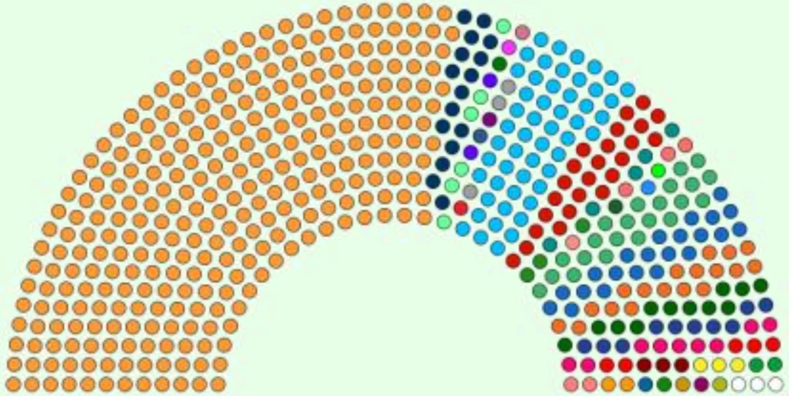
COL749: Computational Social Choice

Lecture 14

Fairness through Randomness

Feb 20, 2025 | Rohit Vaish

INDIVISIBLE



The Model

	A	B	C	D	E
	4	1	2	2	2
	1	0	5	1	1
	1	1	5	1	1

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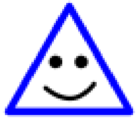
Additive valuations

 {B D E} =  {B} +  {D} +  {E}

= 0+1+1 = 2

Envy-Freeness [Gamow and Stern, 1958; Foley, 1967]

Each agent prefers its own bundle over that of any other agent.

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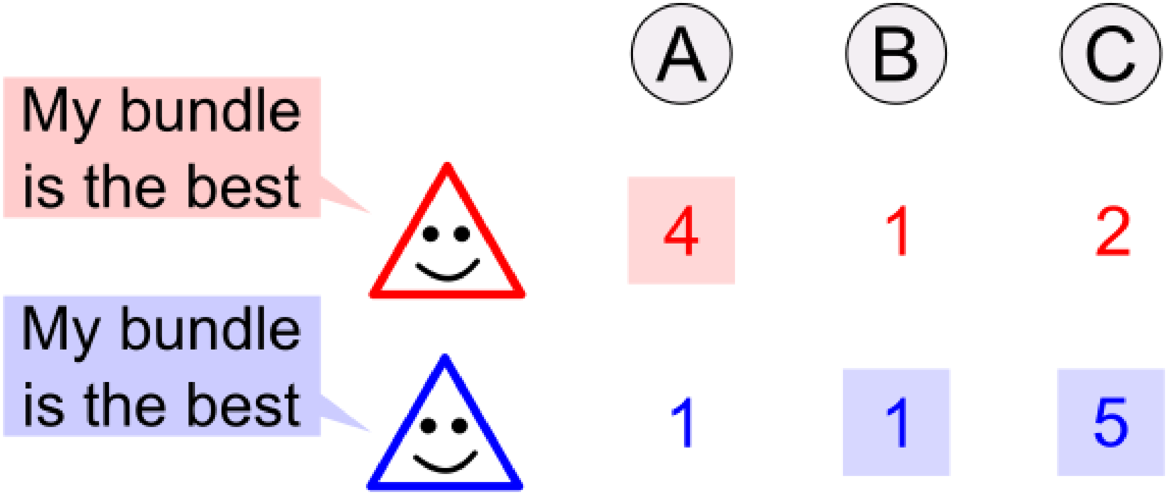

Allocation $A = (A_1, A_2, ..., A_n)$ is EF if for every pair of agents i, k, we have $v_i(A_i) \geq v_i(A_k)$.

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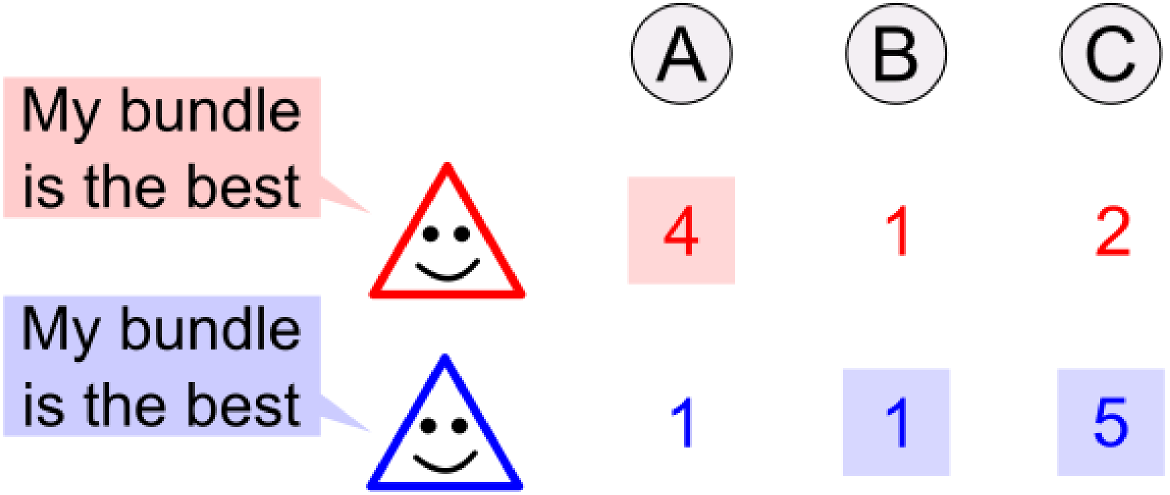

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- ☹ Not guaranteed to exist (two agents, one good)
- ☹ Checking whether an EF allocation exists is NP-complete

Envy-Freeness Up To One Good [Budish, 2011]

Envy can be eliminated by removing some good in the envied bundle.

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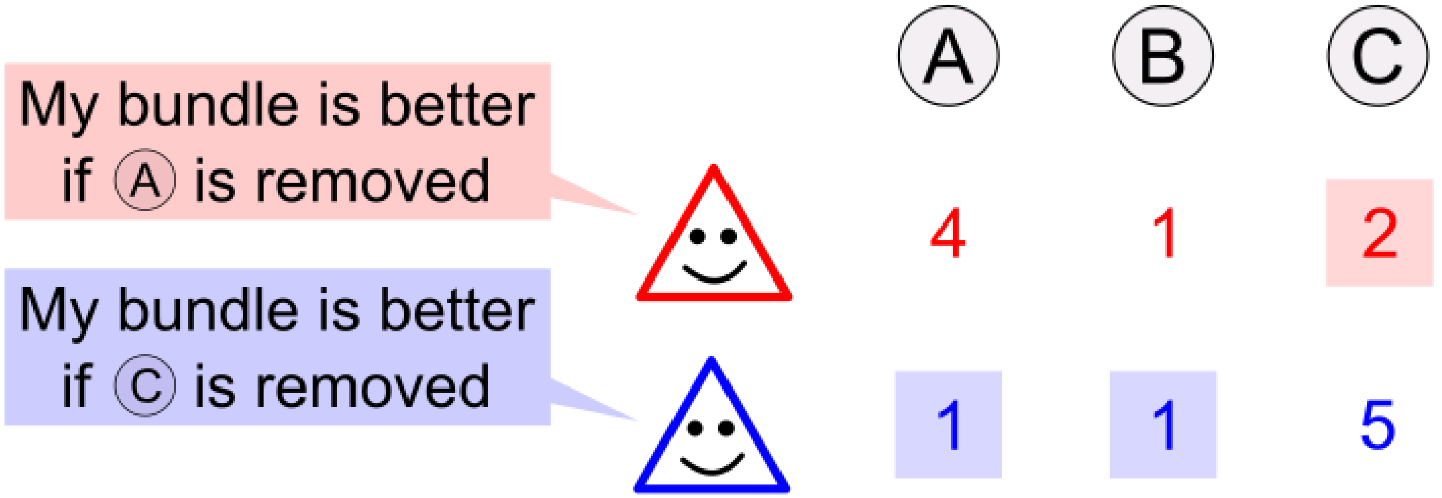

Allocation $A = (A_1, \ldots, A_n)$ is EF1 if for every pair of agents i, k, there exists a good $j \in A_k$ such that $v_i(A_i) \geq v_i(A_k \setminus \{j\})$.

Guaranteed to exist and efficiently computable

Round-robin algorithm

- Fix an ordering of the agents, say a_1, a_2, a_3, ..., a_n.
- Agents take turns according to the ordering (a_1, a_2, ..., a_n, a_1, a_2, ..., a_n,...) to pick their favorite item from the set of remaining items.

For additive valuations, the allocation computed by round-robin algorithm satisfies EF1.

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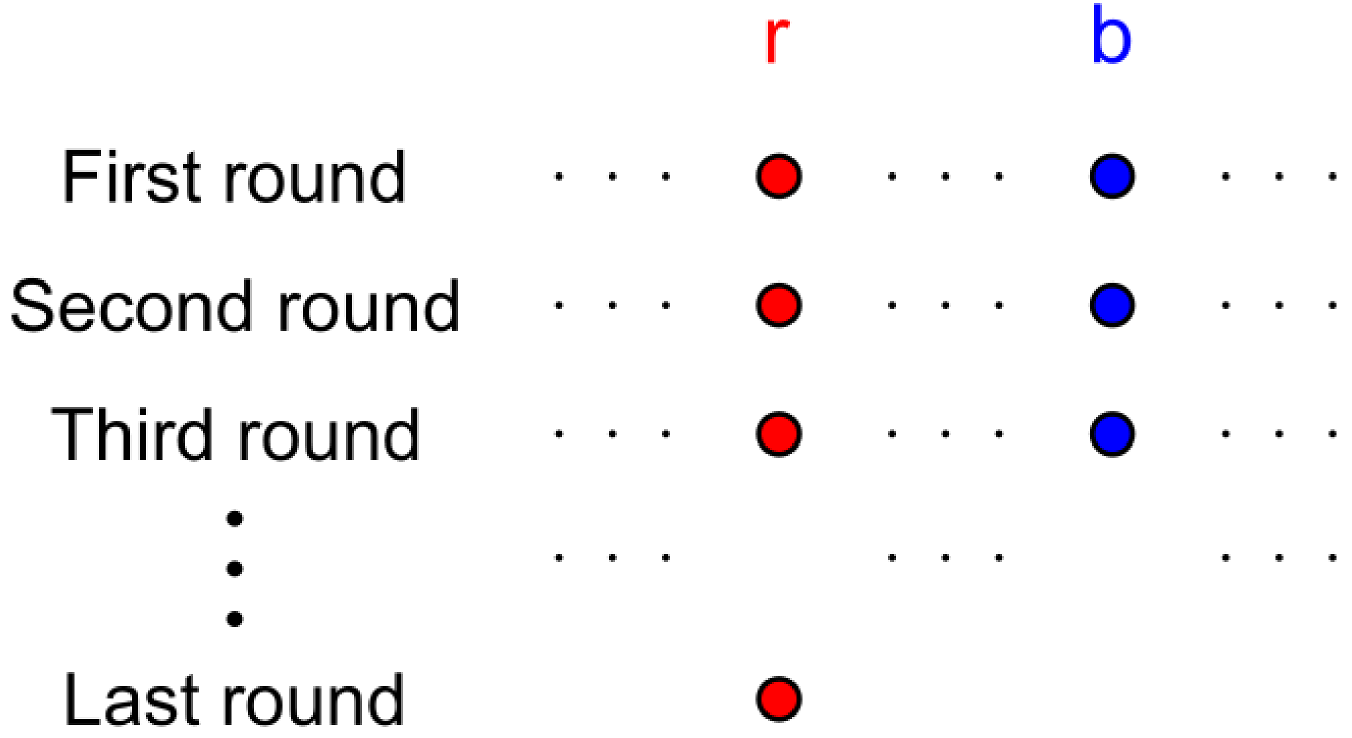
Fix a pair of agents (r,b). Analyze envy of r towards b.



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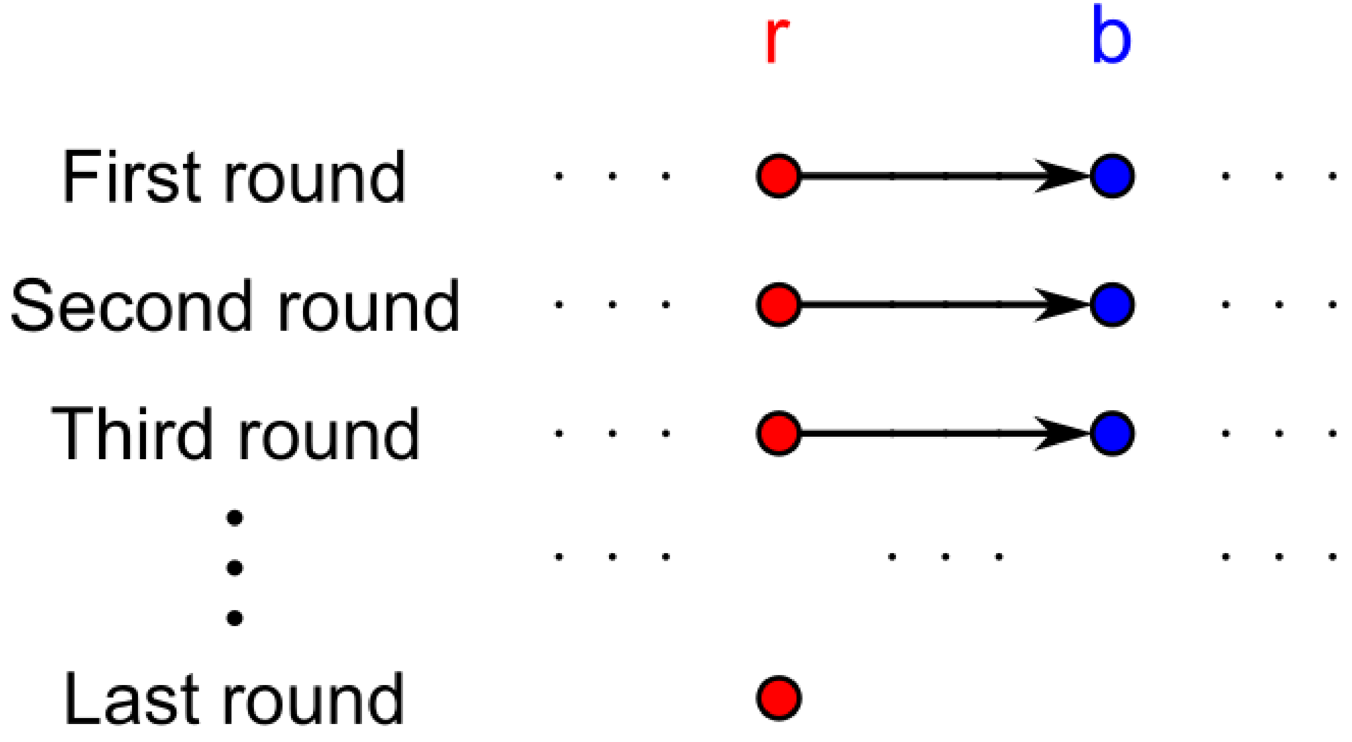

If r precedes b: Then, by additivity, $v_r(A_r) \geq v_r(A_b)$.

