Fairness for a Mixture of Resources

Jatin Yadav

The model

n agents with additive valuations over:




Indivisible

Divisible

EFM: A fusion of EF and EF1

For any two agents a_i and a_j:

- If a_j is given some cake, then a_i does not envy a_j (**EF**).
- Else, a_i does not envy a_j upto one good (**EF1**).

Theorem: An EFM allocation always exists!

AAAI-20 Outstanding Student Paper Award

Fair Division of Mixed Divisible and Indivisible Goods

Xiaohui Bei, Zihao Li, Jinyan Liu, Shengxin Liu, Xinhang Lu

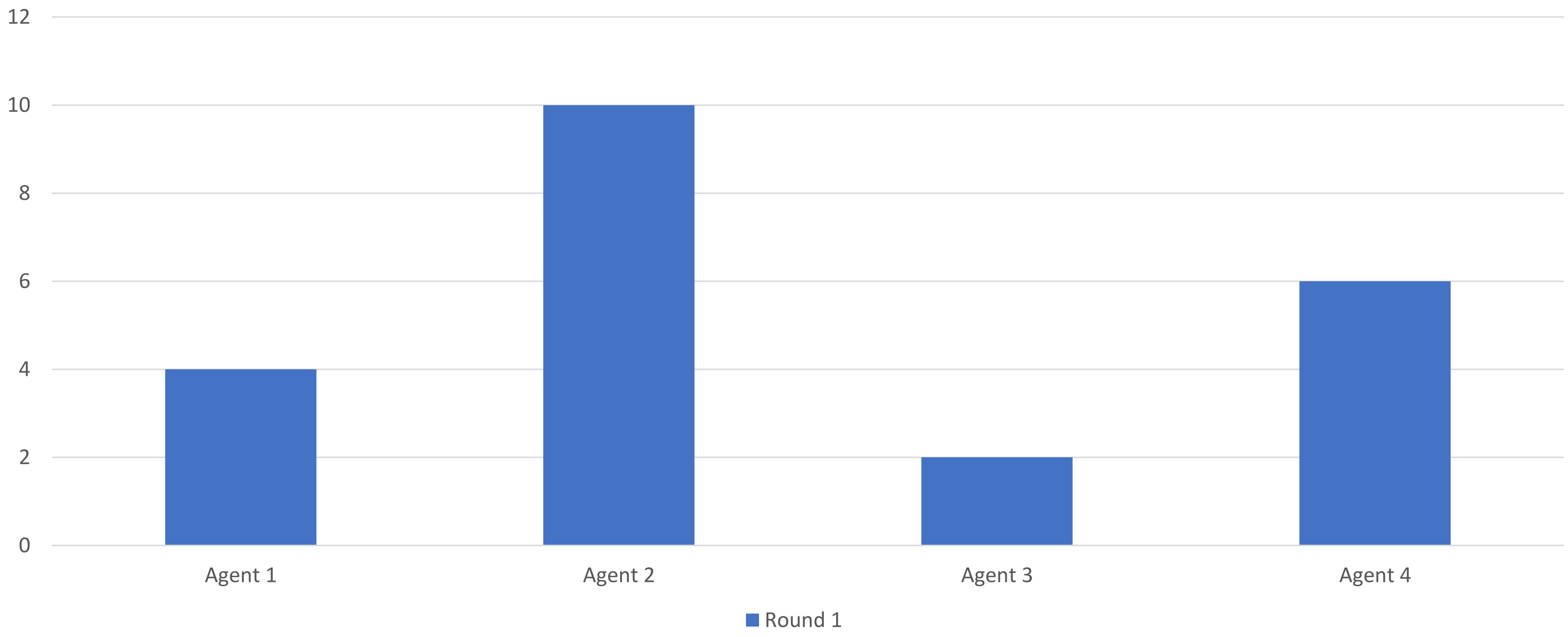
Warmup: Identical Agents

Algorithm:

