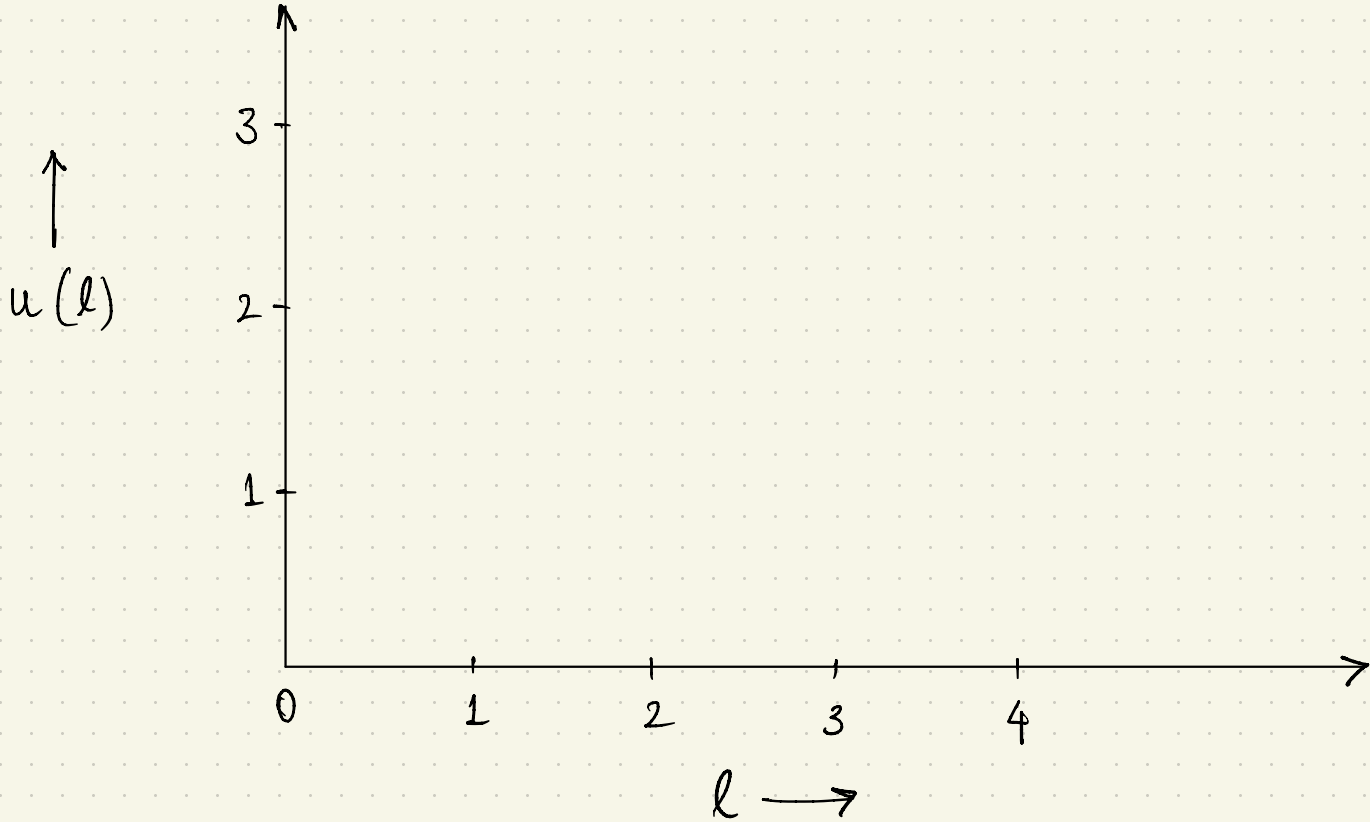
Approval Voting (AV) : 1, 1, 1, ...

utility function $u: N \rightarrow \mathbb{R}$
 $u(i) := w_1 + w_2 + \dots + w_k$

