COL749: Computational Social Choice

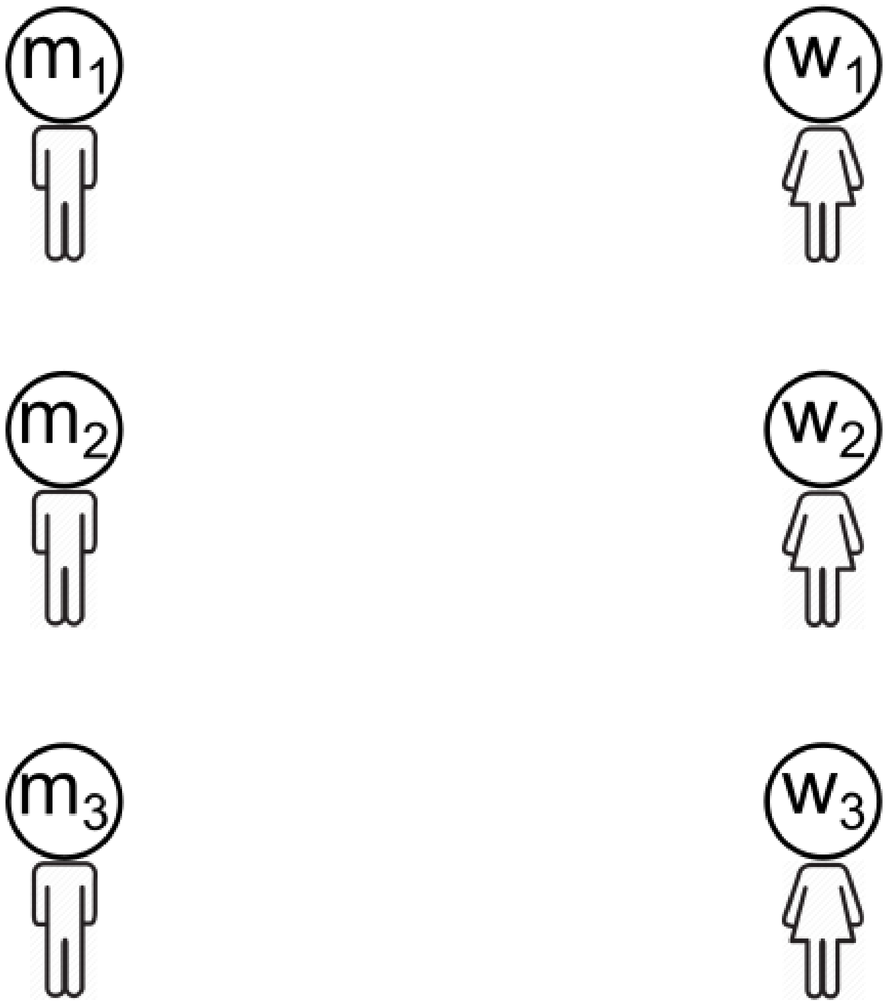
Lecture 2

Structure of Stable Matchings

Jan 06, 2025 | Rohit Vaish

Stable Matching Problem

Stable Matching Problem



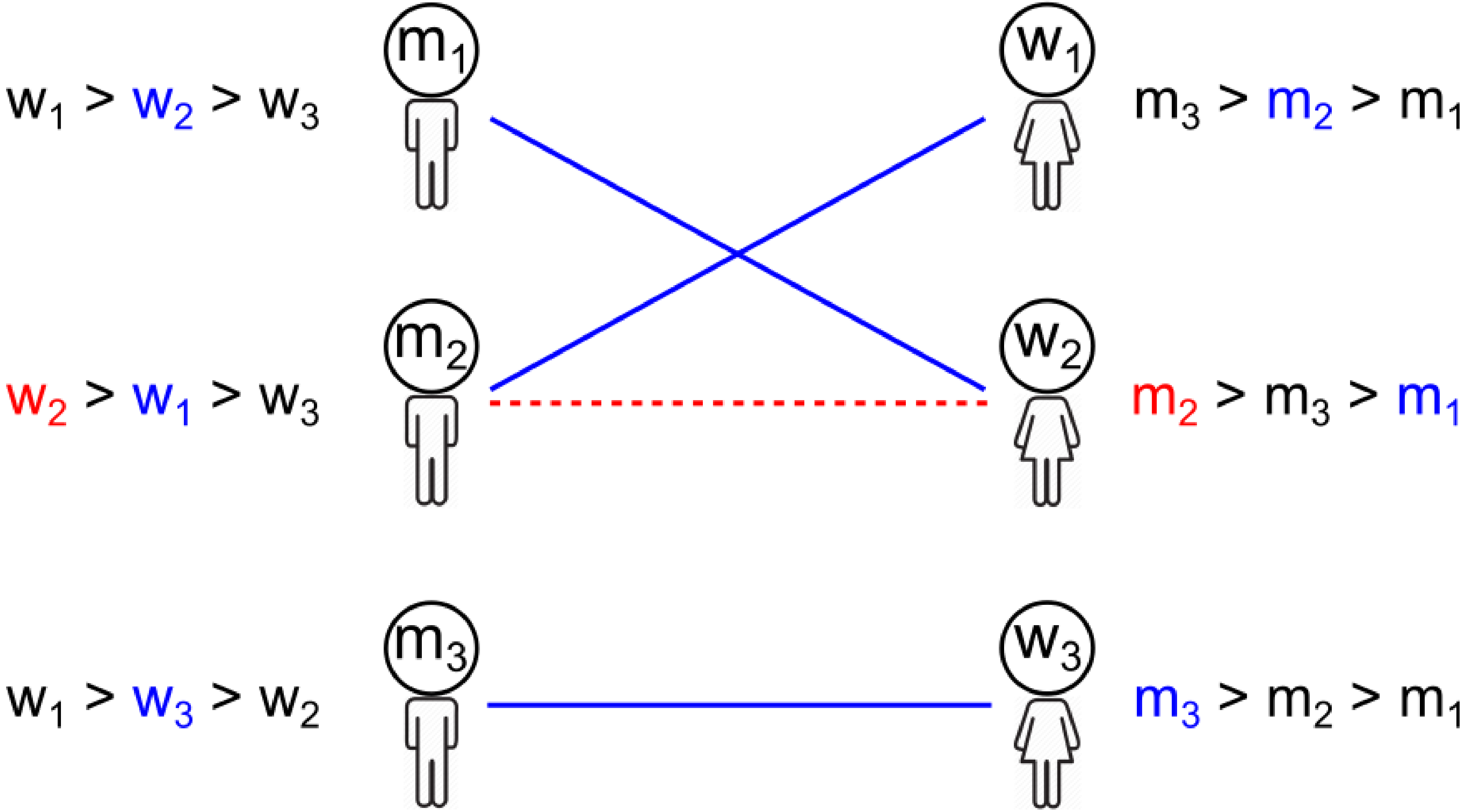
Stable Matching Problem

$w_1 > w_2 > w_3$ m_1 w_1 $m_3 > m_2 > m_1$

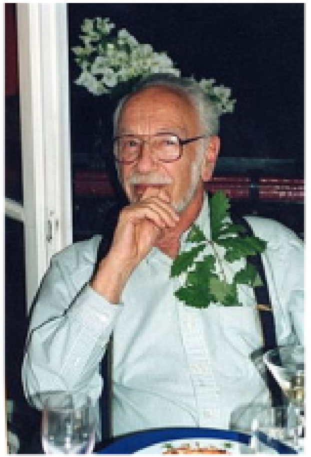
$w_2 > w_1 > w_3$ m_2 w_2 $m_2 > m_3 > m_1$

$w_1 > w_3 > w_2$ m_3 w_3 $m_3 > m_2 > m_1$

Stable Matching Problem



A matching is stable if there is no blocking pair.



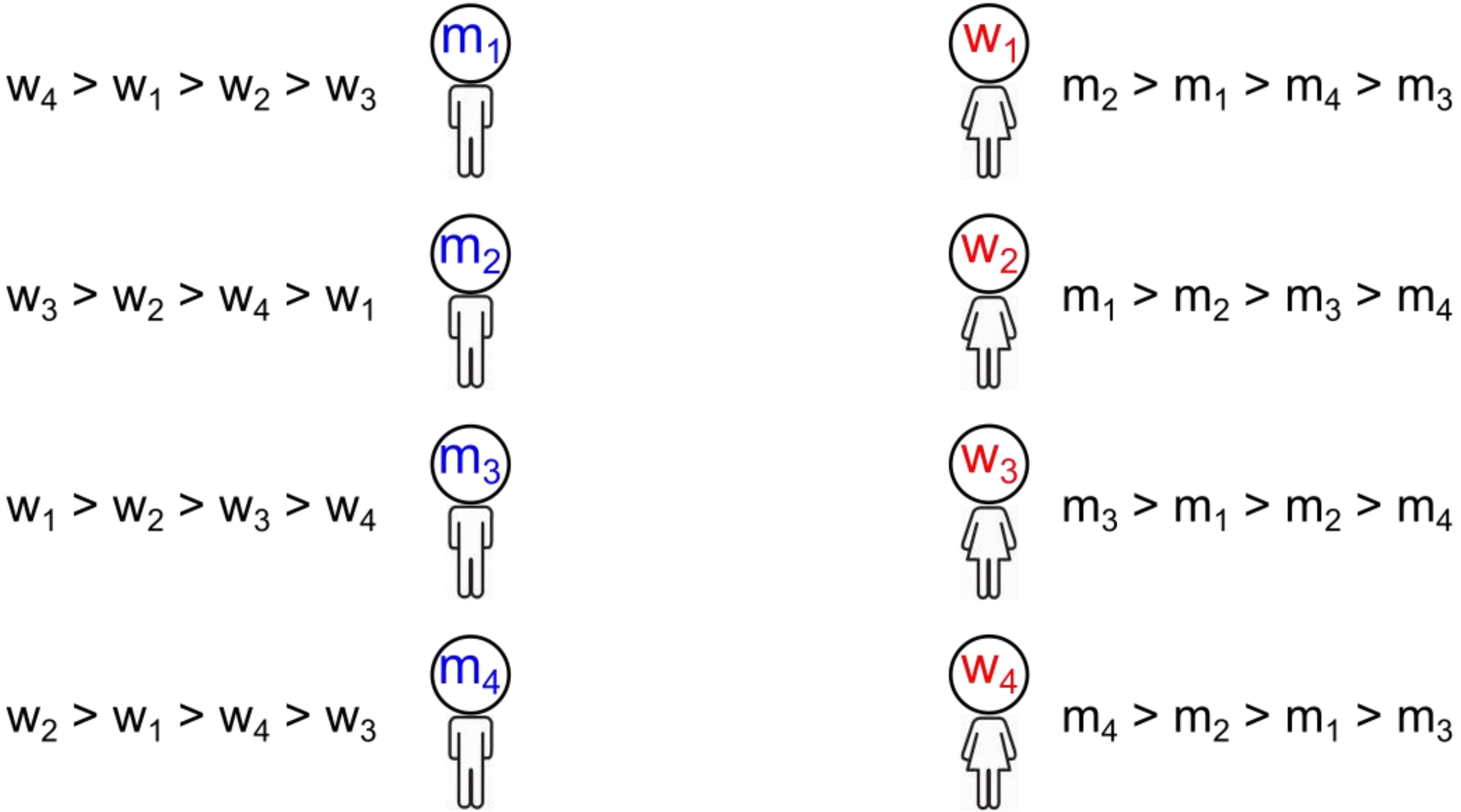
COLLEGE ADMISSIONS AND THE STABILITY OF MARRIAGE

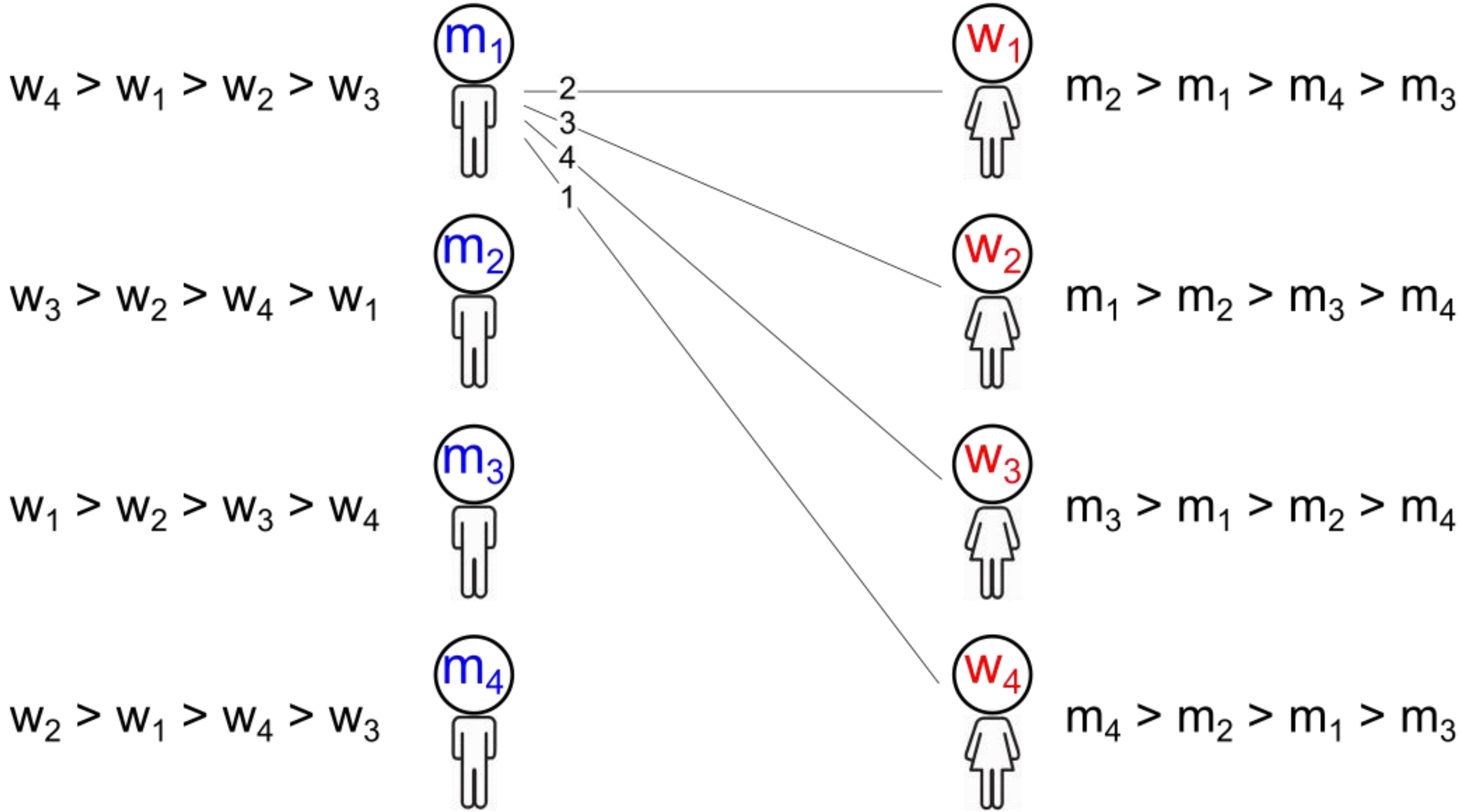
D. GALE* AND L. S. SHAPLEY, Brown University and the RAND Corporation

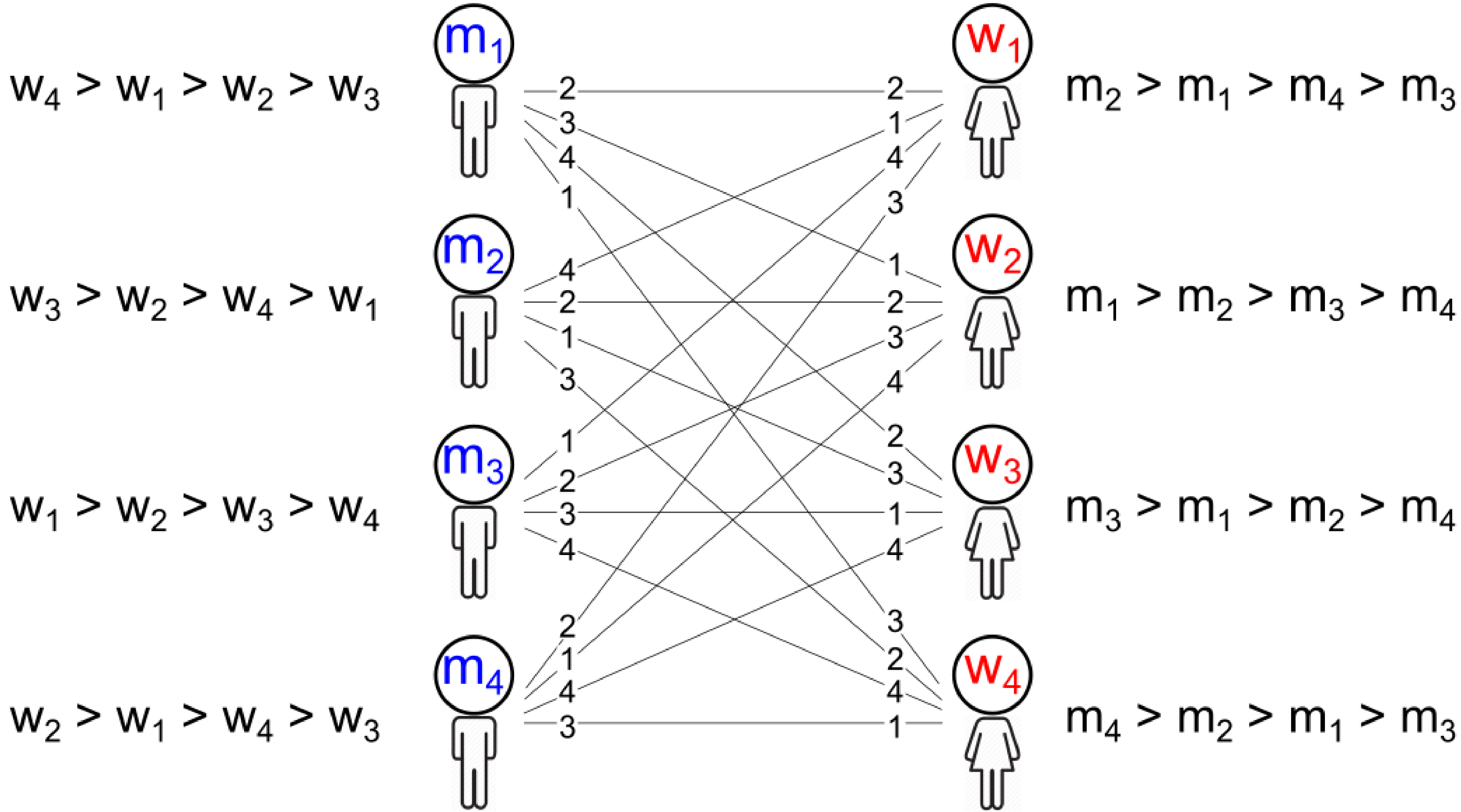
Source: *The American Mathematical Monthly*, Jan., 1962, Vol. 69, No. 1 (Jan., 1962), pp. 9–15

Given any preference profile, a stable matching for that profile always exists and can be computed in polynomial time.

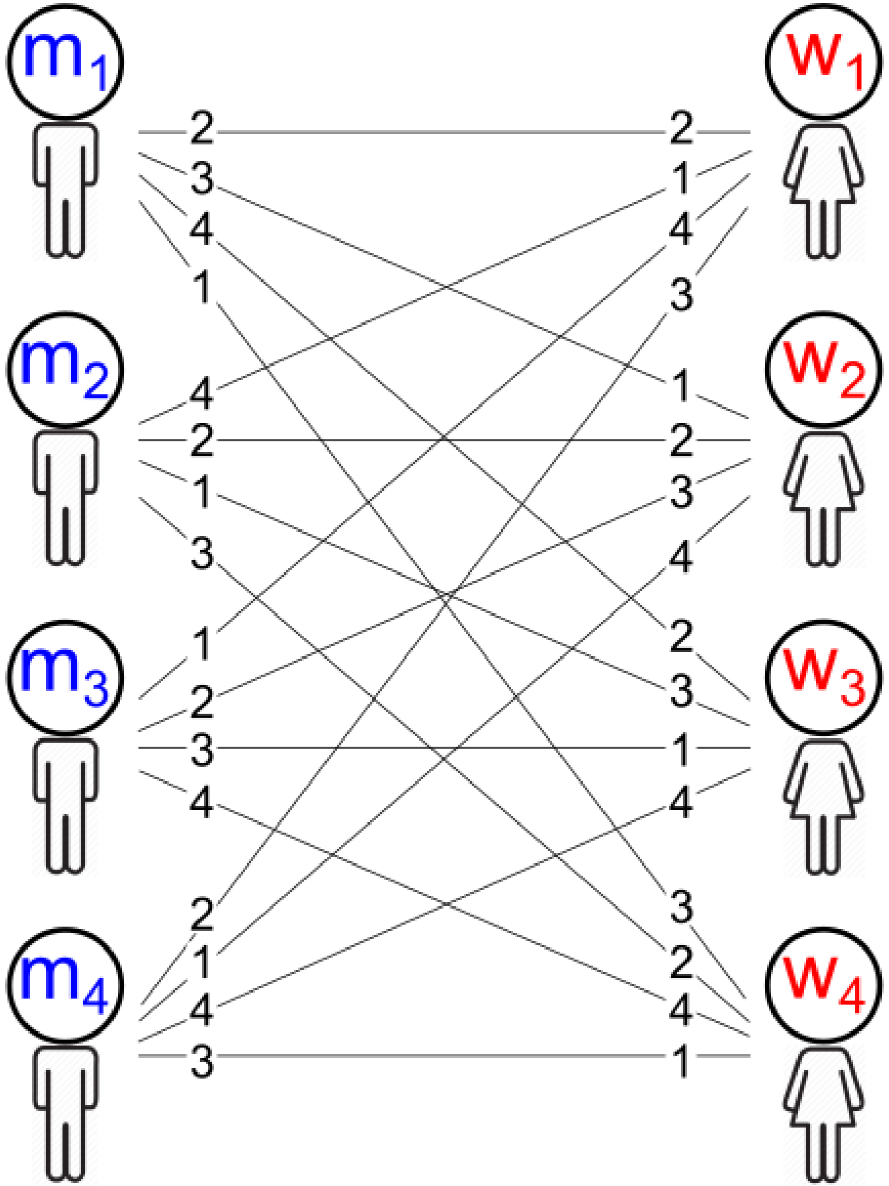
Structure of the Set of Stable Matchings


$w_4 > w_1 > w_2 > w_3$
m_1
w_1
$m_2 > m_1 > m_4 > m_3$
$w_3 > w_2 > w_4 > w_1$
m_2
w_2
$m_1 > m_2 > m_3 > m_4$
$w_1 > w_2 > w_3 > w_4$
m_3
w_3
$m_3 > m_1 > m_2 > m_4$
$w_2 > w_1 > w_4 > w_3$
m_4
w_4
$m_4 > m_2 > m_1 > m_3$