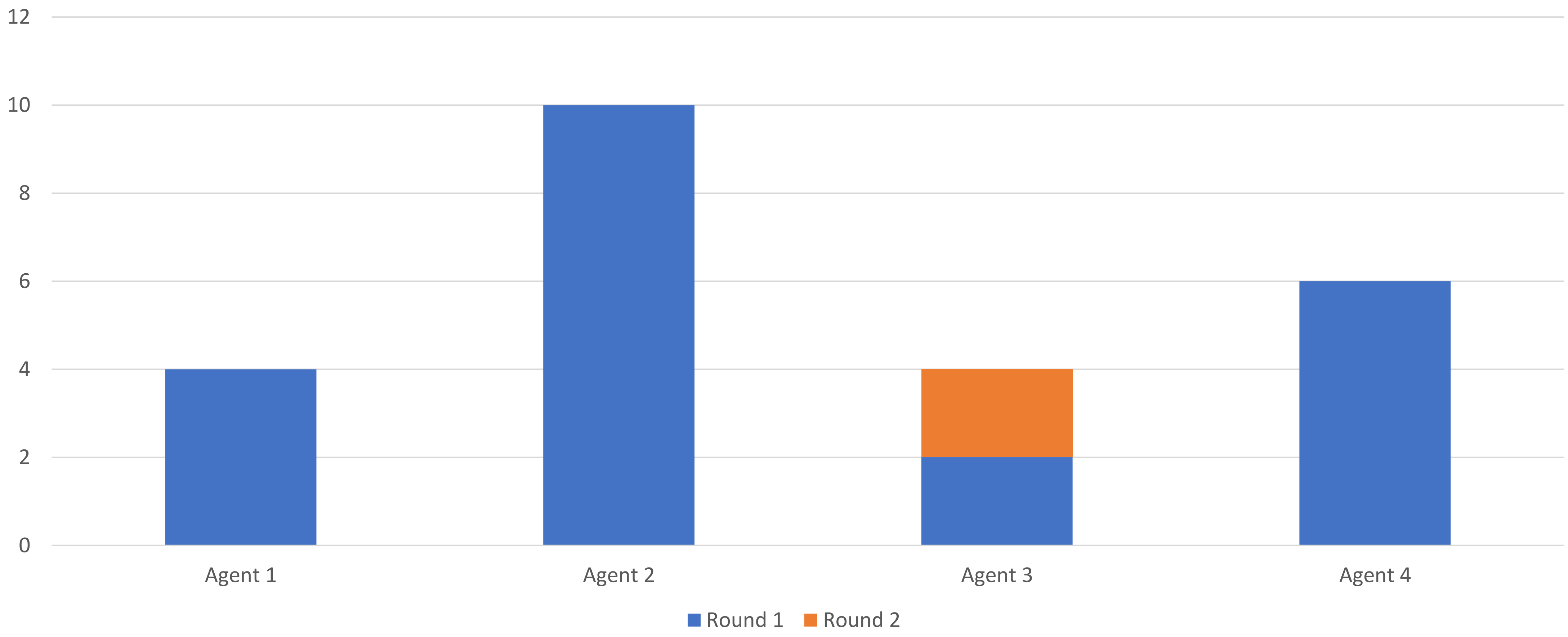
1. **Round robin:** Find an EF1 allocation of the indivisible goods.

2. **Water filling:** Keep on allocating the cake **equally** to the set of poorest agents.

Water filling algorithm



Water filling algorithm



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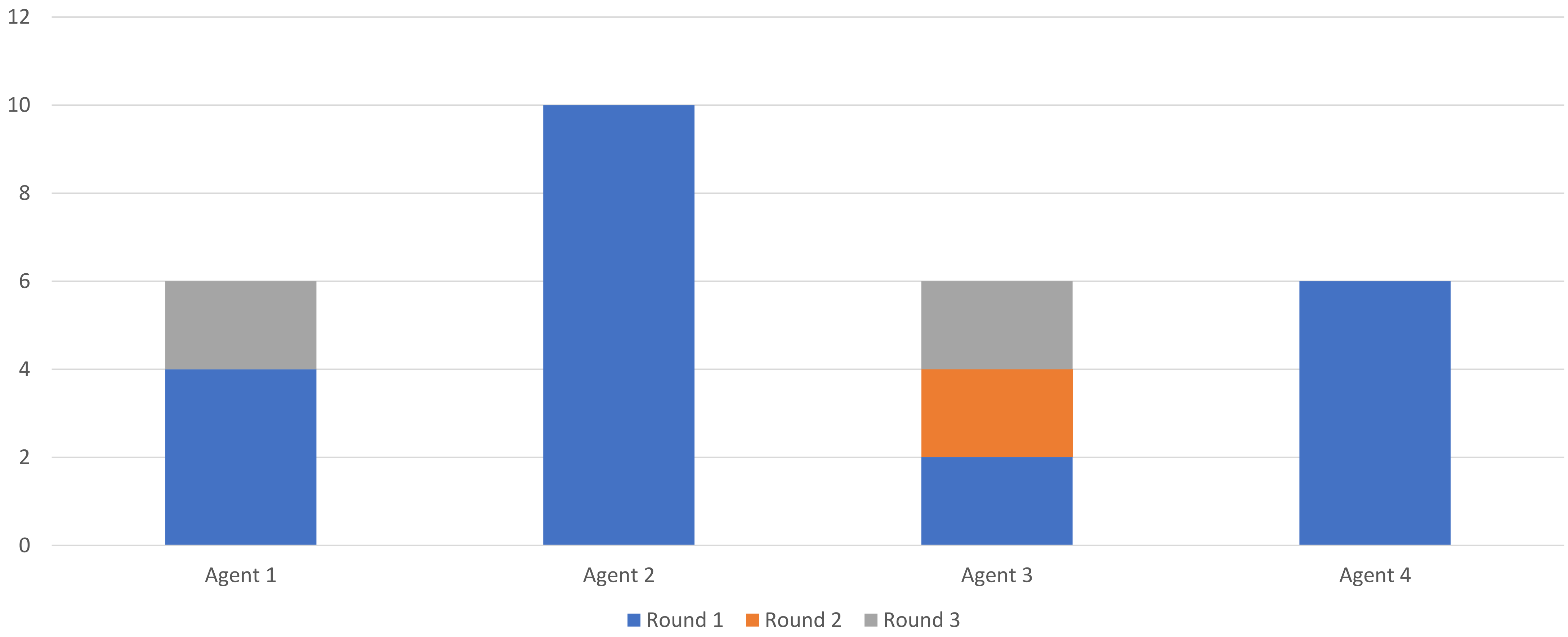