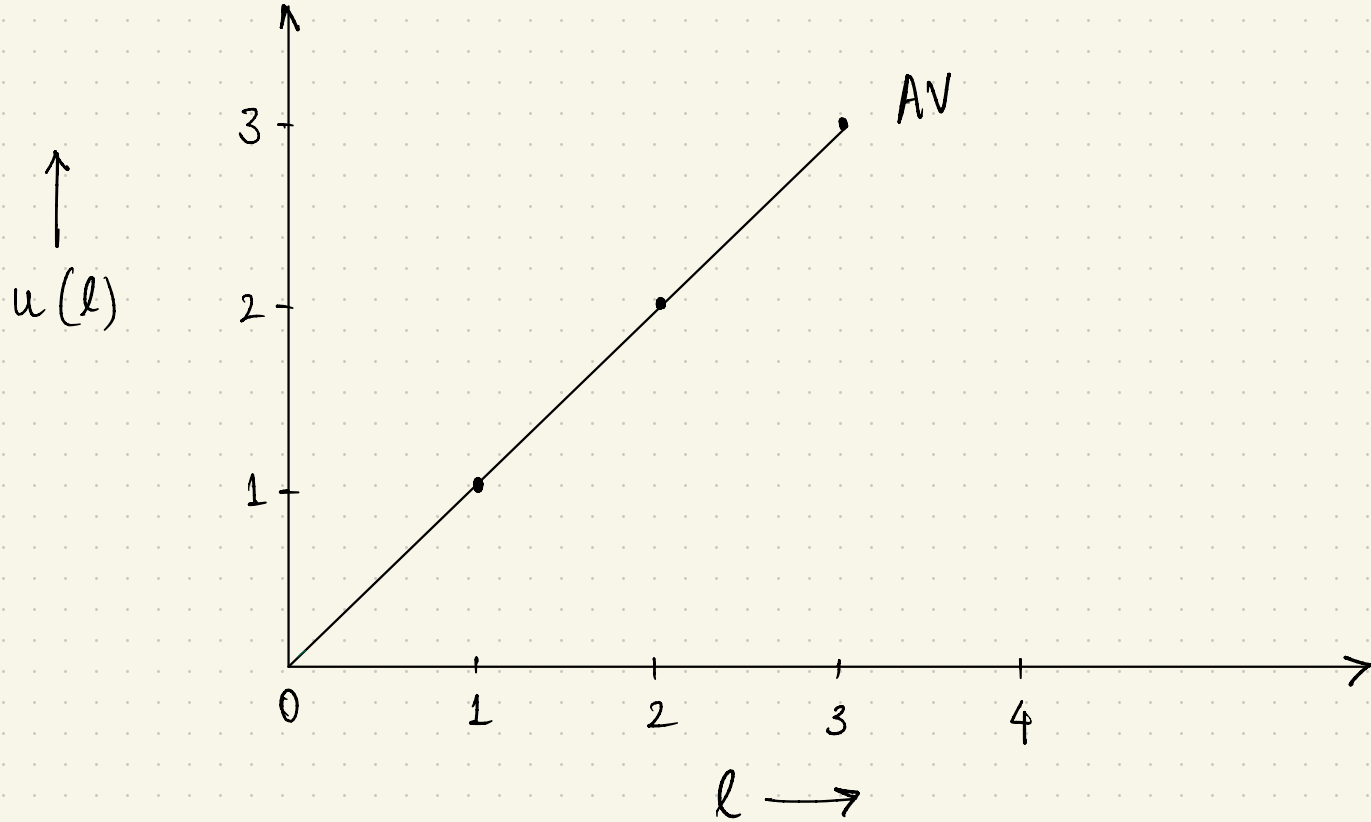
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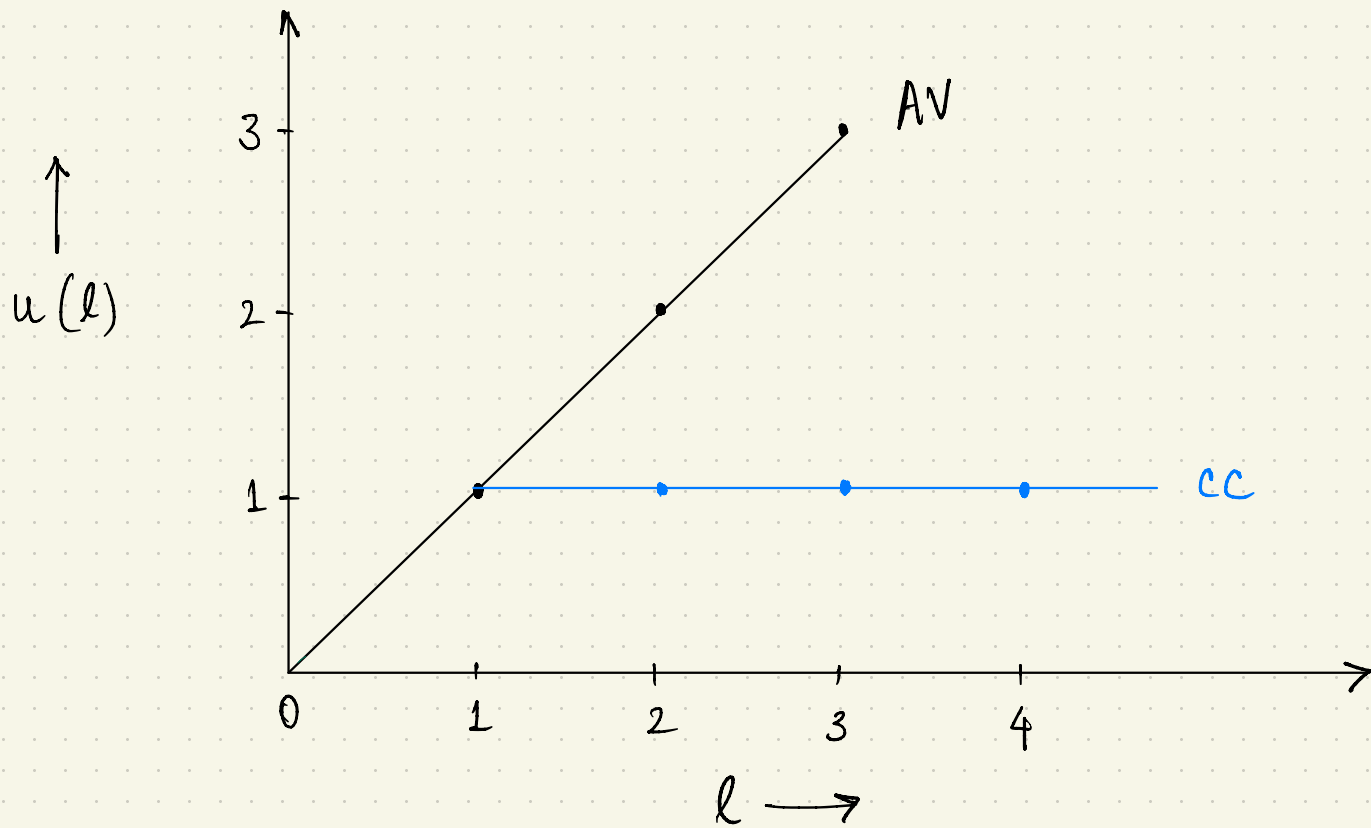
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