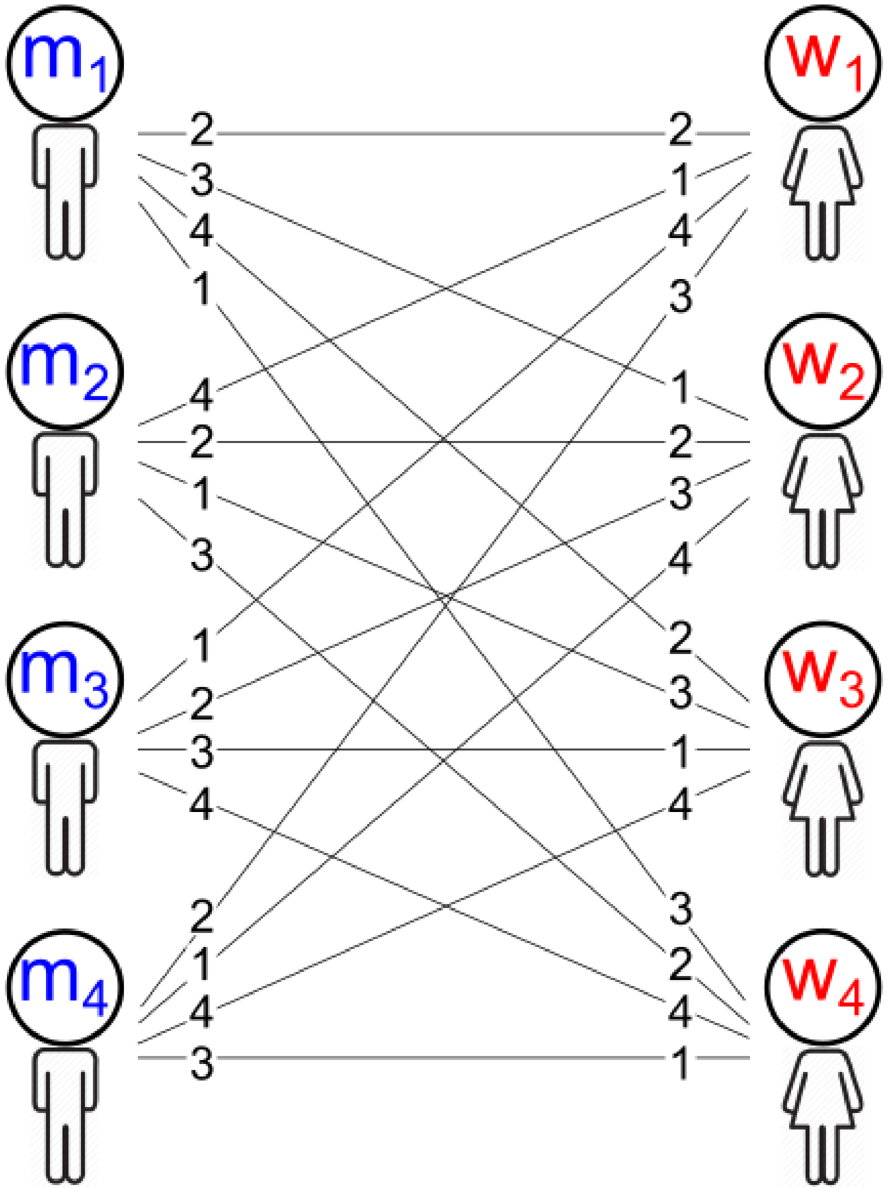

w4 > w1 > w2 > w3
m1
w3 > w2 > w4 > w1
m2
w1 > w2 > w3 > w4
m3
w2 > w1 > w4 > w3
m4
2
3
4
1
w1
m2 > m1 > m4 > m3
w2
m1 > m2 > m3 > m4
w3
m3 > m1 > m2 > m4
w4
m4 > m2 > m1 > m3



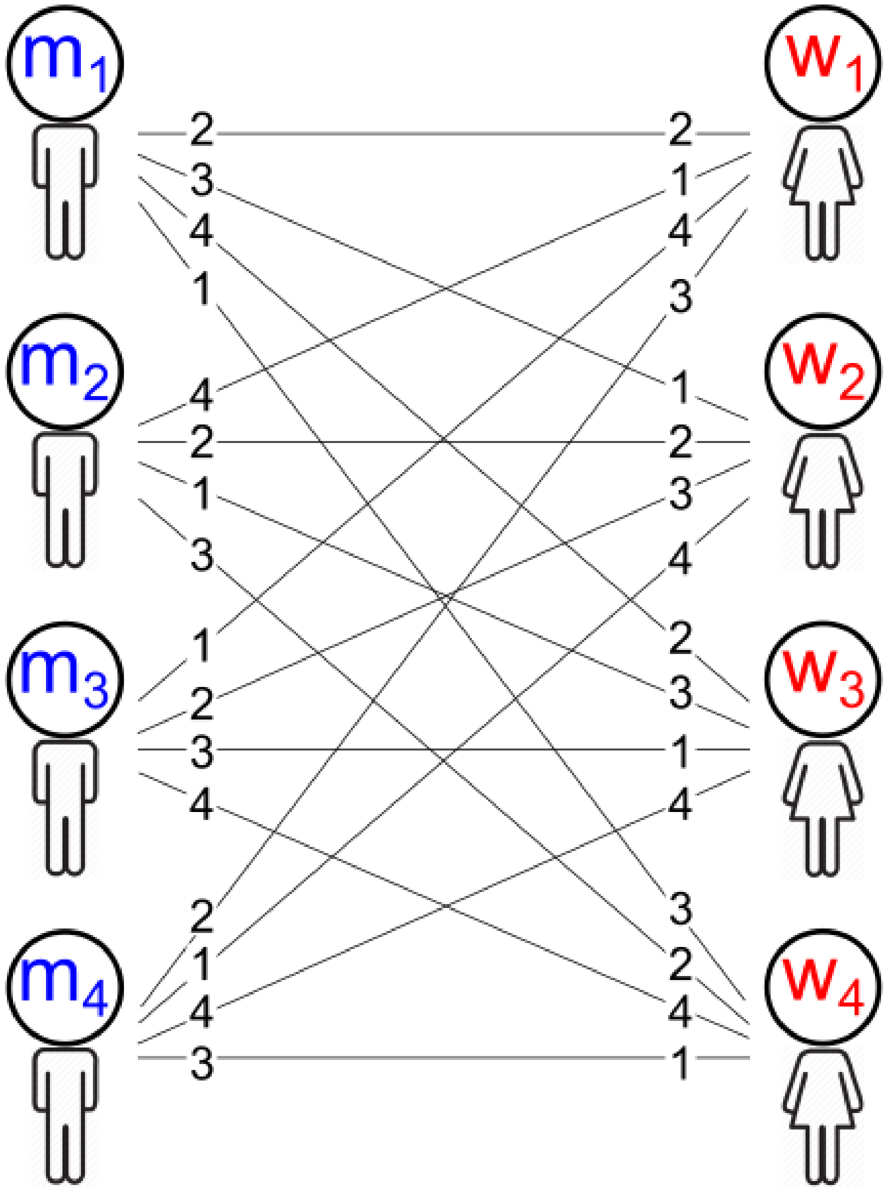
w4 > w1 > w2 > w3
m1
w1
m2 > m1 > m4 > m3
w3 > w2 > w4 > w1
m2
w2
m1 > m2 > m3 > m4
w1 > w2 > w3 > w4
m3
w3
m3 > m1 > m2 > m4
w2 > w1 > w4 > w3
m4
w4
m4 > m2 > m1 > m3

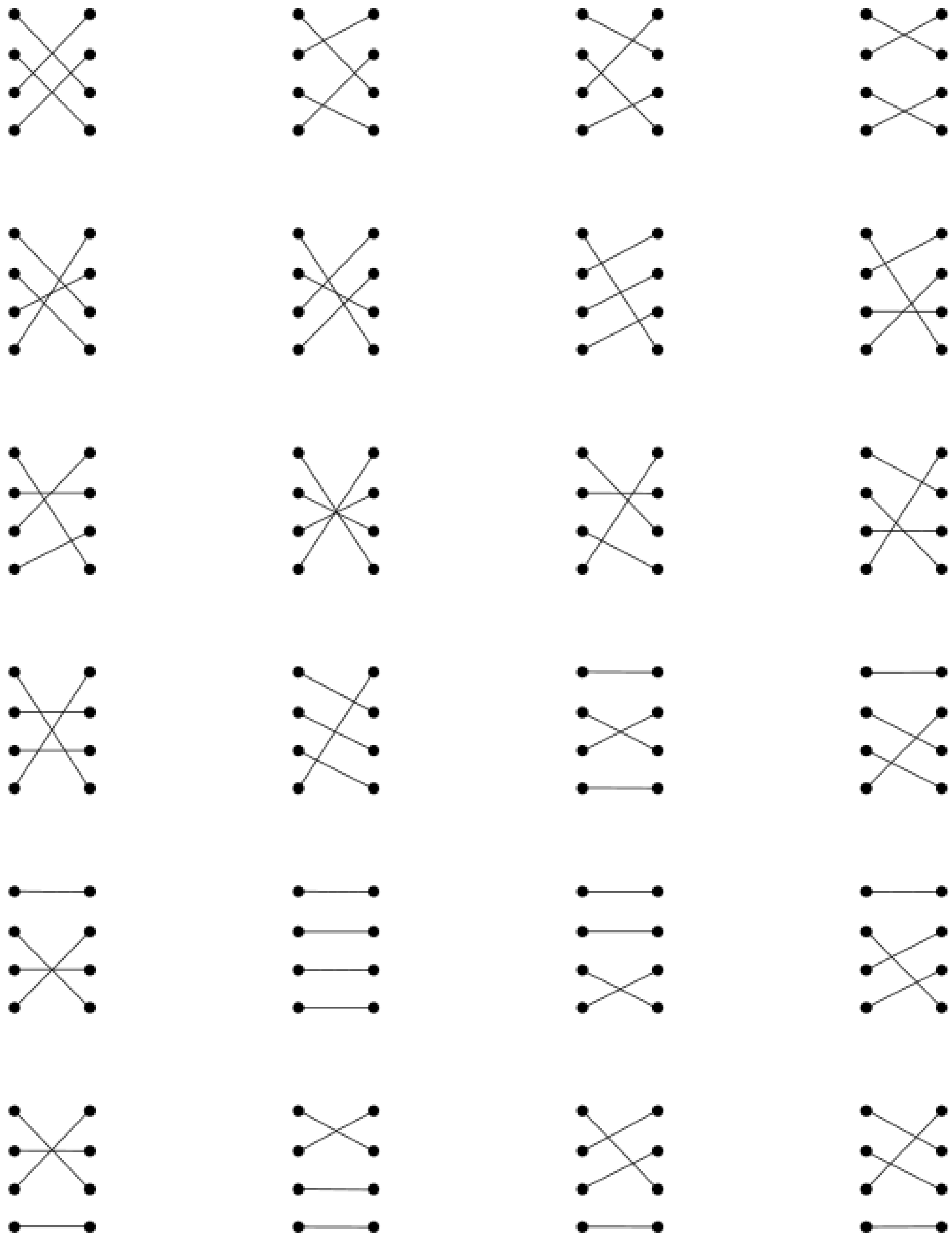


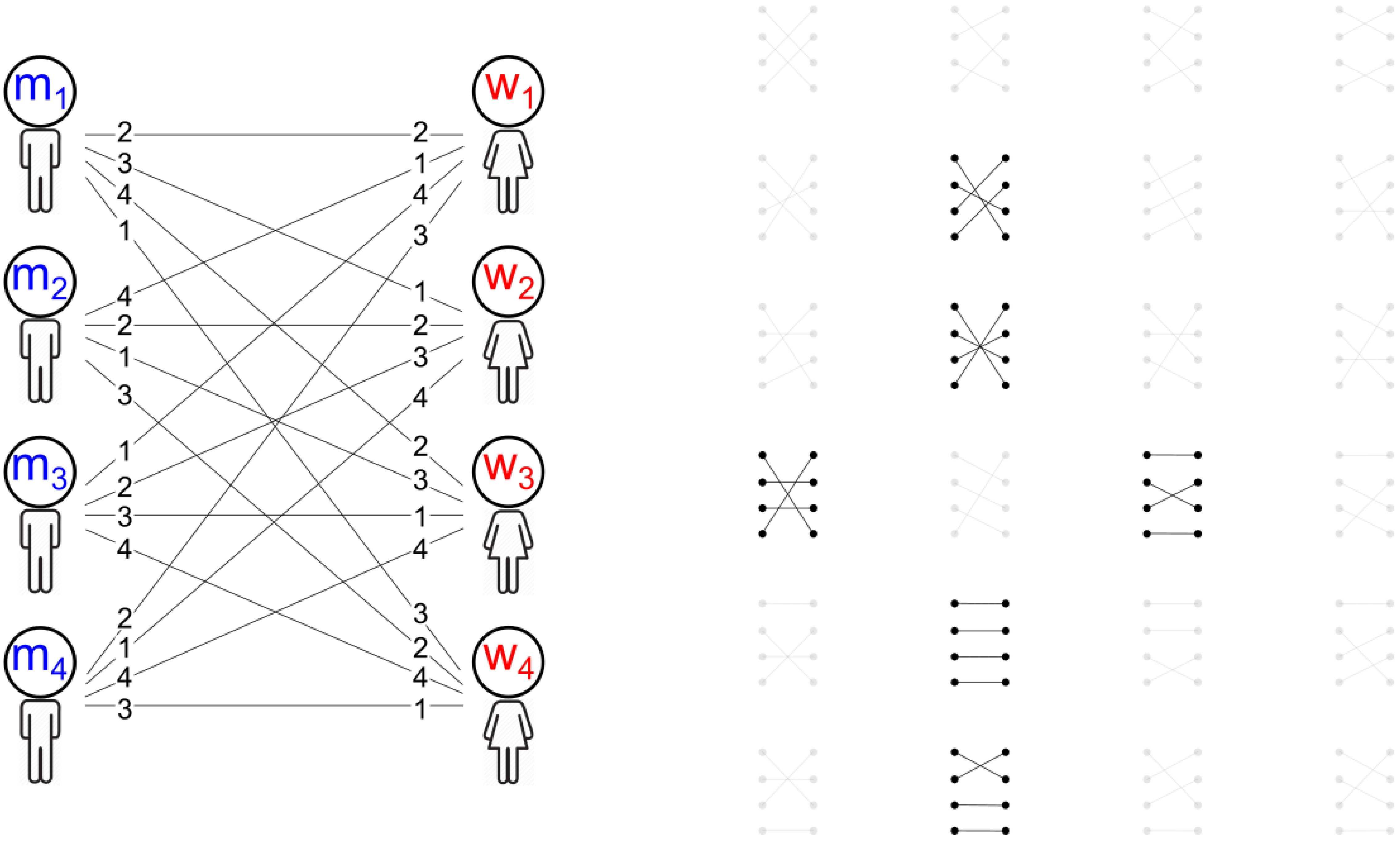
m1
2 3 4 1
m2
4 2 1 3
m3
1 2 3 4
m4
2 1 4 3
w1
2 1 4 3
w2
1 2 3 4
w3
2 3 1 4
w4
3 2 4 1



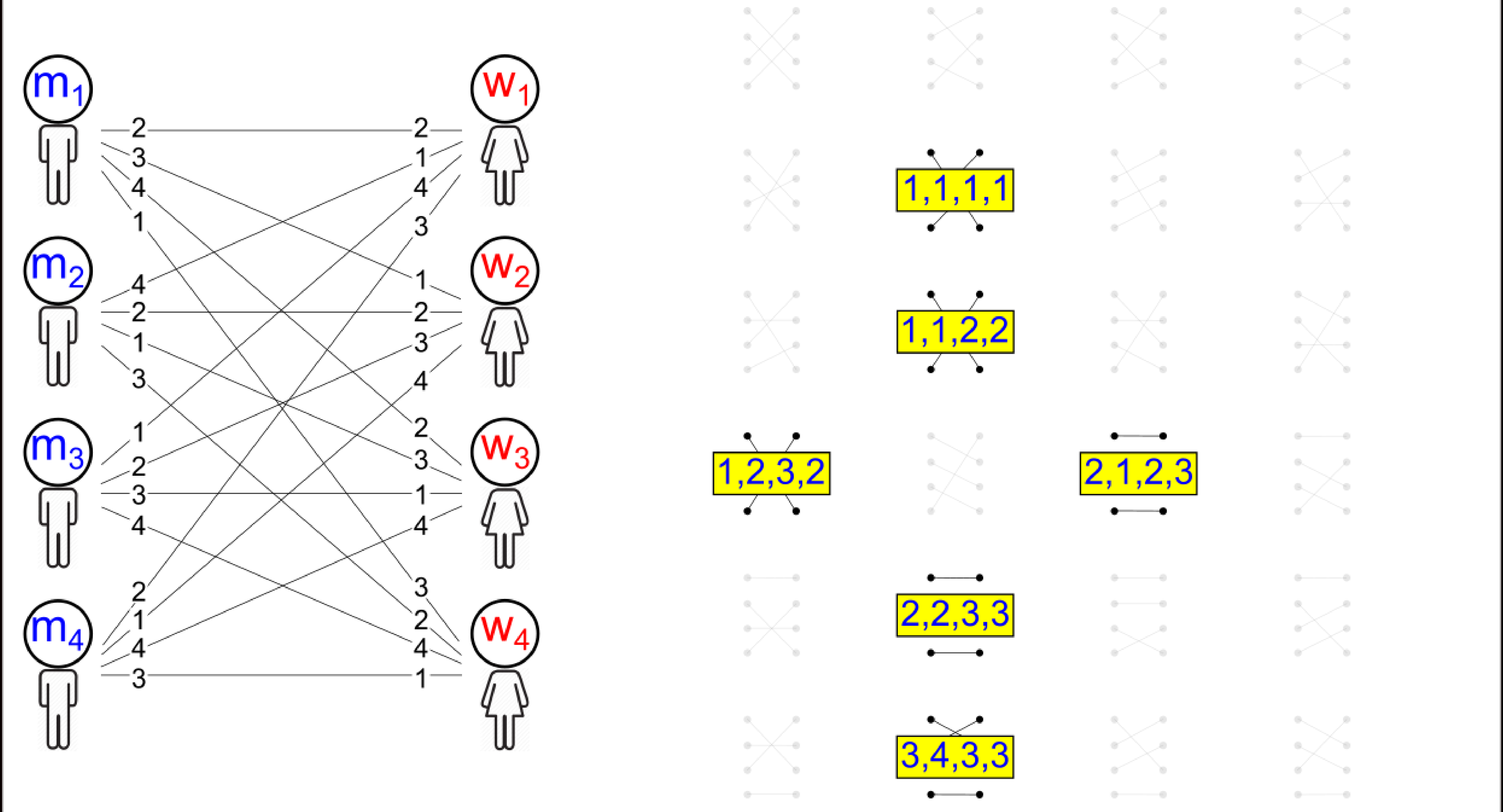
m_1
m_2
m_3
m_4
w_1
w_2
w_3
w_4


m1
2 3 4 1
m2
4 2 1 3
m3
1 2 3 4
m4
2 1 4 3
w1
2 1 4 3
w2
1 2 3 4
w3
2 3 1 4
w4
3 2 4 1