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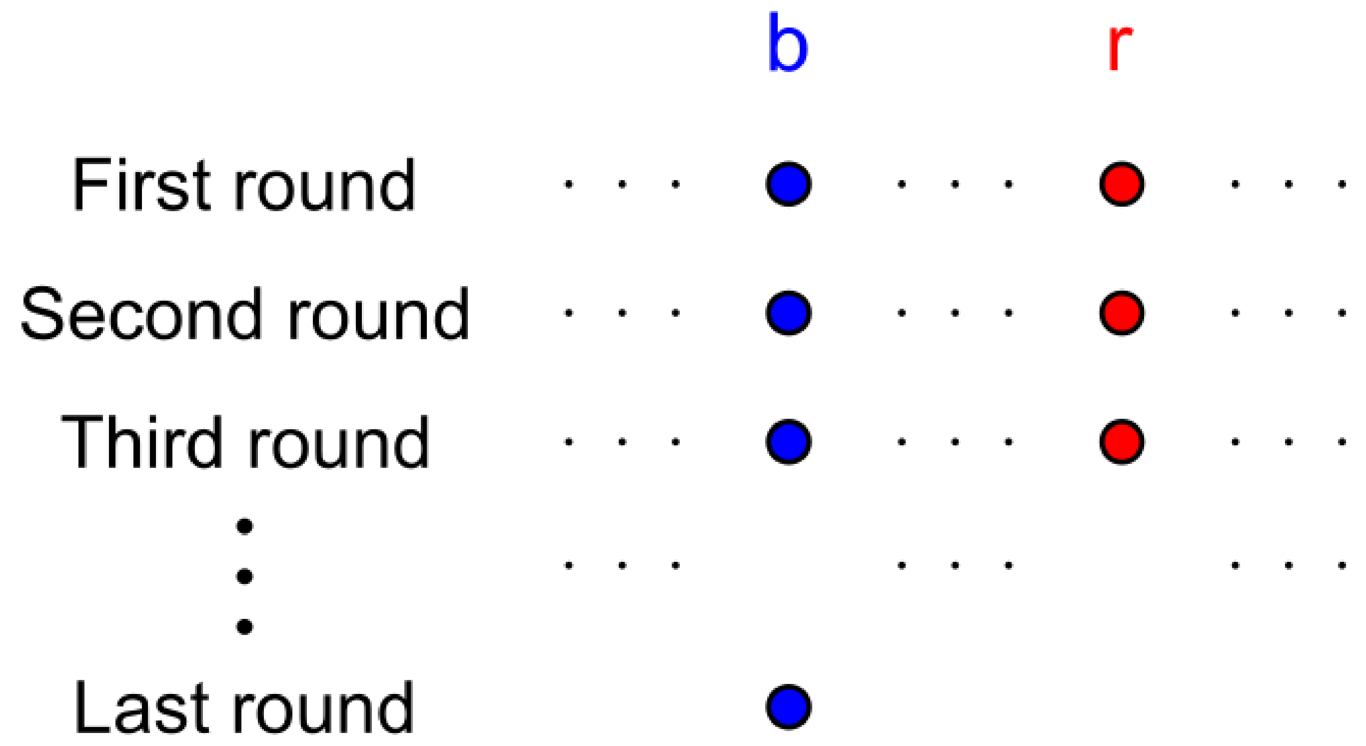

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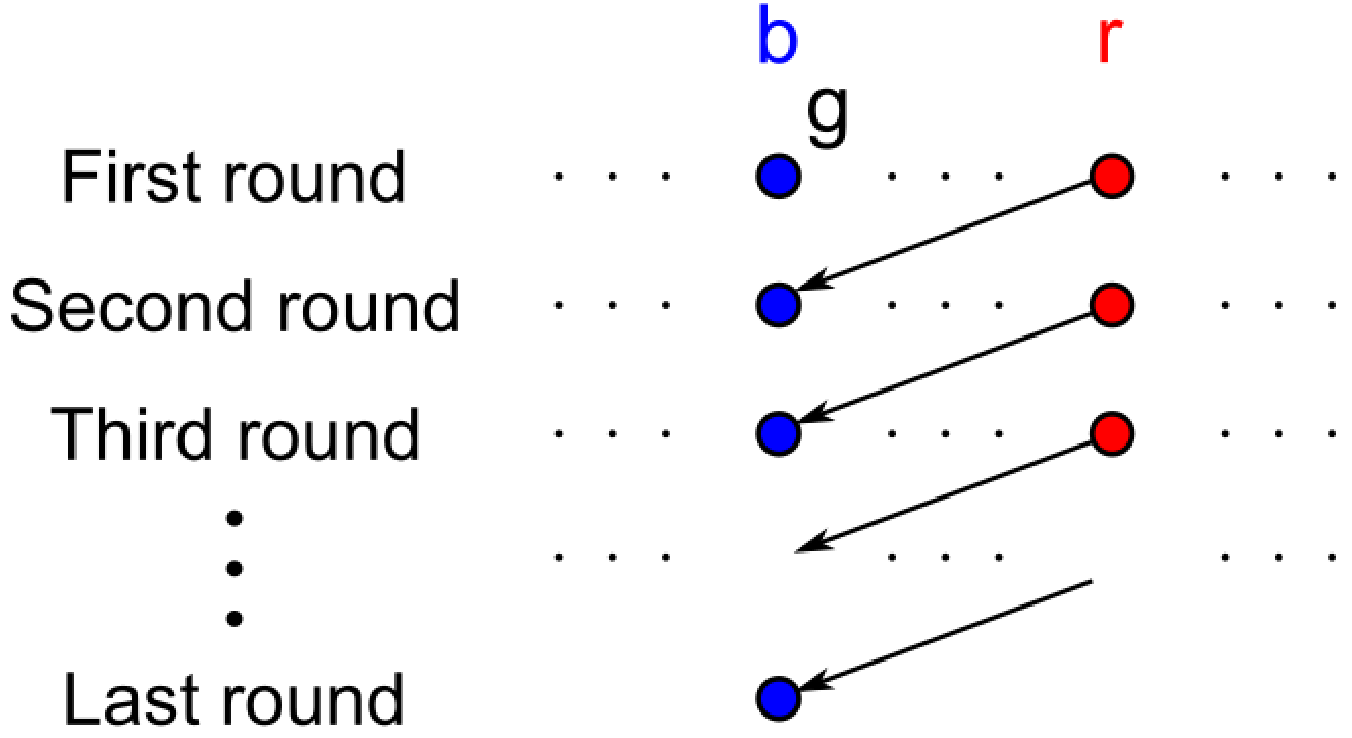
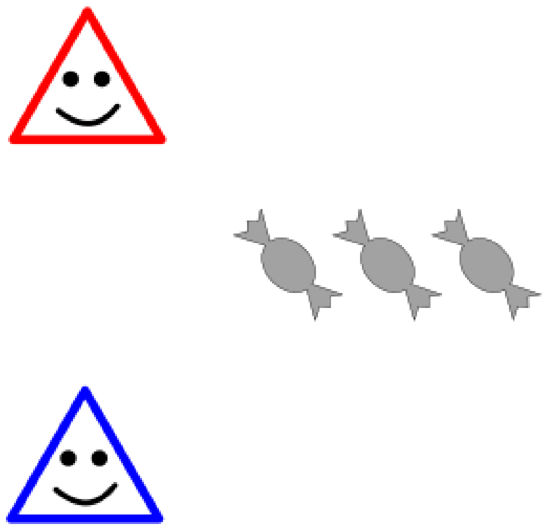


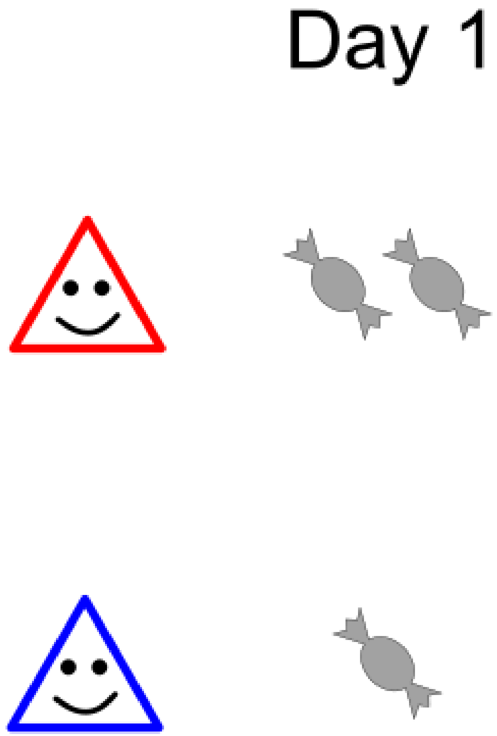
If b precedes r: Again, by additivity, $v_r(A_r) \geq v_r(A_b \setminus \{g\})$.

WHEN APPROXIMATE ENVY-FREENESS

SIMPLY ISN'T ENOUGH

Day 1

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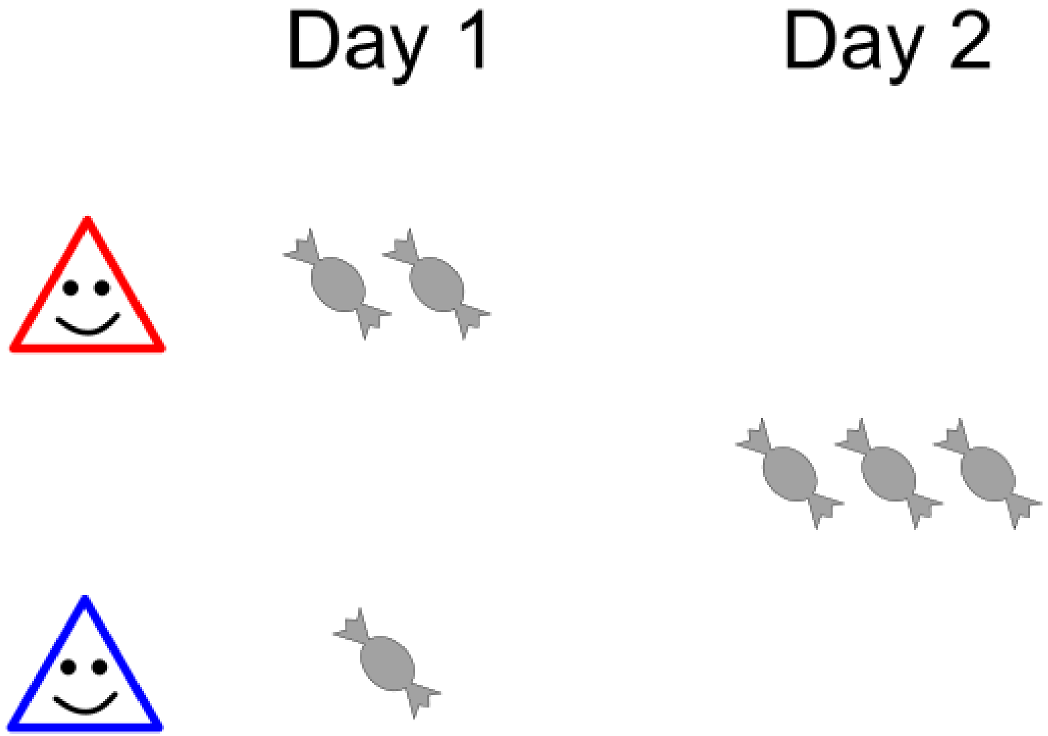
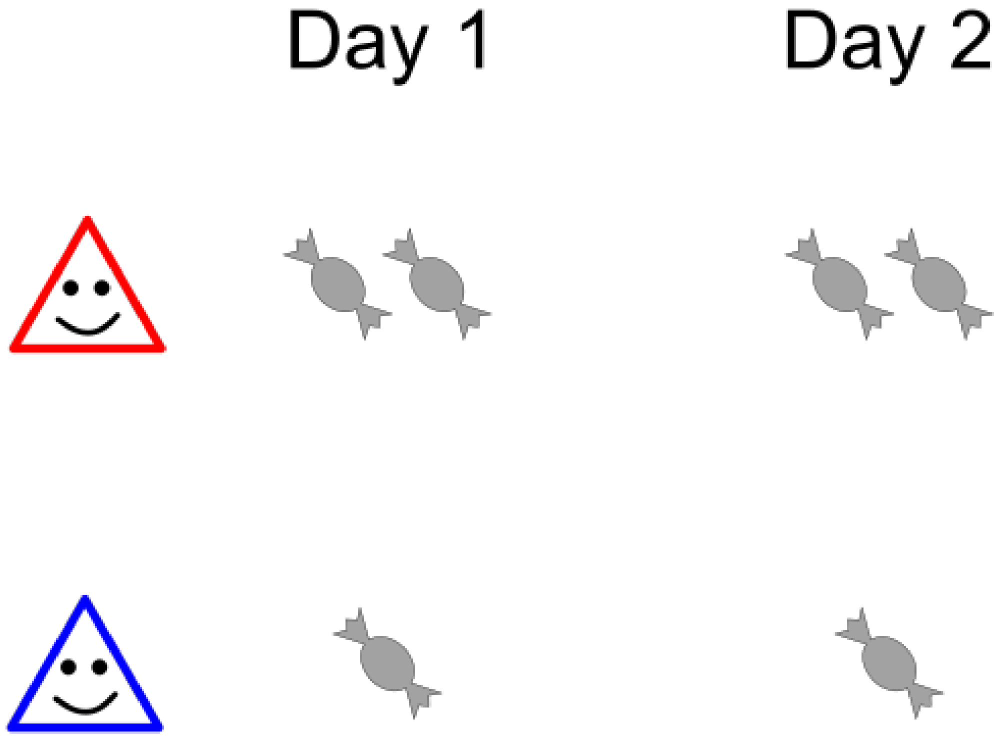