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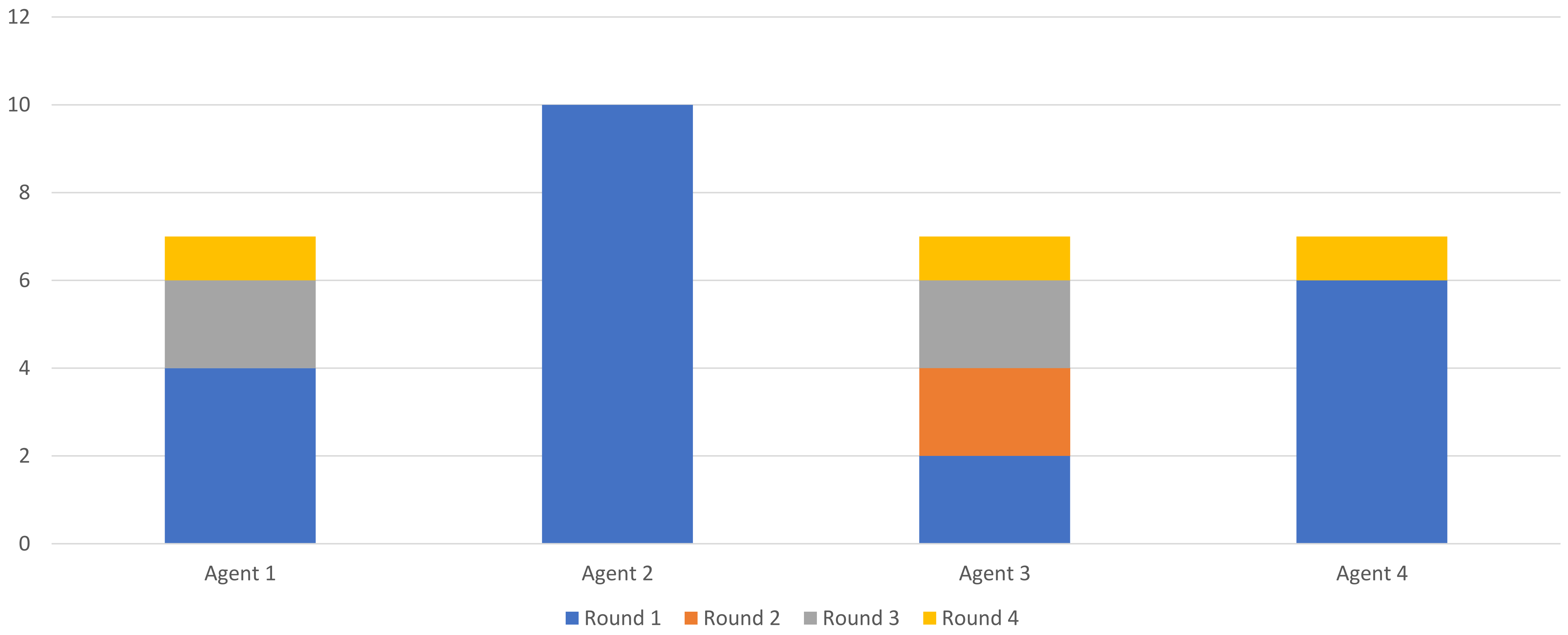


Generalizing to non-identical agents

1. **Round robin:** EF1 allocation of indivisible goods - Still works! ☺

2. **Water filling:** Keep allocating cake **equally** to the set of poorest agents:
   1. How to allocate **equally**? ☹
   2. Which agents are the poorest? ☹

Allocating equally: Perfect cake division

Theorem: Given a cake C, n **(non-identical)** agents and a positive integer k, a perfect division into k pieces exists:

For each agent, all k pieces have the same value.

$$v_i(C_j) = \frac{v_i(C)}{k} \ \forall \ i \in [n], j \in [k]$$

Recap: The envy cycle algorithm

Maintain a partial-allocation that is EF1.