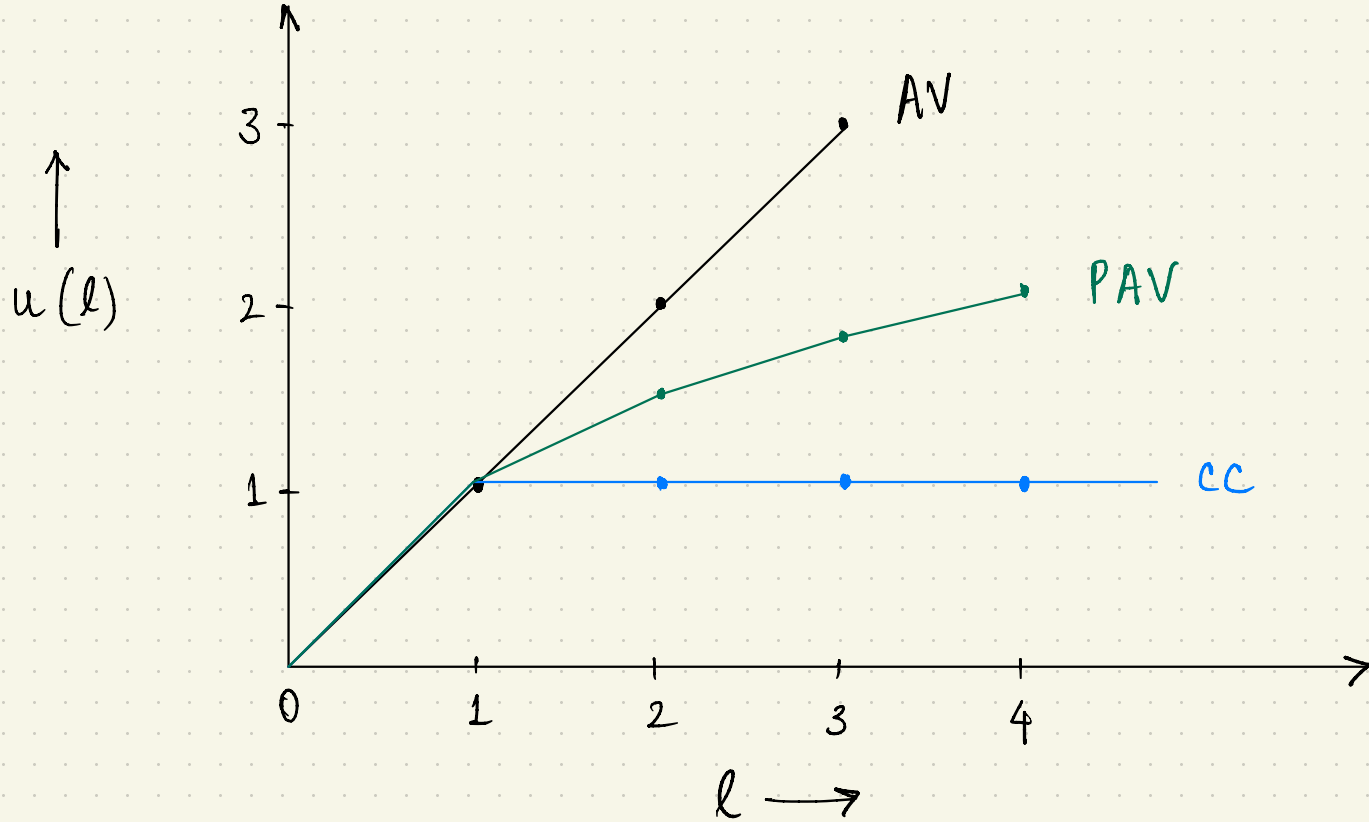
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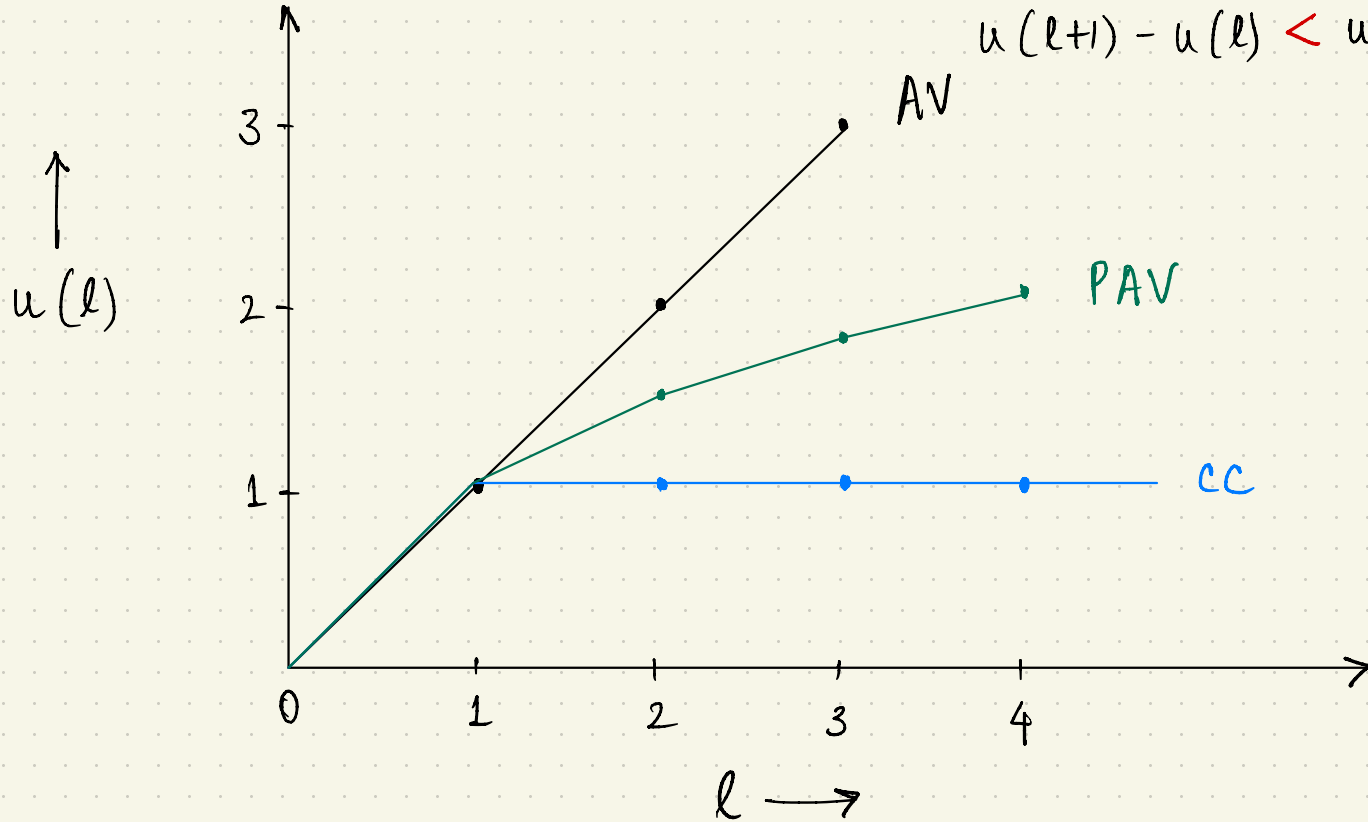