Day 1
Day 2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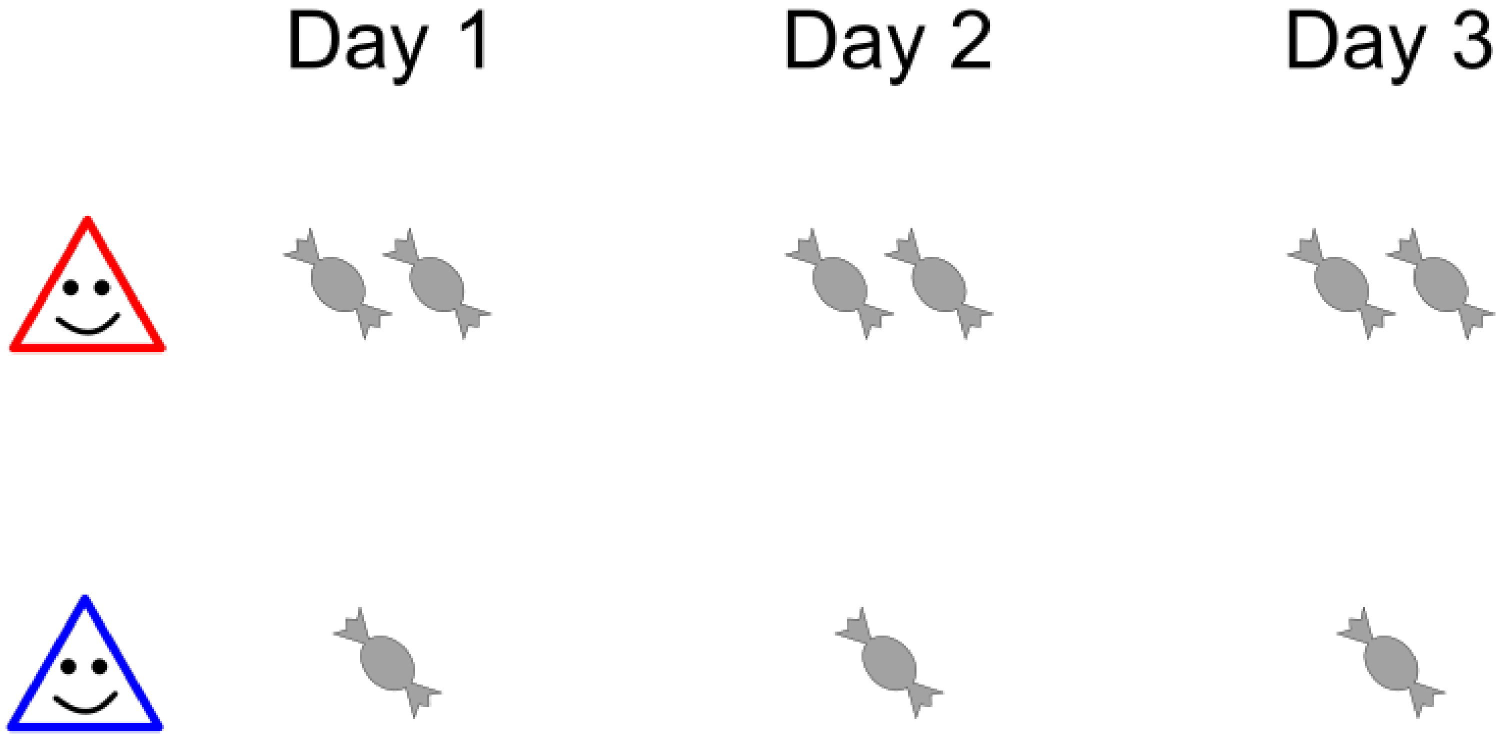
Day 1
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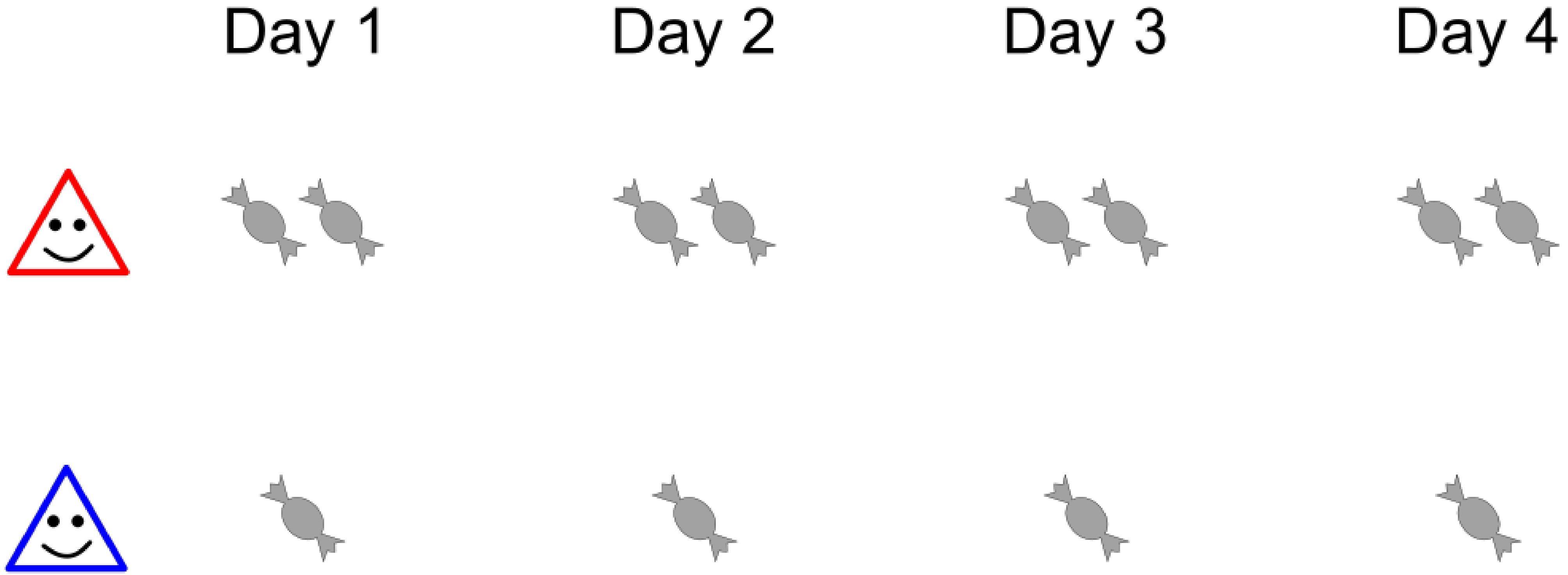
Day 3



Deterministic algorithms can systematically disadvantage certain agents.

A natural workaround: Randomization

A natural workaround: Randomization

Pick a uniform distribution over all round-robin orderings

A natural workaround: Randomization

Pick a uniform distribution over all round-robin orderings

☹ Can still be unfair

Uniform round-robin is unfair

[Bogomolnaia and Moulin, 2001]

	A	B	C
	7	6	0
	2	0	1
	1	2	1

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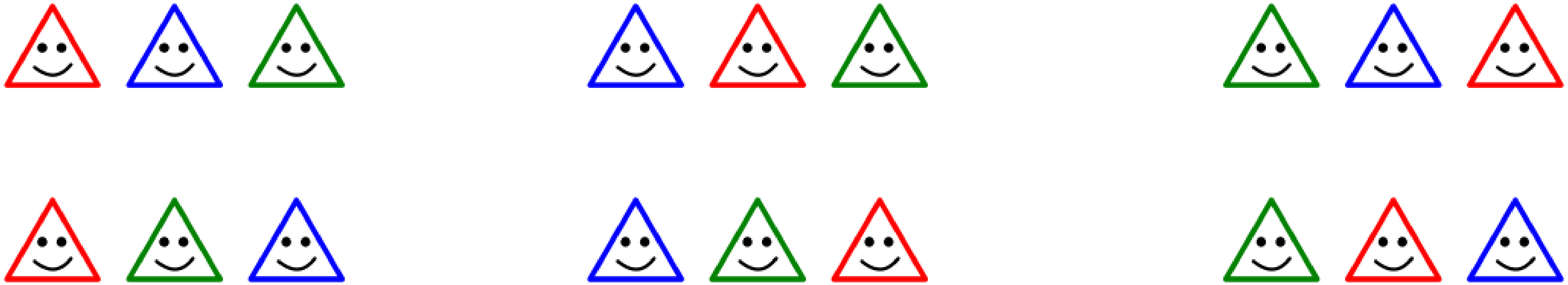

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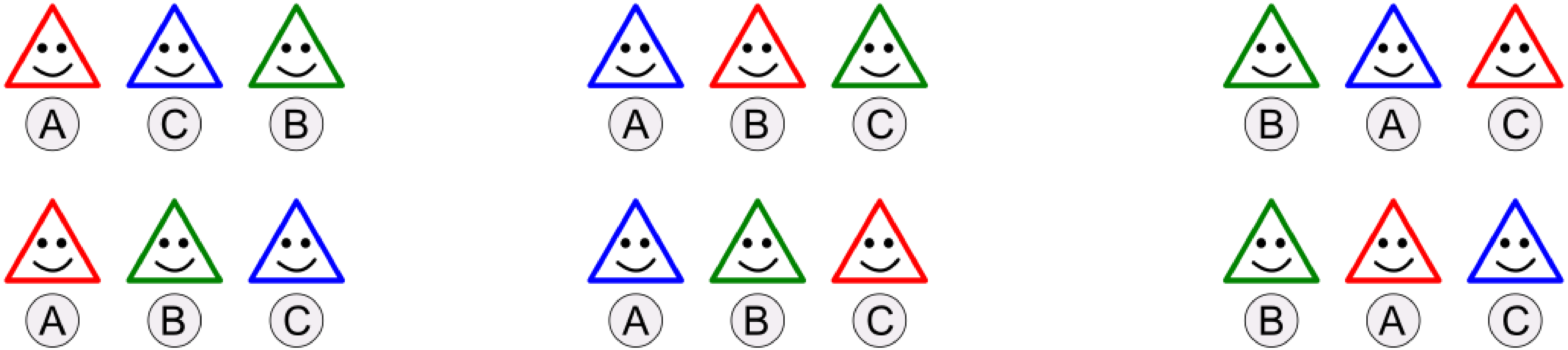

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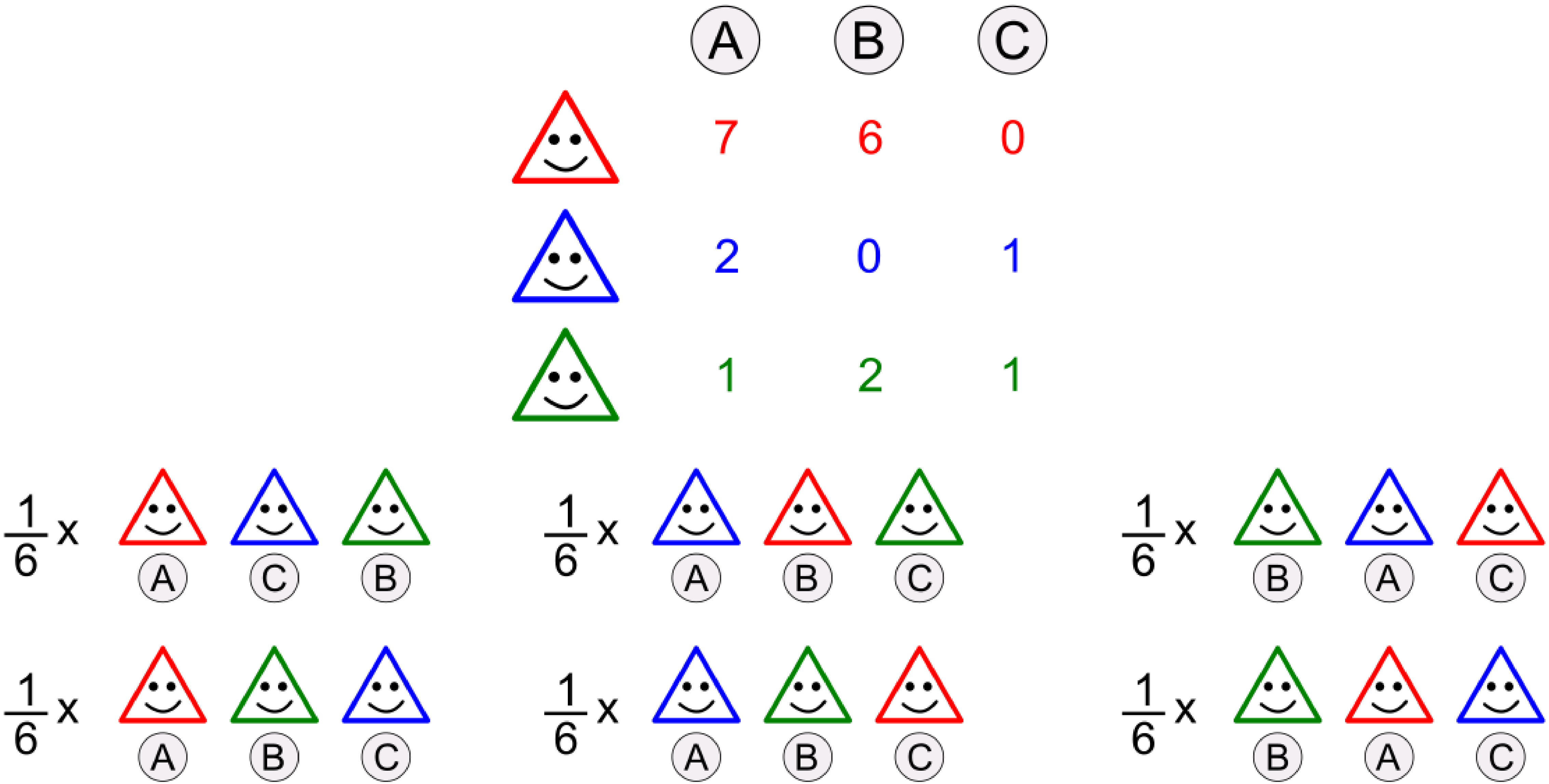

Uniform round-robin is unfair

[Bogomolnaia and Moulin, 2001]

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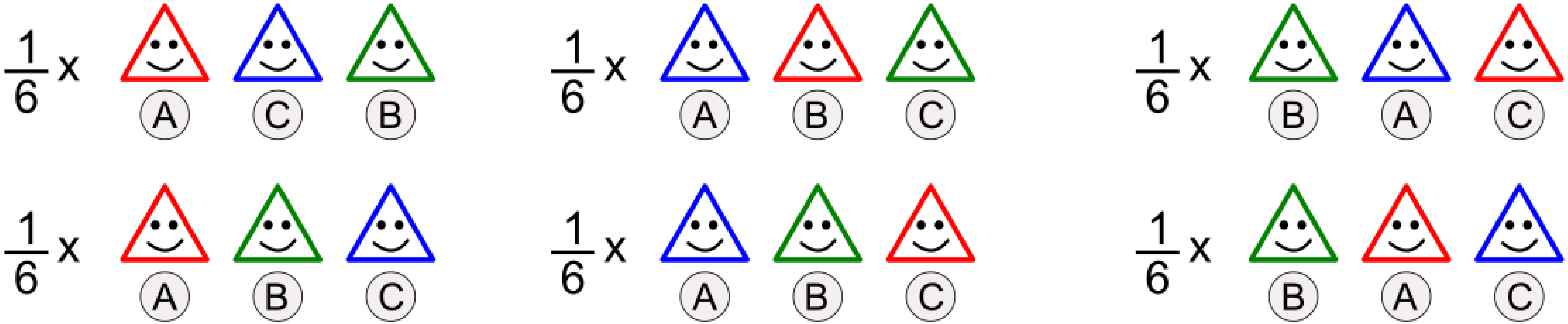

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's expected value for its own bundle = 7*1/2 + 6*1/6 + 0*1/3 = 4.5

's expected value for 's bundle

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's expected value for its own bundle = 7*1/2 + 6*1/6 + 0*1/3 = 4.5

's expected value for 's bundle = 7*0 + 6*5/6 + 0*1/6 = 5

Uniform round-robin is unfair

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envies in expectation

's expected value for its own bundle = 7*1/2 + 6*1/6 + 0*1/3 = 4.5

's expected value for 's bundle = 7*0 + 6*5/6 + 0*1/6 = 5

Fairness of Randomized Allocations

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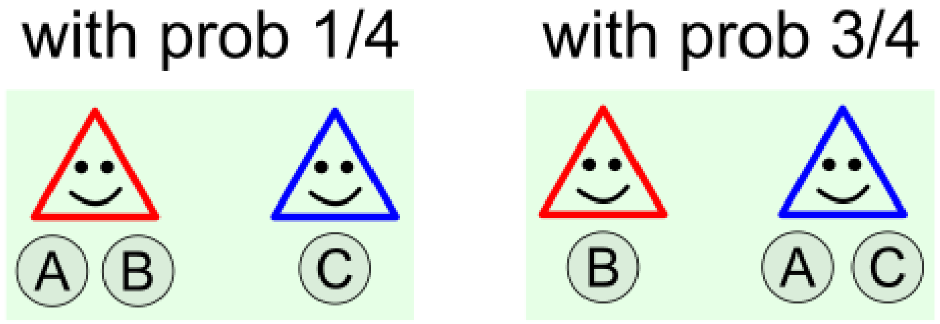
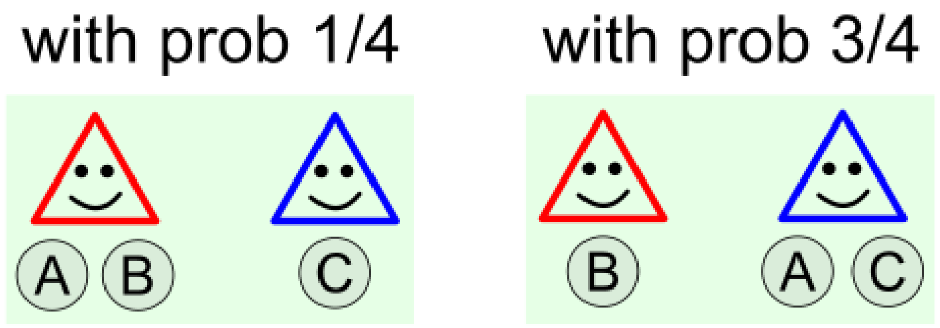


probability distribution over deterministic allocations

Fairness of Randomized Allocations



probability distribution over deterministic allocations

ex-ante fairness | no agent envies another in expectation

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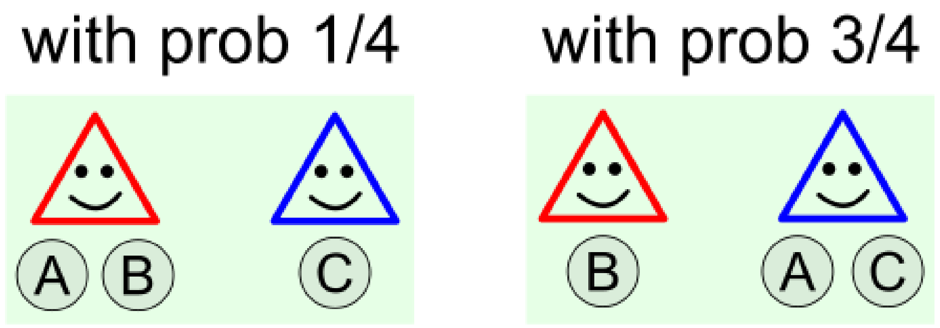
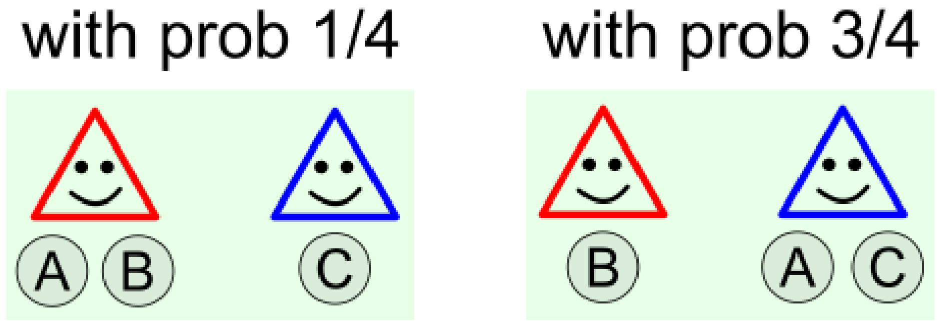


probability distribution over deterministic allocations

ex-ante fairness | no agent envies another in expectation

- Uniform round-robin fails ex-ante fairness.
- "Bundle everything together and assign uniformly randomly" is ex-ante fair.