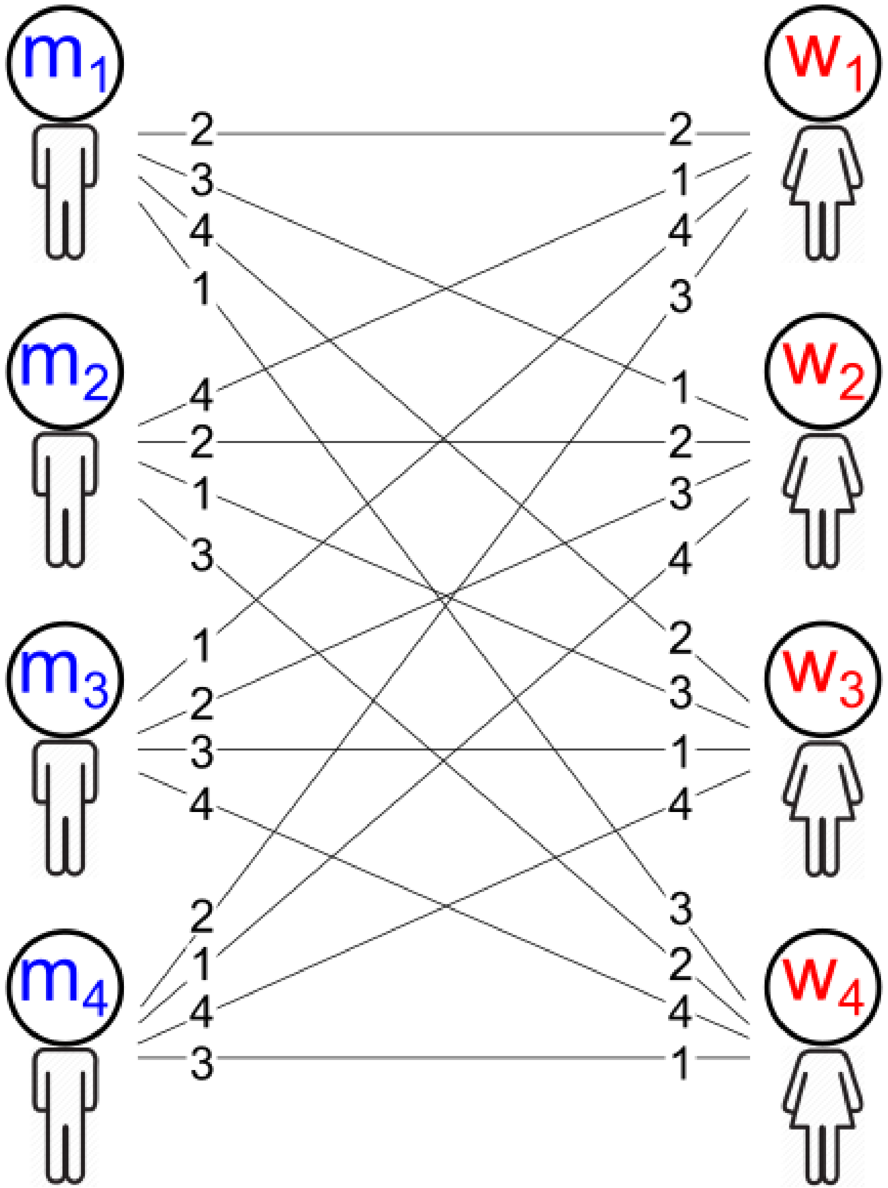




m1
m2
m3
m4
w1
w2
w3
w4



m1
m2
m3
m4
w1
w2
w3
w4
1,1,1,1
1,1,2,2
1,2,3,2
2,1,2,3
2,2,3,3
3,4,3,3



1,1,1,1

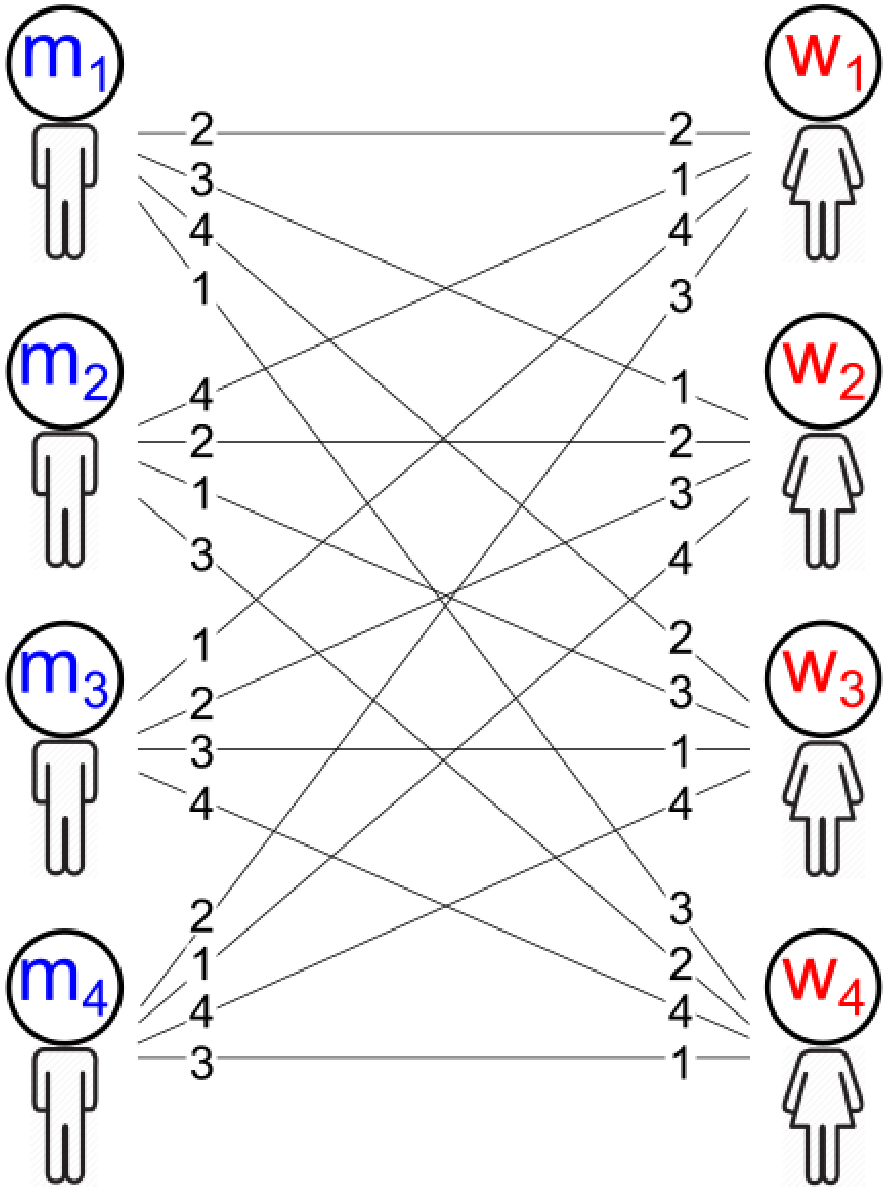
1,1,2,2

1,2,3,2

2,1,2,3

2,2,3,3

3,4,3,3



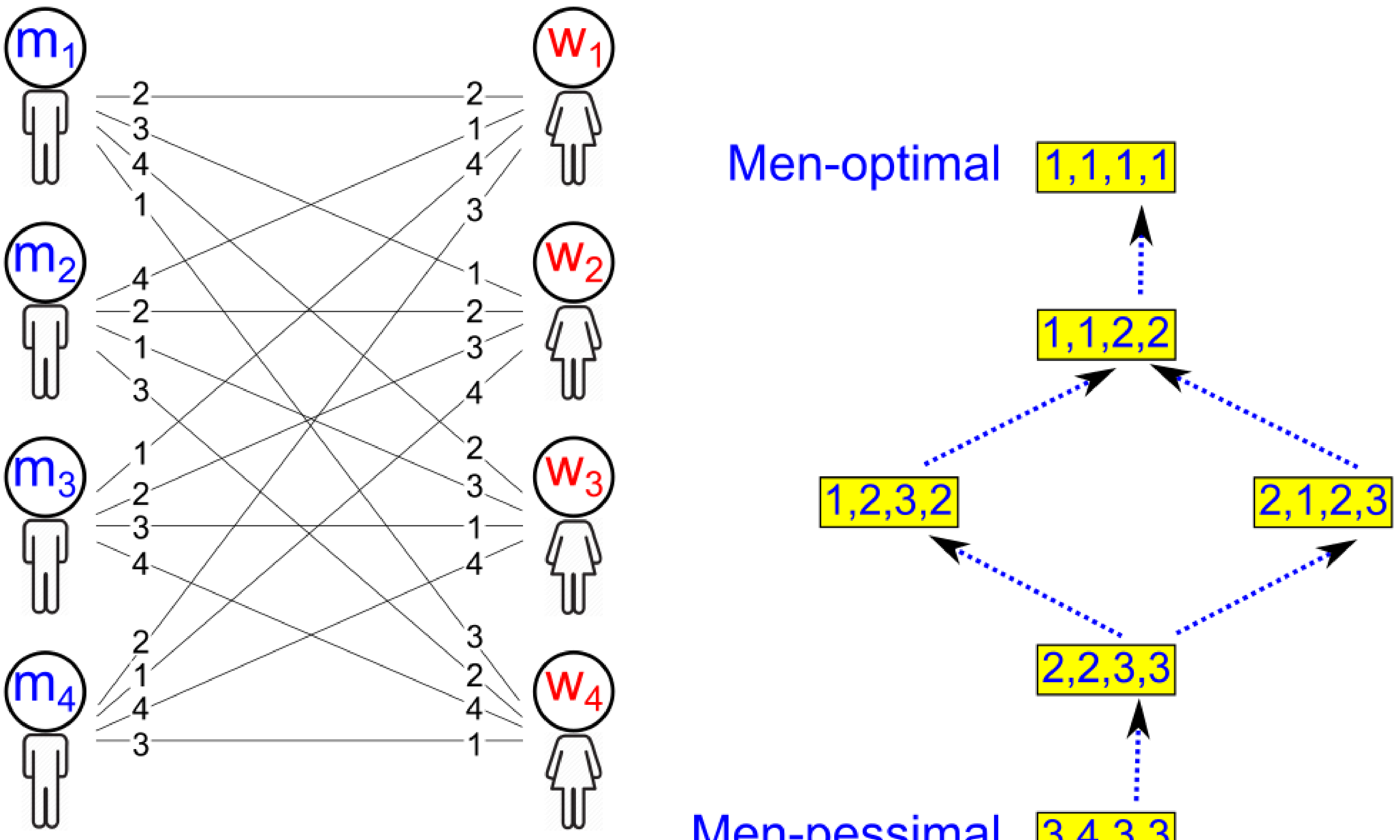
Men-optimal 1,1,1,1

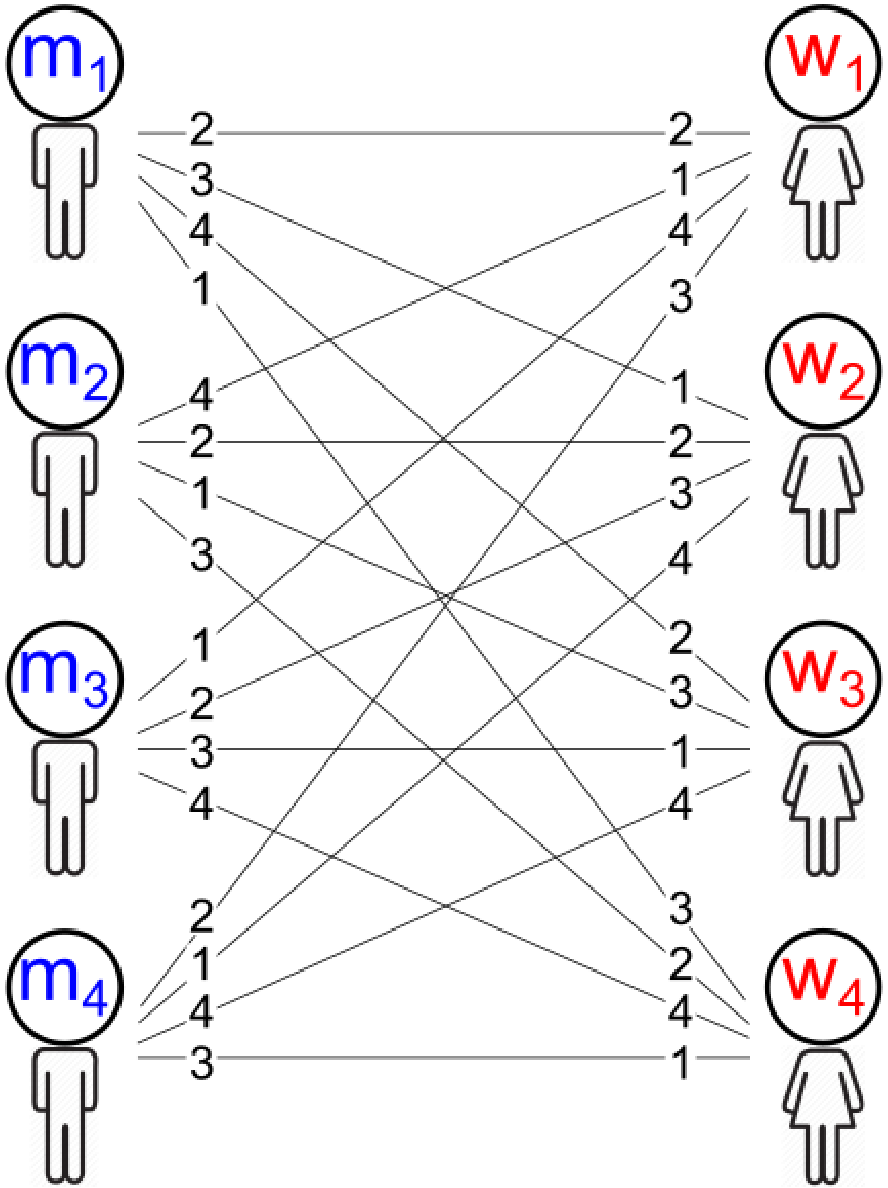
1,1,2,2

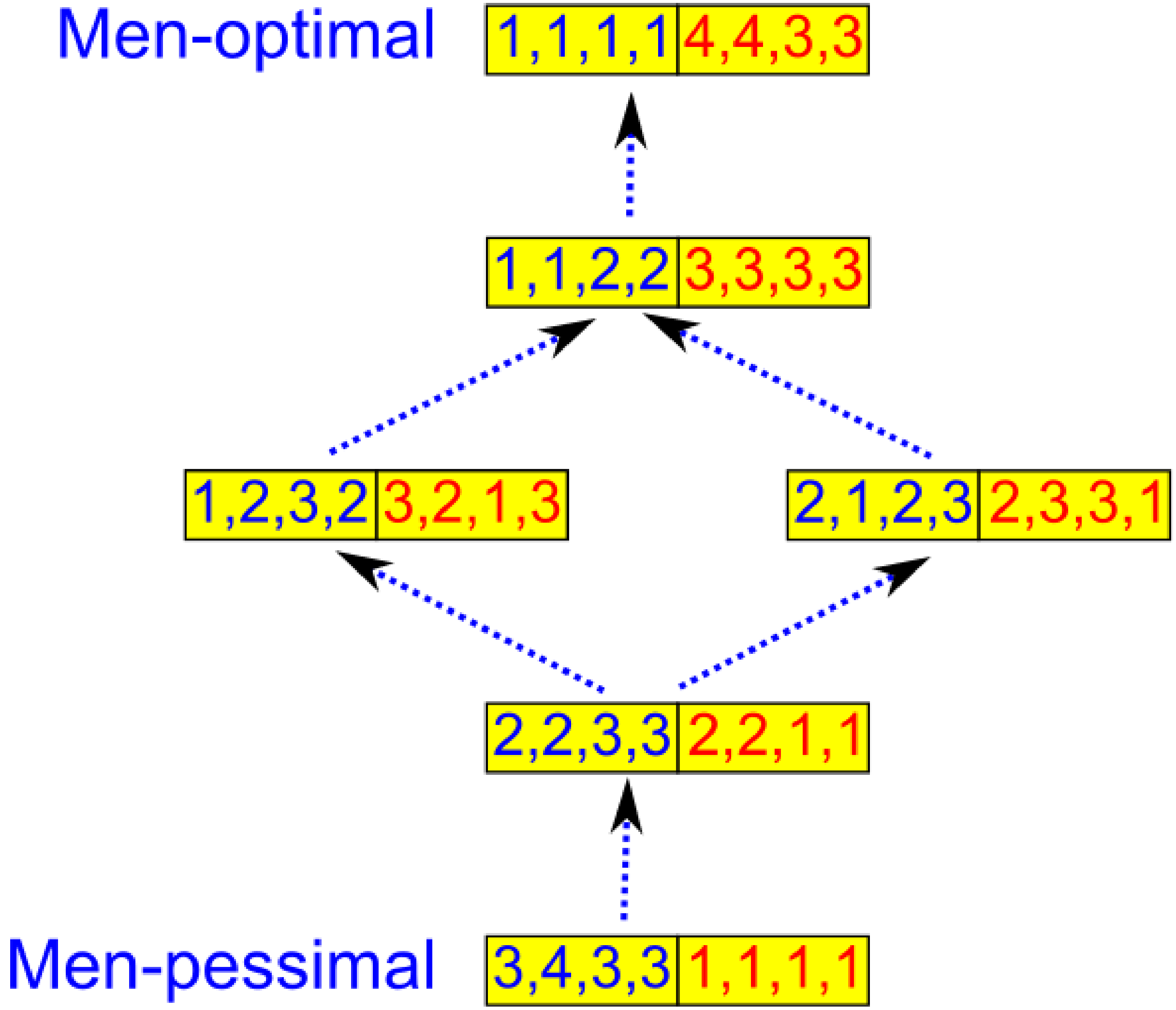
1,2,3,2 2,1,2,3

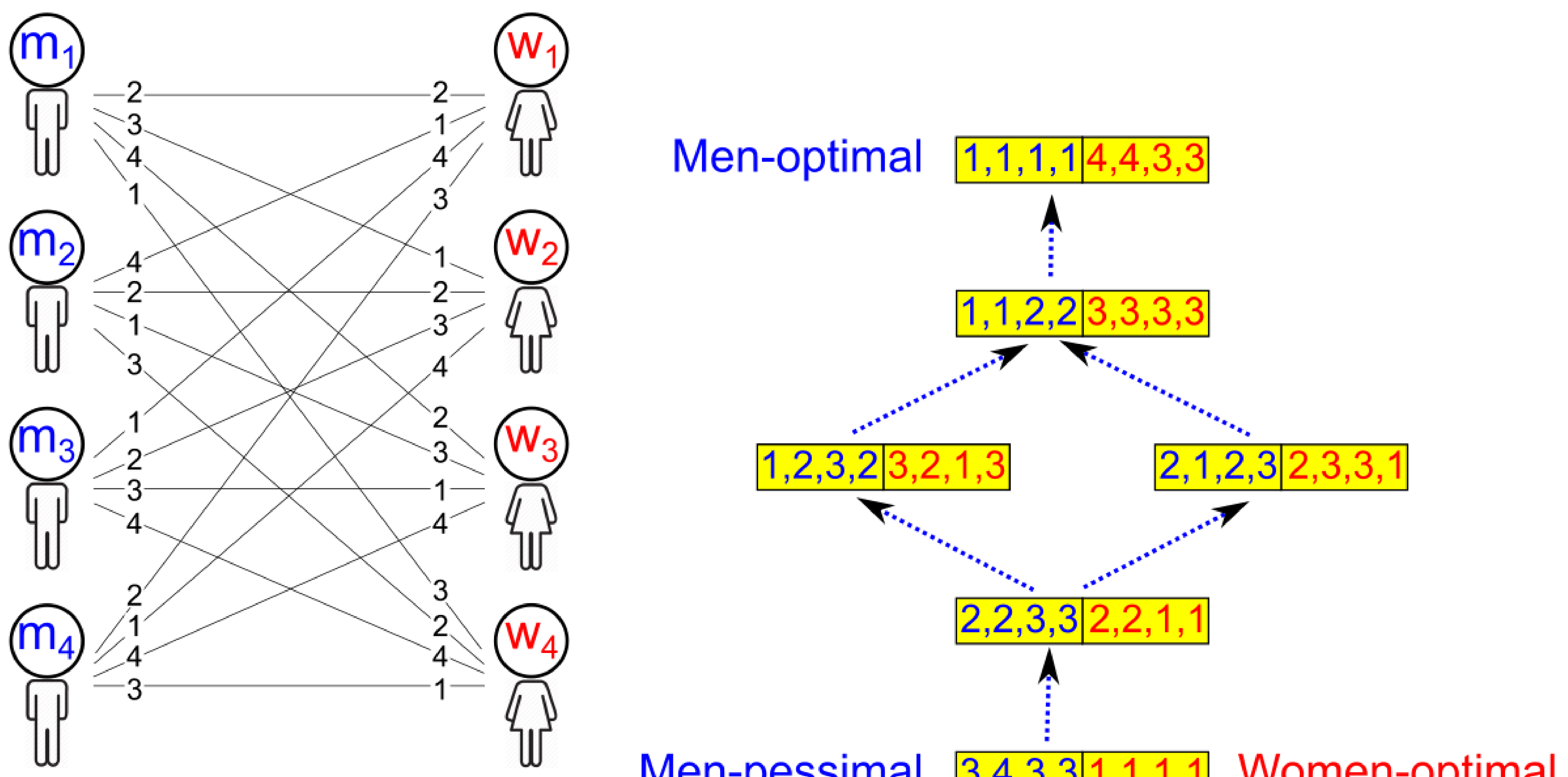
2,2,3,3

Men-pessimal 3,4,3,3

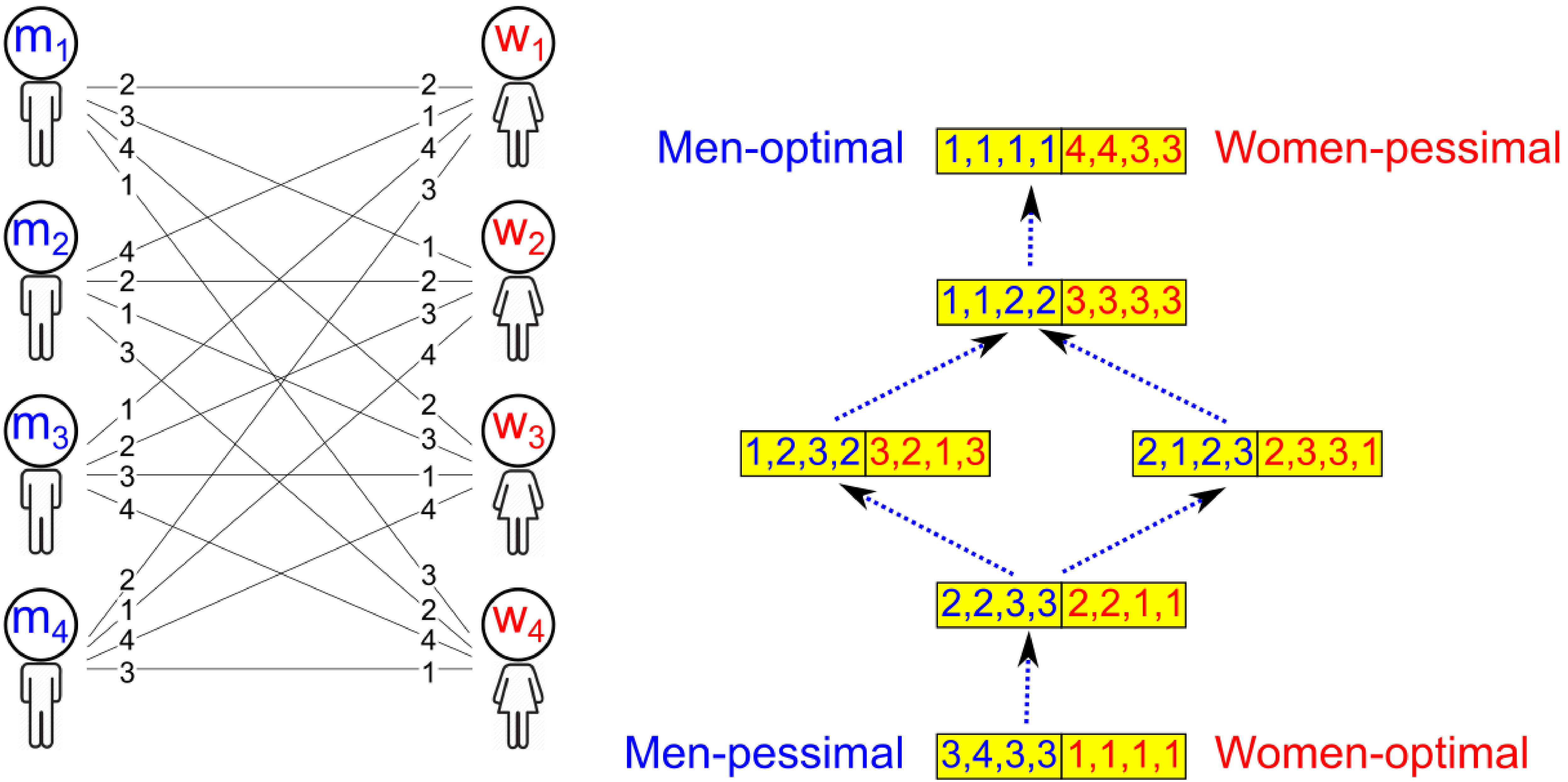

m1
m2
m3
m4
w1
w2
w3
w4
Men-optimal
1,1,1,1
1,1,2,2
1,2,3,2
2,1,2,3
2,2,3,3
Men-pessimal
3,4,3,3


m1
m2
m3
m4
w1
w2
w3
w4

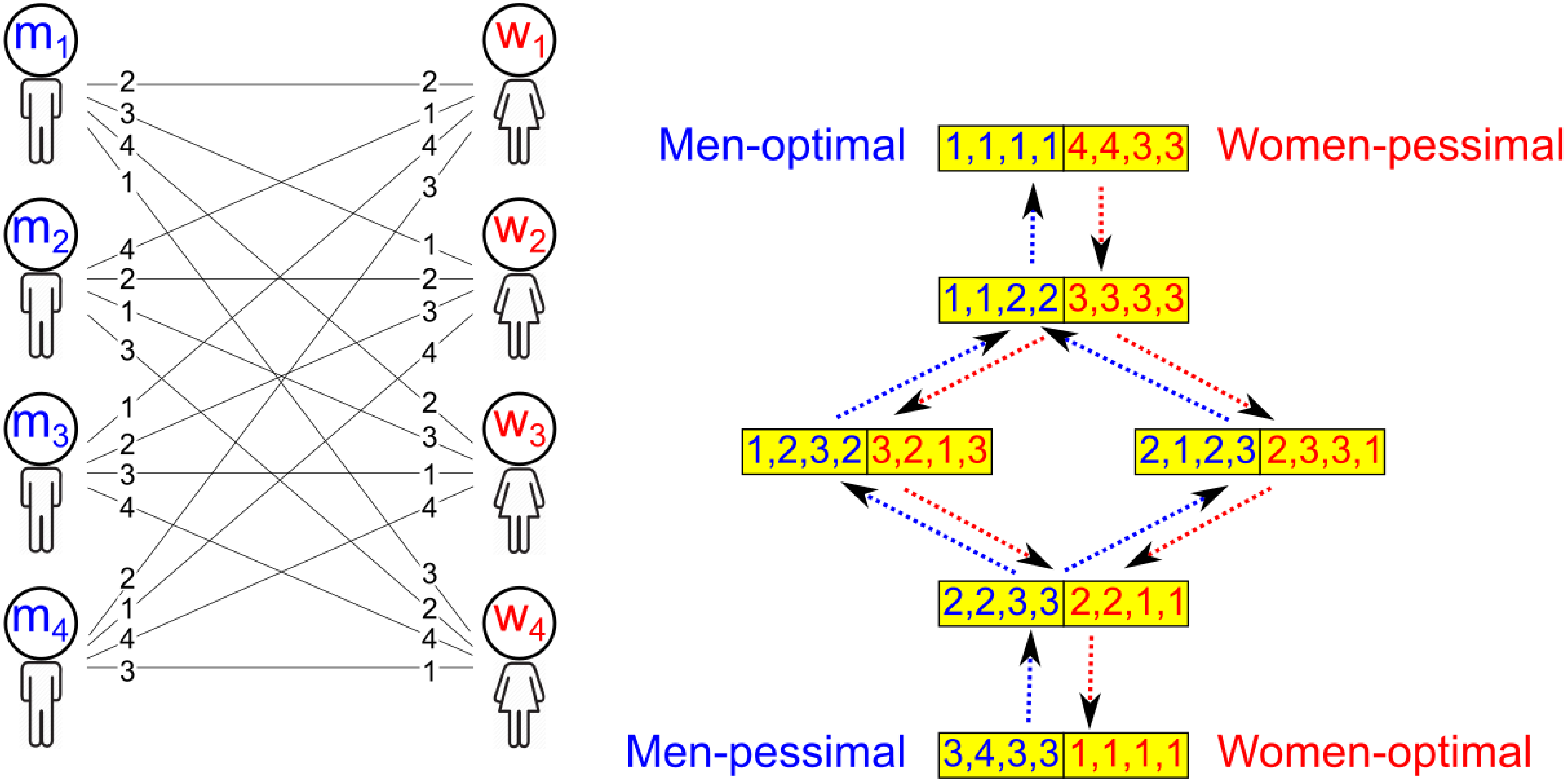

Men-optimal
1,1,1,1 4,4,3,3
1,1,2,2 3,3,3,3
1,2,3,2 3,2,1,3
2,1,2,3 2,3,3,1
2,2,3,3 2,2,1,1
Men-pessimal
3,4,3,3 1,1,1,1