THIELE METHODS

[Skowron, Faliszewski, Lang AIJ 2016]

NP-hard whenever $\exists l$ s.t.

$$u(l+1) - u(l) < u(l) - u(l-1)$$

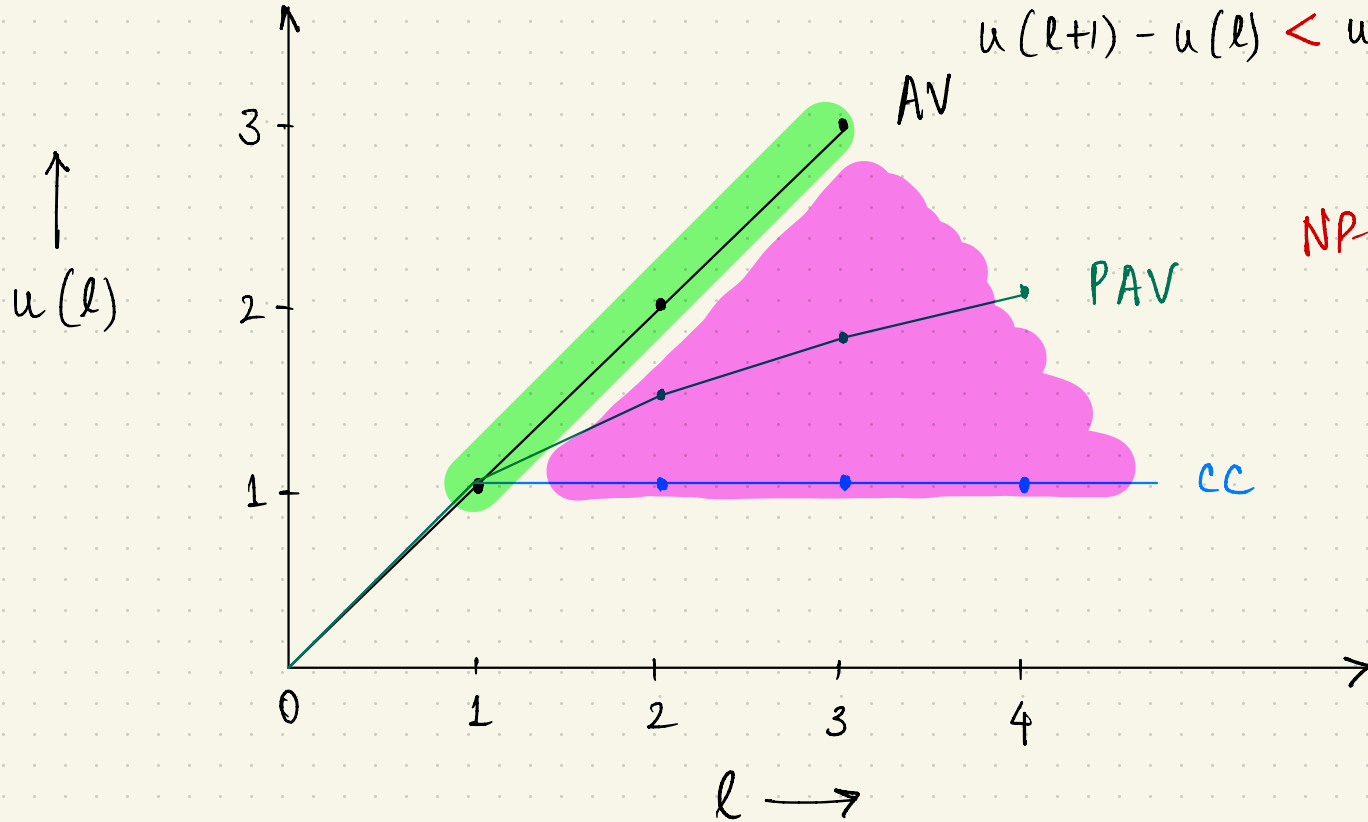


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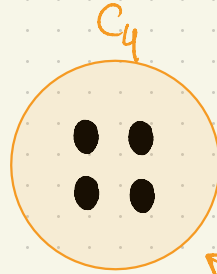
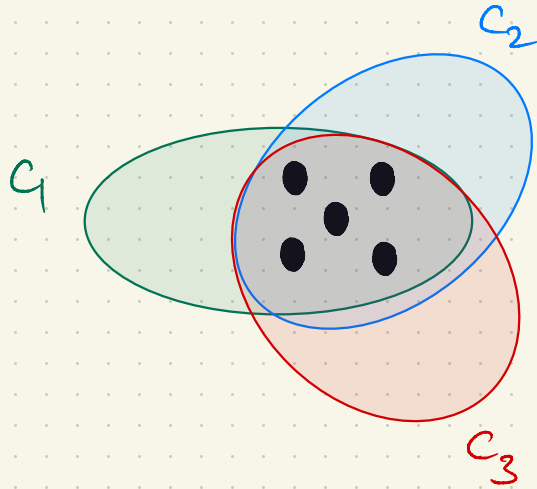
NP-hard

PAV

CC

WHEN IS A COMMITTEE "REPRESENTATIVE"?

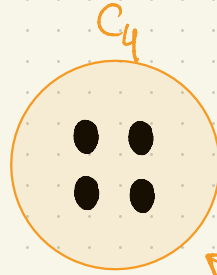
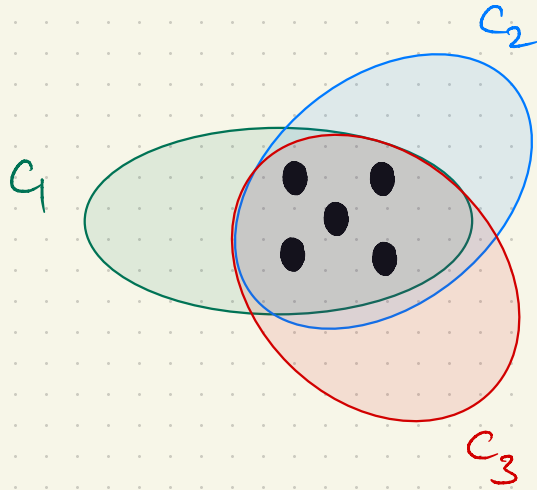
WHEN IS A COMMITTEE "REPRESENTATIVE"?



$k=3$: AV picks $\{C_1, C_2, C_3\}$

← Not represented

WHEN IS A COMMITTEE "REPRESENTATIVE"?



$k=3$: AV picks $\{C_1, C_2, C_3\}$

Not represented

Any large and cohesive set of voters deserves representation

WHEN IS A COMMITTEE "REPRESENTATIVE"?

Large

Cohesive

Representation

Strong justified reprⁿ

Justified representation

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$$|S| \geq \frac{n}{k}$$

$$|\bigcap_{i \in S} A_i| \geq 1$$

$$|W \cap \bigcap_{i \in S} A_i| \geq 1$$

Justified representation

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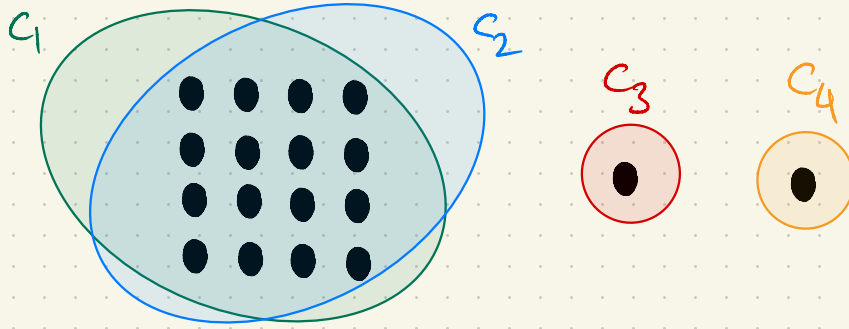
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JR CAN BE TOO PERMISSIVE

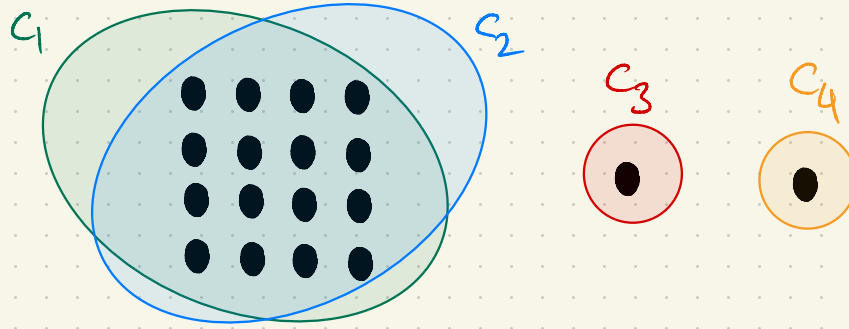
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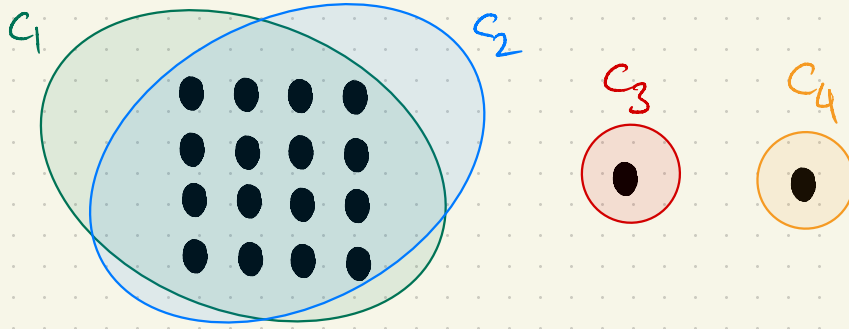