Fairness of Randomized Allocations



probability distribution over deterministic allocations

ex-ante fairness	no agent envies another in expectation
ex-post fairness	each deterministic allocation in the support is EF1

Does there always exist a randomized allocation that gives "best of both worlds", i.e., is ex-ante and ex-post fair?

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[Aziz, Freeman, Shah, Vaish, *Operations Research* 2023]

For additive valuations, there always exists a randomized allocation that is ex-ante envy-free and ex-post EF1. Such an allocation can be constructed in polynomial time.

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[Aziz, Freeman, Shah, Vaish, *Operations Research* 2023]

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Proof by "eating".

Probabilistic Serial/ "Eating" Algorithm

[Bogomolnaia and Moulin, 2001]

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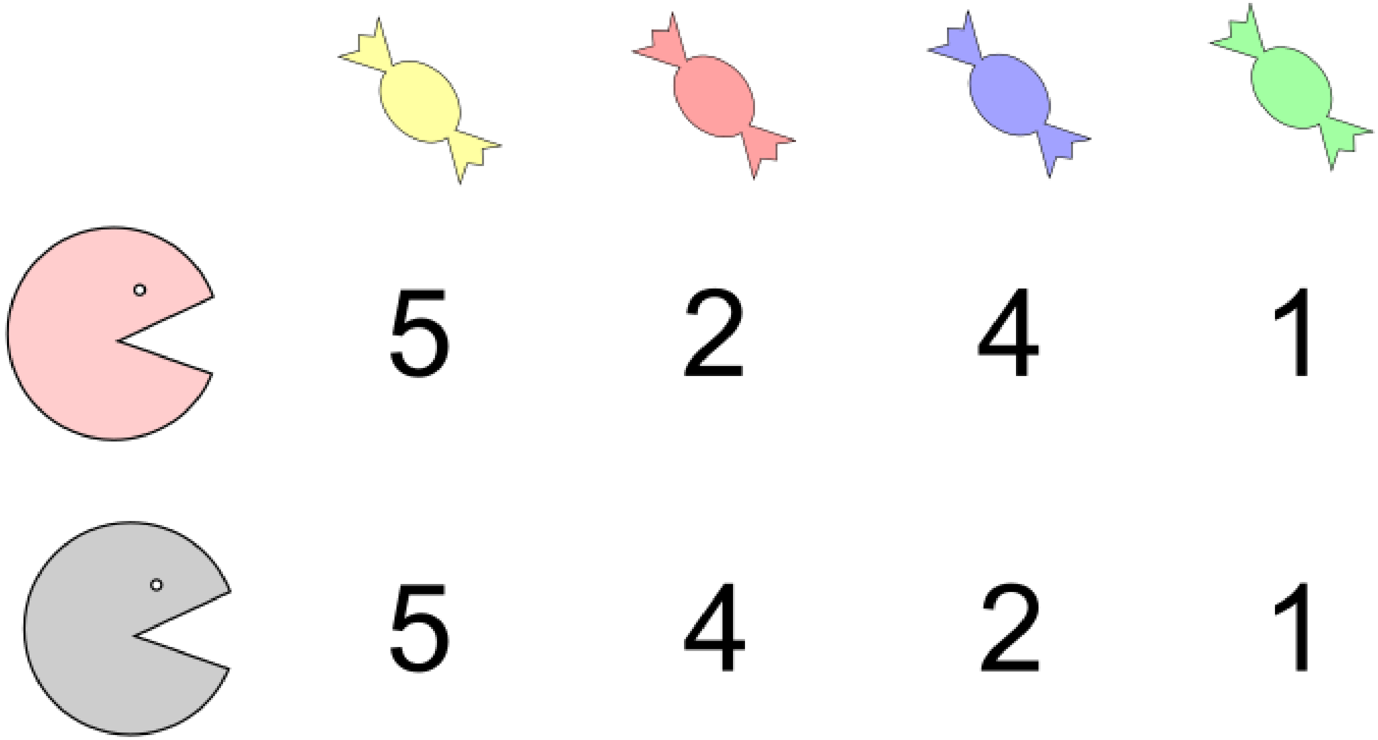

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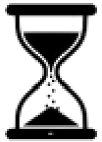
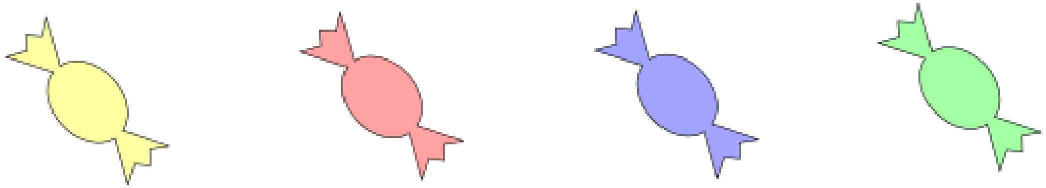
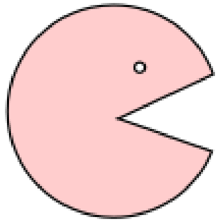
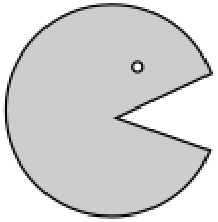


t=0 t=0.5 t=1 t=1.5 t=2



	5	2	4	1
	5	4	2	1

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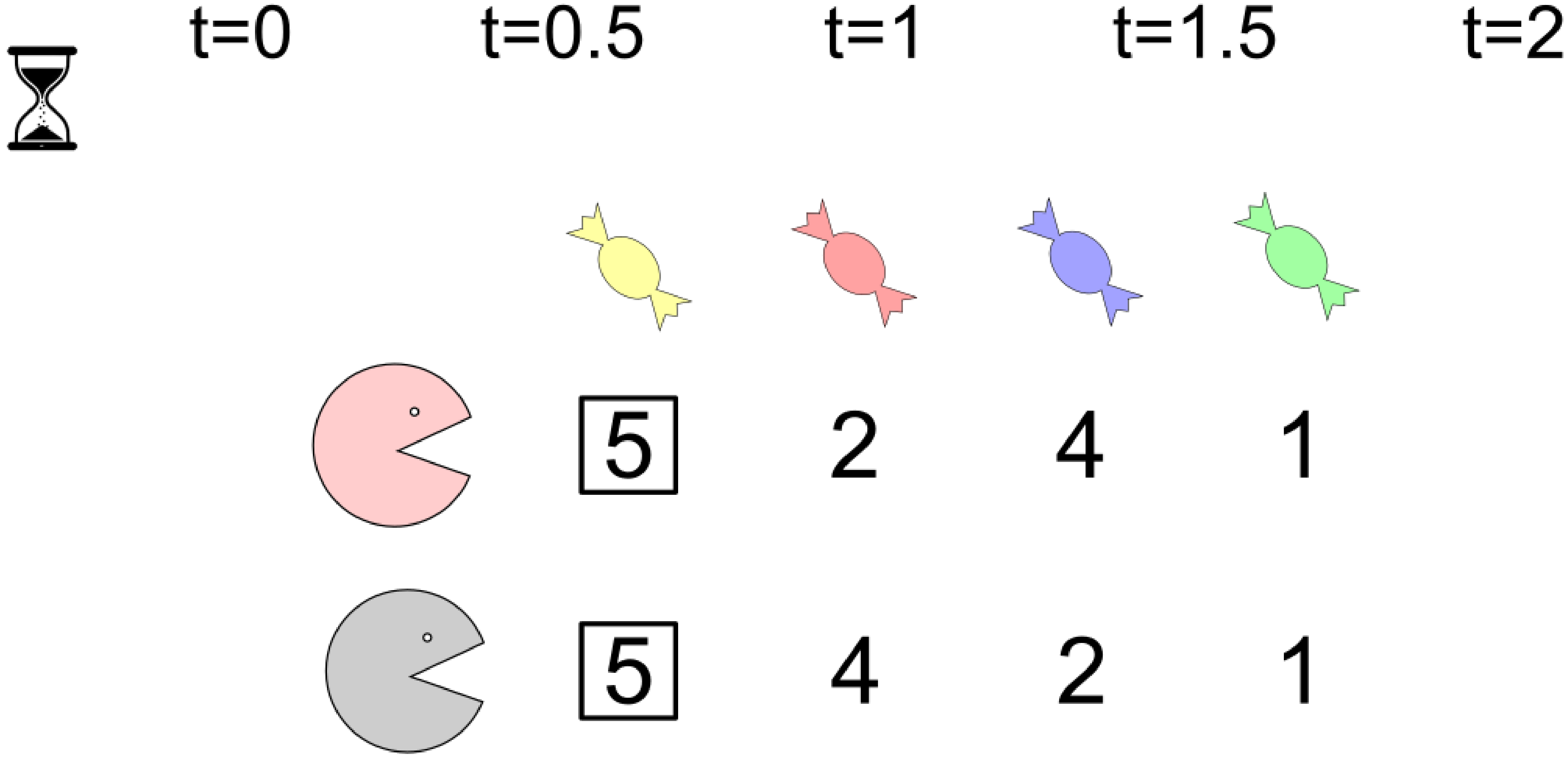

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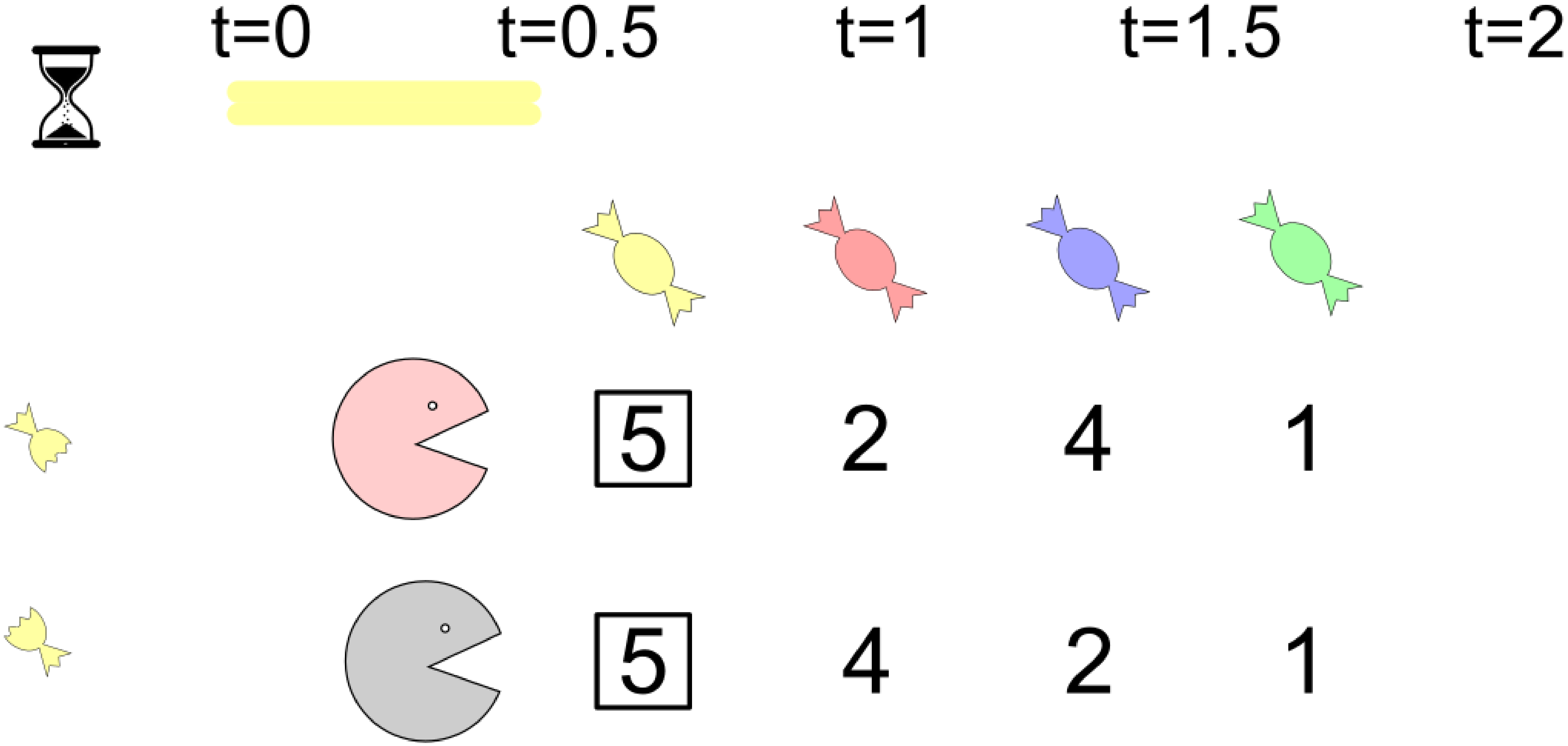