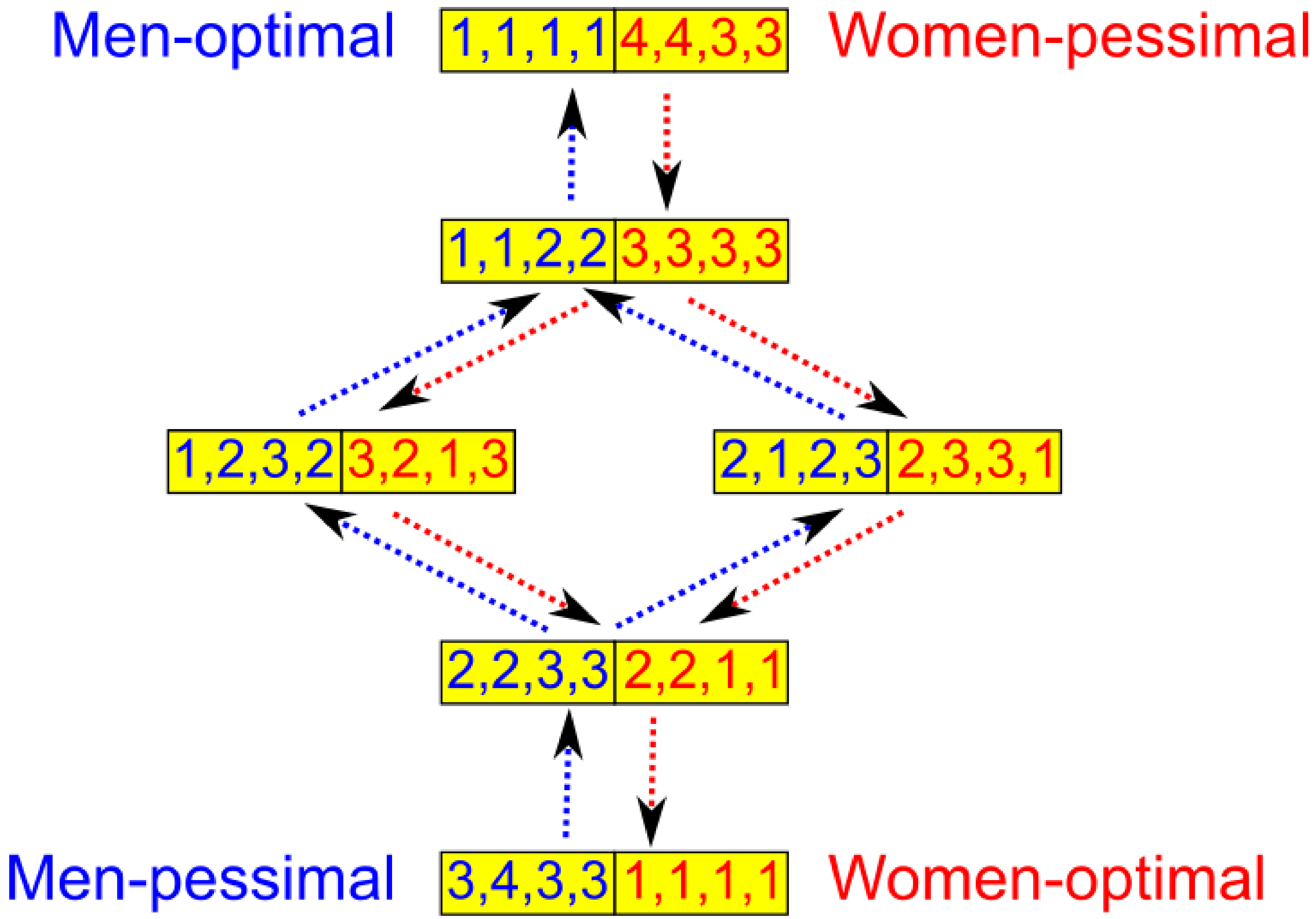


m1
m2
m3
m4
w1
w2
w3
w4
Men-optimal
1,1,1,1 4,4,3,3
1,1,2,2 3,3,3,3
1,2,3,2 3,2,1,3
2,1,2,3 2,3,3,1
2,2,3,3 2,2,1,1
Men-pessimal
3,4,3,3 1,1,1,1
Women-optimal

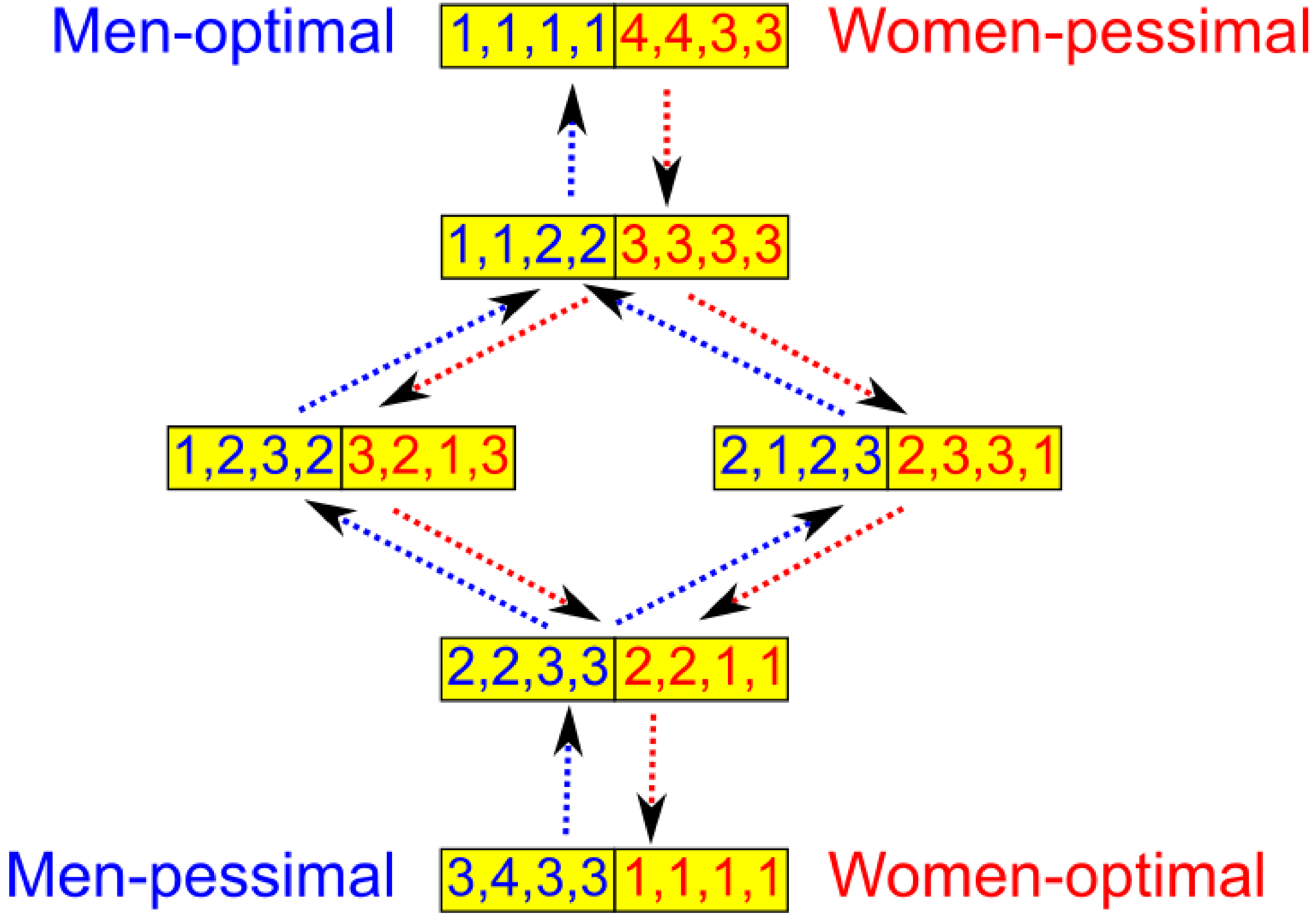


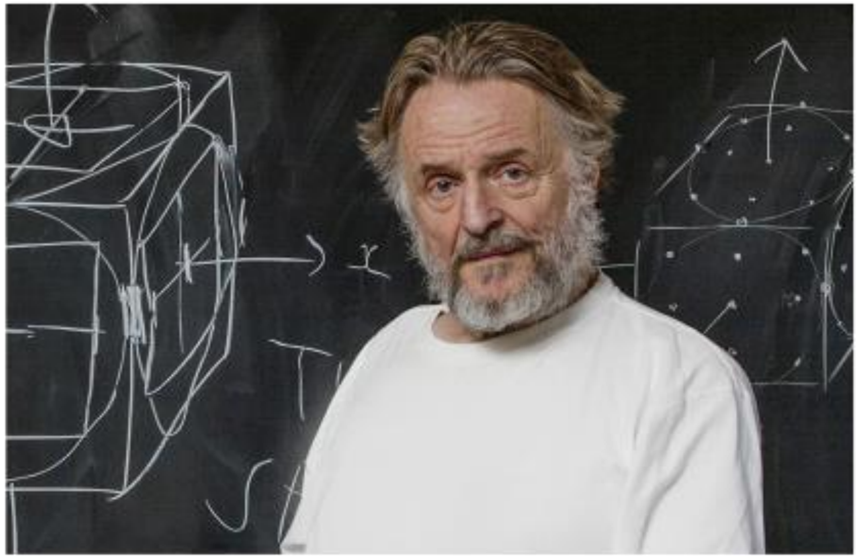
m_1 m_2 m_3 m_4
w_1 w_2 w_3 w_4
Men-optimal
1,1,1,1 4,4,3,3
Women-pessimal
1,1,2,2 3,3,3,3
1,2,3,2 3,2,1,3
2,1,2,3 2,3,3,1
2,2,3,3 2,2,1,1
Men-pessimal
3,4,3,3 1,1,1,1
Women-optimal


m1
2 3 4 1
m2
4 2 1 3
m3
1 2 3 4
m4
2 1 4 3
w1
2 1 4 3
w2
1 2 3 4
w3
2 3 1 4
w4
3 2 4 1
Men-optimal
1,1,1,1 4,4,3,3
Women-pessimal
1,1,2,2 3,3,3,3
1,2,3,2 3,2,1,3
2,1,2,3 2,3,3,1
2,2,3,3 2,2,1,1
Men-pessimal
3,4,3,3 1,1,1,1
Women-optimal


Men-optimal
1,1,1,1
4,4,3,3
Women-pessimal
1,1,2,2
3,3,3,3
1,2,3,2
3,2,1,3
2,1,2,3
2,3,3,1
2,2,3,3
2,2,1,1
Men-pessimal
3,4,3,3
1,1,1,1
Women-optimal

The Lattice of Stable Matchings





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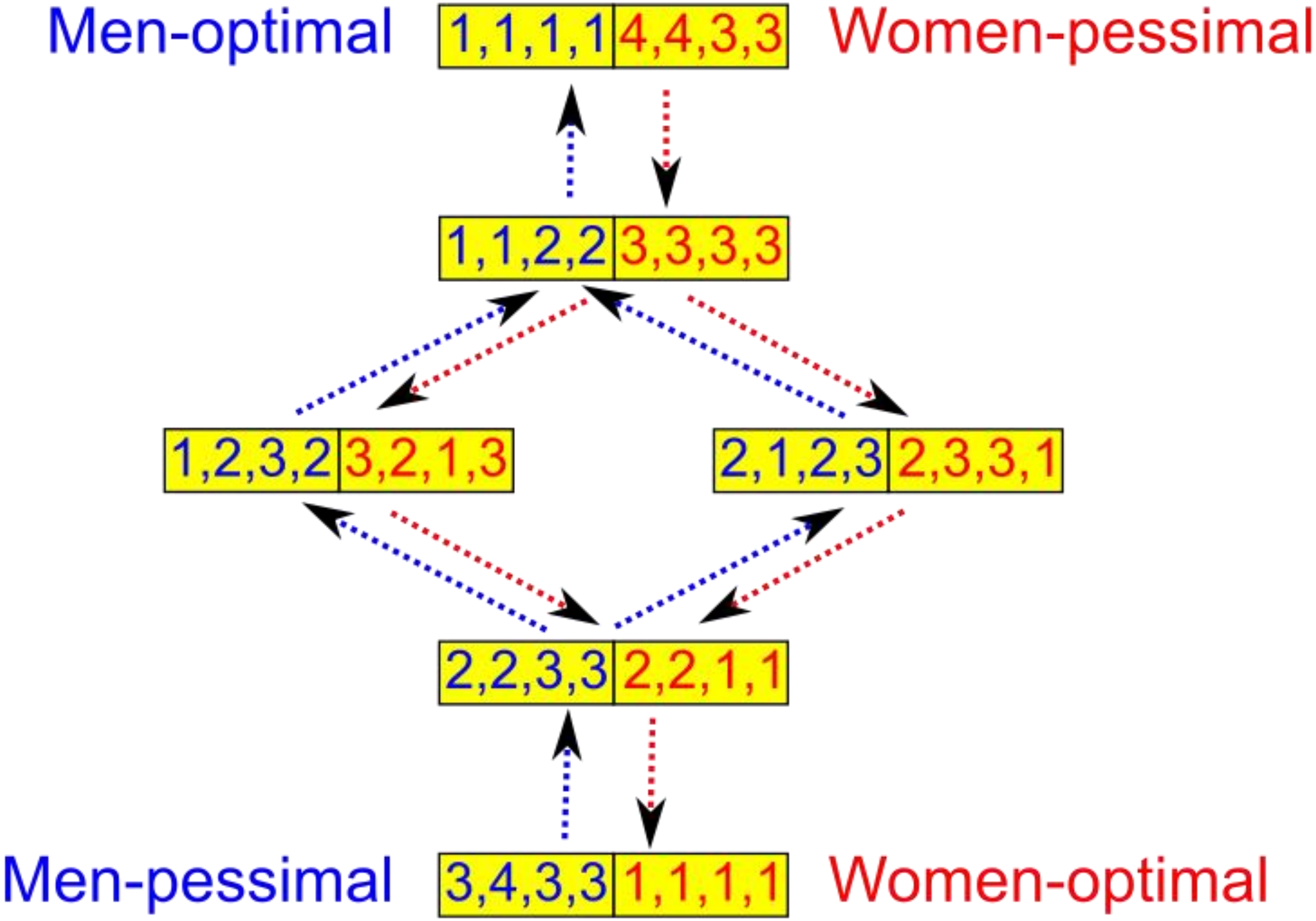
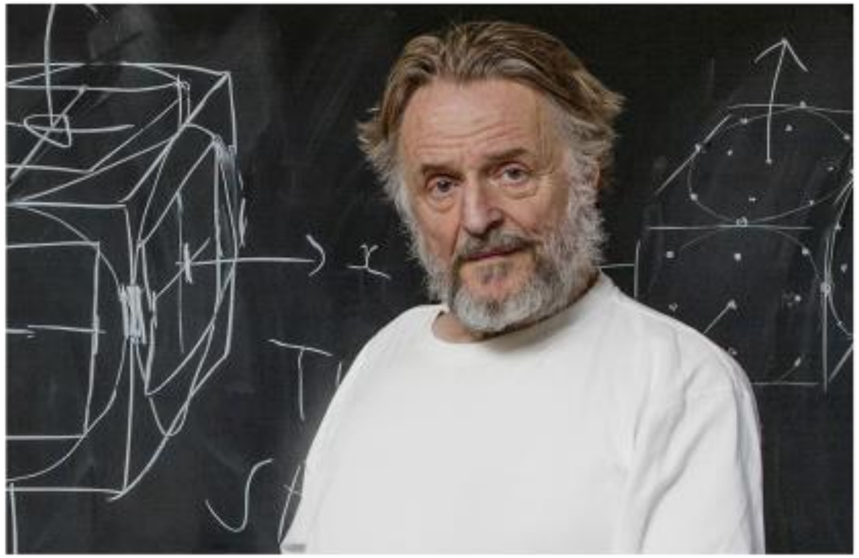
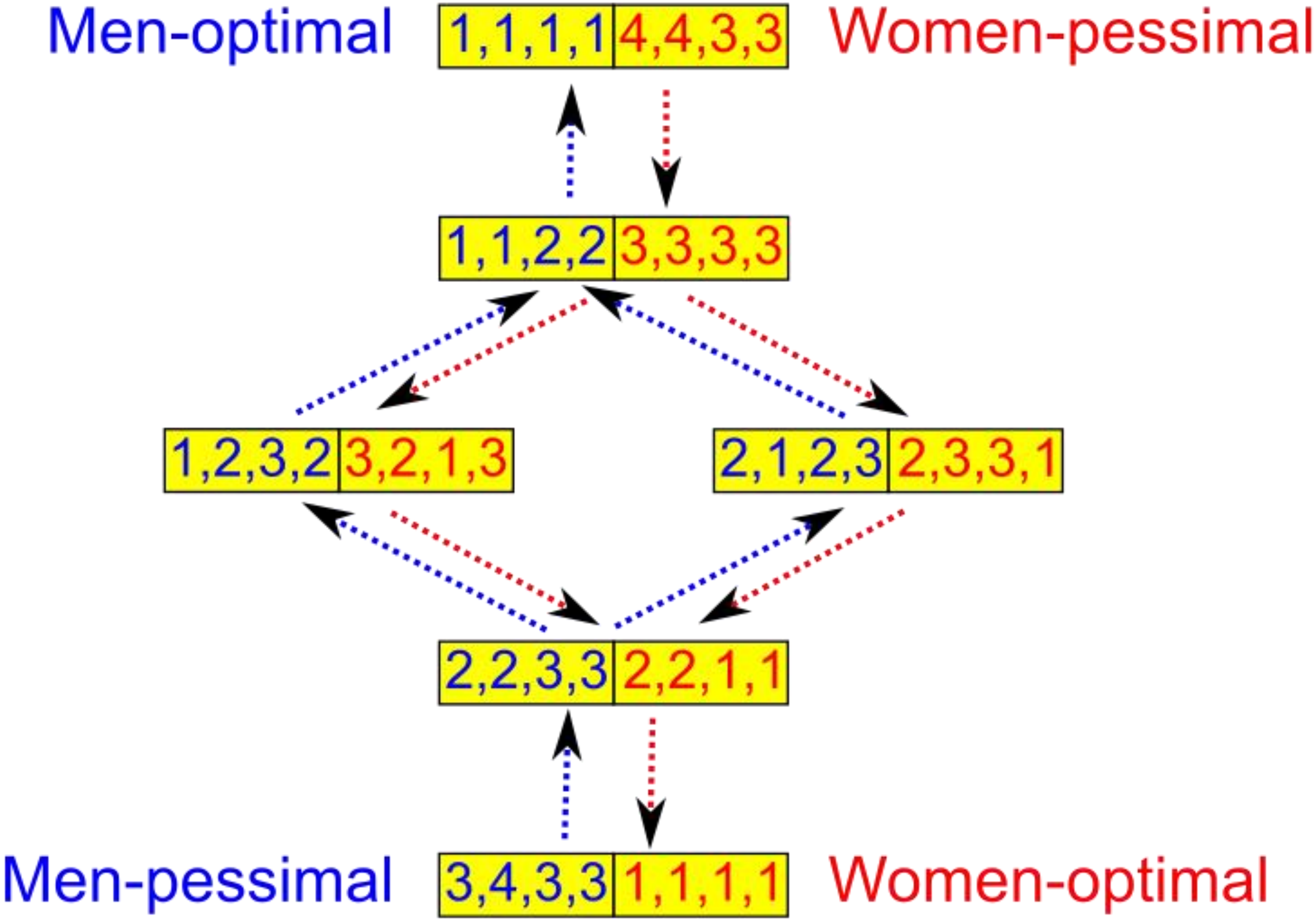
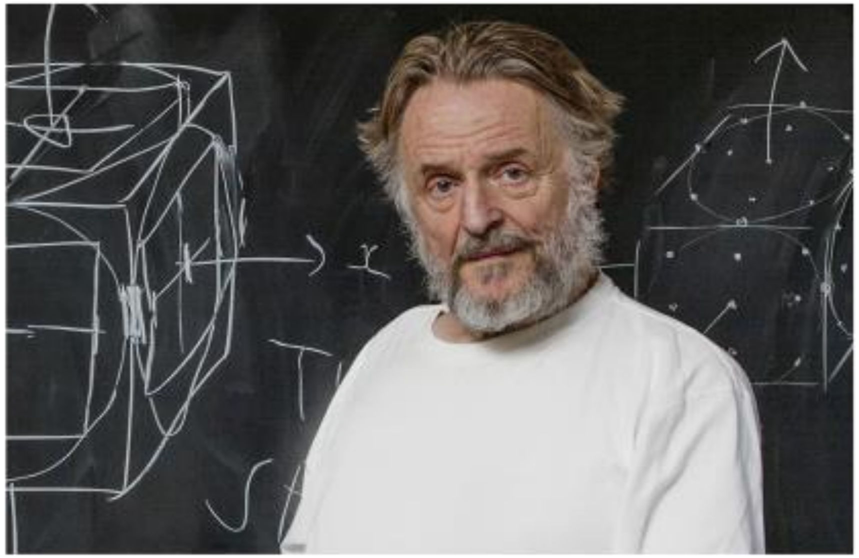