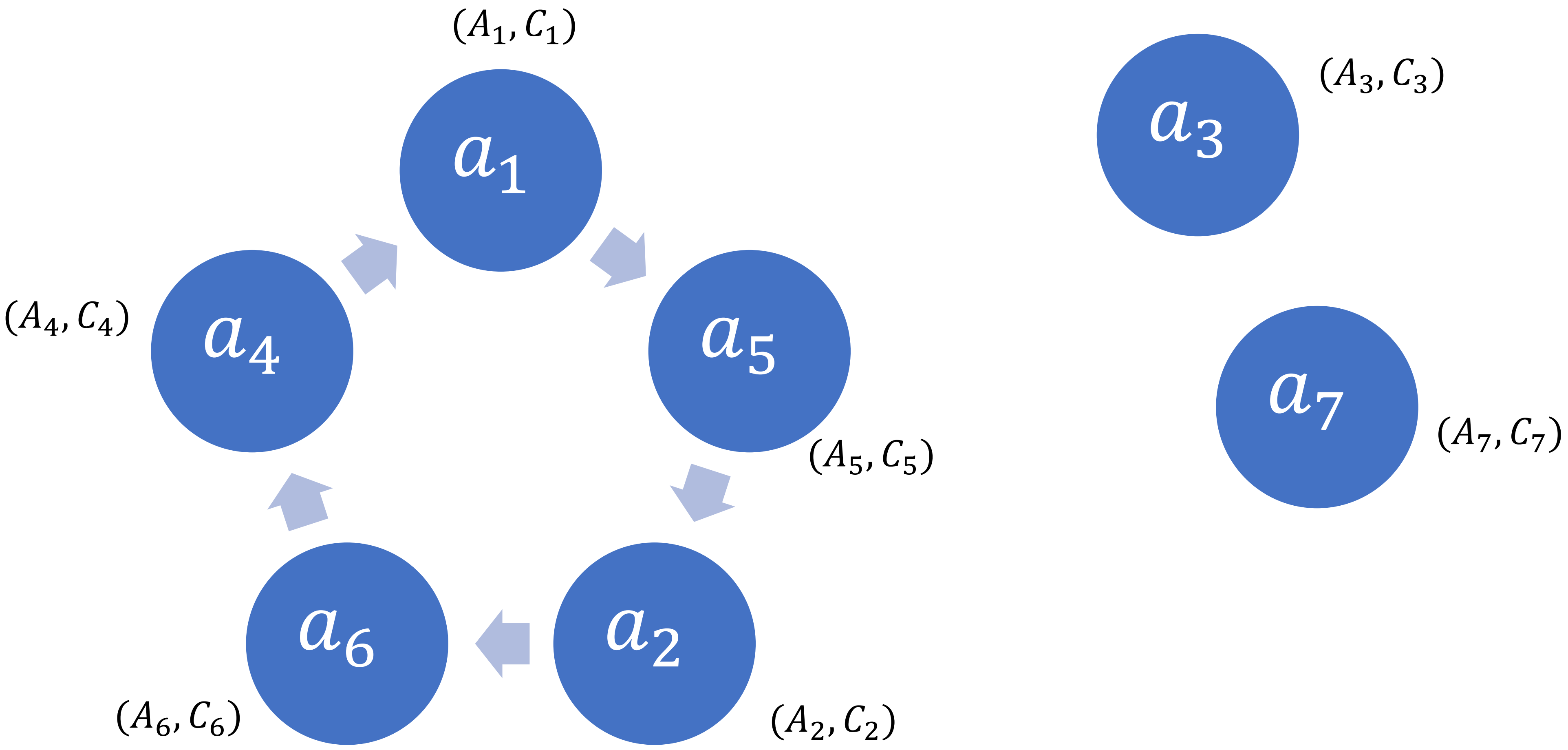
In the envy graph:

- Either $\exists$ a source $\Rightarrow$ give it a good.
- Or $\exists$ a cycle $\Rightarrow$ do a cyclic shift of bundles.