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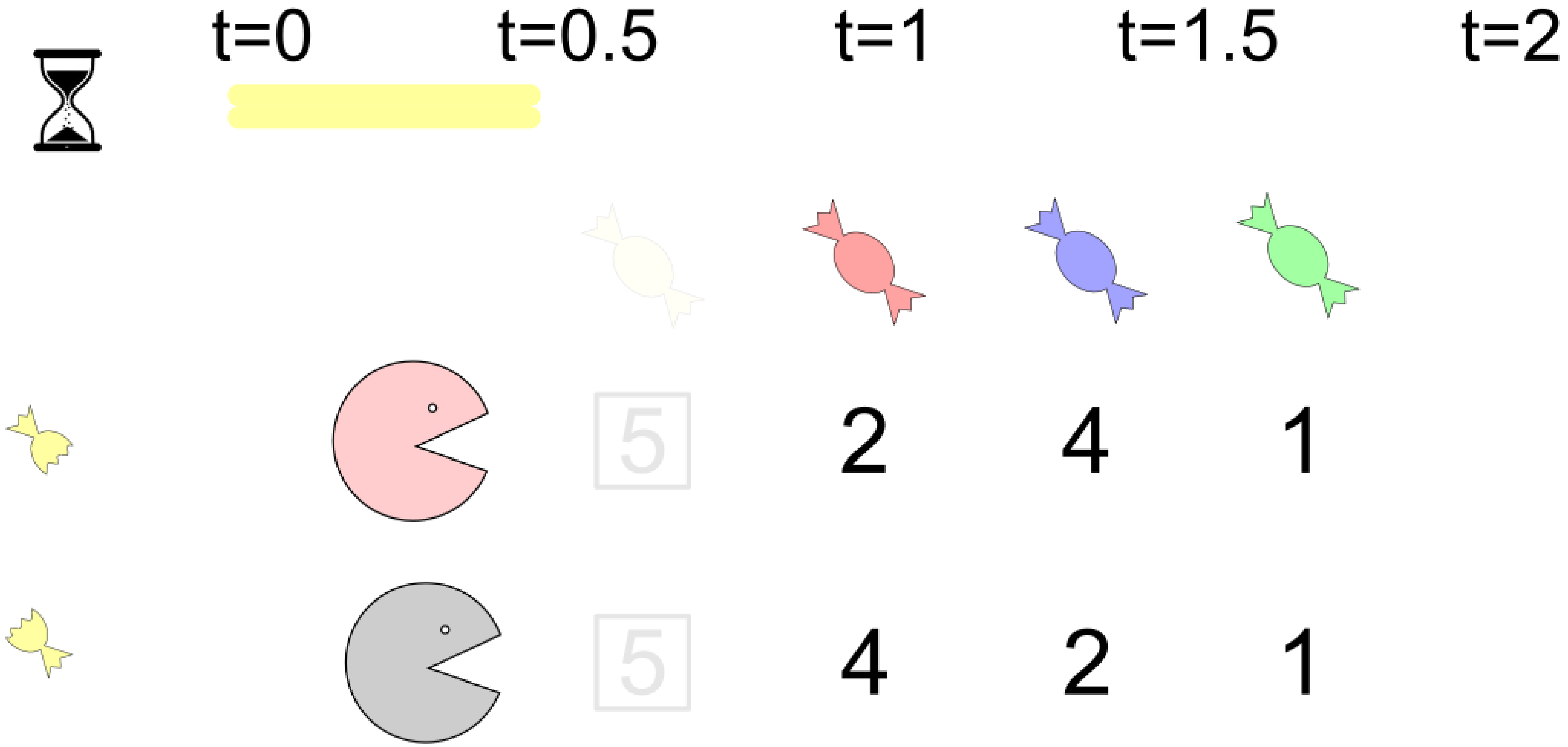

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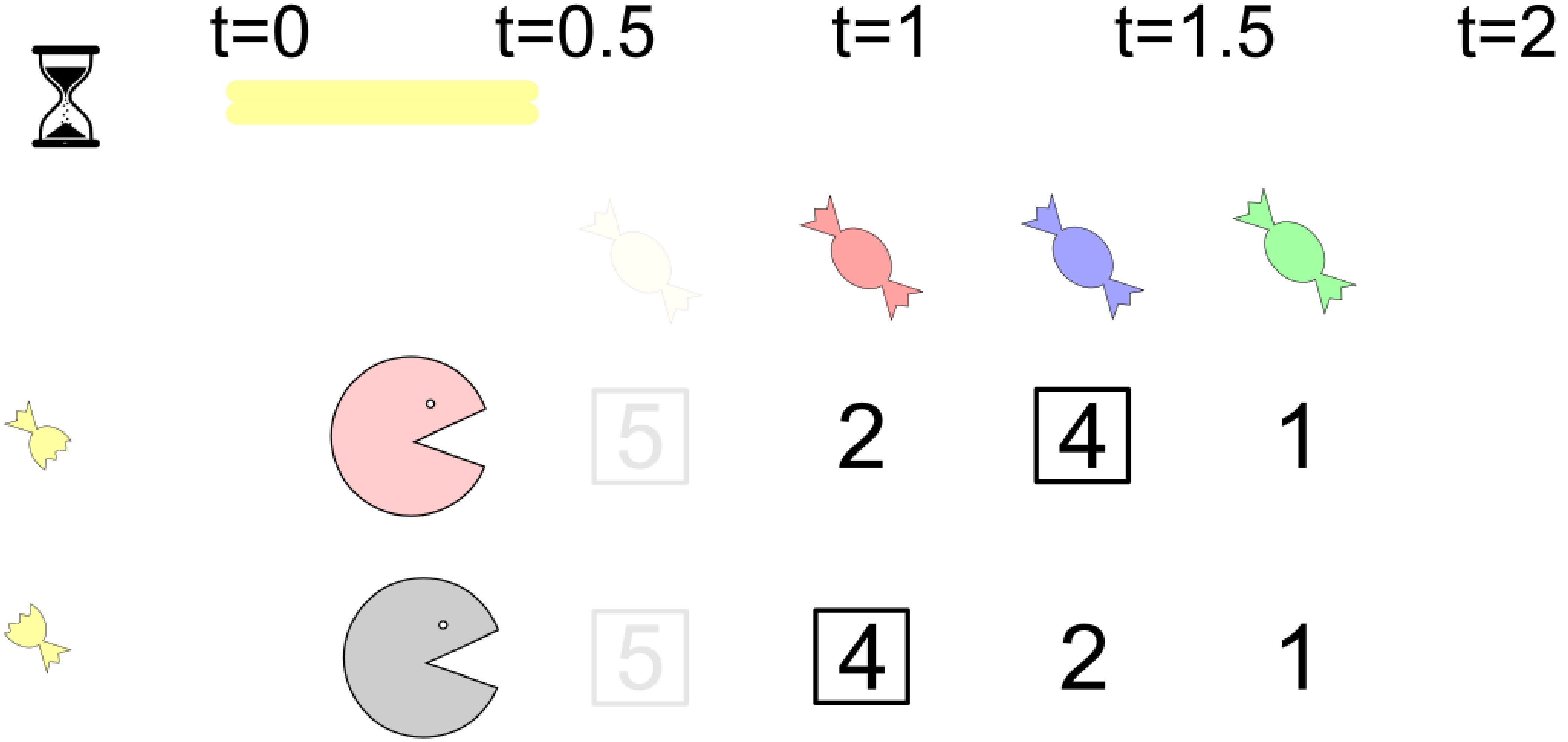

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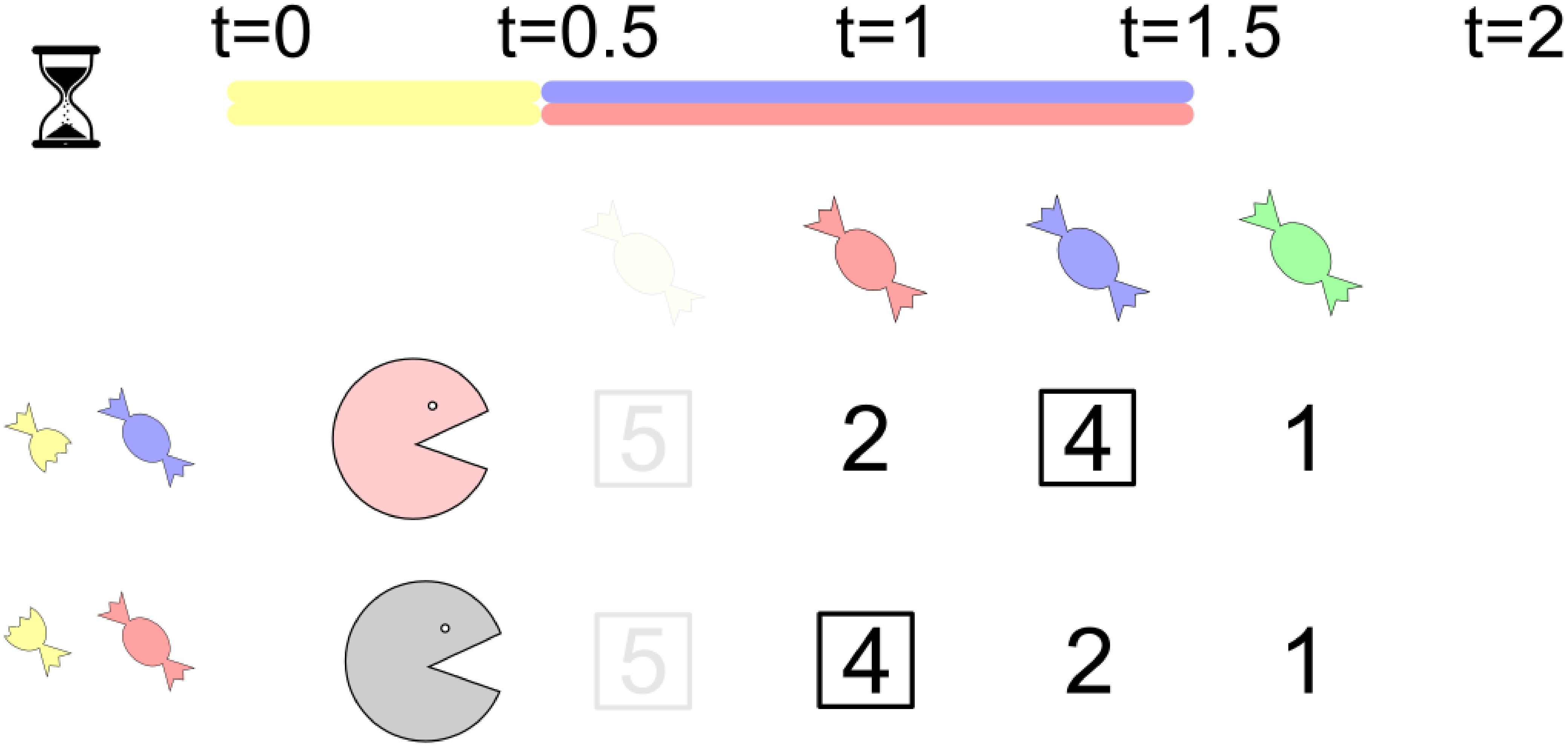

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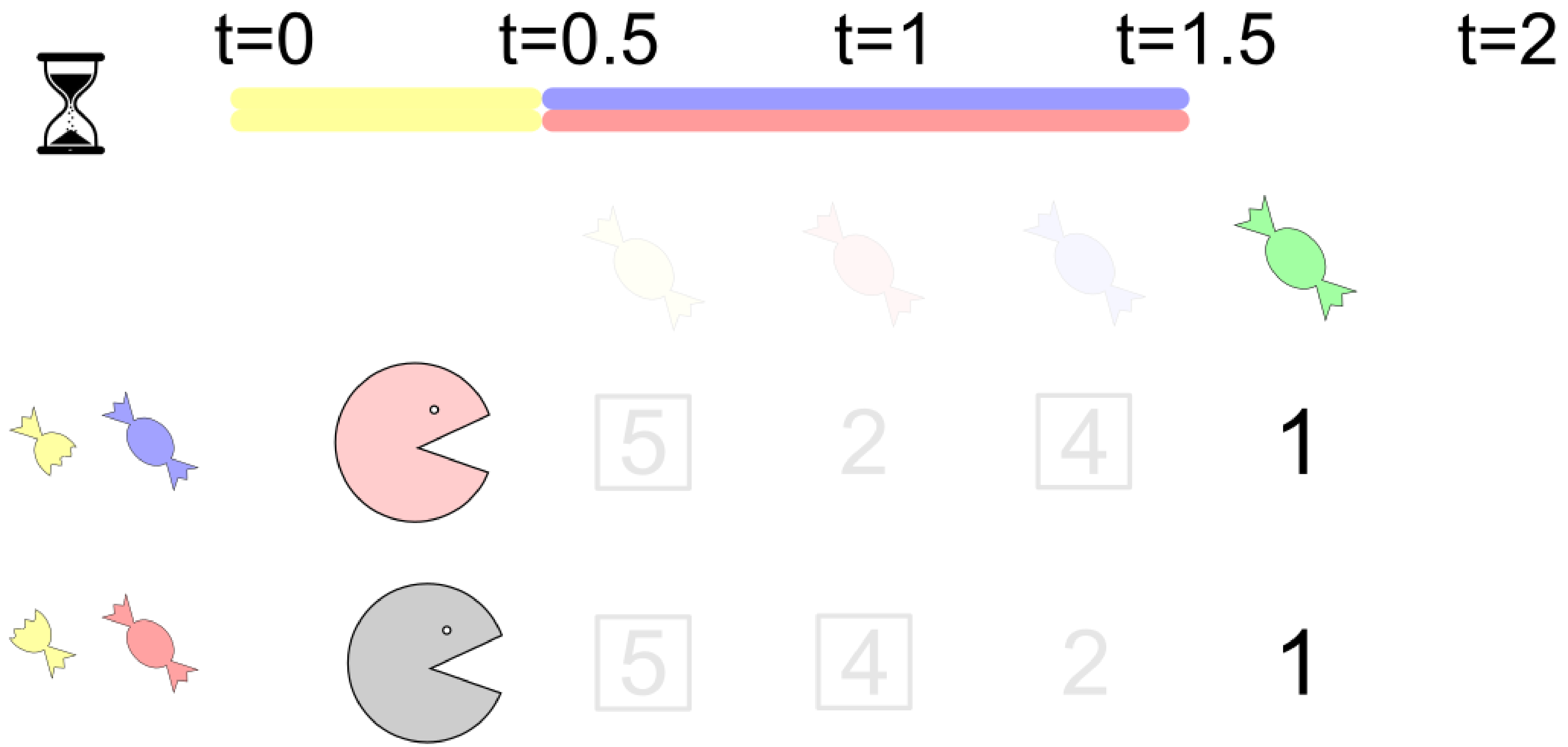

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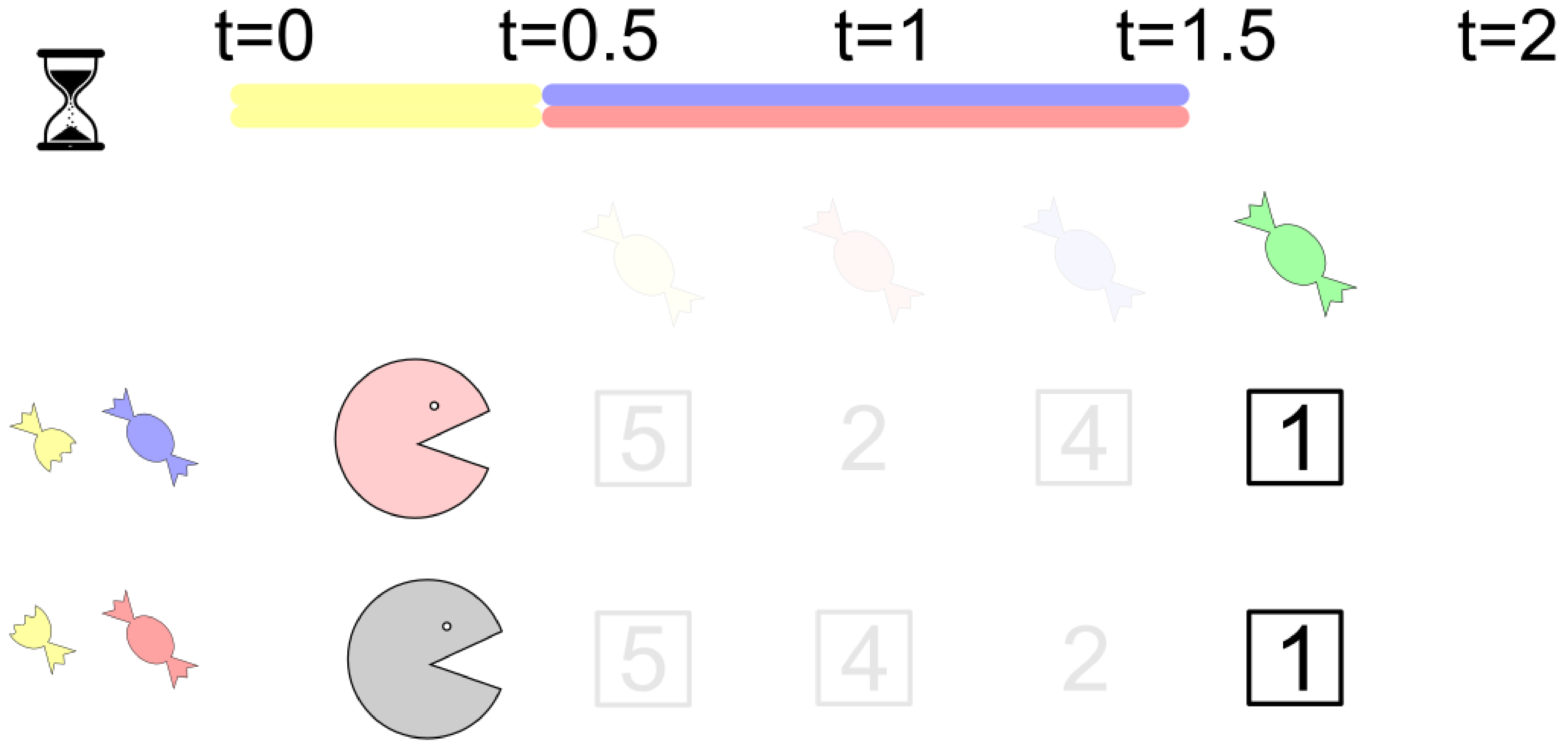