John H. Conway

The Lattice of Stable Matchings





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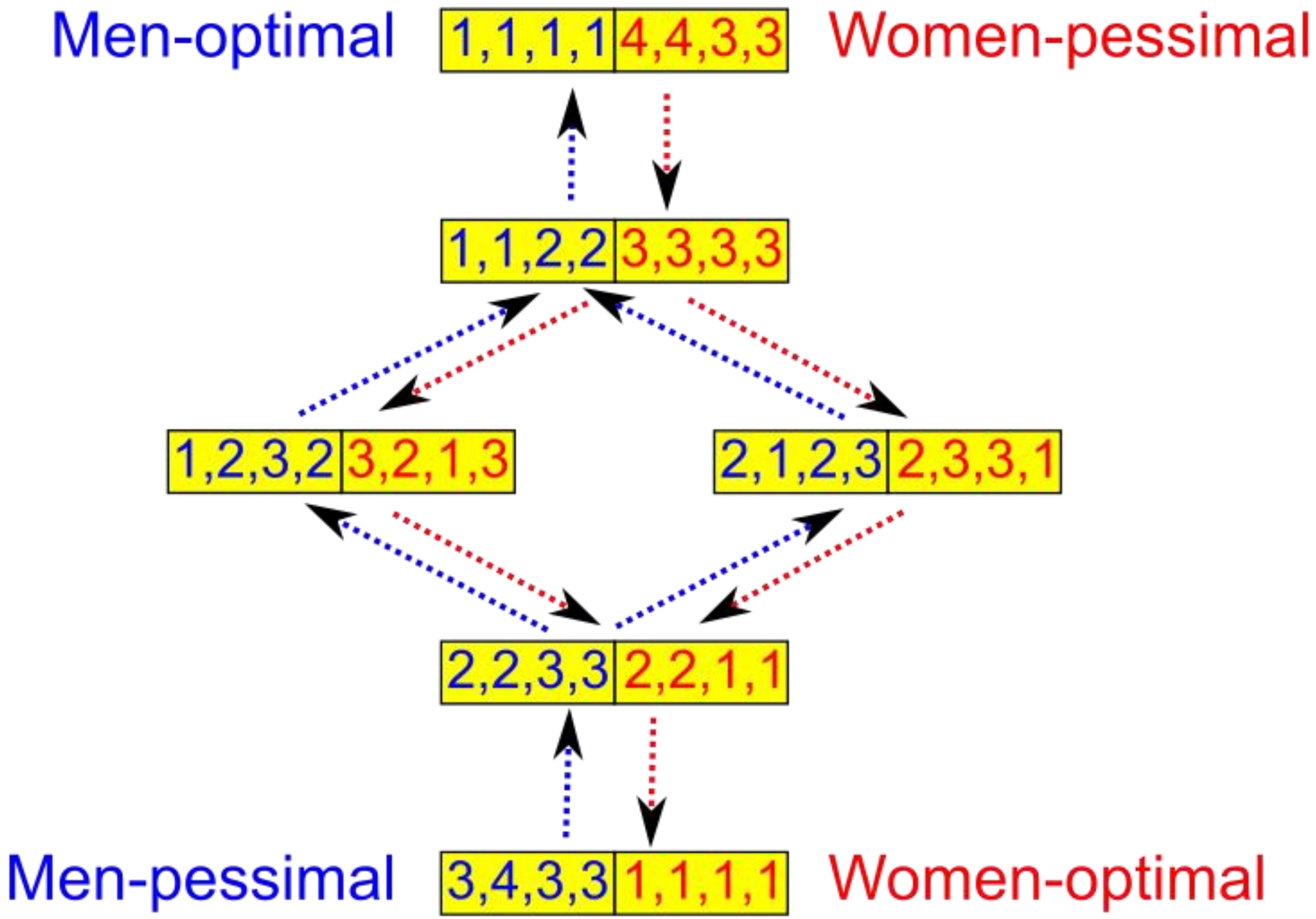

Some Observations

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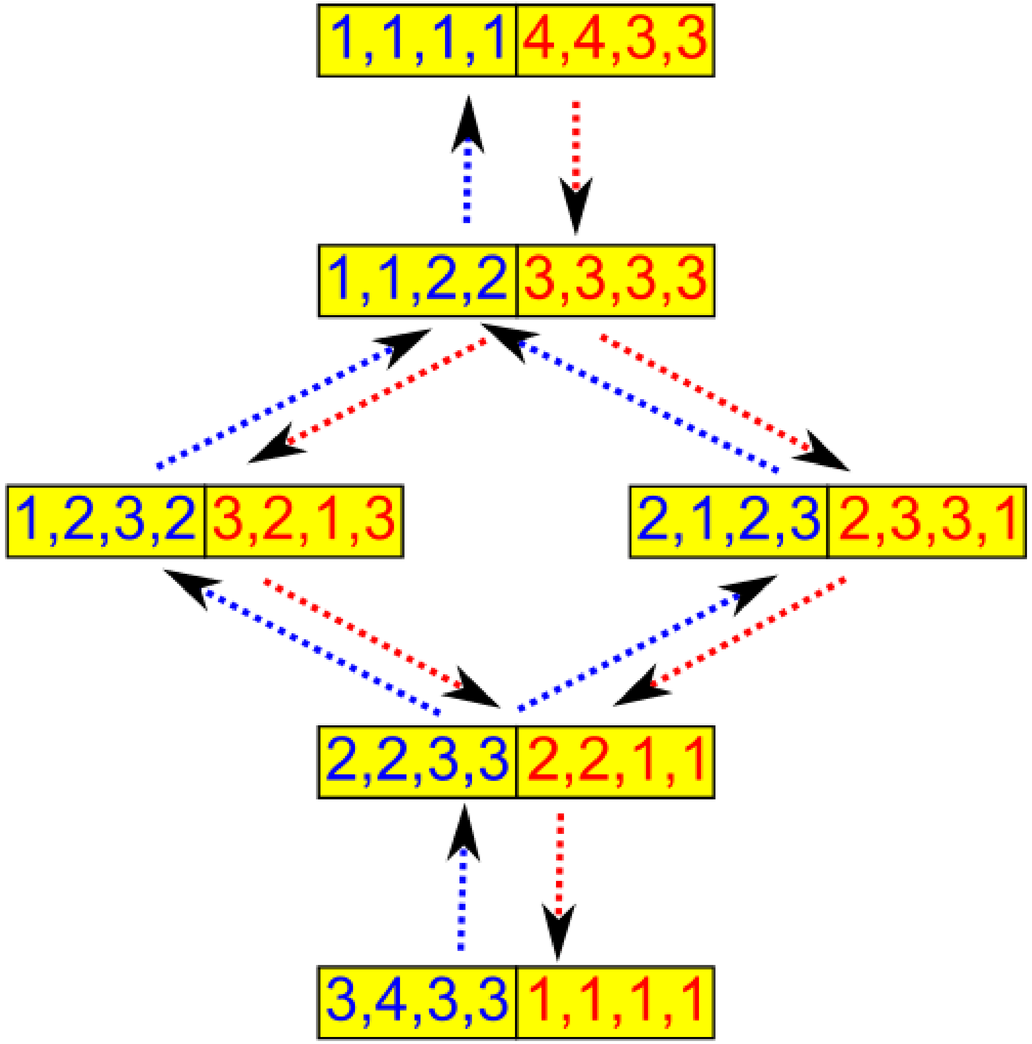
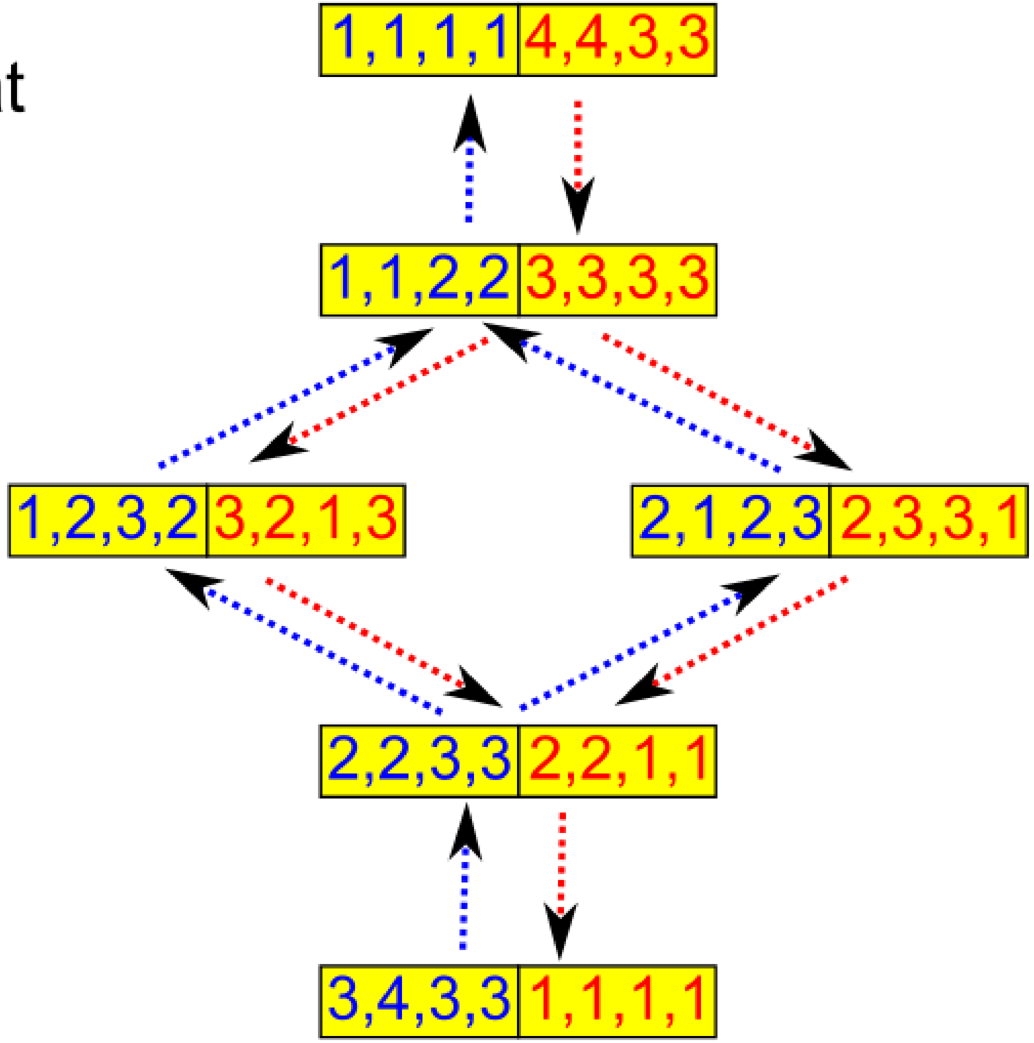


Some Observations

Consensus

There is a stable matching that all men find at least as good as *any other* stable matching, and one that they find at least as bad. (Analogously for the women.)



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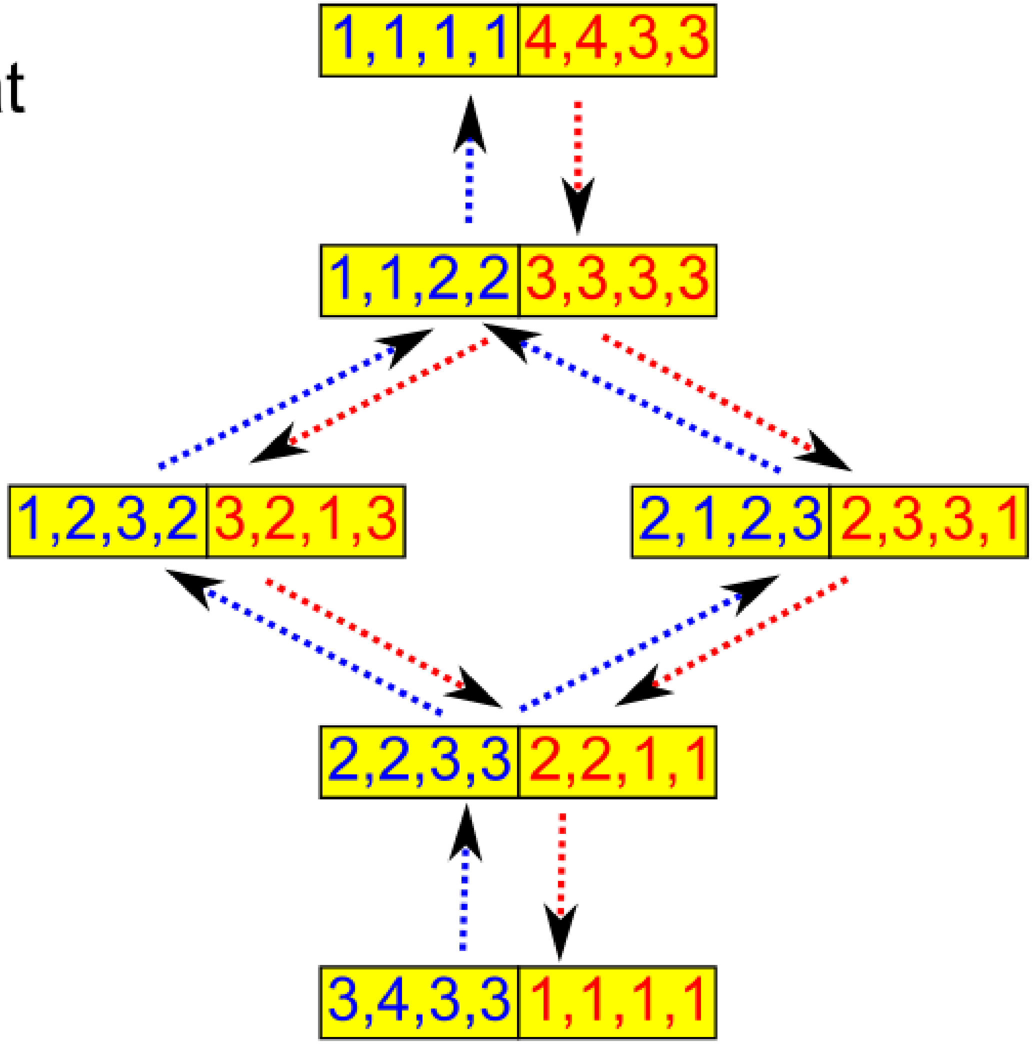
Conflict

For any distinct stable matchings P and Q, if all men find P at least as good as Q, then all women find Q at least as good as P (and vice versa).



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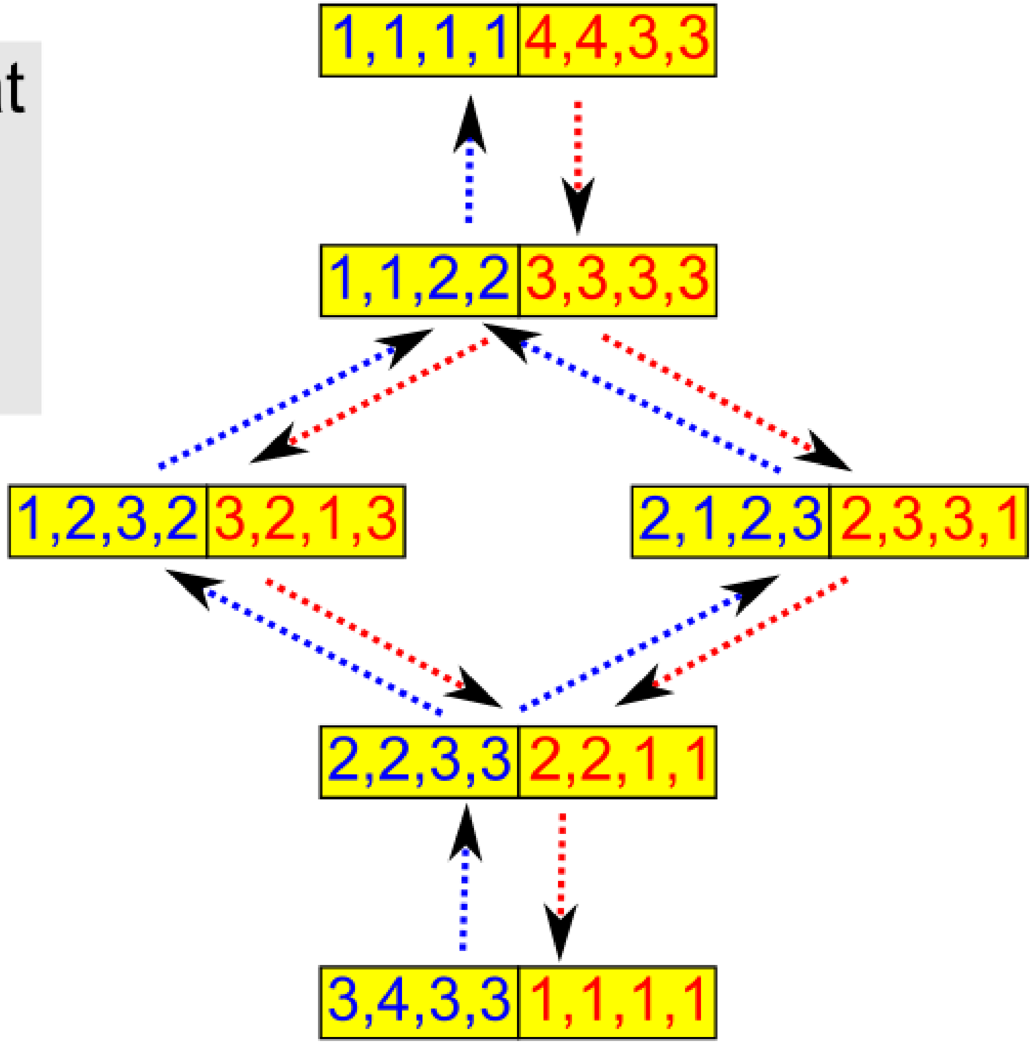

A man m and a woman w are **achievable** for each other if there is some stable matching in which they are matched with each other.

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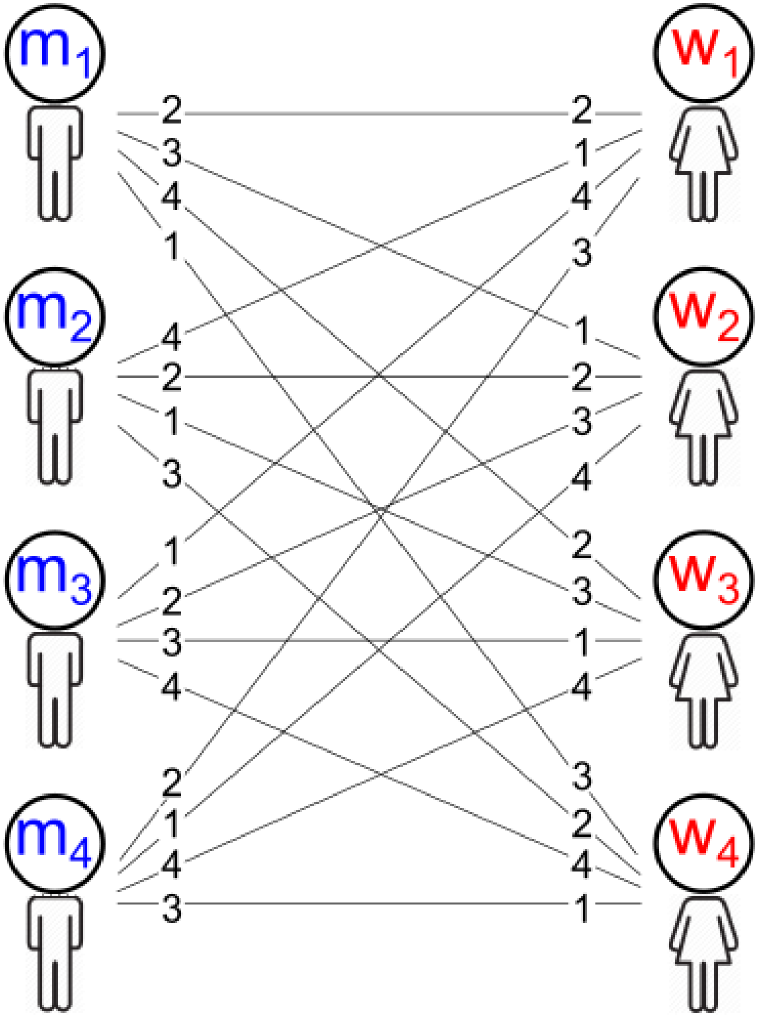

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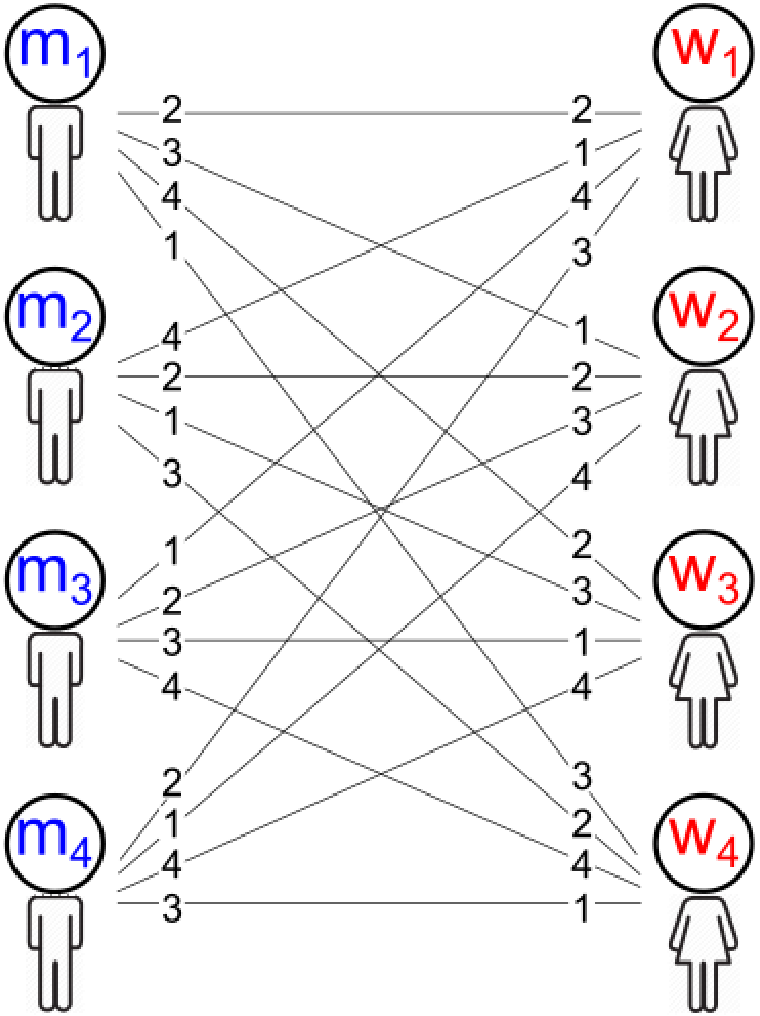