$\{C_1, C_3, C_4\}$ satisfies JR

but $\{C_1, C_2, C_3\}$ may be more representative

JR CAN BE TOO PERMISSIVE

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$\{C_1, C_3, C_4\}$ satisfies JR

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Idea: Very large cohesive group of voters deserves many representatives

n/k voters $\longrightarrow$ 1 representative
 $\ell \cdot n/k$ voters $\longrightarrow$ ℓ representatives

EXTENDED JUSTIFIED REPRESENTATION (EJR)

[Aziz, Brill, Conitzer, Elkind, Freeman, Walsh Soc. Choice & Welfare (2017)]

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$\hookrightarrow$ at least someone is represented l times

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for every $l \in \{1, 2, \dots\}$ $l=1 : \text{JR}$

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☹ AV fails JR $\Rightarrow$ AV fails EJR

☹ CC satisfies JR but fails EJR (Exercise)

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😊 PAV satisfies EJR (Exercise: exchange argument)

EXTENDED JUSTIFIED REPRESENTATION (EJR)

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↳ Only Thiele method that provides EJR

COMPLEXITY OF EJR

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Finding PAV committee is NP-hard

COMPLEXITY OF EJR



Finding PAV committee is NP-hard



Checking whether a committee satisfies EJR is co-NP-complete.

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Recall the proof of "PAV satisfies JR/EJR"

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Swap a candidate

by at least $\frac{n}{k^2}$

COMPLEXITY OF EJR

Local Search PAV

[Aziz, Elkind, Huang, Lackner, Sanchez-Fernandez, Skowron AAI 2018]

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- * Start with an arbitrary committee.
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- otherwise return the current committee.

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Polynomial time

Why?

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Polynomial time

Why? max. PAV score = $n \cdot \ln k$

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	Large	Cohesive	Representation
Strong justified repr ⁿ	$ S \geq \frac{n}{k}$	$ \bigcap_{i \in S} A_i \geq 1$	$ W \cap \bigcap_{i \in S} A_i \geq 1$
Justified representation	$ S \geq \frac{n}{k}$	$ \bigcap_{i \in S} A_i \geq 1$	$ W \cap \bigcup_{i \in S} A_i \geq 1$
Extended justified repr ⁿ	$ S \geq \ell \frac{n}{k}$	$ \bigcap_{i \in S} A_i \geq \ell$	$\exists i \in S \text{ s.t. } W \cap A_i \geq \ell$

WHEN IS A COMMITTEE "REPRESENTATIVE"?

Existence

Computation

Verification

Strong justified reprⁿ

Justified representation

Extended justified reprⁿ

WHEN IS A COMMITTEE "REPRESENTATIVE"?

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May not exist

Justified representation

Always exists
(CC, PAV)

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WHEN IS A COMMITTEE "REPRESENTATIVE"?

	Existence	Computation	Verification
Strong justified repr ⁿ	May not exist		
Justified representation	Always exists (CC, PAV)	Poly time (Local search PAV)	Poly time
Extended justified repr ⁿ	Always exists (PAV)	Poly time (Local search PAV)	co-NP-complete

QUIZ

QUIZ

In the given voting instance, identify a committee that satisfies SJR but fails EJR.

$$C = \{a, b, c, d, e\} \quad k = 4$$

$$v_1 : \{a, b\}$$

$$v_2 : \{a, b\}$$

$$v_3 : \{c\}$$

$$v_4 : \{d, e\}$$

PLAN FOR TODAY

- * More voting rules: Sequential PAV
- * More representation axioms: Core
- * Other properties: Strategy proofness

SEQUENTIAL PAV

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* $W \leftarrow \emptyset$

* while $|W| < k$

find candidate c that maximizes PAV score of $W \cup \{c\}$

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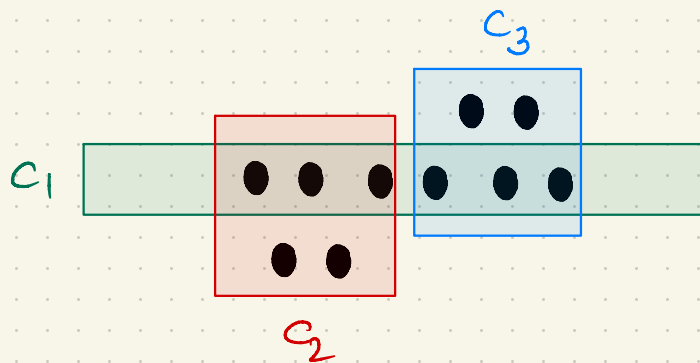
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$k=2$

PAV :

Seq. PAV :

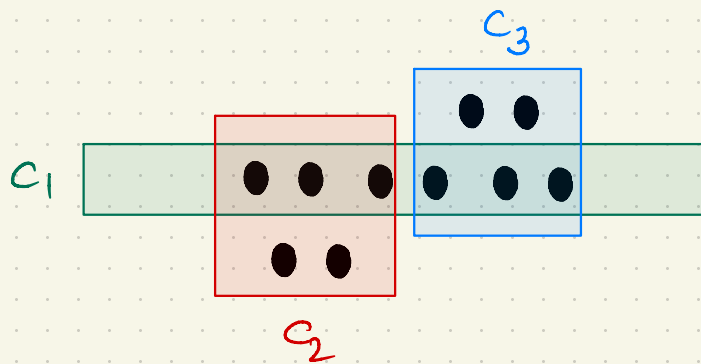
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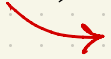


$k=2$

PAV : $\{c_2, c_3\}$

Seq. PAV : $\{c_1, c_2\}$ or $\{c_1, c_3\}$

SEQUENTIAL PAV

Thm: PAV score of seq. PAV committee $\geq \left(1 - \frac{1}{e}\right)$ opt. PAV score
 $\approx 63\%$

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$$f(W \cup \{c\}) - f(W) \geq f(W' \cup \{c\}) - f(W') \\ \text{if } W \subseteq W'$$

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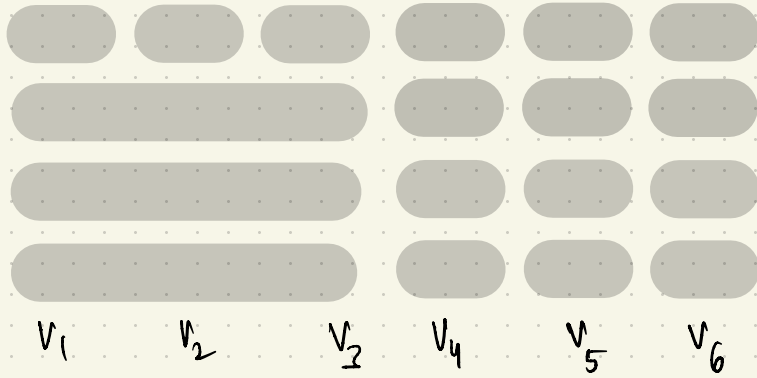
[Sánchez-Fernández, Elkind, Lackner, Fernández, Fisteus, Val, Skowron AAAI 2017]

Seq. PAV **satisfies** JR for $k \leq 5$ and **fails** JR for $k \geq 6$.