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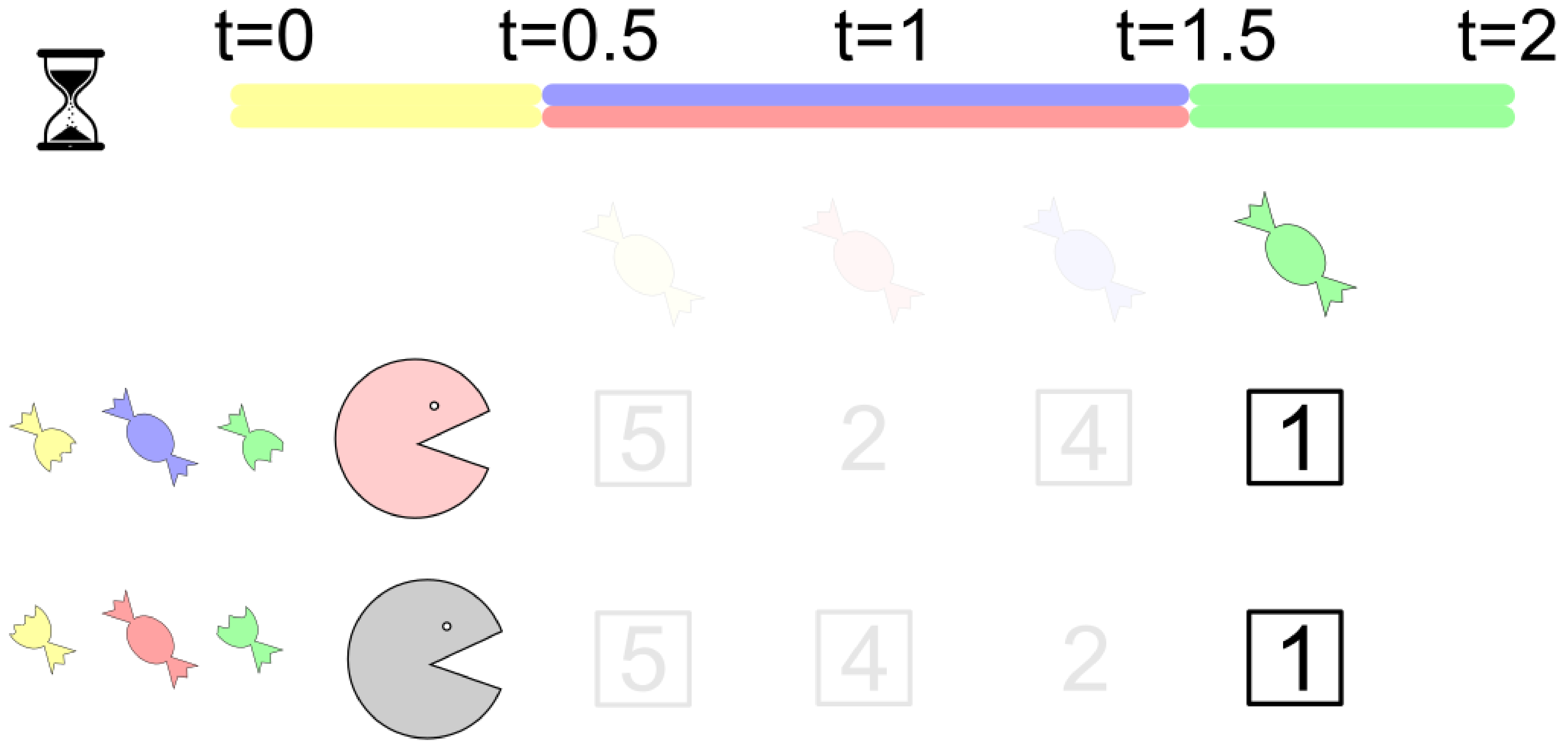

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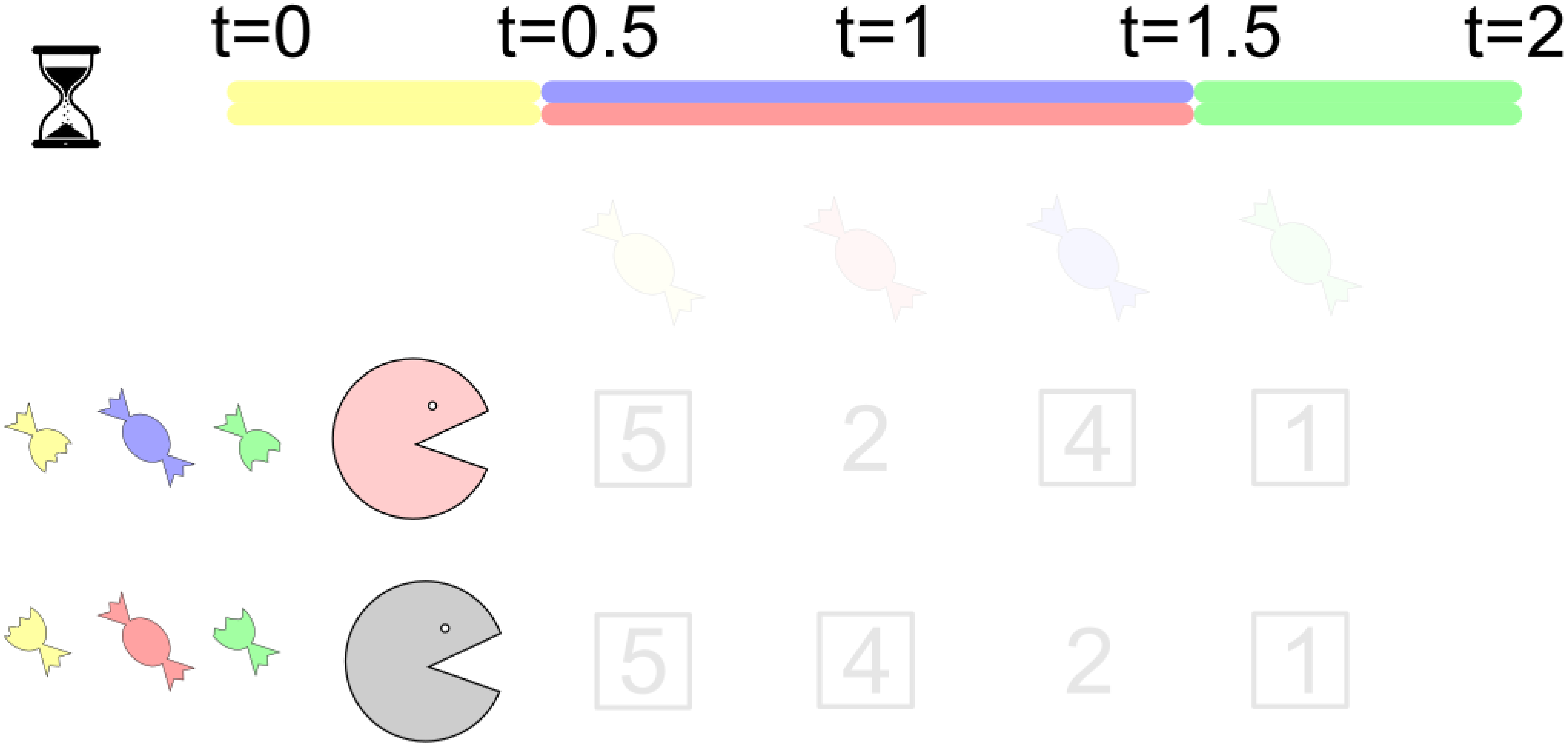

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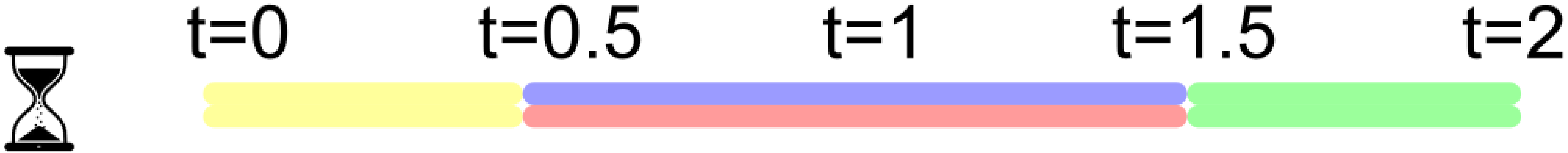

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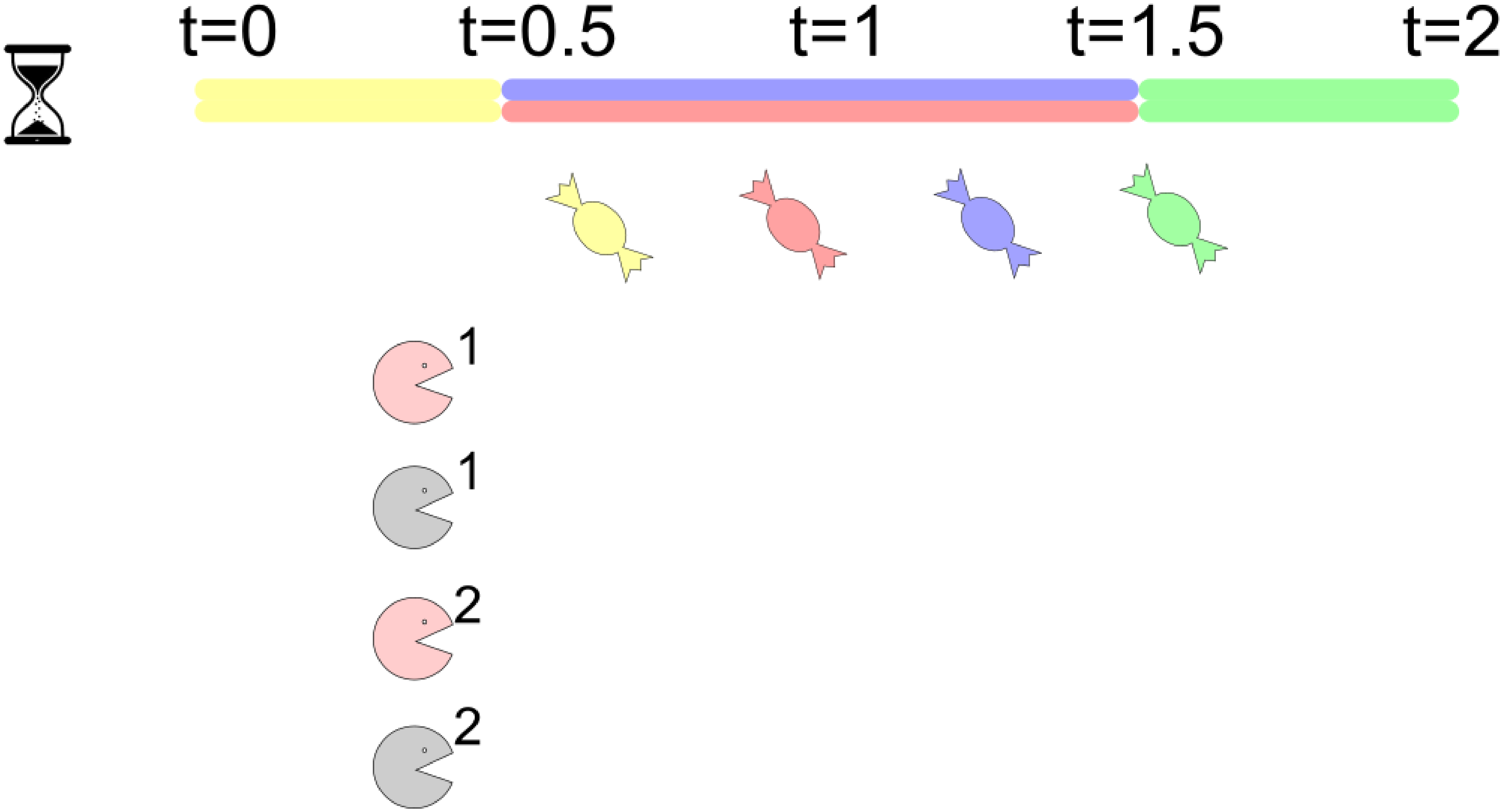

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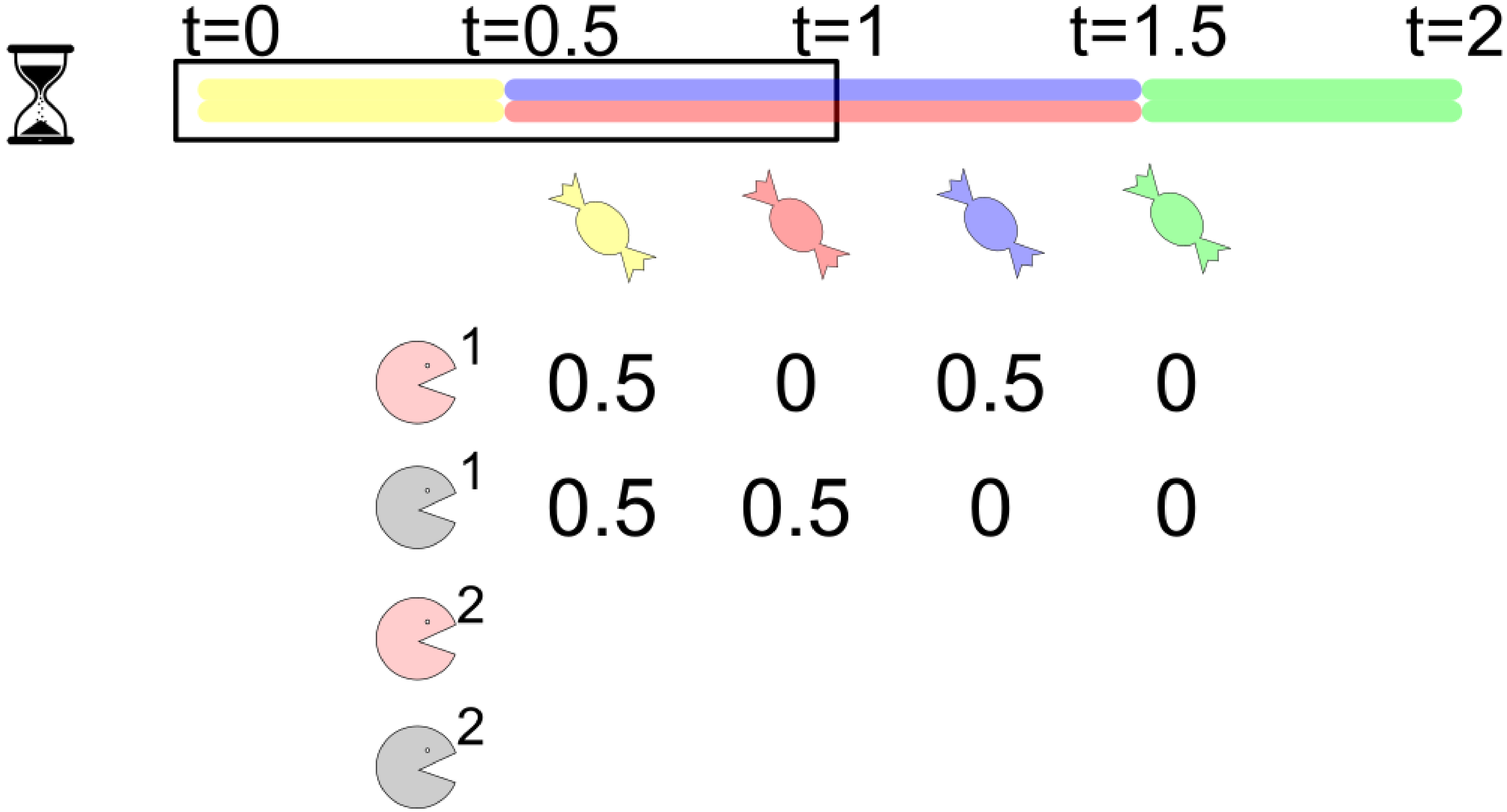