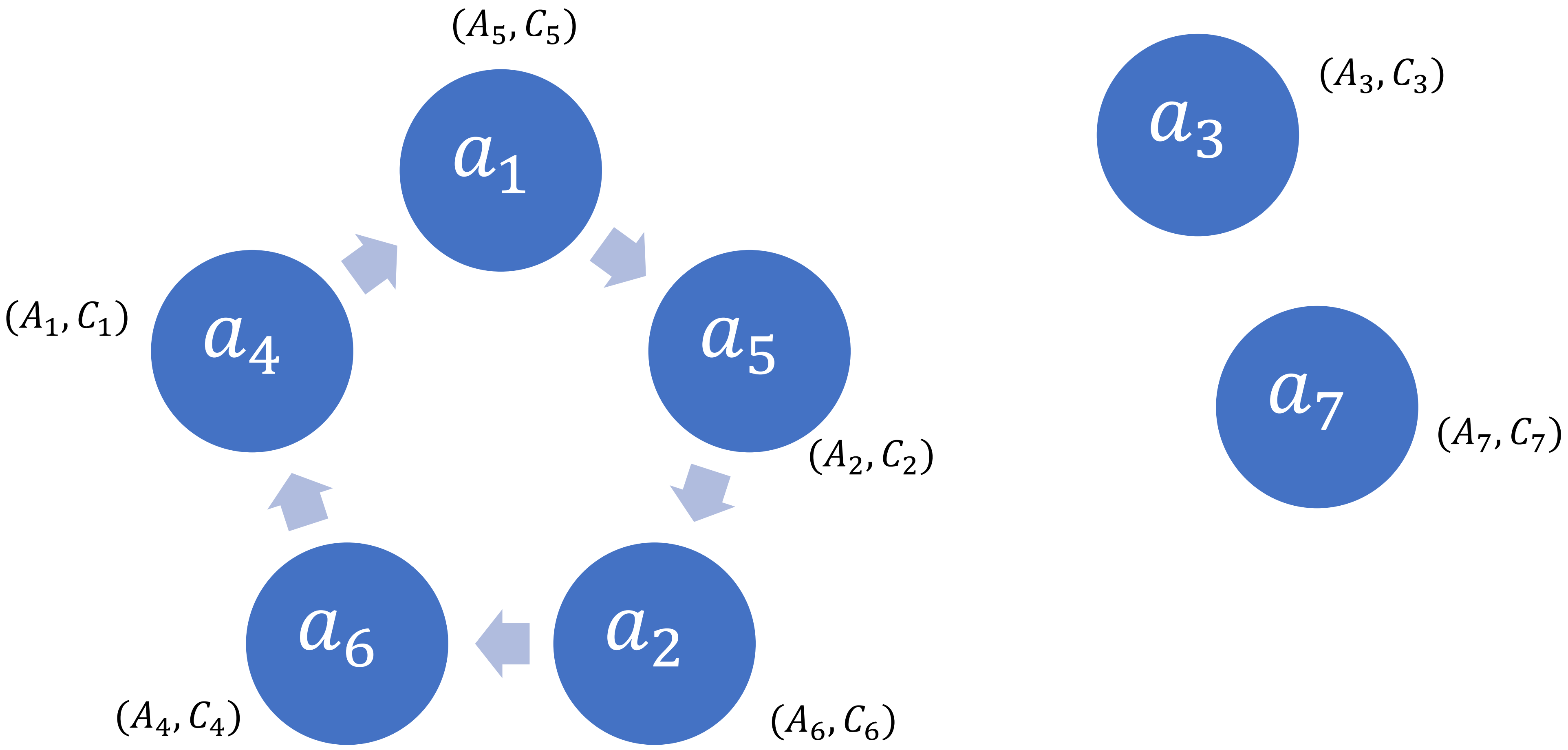
Measure of progress?

- Either a good is allocated.
- Or the number of envy edges **strictly** decreases.