IS EJR TOO PERMISSIVE?

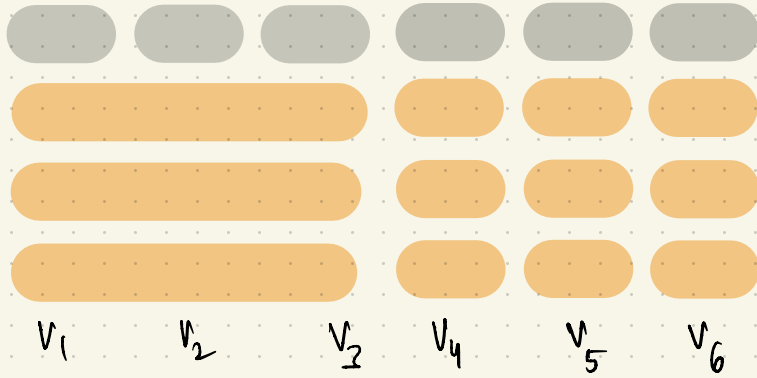
IS EJR TOO PERMISSIVE?

$k=12$



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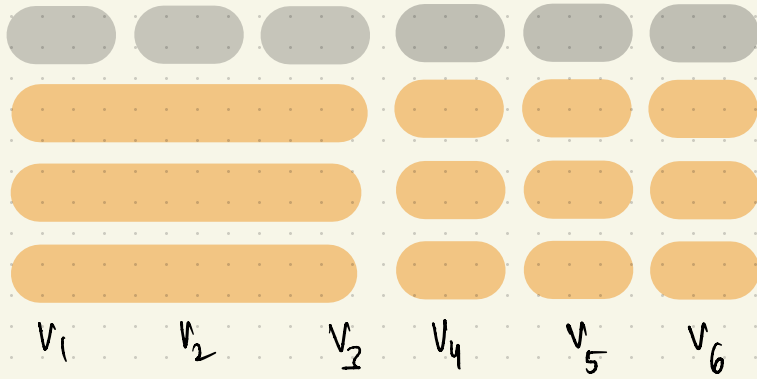
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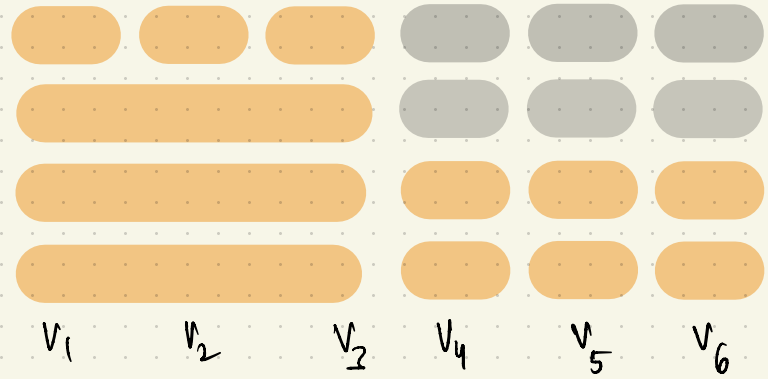
PAV committee

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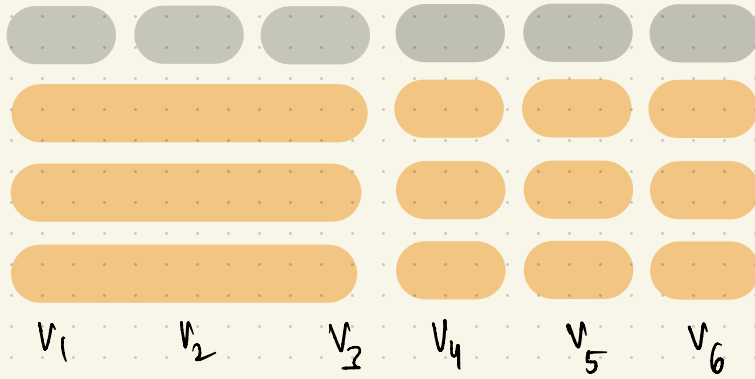
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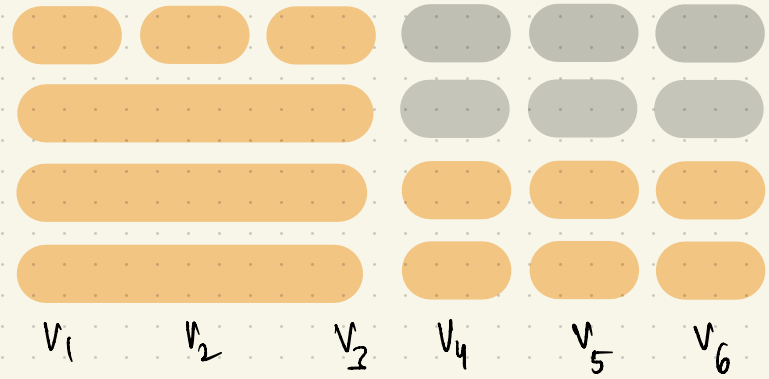
"Fairer" representation

IS EJR TOO PERMISSIVE?

$k=12$



PAV committee



"Fairer" representation

Half of voter population deserves half of the seats.