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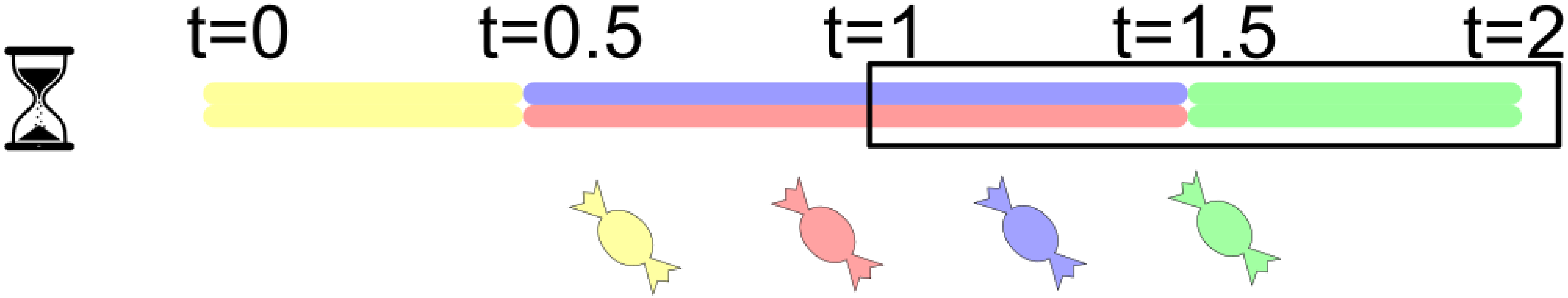

1	0.5	0	0.5	0
1	0.5	0.5	0	0
2	0	0	0.5	0.5
2	0	0.5	0	0.5

Probabilistic Serial/ "Eating" Algorithm

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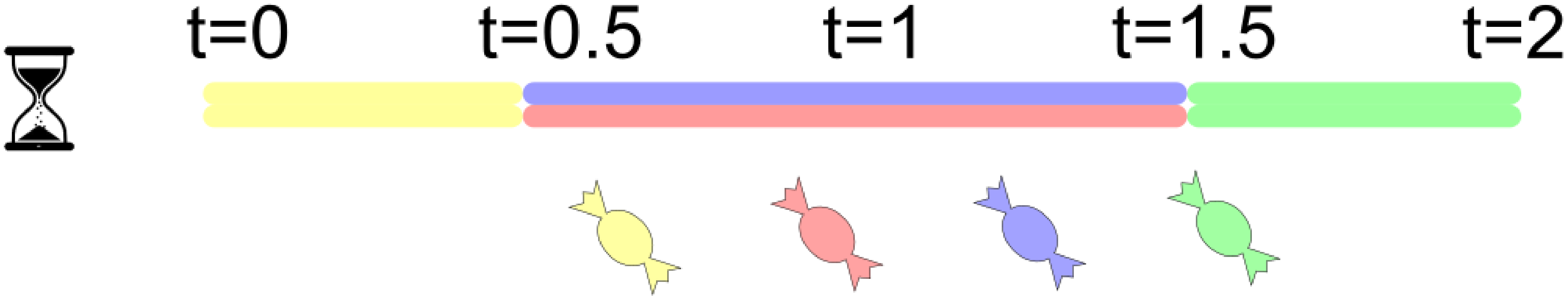

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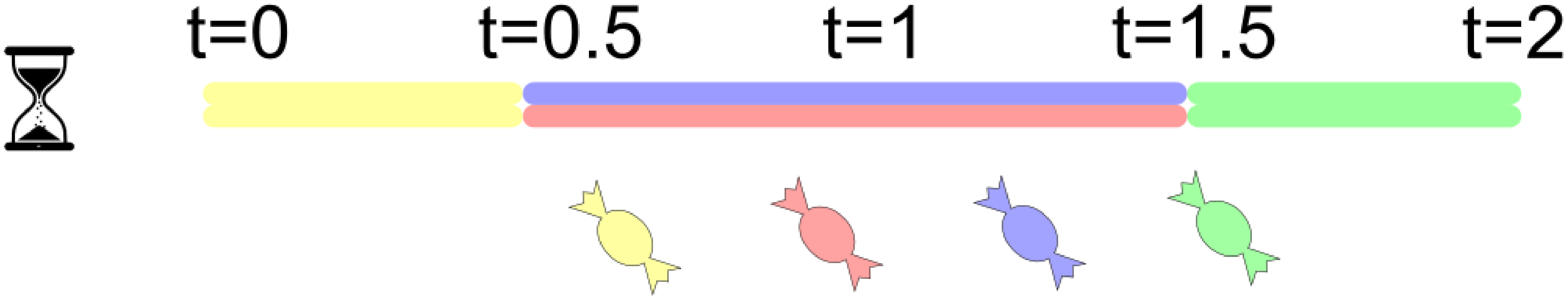

Doubly stochastic

1	0.5	0	0.5	0
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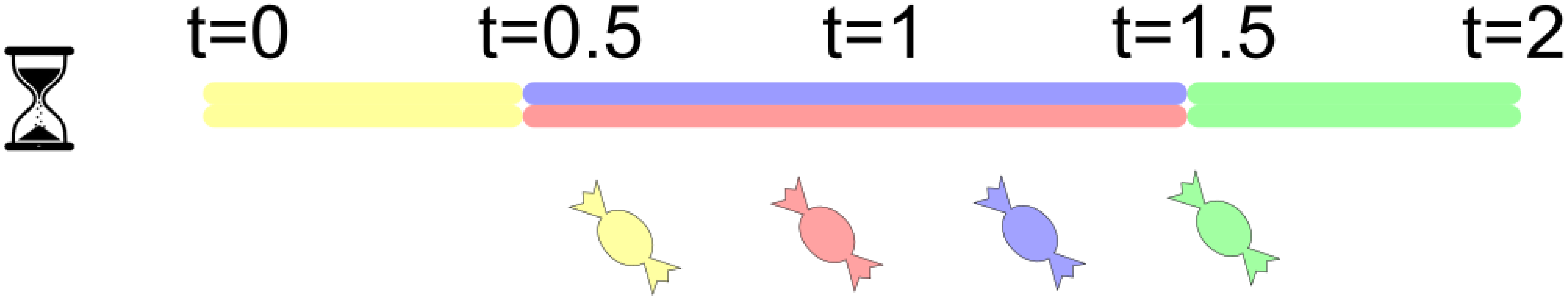


Doubly stochastic

1	0.5	0	0.5	0
1	0.5	0.5	0	0
2	0	0	0.5	0.5
2	0	0.5	0	0.5

Apply Birkhoff-von Neumann decomposition

Probabilistic Serial/ "Eating" Algorithm

[Bogomolnaia and Moulin, 2001]

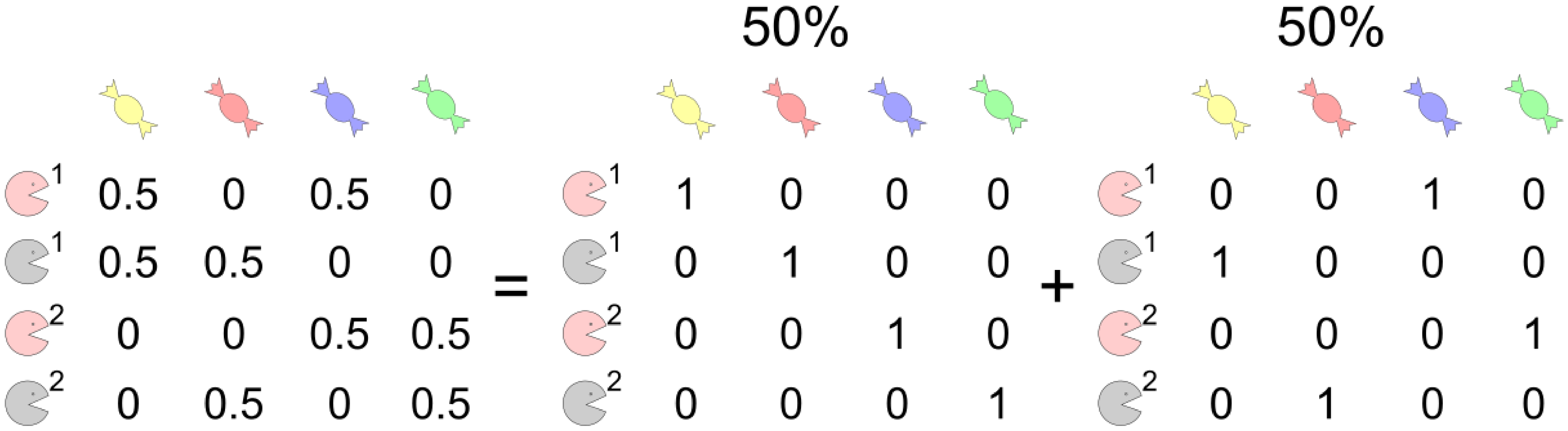
t=0 t=0.5 t=1 t=1.5 t=2

Any doubly stochastic matrix can be expressed as a convex combination of permutation matrices.

Doubly stochastic

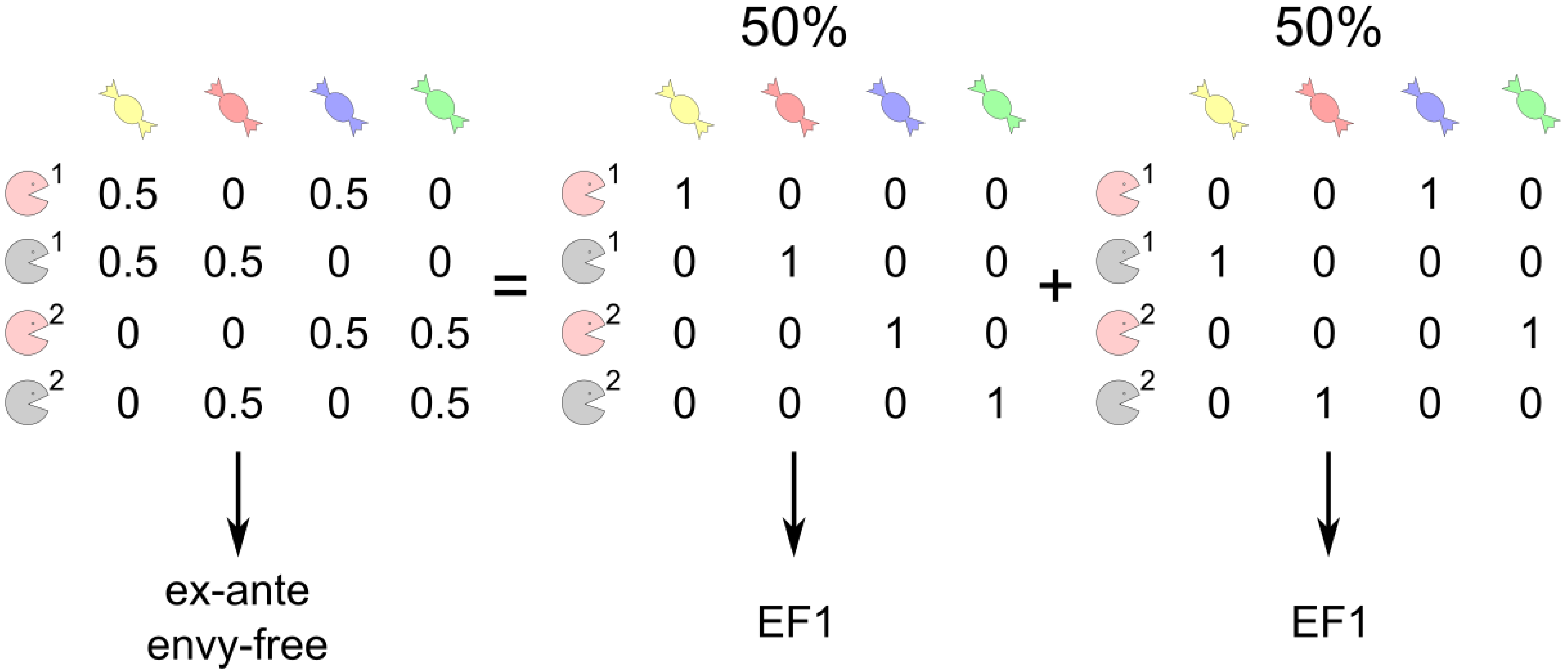
1	0.5	0.5	0	0
2	0	0	0.5	0.5
2	0	0.5	0	0.5

Apply Birkhoff-von Neumann decomposition



50%
50%
0.5 0 0.5 0
0.5 0.5 0 0
0 0 0.5 0.5
0 0.5 0 0.5
=
1 0 0 0
0 1 0 0
0 0 1 0
0 0 0 1
+
0 0 1 0
1 0 0 0
0 0 0 1
0 1 0 0

Claim: This probability distribution is ex-ante and ex-post fair.



Claim: This probability distribution is ex-ante and ex-post fair.

1	0.5	0	0.5	0
1	0.5	0.5	0	0
2	0	0	0.5	0.5
2	0	0.5	0	0.5

↓

ex-ante
envy-free

because each agent eats
its favorite good
at each instant of time

Claim: This probability distribution is ex-ante and ex-post fair.

1	0.5	0	0.5	0
1	0.5	0.5	0	0
2	0	0	0.5	0.5
2	0	0.5	0	0.5

ex-ante
envy-free

because each agent eats
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Claim: This probability distribution is ex-ante and ex-post fair.