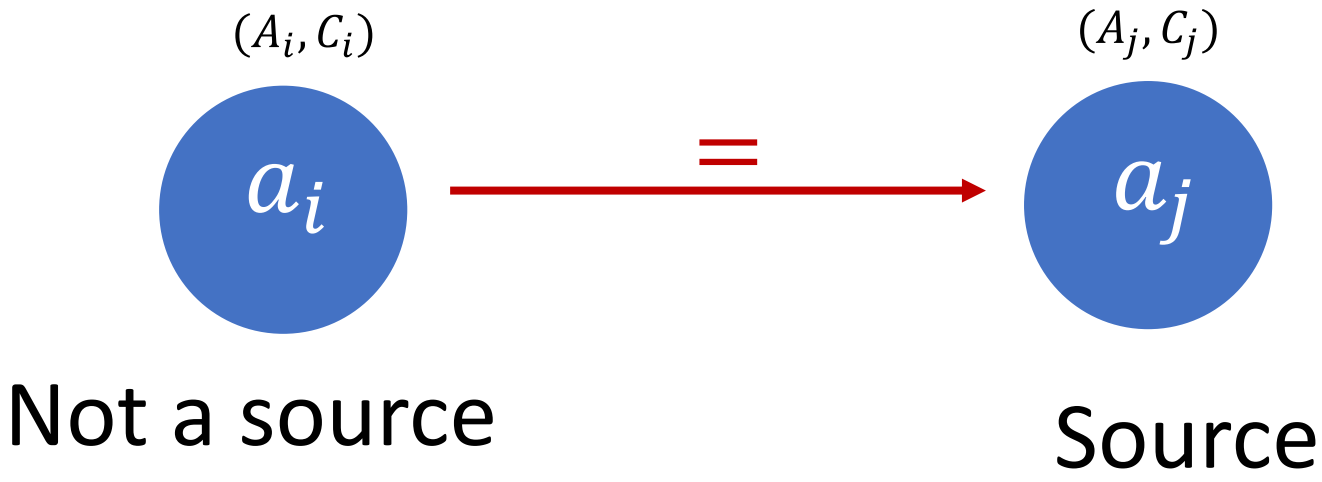
Envy cycle elimination maintains EFM



Envy cycle elimination maintains EFM



Give cake "equally" to sources?



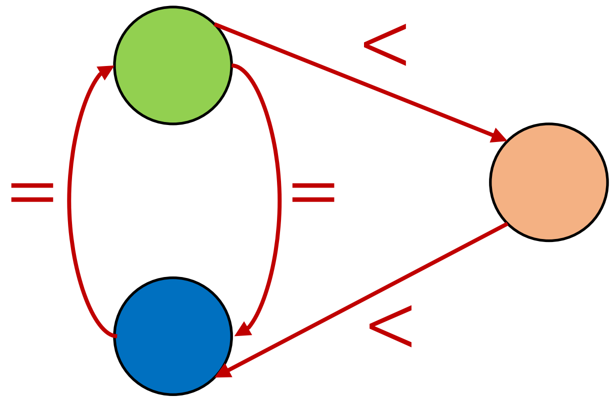
Issue: Can not allocate anything to a_j if $v_i(A_i, C_i) = v_i(A_j, C_j)$ ☹

Fix: Must also consider "equality" edges.

Who to allocate cake to?

Addable subset: A subset S of **sources**, that does **not** have incoming **equality** edges from outside.

Might not exist! ☹



(Envy U equality) cycle elimination

Consider graph with **both** envy and equality edges.

EFM is maintained on cyclic transfer. ☺

But, progress might not be made ☹ …

* unless there is at least one envy edge in the cycle. ☺

So…

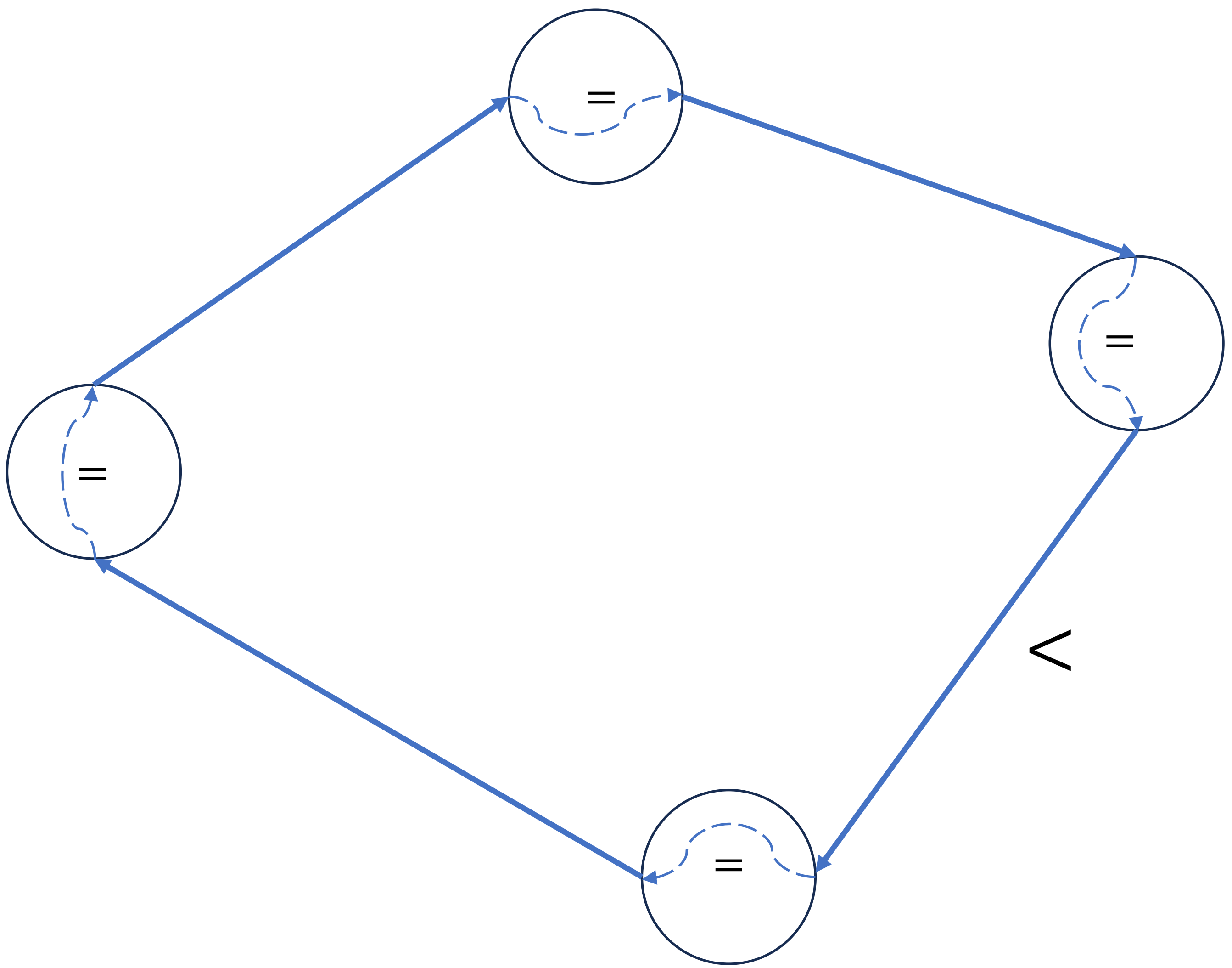
- Cycle with at least one envy edge in the (envy ∪ equality) graph
  ⇒ Do a cyclic transfer.