CORE

CORE

A group of voters S with $|S| \geq \ell \cdot \frac{n}{k}$ blocks committee W

if there is a subset $T \subseteq C$ with $|T| = \ell$ such that

$$\forall i \in S \quad |A_i \cap T| > |A_i \cap W|.$$

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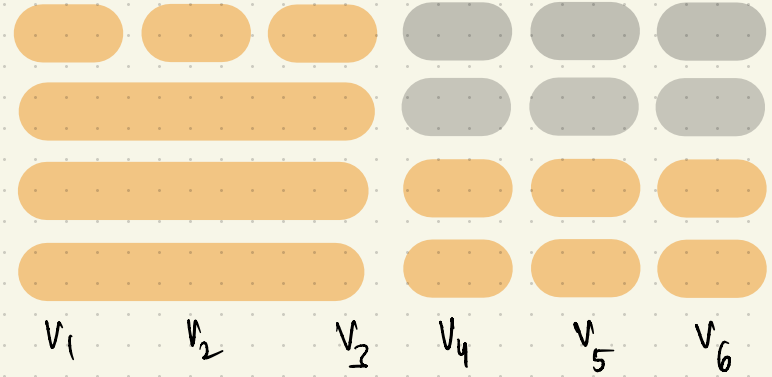
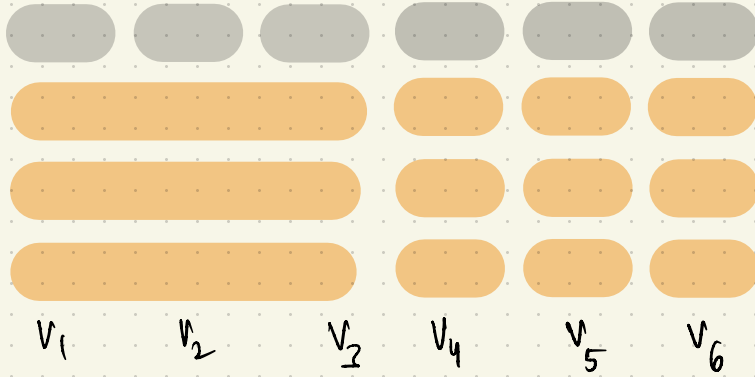
if there is a subset $T \subseteq C$ with $|T| = \ell$ such that

$$\forall i \in S \quad |A_i \cap T| > |A_i \cap W|.$$

Committee W is in the core if it is not blocked.

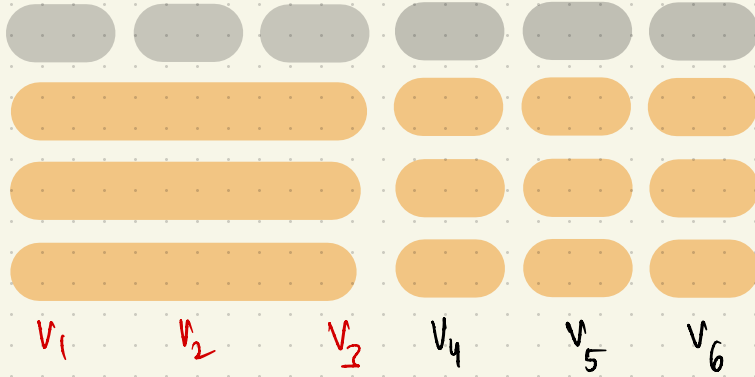
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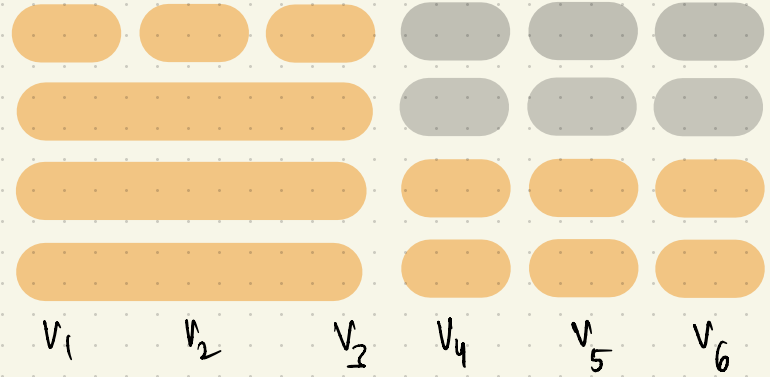


CORE

$k=12$



Not Core

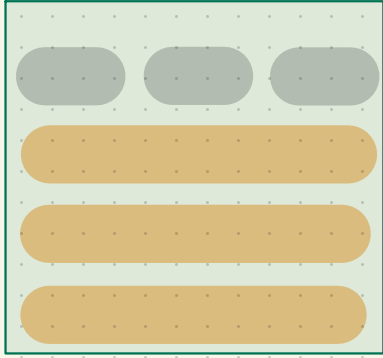


Core

CORE

$k=12$

T



v_1

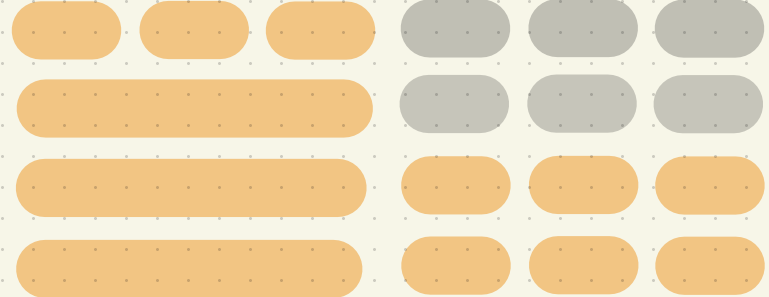
v_2

v_3

v_4

v_5

v_6



v_1

v_2

v_3

v_4

v_5

v_6

Not Core

$$S = \{v_1, v_2, v_3\}$$

$$|S| \geq 6 \cdot \frac{6}{12}$$

$\nearrow l$
 $\nearrow n$
 $\searrow k$

Core

CORE

Observe: Core $\Rightarrow$ EIR.

CORE

Observe: Core $\Rightarrow$ EJR.

Why? Voters violating EJR constitute a blocking coalition with $T \subseteq \bigcap_{i \in S} A_i$