1	0.5	0	0.5	0
1	0.5	0.5	0	0
2	0	0	0.5	0.5
2	0	0.5	0	0.5

=

50%

1	1	0	0	0
1	0	1	0	0
2	0	0	1	0
2	0	0	0	1

+

50%

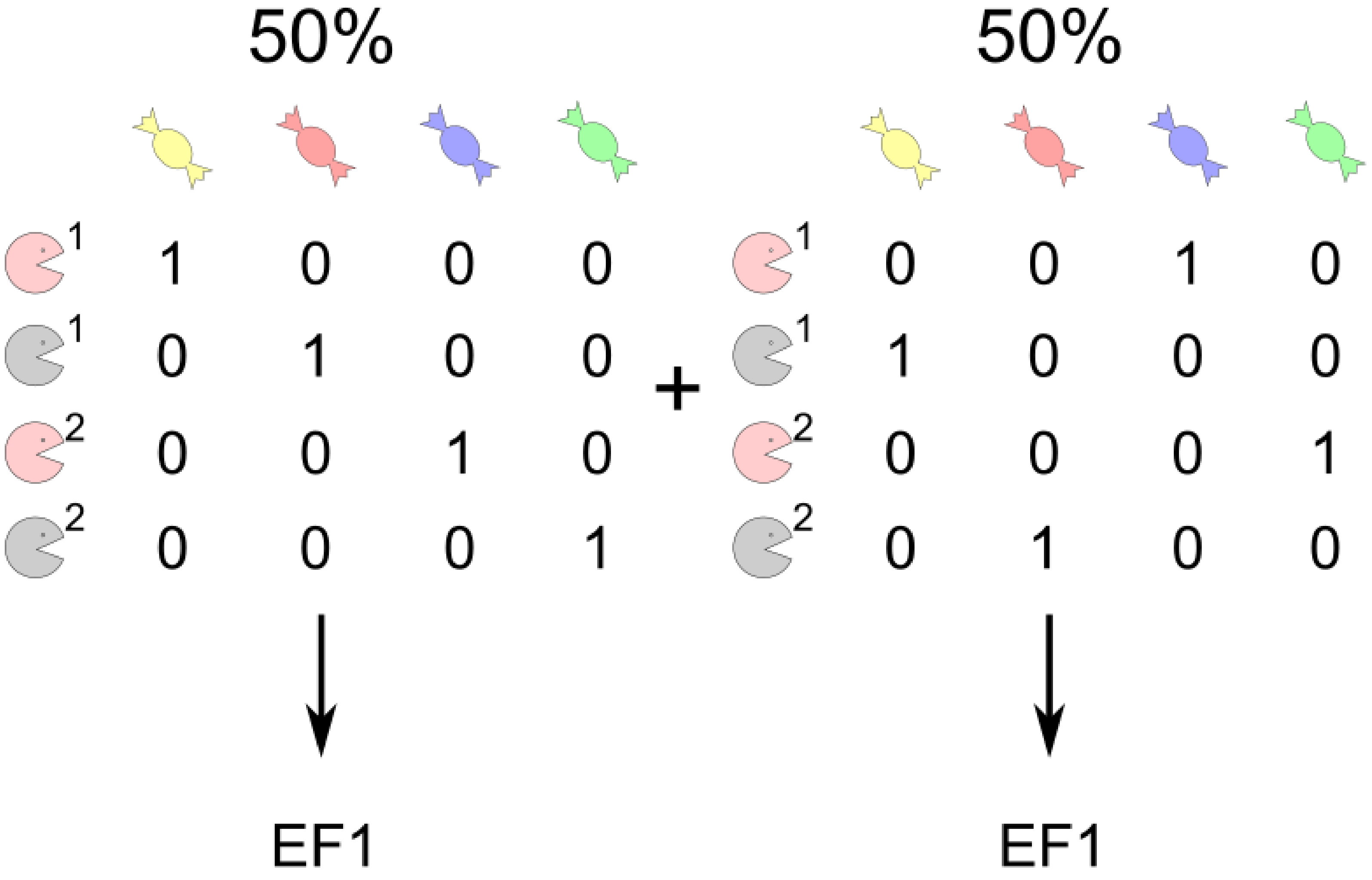
1	0	0	1	0
1	1	0	0	0
2	0	0	0	1
2	0	1	0	0

↓ ex-ante envy-free

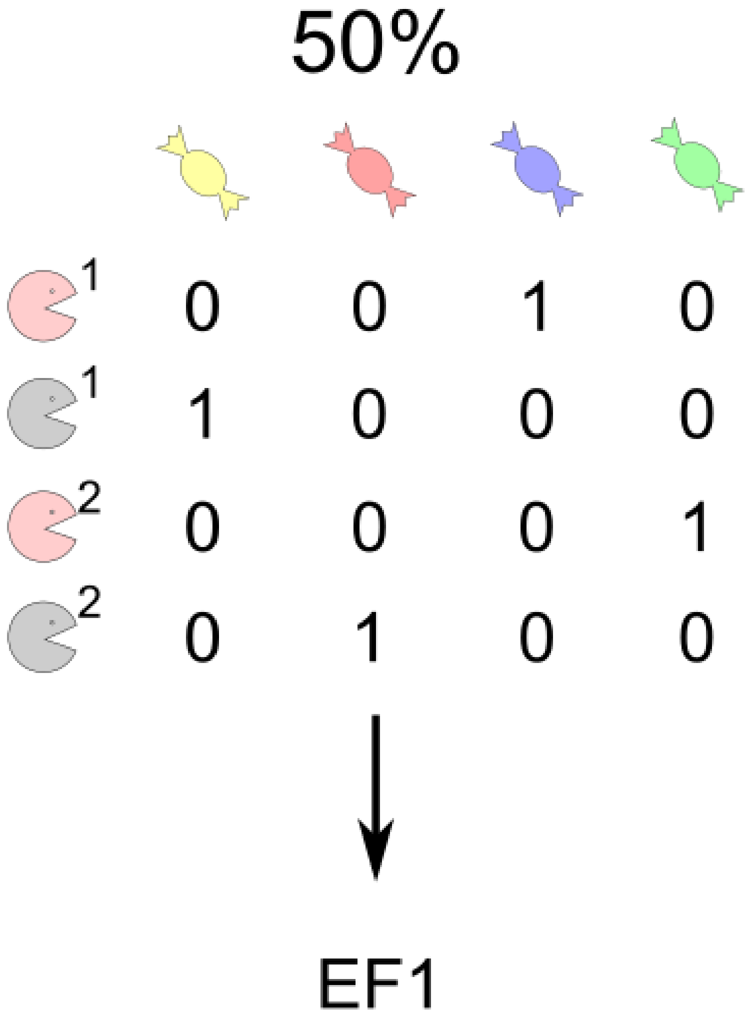
↓ EF1

↓ EF1

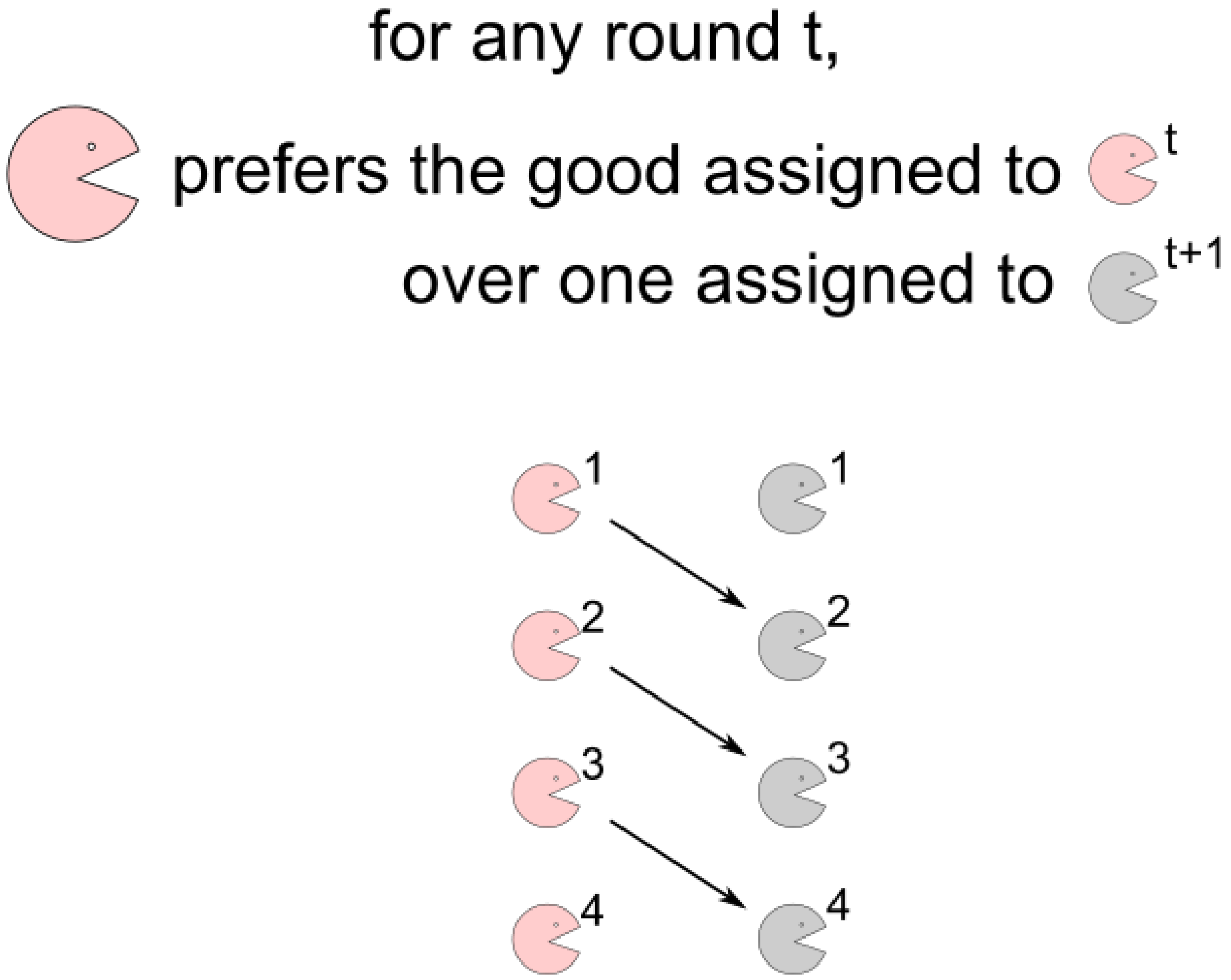
Claim: This probability distribution is ex-ante and ex-post fair.

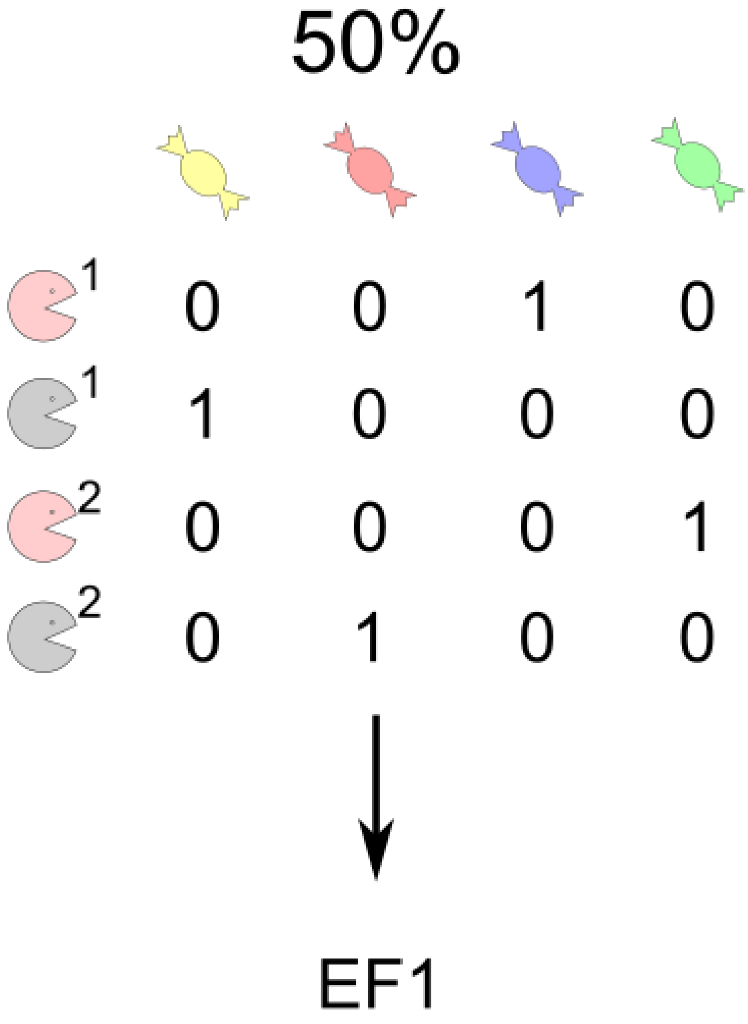


Claim: This probability distribution is ex-ante ✓ and ex-post fair.



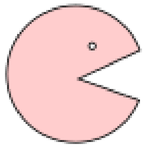
Claim: This probability distribution is ex-ante and ex-post fair.





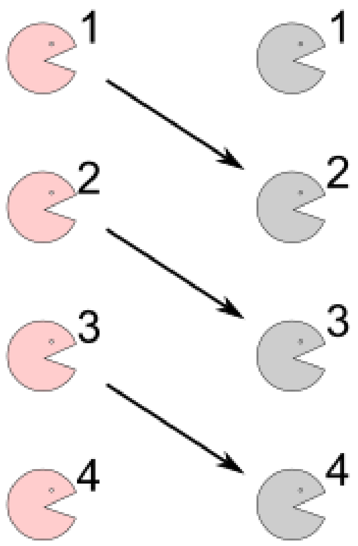
Claim: This probability distribution is ex-ante and ex-post fair.

for any round t,



prefers the good assigned to ^{t}

over one assigned to $^{t+1}$



50%

1	0	0	1	0
1	1	0	0	0
2	0	0	0	1
2	0	1	0	0

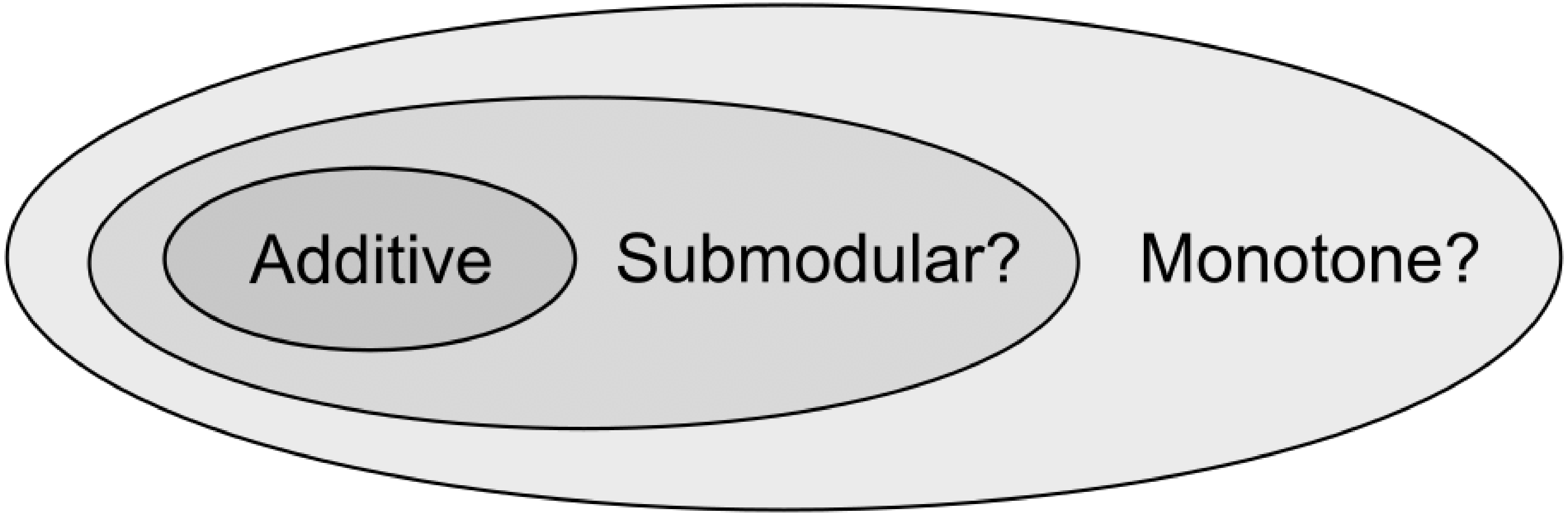
EF1

■

Ex-ante EF and ex-post EF1 for additive goods

Ex-ante EF and ex-post EF1 for additive goods

Ex-ante EF and ex-post EF1 for additive goods



EF1 exists

[Lipton, Markakis, Mossel, and Saberi, *EC* 2004]

Ex-ante EF and ex-post EF1 for additive goods

Envy-free up to *any* good (EFX)?

Ex-ante EF and ex-post EF1 for additive goods

Envy-free up to *any* good (EFX)?

Ex-post EFX alone unresolved for 4+ agents

Ex-ante EF and ex-post EF1 for additive goods

and Pareto optimal (PO)?

Ex-ante EF and ex-post EF1 for additive goods

and Pareto optimal (PO)?

Ex-post EF1 + PO alone always exists for additive valuations

[Caragiannis, Kurokawa, Moulin, Procaccia, Shah, and Wang, *EC* 2016, *TEAC* 2019]

Ex-ante EF and ex-post EF1 for additive goods

and chores?

Ex-ante EF and ex-post EF1 for additive goods

and chores?

EF1 alone always exists for monotone chores

[Barman, Sricharan, and Vaish, *APPROX* 2021]

Next Time

Mixture of
divisible and indivisible

Quiz

Quiz

Find an ex-ante EF and ex-post EF1 allocation.

	A	B	C	D
	4	3	1	1
	5	2	1	1
	0	1	2	3