- Else, allocate *as much cake as possible* to the **maximal** addable subset of agents:
  - Why does an addable subset exist?
  - Why is the maximum addable subset unique?
  - How much cake to allocate?

Existence of an addable subset

1. Compress the SCC's of the equality graph.
2. Consider the envy edges.
3. Claim:
   1. No envy edge within the same component.
   2. No cycle of components through (envy ∪ equality) edges.

Why no cycle of components through (envy ∪ equality) edges?



Existence of an addable subset

1. Compress the SCC's of the equality graph.
2. Consider the envy edges.
3. Claim:
    1. No envy edge within the same component.
    2. No cycle of components through (envy $\cup$ equality) edges.

The source component is an addable subset!

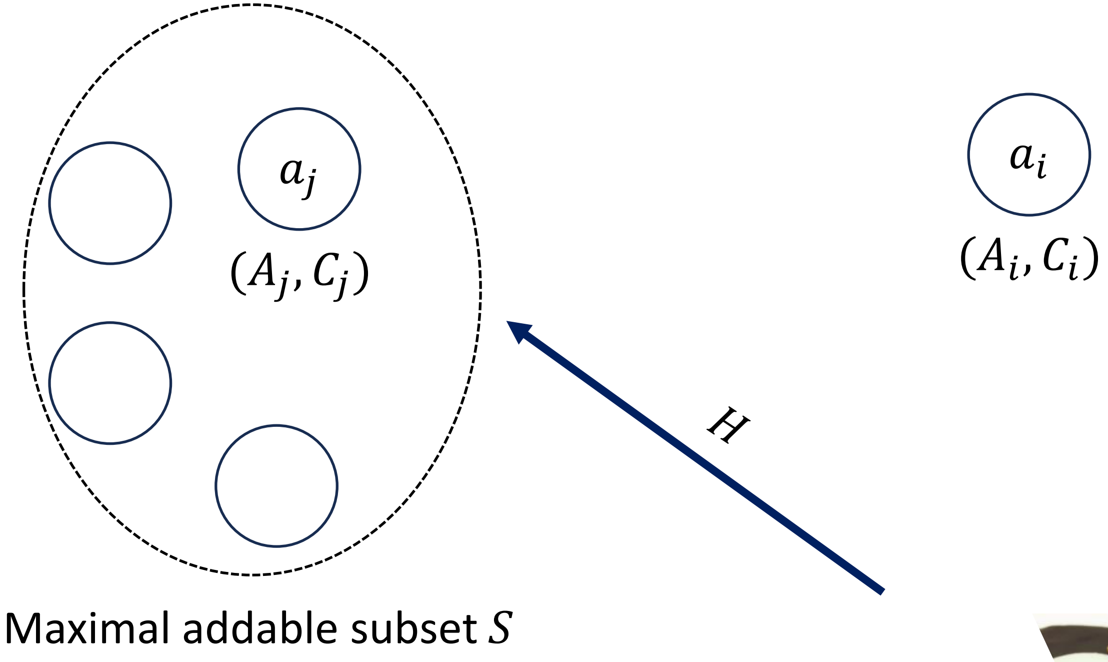
Uniqueness of maximal addable subset

If S and T are addable, then so is $S \cup T$:

S and T are subsets of sources in the envy graph
$\Rightarrow S \cup T$ is a subset of sources.