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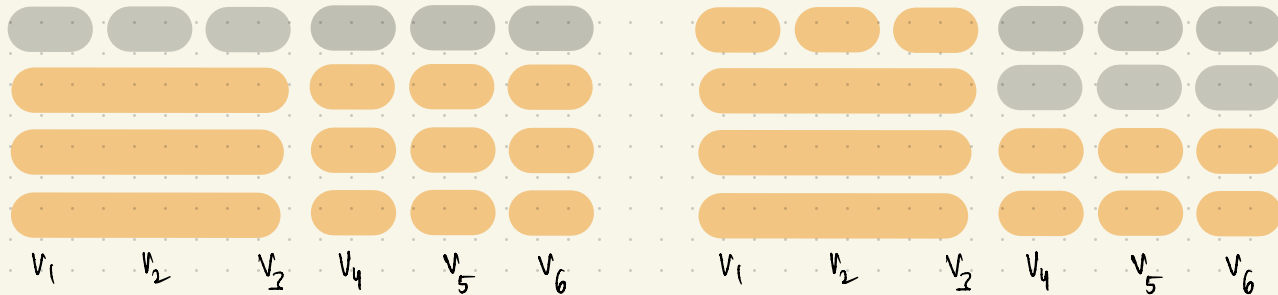
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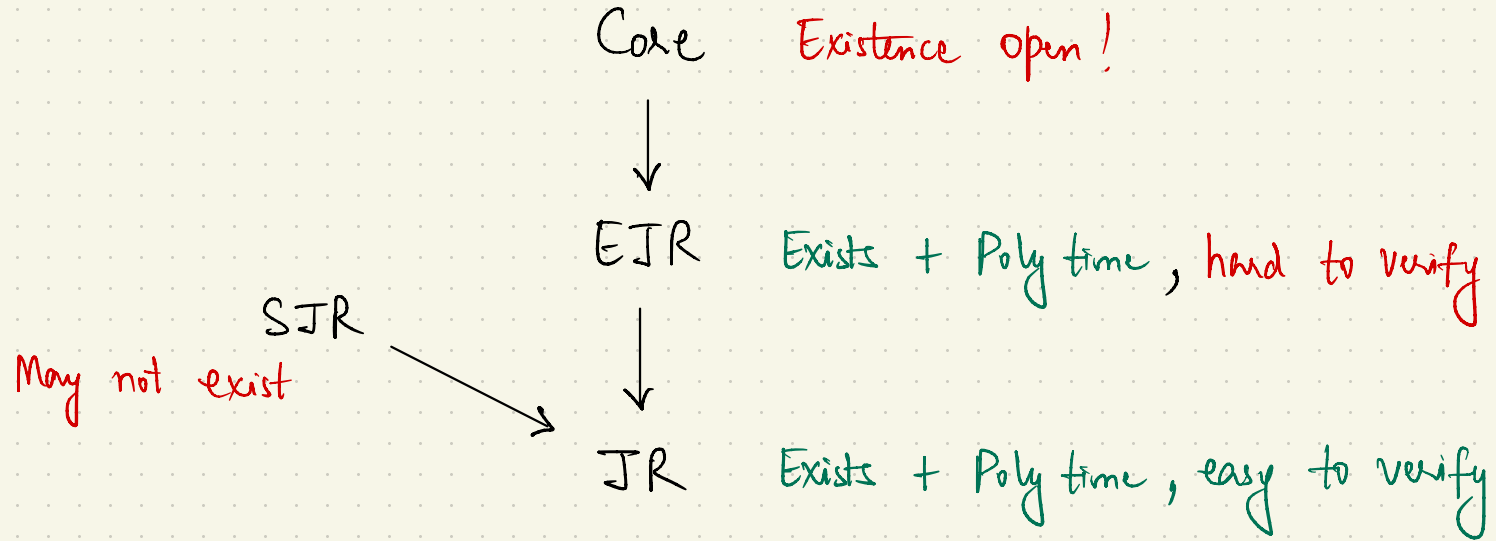
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Core v/s egalitarian (Pigou-Dalton)



PICTURE SO FAR



STRATEGY PROOFNESS

STRATEGY PROOFNESS

approved candidates
w/ true prefs

No voter can derive a higher utility
by misreporting its approved set of candidates.

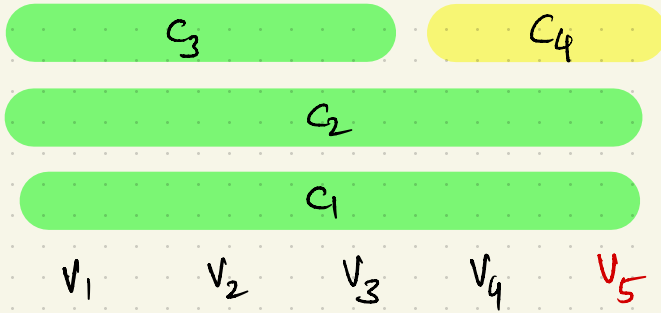
STRATEGY PROOFNESS

PAV is **not** strategy proof.

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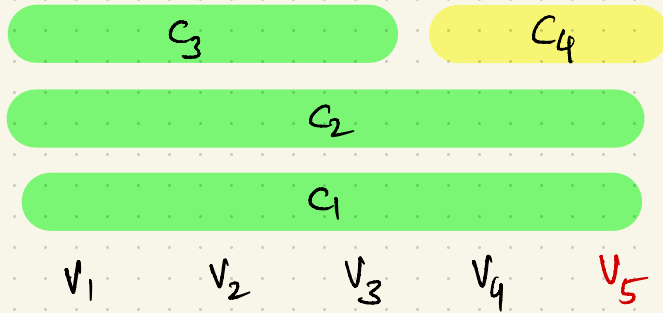
$$k=3$$



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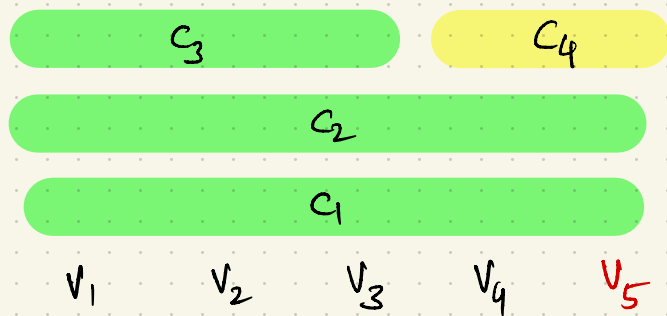


PAV selects $\{C_1, C_2, C_3\}$

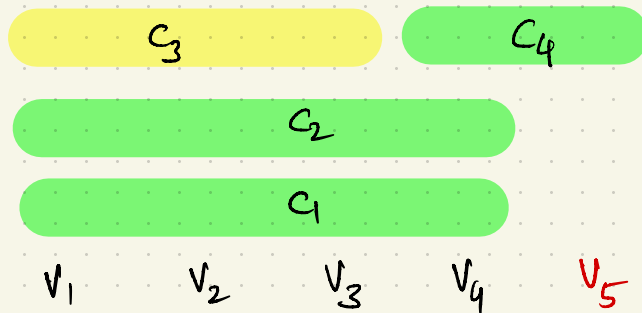
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$k=3$



PAV selects $\{c_1, c_2, c_3\}$



PAV selects $\{c_1, c_2, c_4\}$

STRATEGY PROOFNESS

[Peterson, AAMAS 2018]

Thm: No voting rule is strategyproof and satisfies JR.

STRATEGY PROOFNESS

[Peterson, AAMAS 2018]

Thm: No voting rule is strategyproof and satisfies IR.

Proof: By SAT solving.

STRATEGY PROOFNESS

[Peterson, AAMAS 2018]

Thm: No voting rule is strategyproof and satisfies IR.

Proof: By SAT solving.

* Approval voting is strategyproof but fails IR.

COURSE RECAP

* Interplay of computation and incentives

Matching

Fair Division

Voting

COURSE RECAP

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Matching

Fair Division

Voting

- * Deferred-acceptance
- * Stable lattice
- * Stability v/c truthfulness
- * Fair matchings
- * Top-trading cycles
and kidney exchange

COURSE RECAP

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- * Cake and rent division
- * EPI and envy cycles
- * Nash social welfare
- * Top-trading envy cycles
- * Best of both worlds

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Voting

- * Rules, rules, rules!
- * Manipulation is inevitable
- * ... but CS saves the day!
- * Structure helps
- * Axioms $\rightarrow$ Quantitative

TAKE AWAYS

- * Collective decision-making is everywhere!
- * Algorithms affect and are affected by users' incentives
- * Computation  Economics

NEXT TIME

Guest lecture on "SAT solving"
by Priyanka Golia.