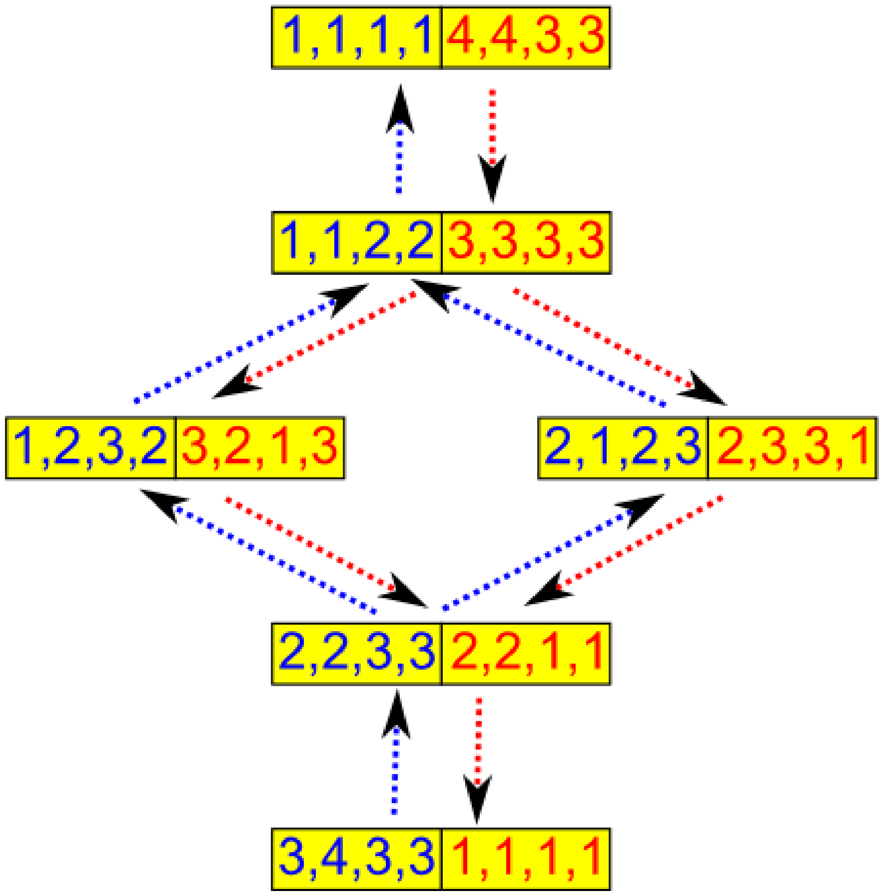

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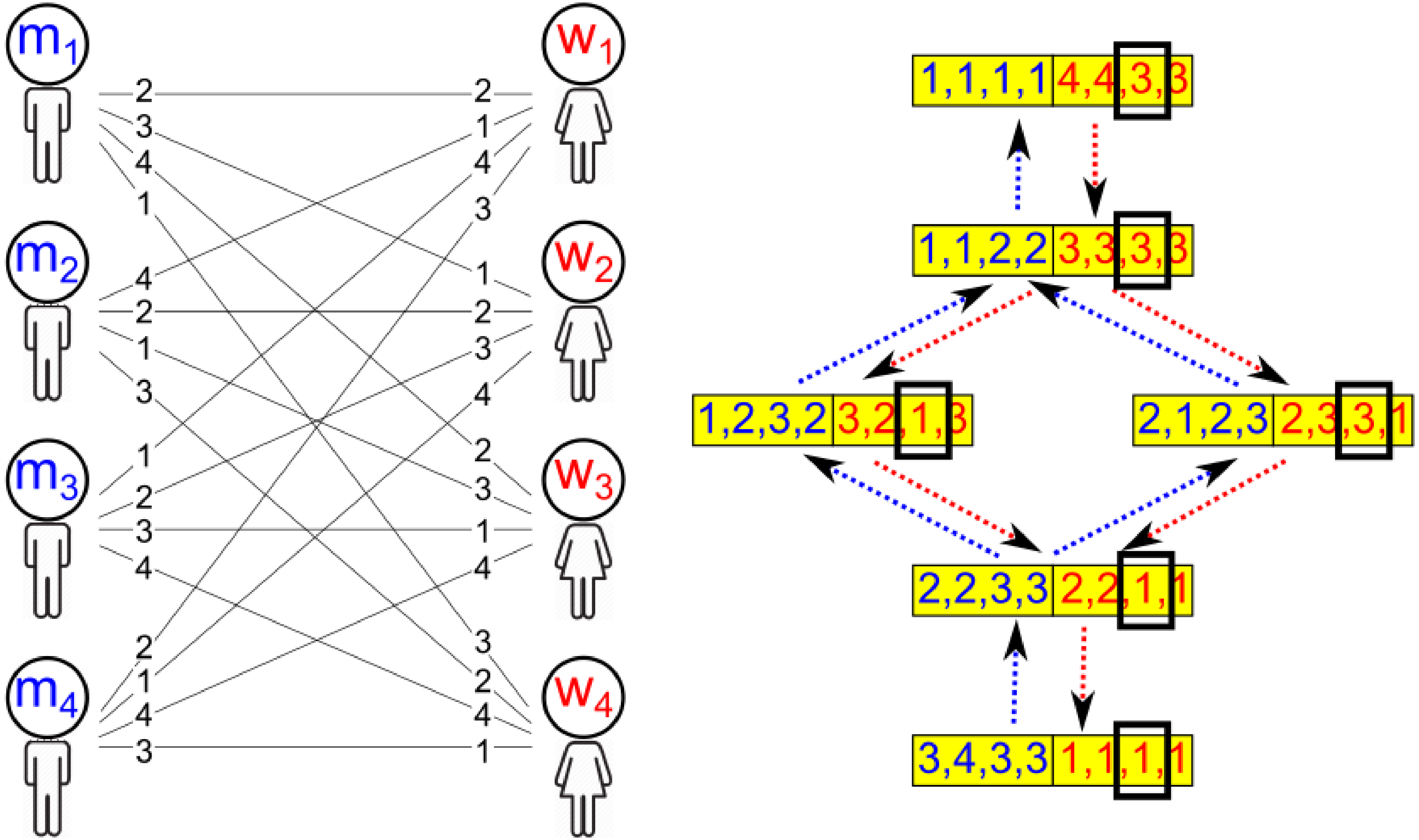

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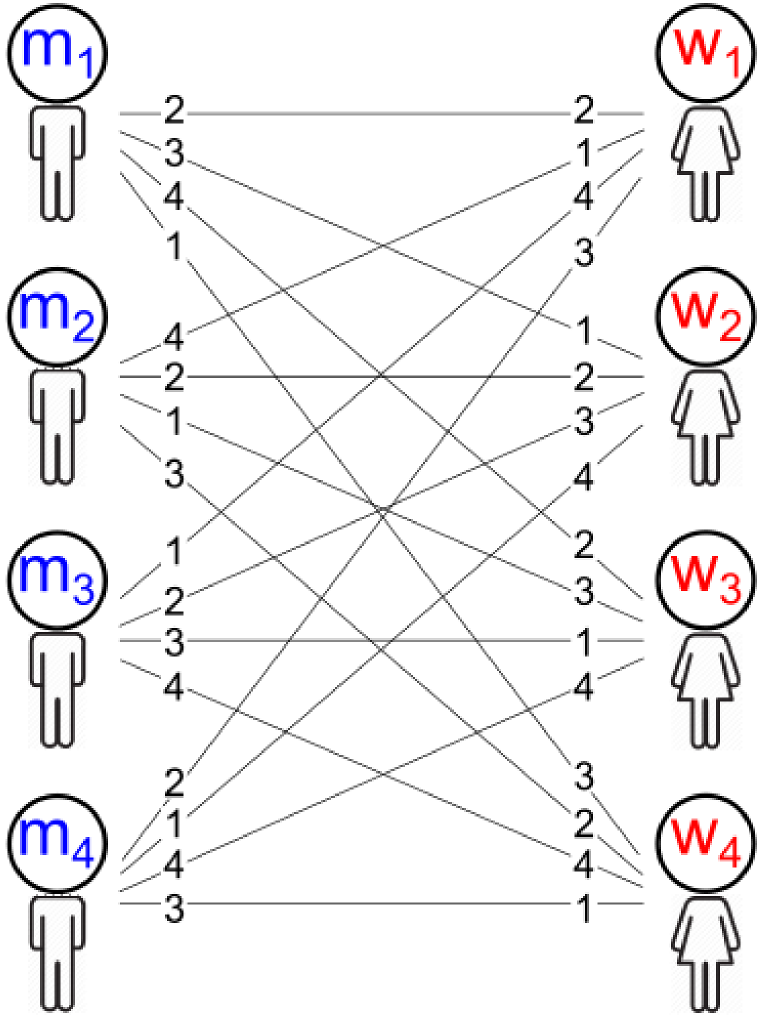

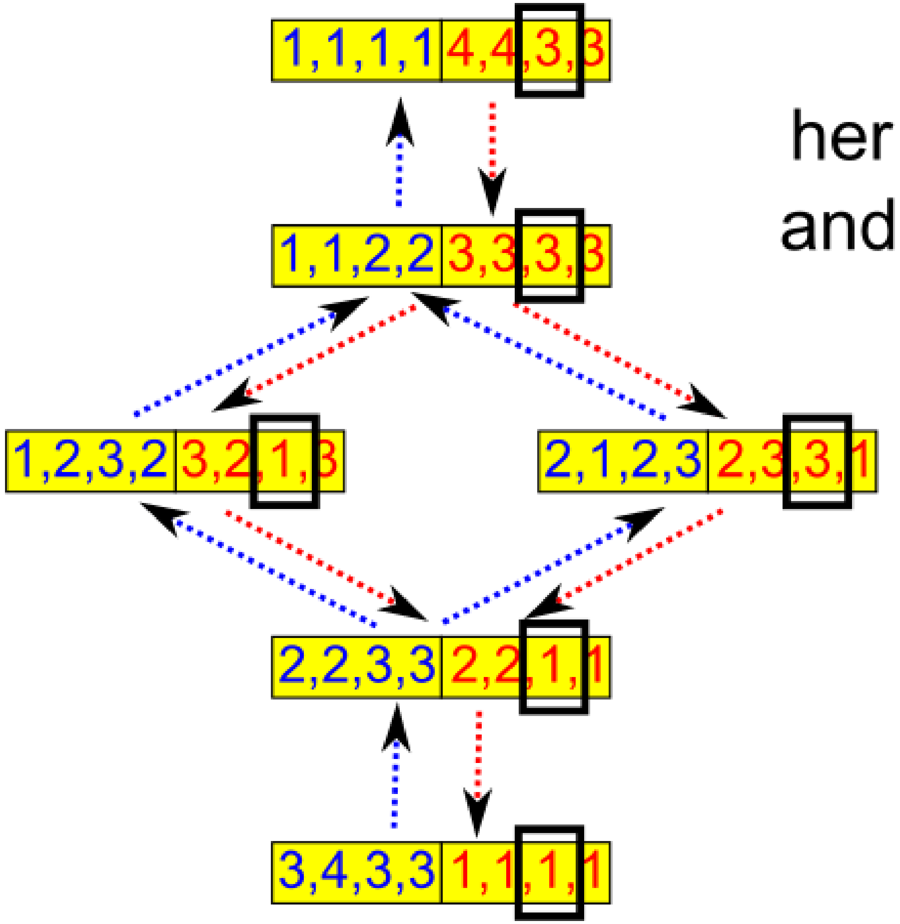

For w_3,
her first choice man m_3
and third choice man m_2
are **achievable**

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Strict preferences ⇒
Each man/woman has *exactly one* favorite achievable woman/man

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Define:
Men-optimal mapping: Each man points to his favorite achievable woman
Women-optimal mapping: Each woman points to her favorite achievable man

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Strict preferences ⇒
Each man/woman has *exactly one* favorite achievable woman/man

Define:
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We will show that men/women-optimal mappings are *one-to-one*.

Men-optimal mapping is one-to-one.

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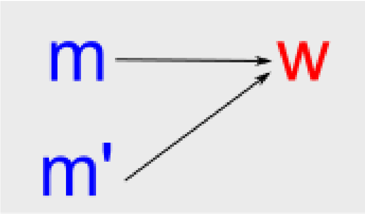
Suppose not. Then two men m and m' must map to the same woman w.



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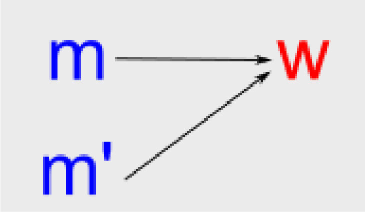

Suppose w prefers m over m'.

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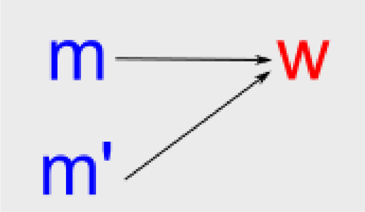
Suppose not. Then two men m and m' must map to the same woman w.

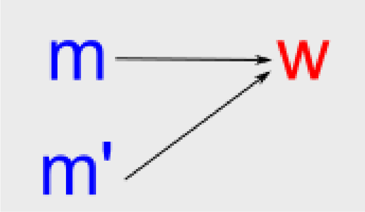


Suppose w prefers m over m'.

There must be a stable matching P where m' and w are matched.

Men-optimal mapping is one-to-one.

Suppose not. Then two men m and m' must map to the same woman w.



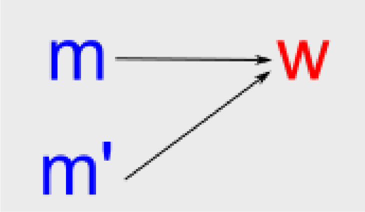
Suppose w prefers m over m'.

There must be a stable matching P where m' and w are matched.

In P, m must be matched to a woman he likes *less* than w (because w is m's favorite achievable woman).

Men-optimal mapping is one-to-one.

Suppose not. Then two men m and m' must map to the same woman w.



Suppose w prefers m over m'.

There must be a stable matching P where m' and w are matched.

In P, m must be matched to a woman he likes *less* than w (because w is m's favorite achievable woman).

But then, m and w will block P.

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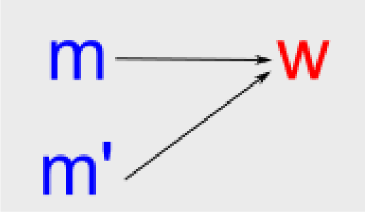

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But then, m and w will block P.

■

Algorithm for computing the men-optimal (or women-optimal) matching?

[Gale and Shapley, 1962]

Given any preference profile, the matching computed by the men-proposing deferred-acceptance algorithm is men-optimal. Similarly, a women-optimal matching is obtained when women propose.

[Gale and Shapley, 1962]

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Suffices to show that in the men-proposing DA algorithm, a man is never rejected by his favorite achievable woman.

[Gale and Shapley, 1962]

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Suffices to show that in the men-proposing DA algorithm, a man is never rejected by his favorite achievable woman.

By way of contradiction, suppose man m is the first man to be rejected by his favorite achievable woman w.

[Gale and Shapley, 1962]

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Suffices to show that in the men-proposing DA algorithm, a man is never rejected by his favorite achievable woman.

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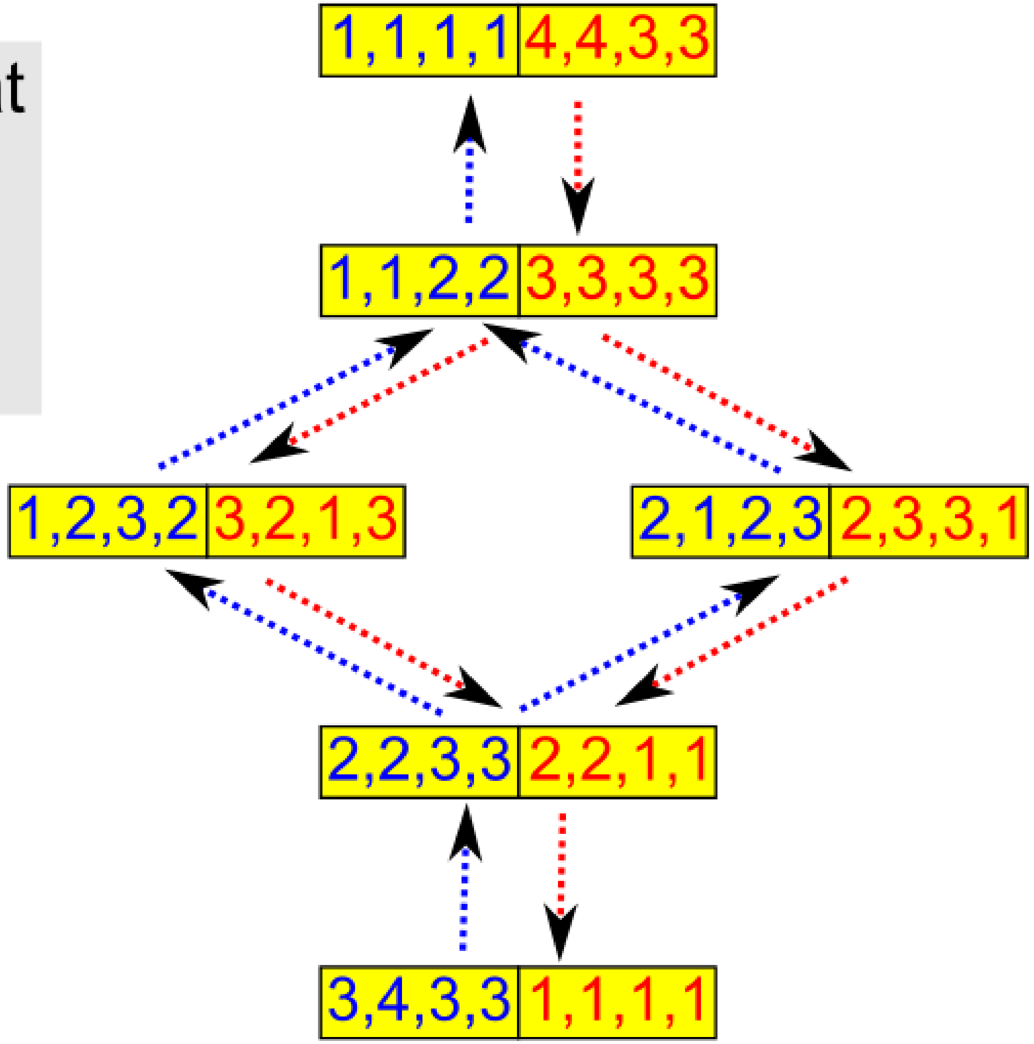
Some Observations

Consensus

There is a stable matching that all men find at least as good as *any other* stable matching, and one that they find at least as bad. (Analogously for the women.)

Conflict

For any distinct stable matchings P and Q, if all men find P at least as good as Q, then all women find Q at least as good as P (and vice versa).



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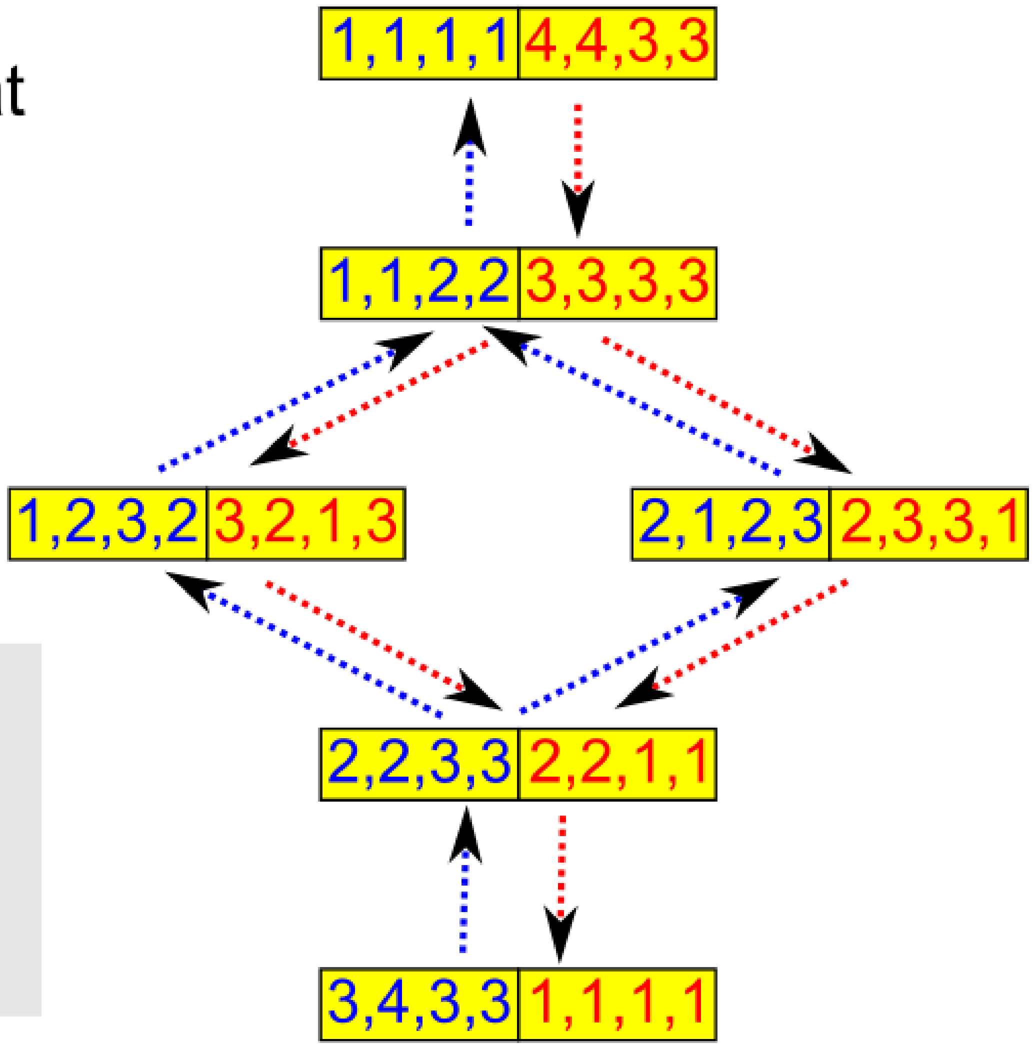

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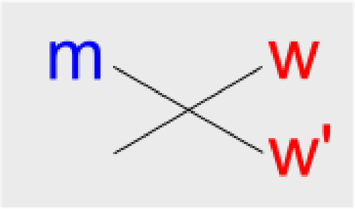
Q: 

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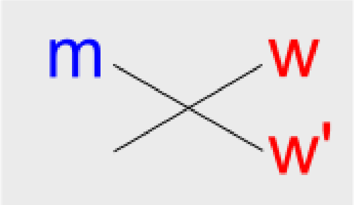
Q: 

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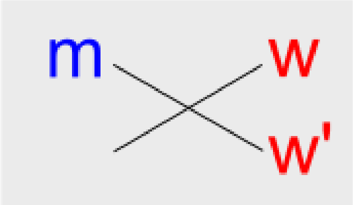
Q: 

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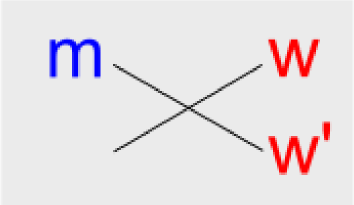
Q: 