No equality edge from $\bar{S}$ to S, from $\bar{T}$ to T
$\Rightarrow$ no equality edge from $\overline{S \cup T}$ to $S \cup T$

How much cake?



$$\frac{v_i(H)}{|S|} \leq v_i(A_i, C_i) - v_i(A_j, C_j)$$

Progress?

Either the number of envy edges decreases

OR

The size of the maximal addable subset decreases

Progress?

Say the number of envy edge does not decrease.

⇒ The set of envy edges remains the same.

⇒ The set of sources in the envy graph remains the same.

Let T be the new maximal addable subset:

- $T \neq S$, as S now has an incoming equality edge.
- $S \cup T$ must also have been an addable subset to begin with!

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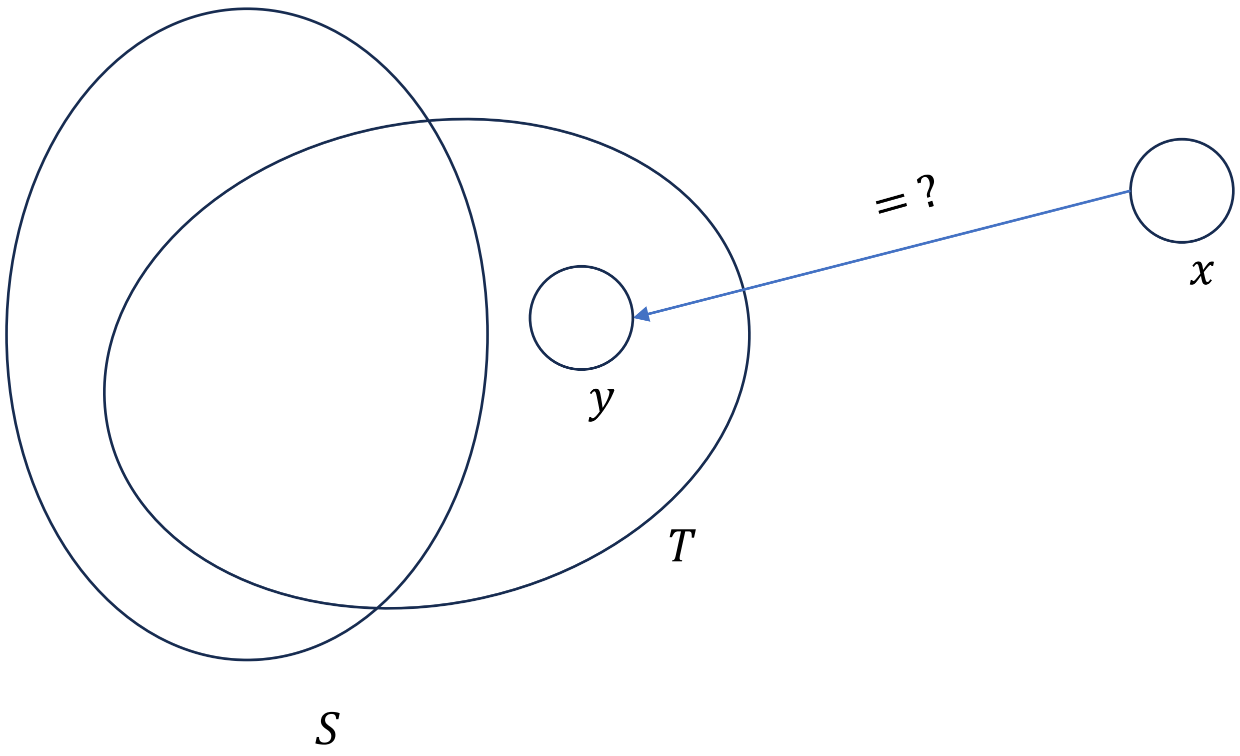

Progress?

Say the number of envy edge does not decrease.

⇒ The set of envy edges remains the same.

⇒ The set of sources in the envy graph remains the same.

Let T be the new maximal addable subset:

- $T \neq S$, as S now has an incoming equality edge.
- $S \cup T$ must also have been an addable subset to begin with!
- Maximality of S is contradicted!

Generalization to chores

Divisible / Indivisible	Cake	Bad Cake
Goods	✓	✓
Chores	? • Identical rankings ✓ • $m \leq n + 1$ ✓	✓

[Bhaskar, Sricharan and Vaish, APPROX 2021]