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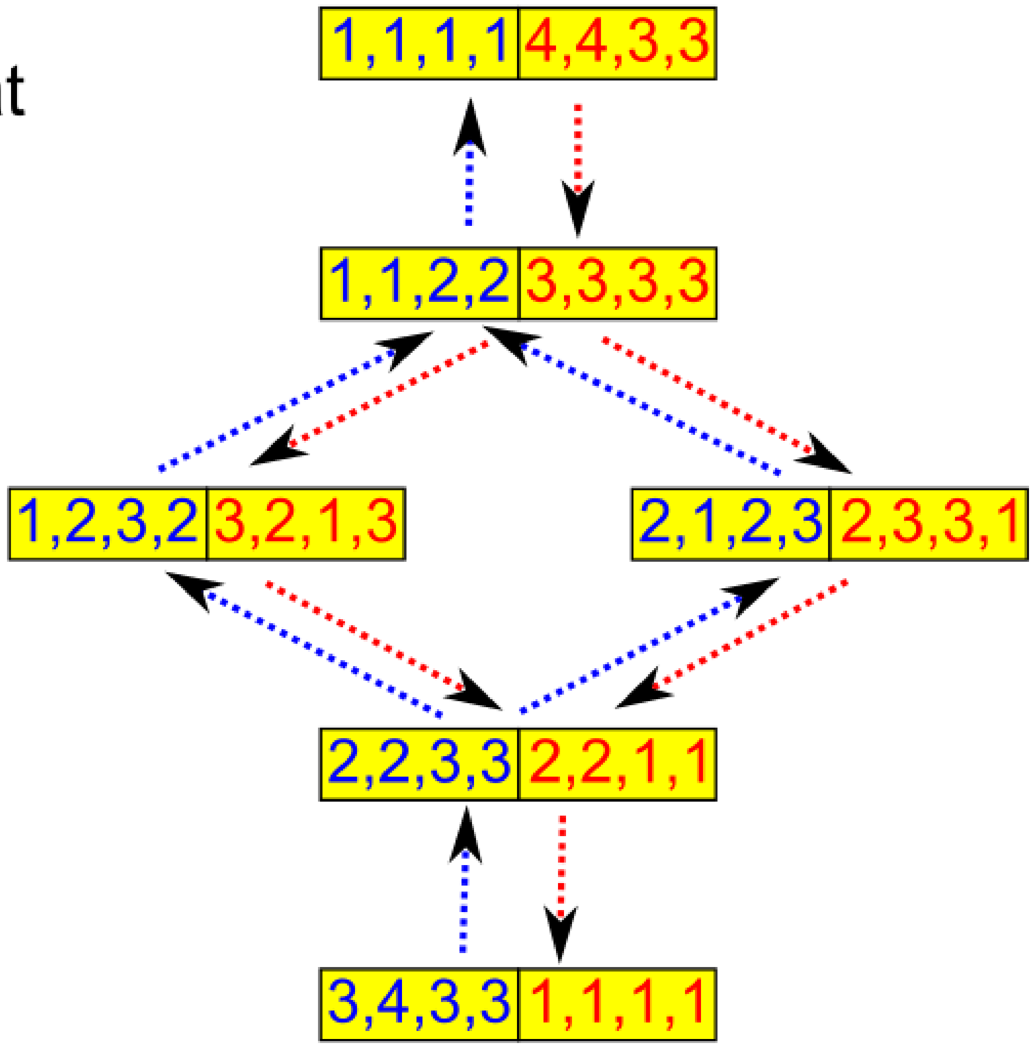

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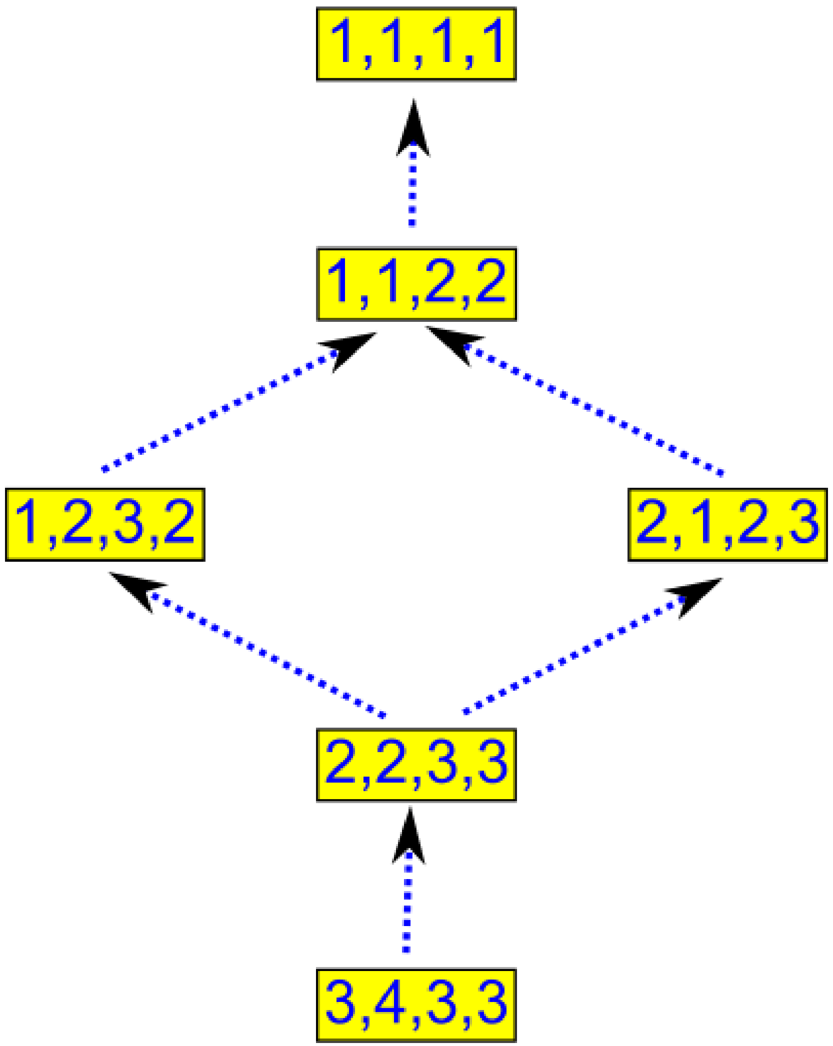

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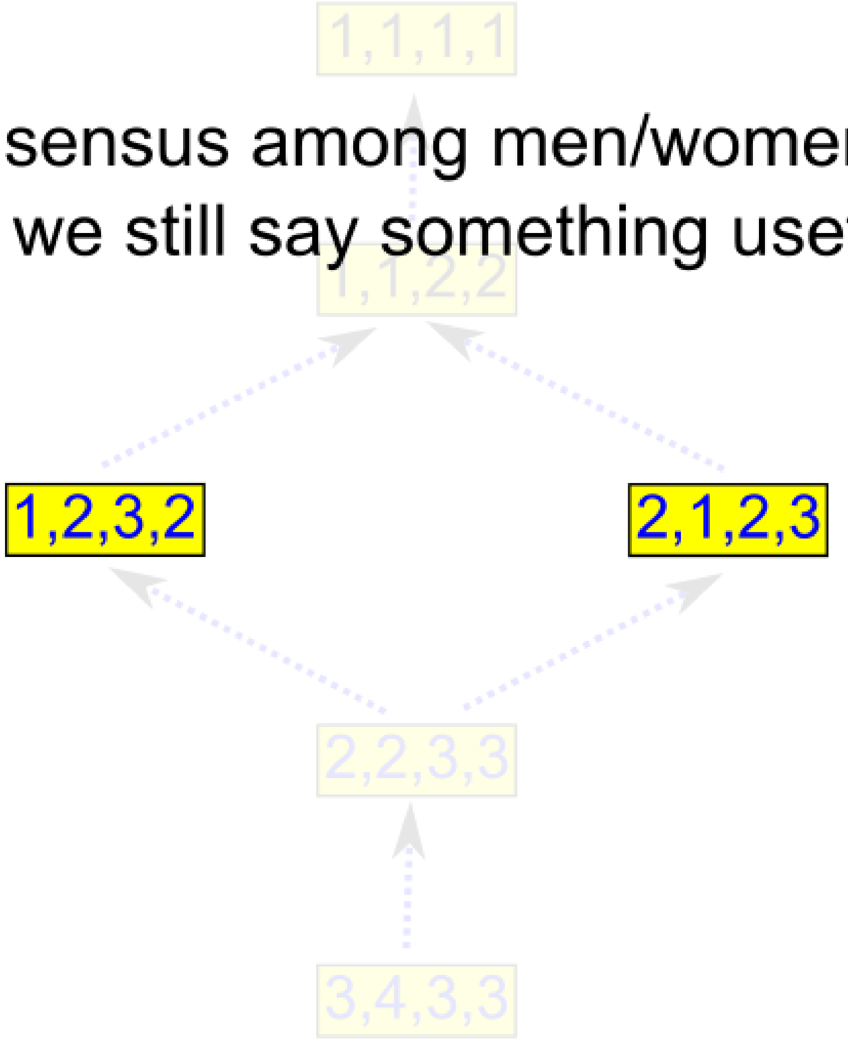
When there isn't a consensus among men/women w.r.t. two matchings, can we still say something useful?



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Let's generalize this idea to arbitrary pairs of stable matchings.

Let P and Q be any pair of stable matchings (not necessarily distinct).

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Suffices to show that for any m and w, $\max_{P,Q}(m) = w \Leftrightarrow \max_{P,Q}(w) = m$.

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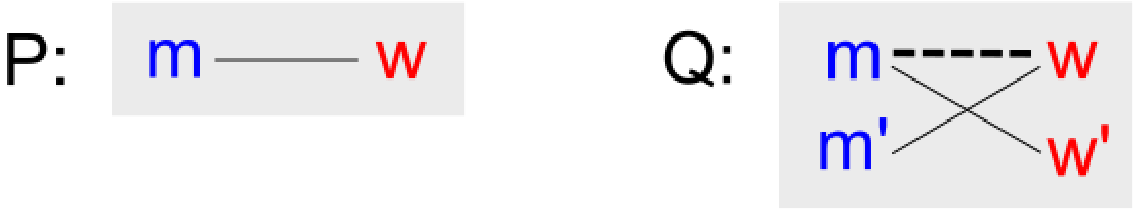

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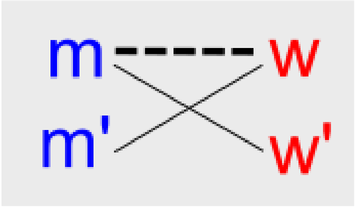

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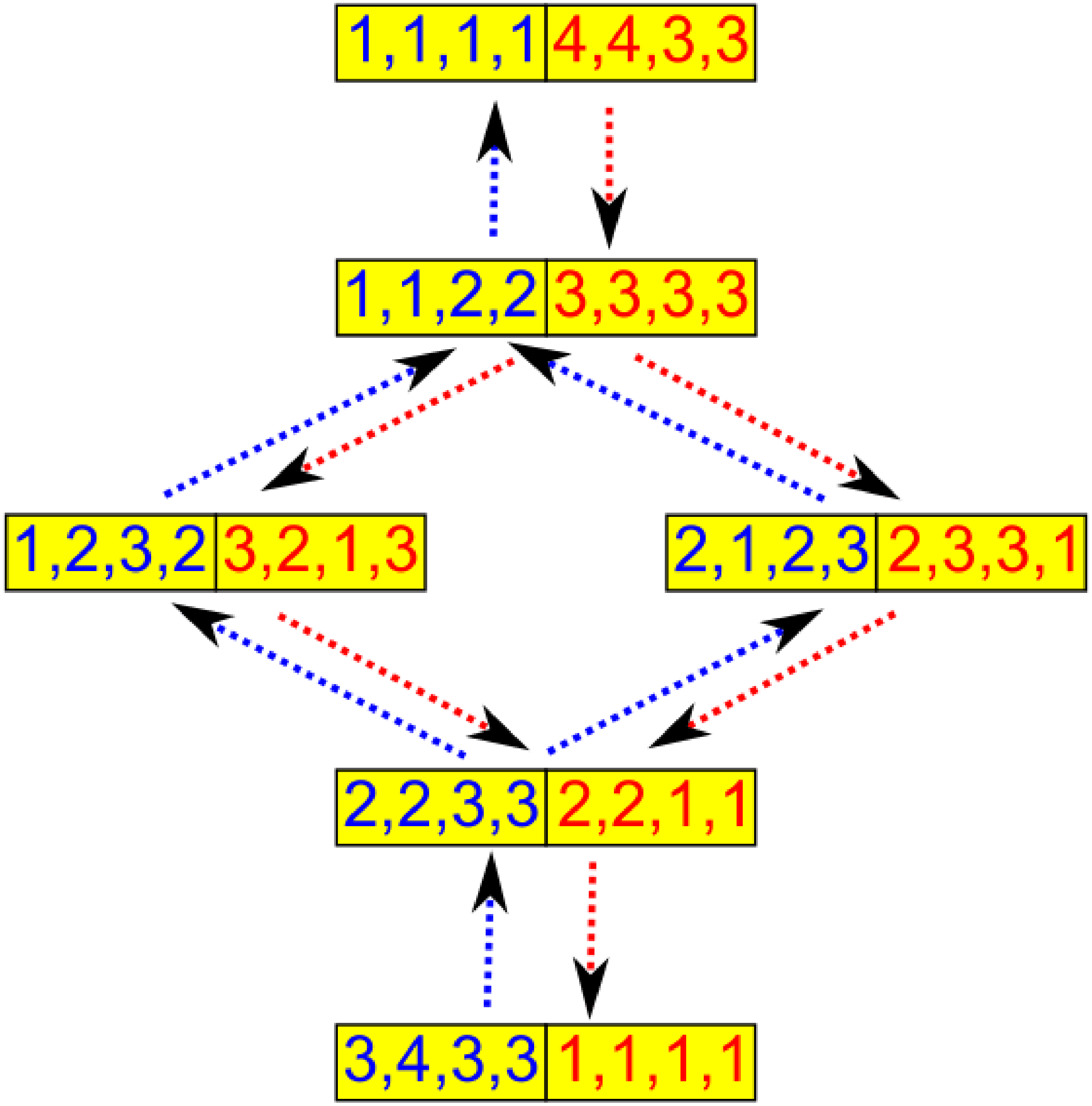
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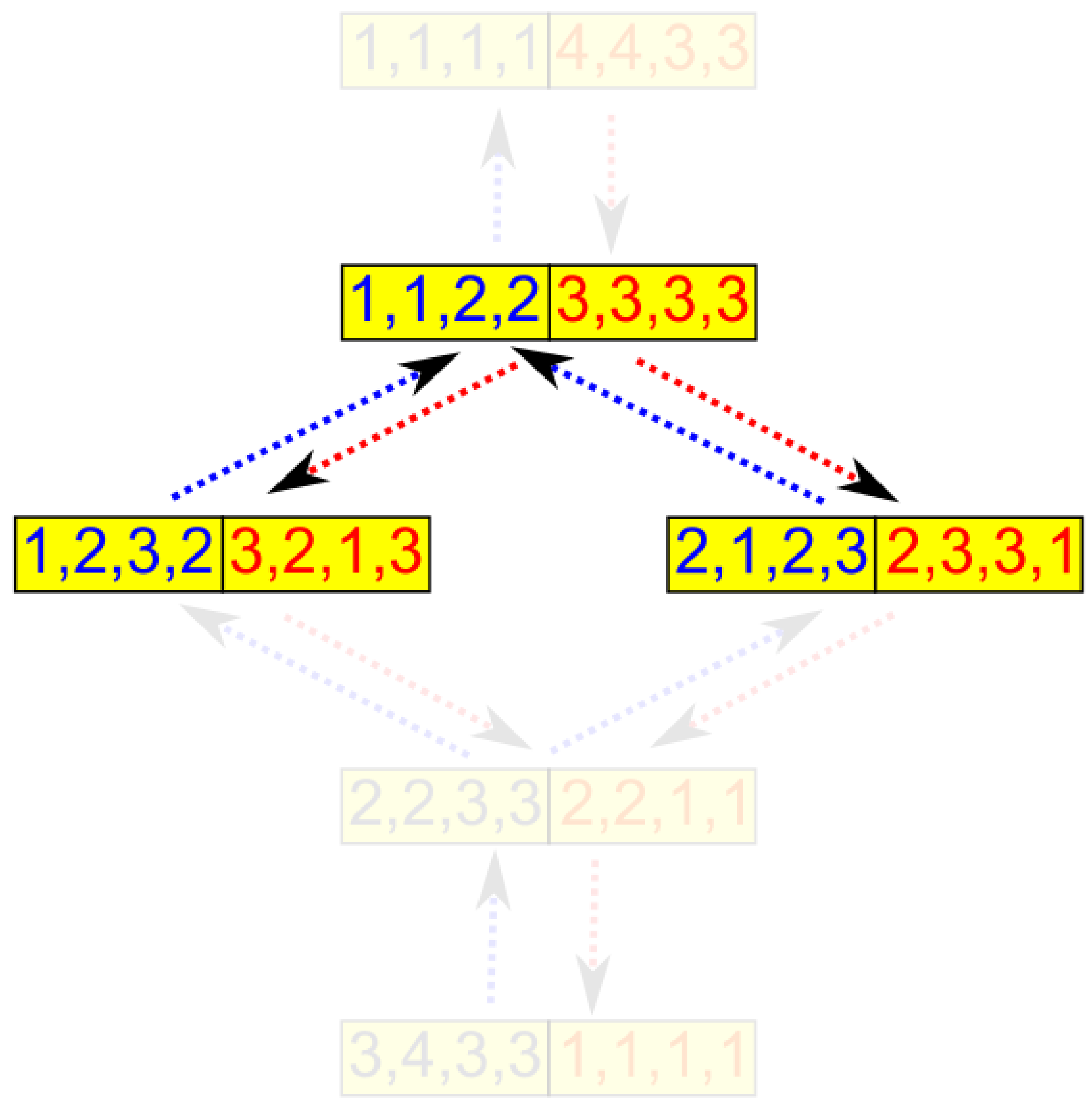

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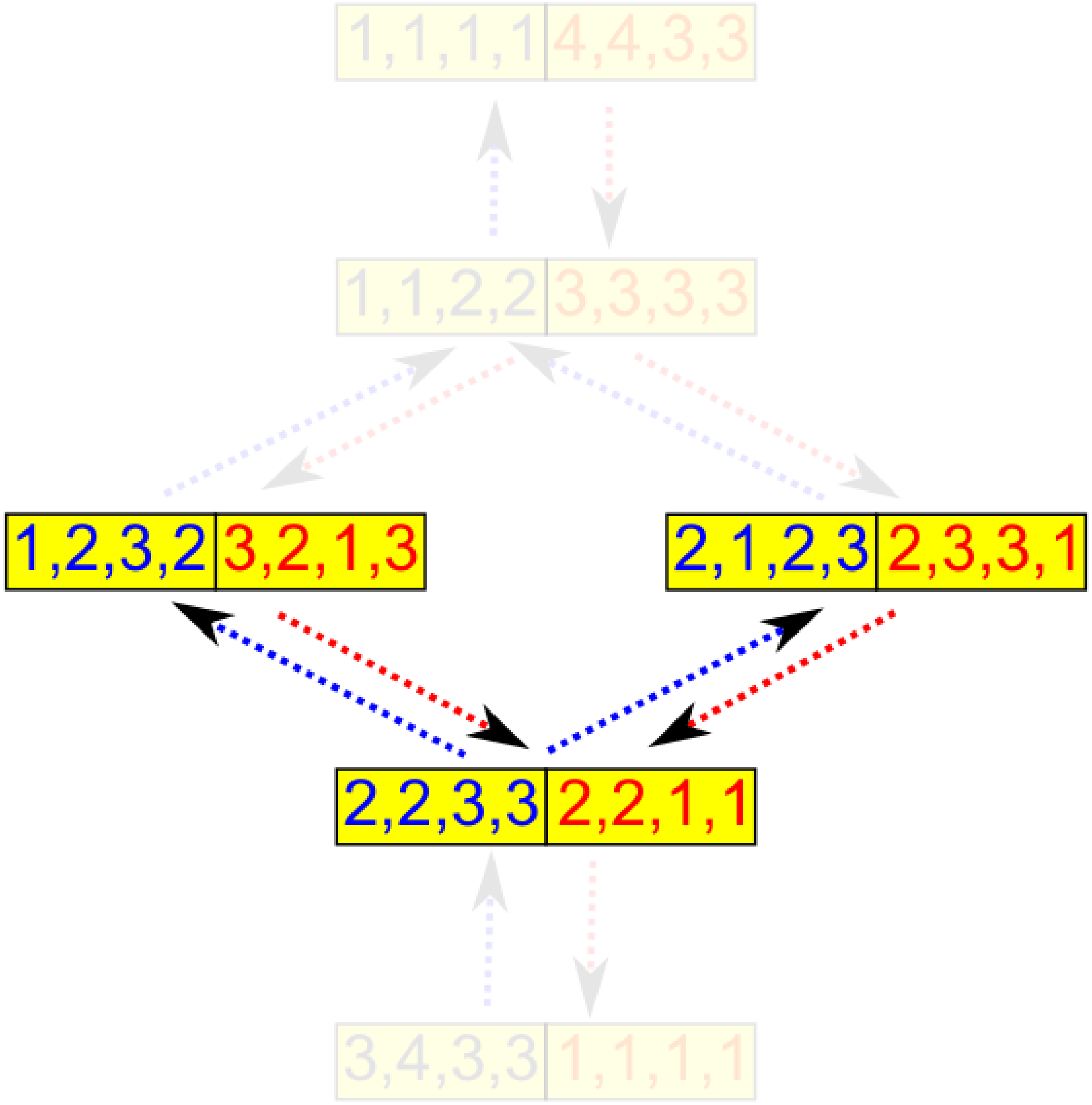

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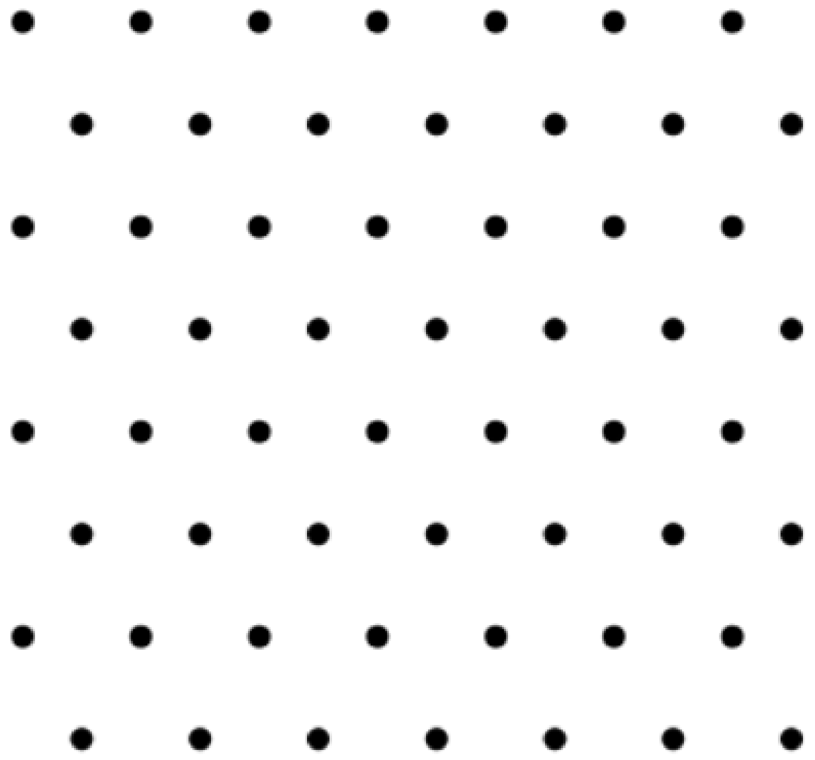

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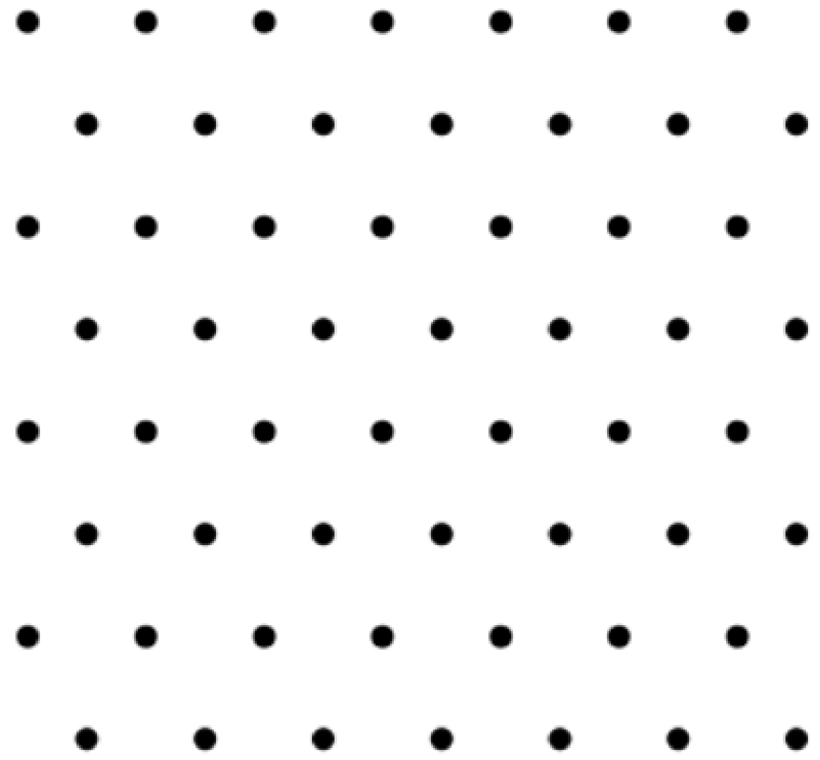

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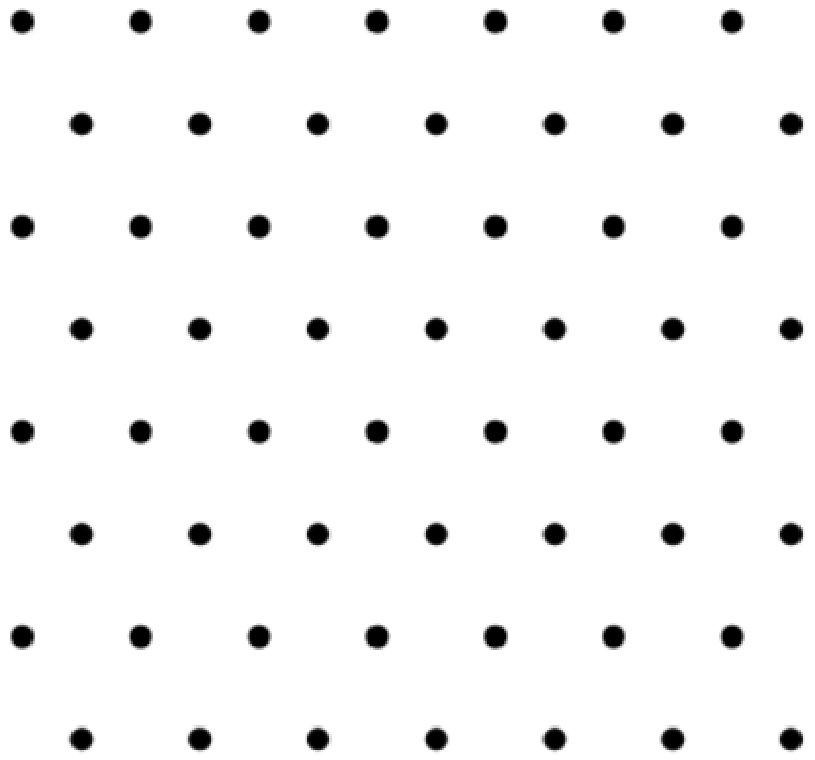

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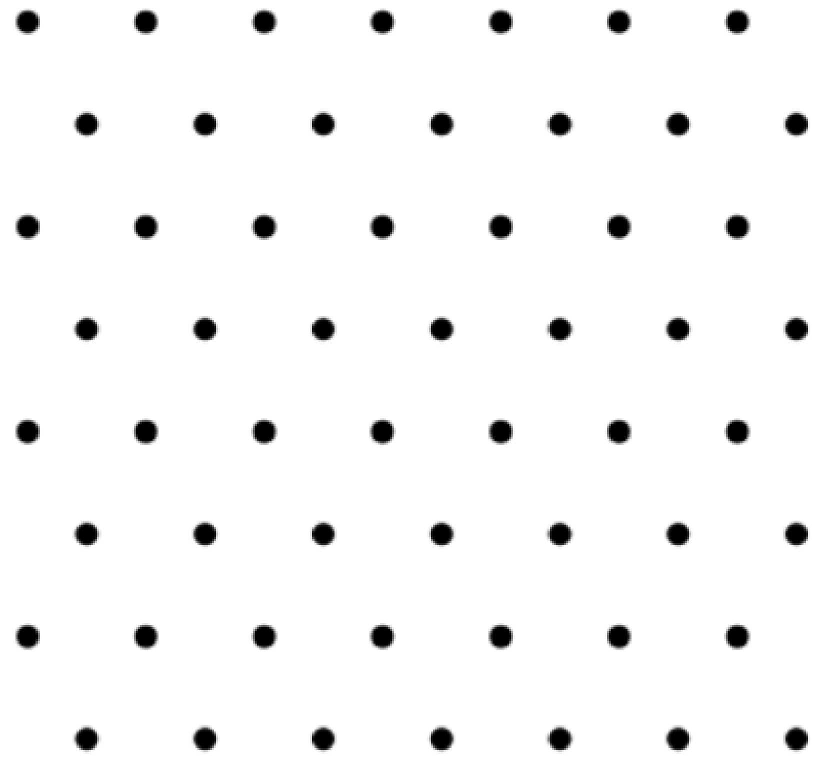
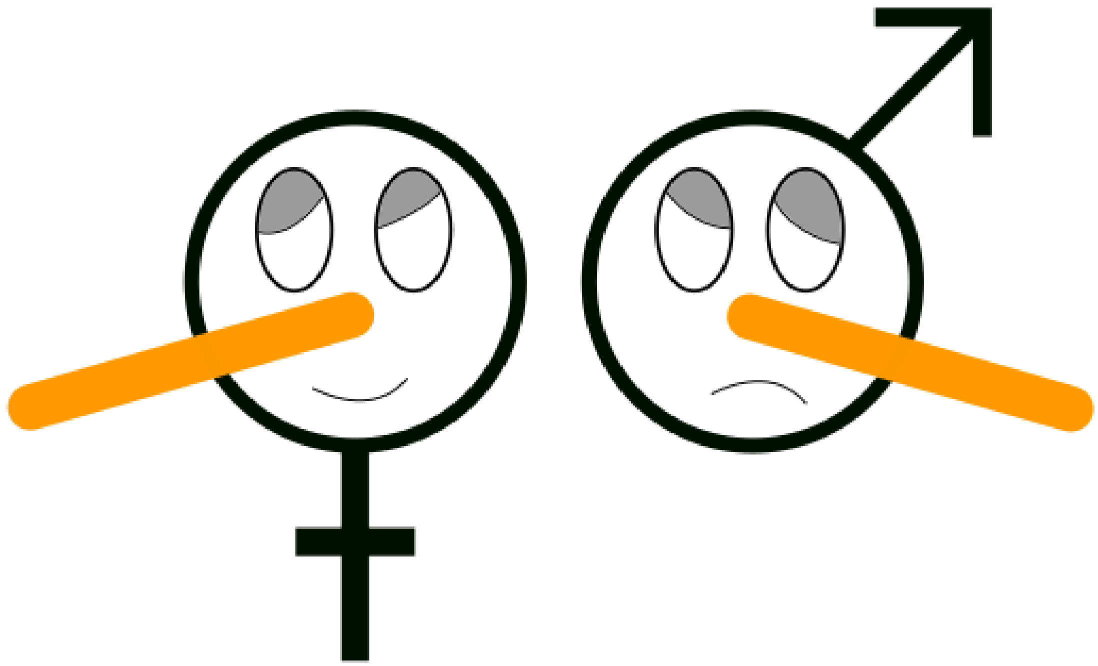


Next Time

Incentives in the Stable Matching Problem



Quiz

Quiz

Prove that an instance has a unique stable matching
if and only if
the men-optimal and women-optimal matchings are the same.

References

- Structure of the Set of Stable Matchings

Alvin Roth and Marilda Sotomayor
"Two-Sided Matching: A Study in Game-Theoretic Modeling and Analysis"
Econometric Society Monograph Series, 1990

Stable Roommates

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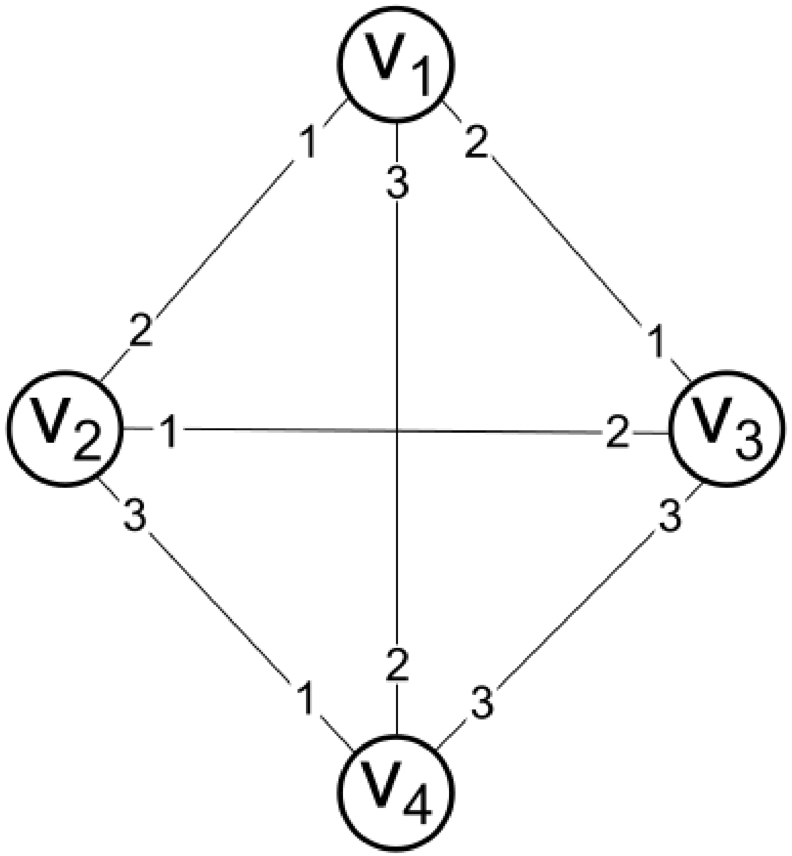

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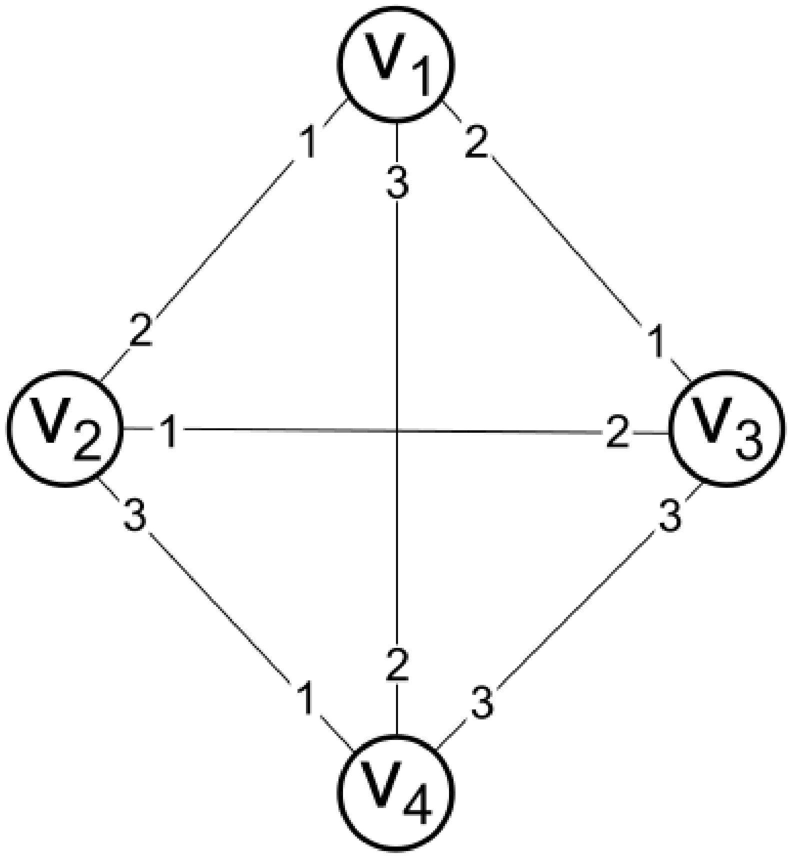

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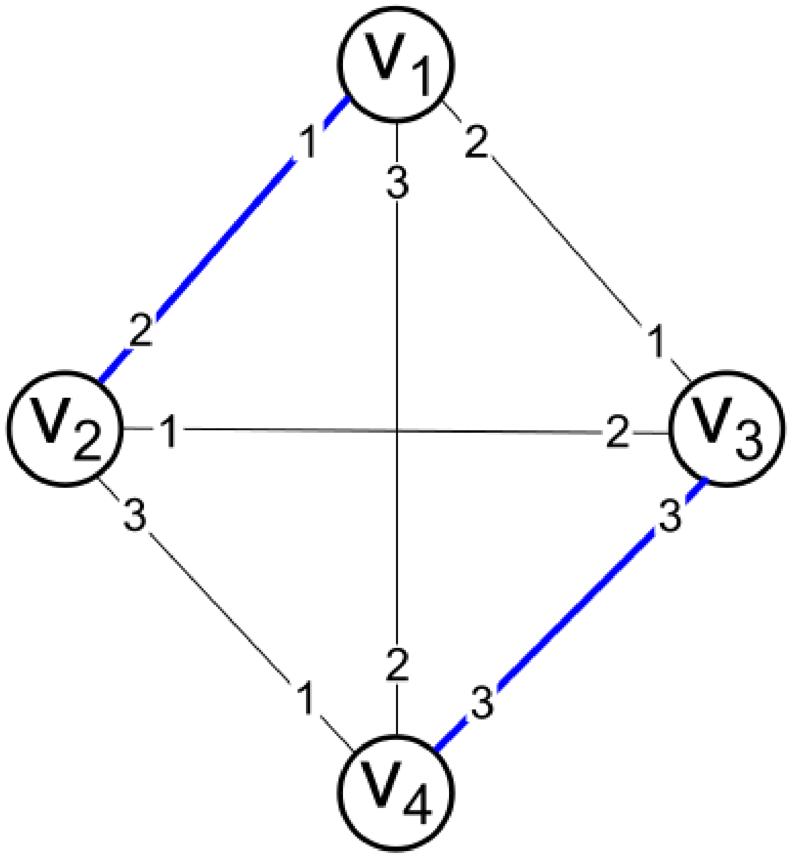

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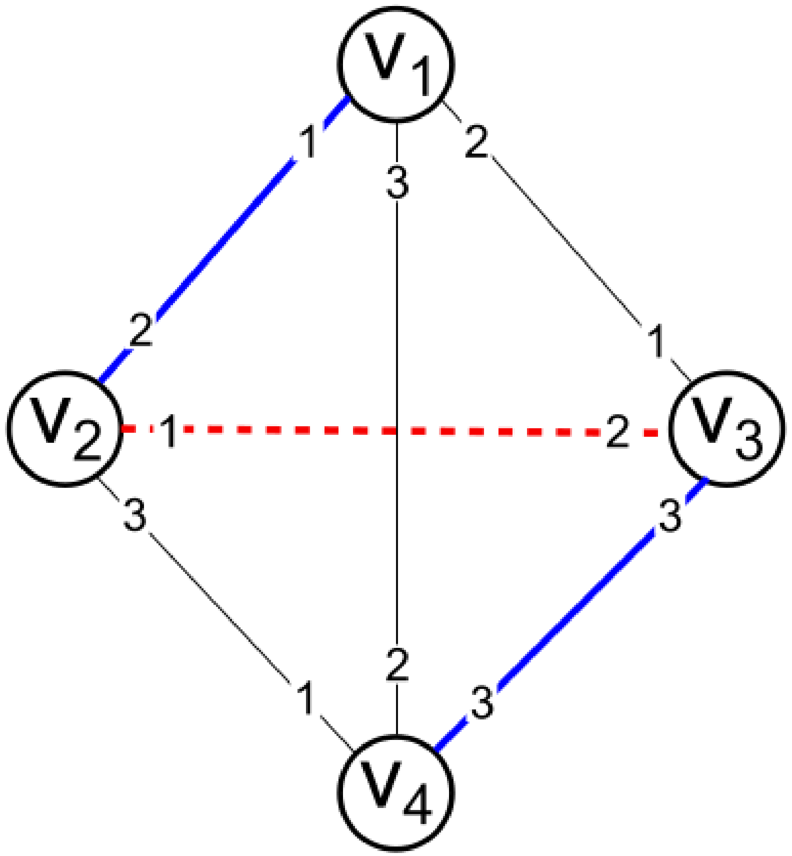

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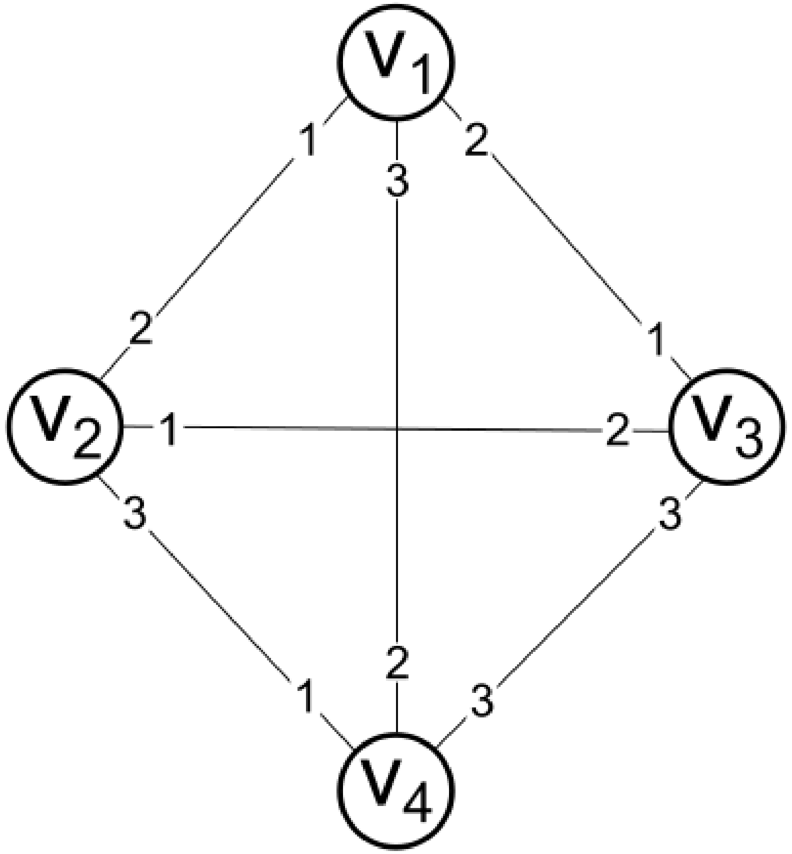
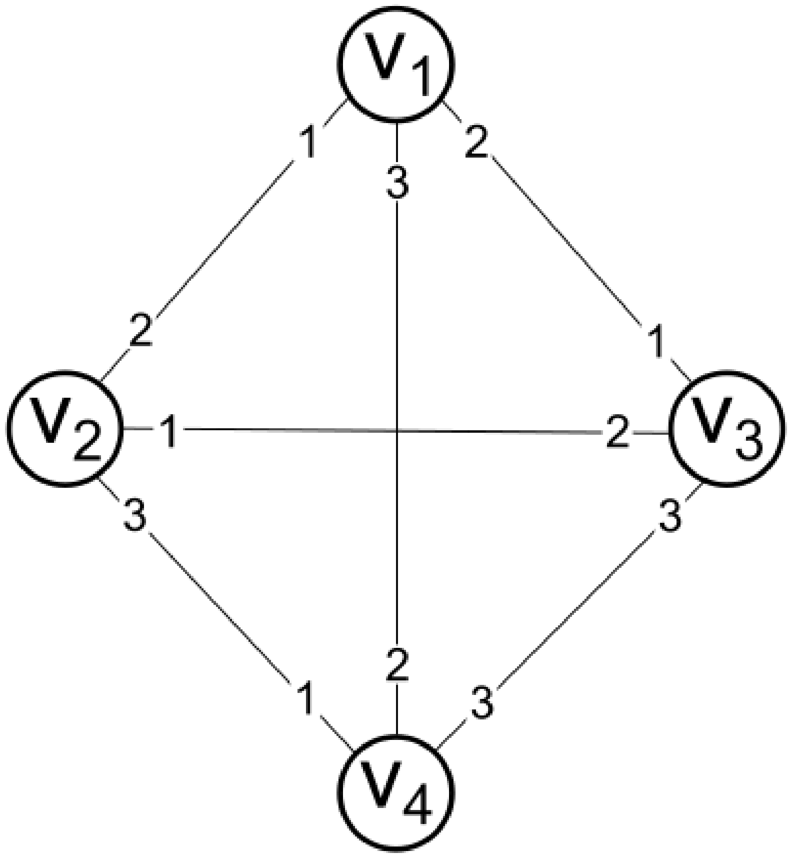
[Gale and Shapley, 1962]



Stable Roommates

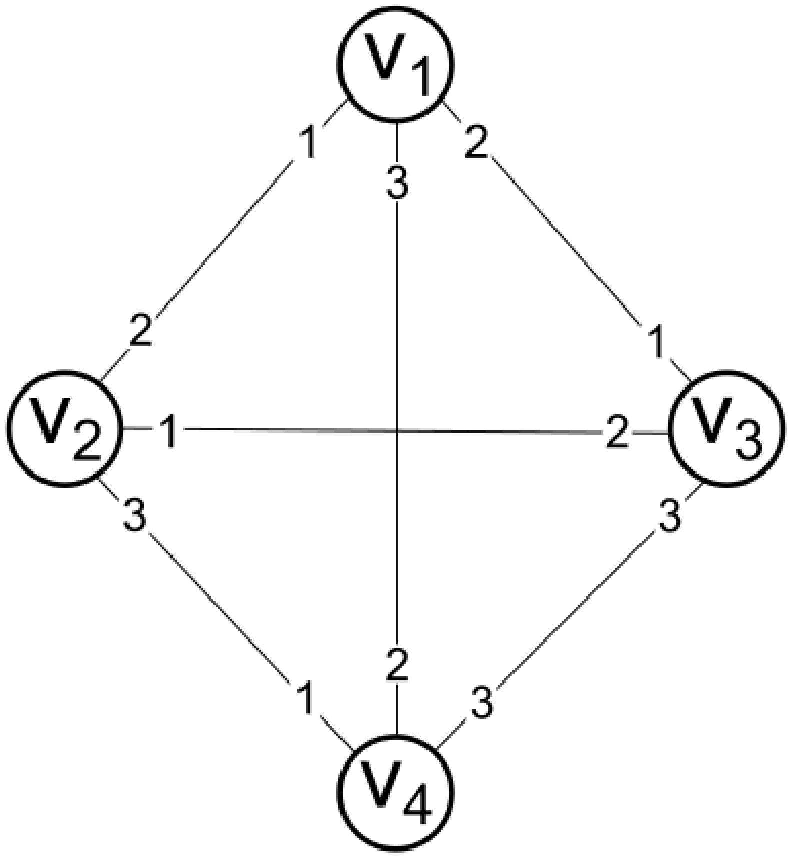
[Gale and Shapley, 1962]



There is no stable matching in the above instance.

Stable Roommates

[Gale and Shapley, 1962]



There is no stable matching in the above instance.

Whoever is matched with v_4 will block with one of the other two agents.