COL749: Computational Social Choice

# Lecture 12

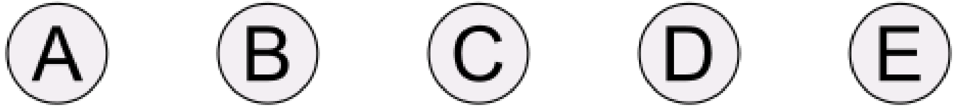
# Fair Allocation of Indivisible Chores

Feb 13, 2025 | Rohit Vaish

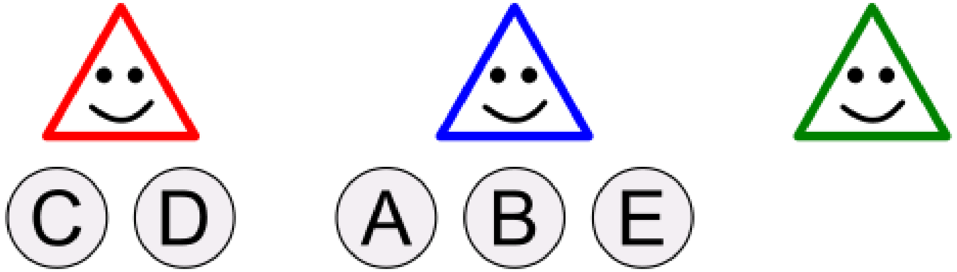
# The Model

Set of agents

Set of indivisible items



Allocation



# Valuation Function

For three items Ⓐ Ⓑ Ⓒ

△{ } = 0

△{Ⓐ} = 1

△{Ⓑ} = 0

△{Ⓒ} = 2

△{Ⓐ Ⓑ} = 1

△{Ⓑ Ⓒ} = 2

△{Ⓐ Ⓒ} = 3

△{Ⓐ Ⓑ Ⓒ} = 3

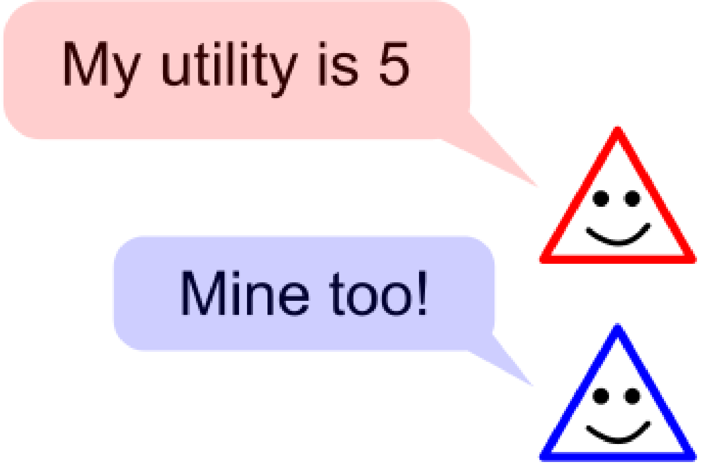
Description grows *exponentially* with the number of items!

# Additive Valuations

△{Ⓐ Ⓑ Ⓒ} = △{Ⓐ} + △{Ⓑ} + △{Ⓒ}

# Additive Valuations

$$\triangle\{Ⓐ Ⓑ Ⓒ\} = \triangle\{Ⓐ\} + \triangle\{Ⓑ\} + \triangle\{Ⓒ\}$$



| | Ⓐ | Ⓑ | Ⓒ |
|---|---|---|---|
| △ | 4 | 1 | 2 |
| △ | 1 | 0 | 5 |

Description grows *linearly* with the number of items.

# Marginal Value

Marginal value of Ⓐ for △ with respect to {Ⓑ Ⓒ}

△ Ⓐ | {Ⓑ Ⓒ} = △ {Ⓐ Ⓑ Ⓒ} - △ {Ⓑ Ⓒ}

# Types of Resources

# Types of Resources

The item Ⓐ is a good for △ if for *all* subsets of items S

△ Ⓐ | S ≥ 0

# Types of Resources

The item Ⓐ is a good for △ if for *all* subsets of items S

$$\triangle\, Ⓐ \,\big|\, S \geq 0$$

E.g., an extra GB of cloud storage

2GB 5GB 15GB 10GB

# Types of Resources

The item Ⓑ is a chore for △ if for *all* subsets of items S

△ Ⓑ | S ≤ 0

E.g., a dish that you forgot to wash

# Types of Resources

Good for one agent, chore for another: Mixed items

E.g., service charge in restaurant bills

# Types of Resources

If all items are goods for all agents: Goods instance

If all items are chores for all agents: Chores instance

Otherwise: Mixed instance

# Types of Resources

## Goods

## Chores

## Mixed

# Types of Valuation Functions

## Goods

## Chores

## Mixed

# Types of Valuation Functions

## Goods

=

Monotone↑

△ S ≥ △ T

whenever S ⊇ T

## Chores

## Mixed

# Types of Valuation Functions

## Goods

=

Monotone↑

$\triangle S \geq \triangle T$

whenever $S \supseteq T$

## Chores

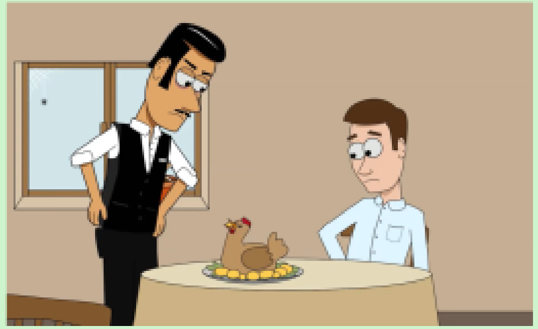
=

Monotone↓

$\triangle S \leq \triangle T$

whenever $S \supseteq T$

## Mixed



# Types of Valuation Functions

## Goods

=

Monotone↑

△ S ≥ △ T

whenever $S \supseteq T$

## Chores

=

Monotone↓

△ S ≤ △ T

whenever $S \supseteq T$

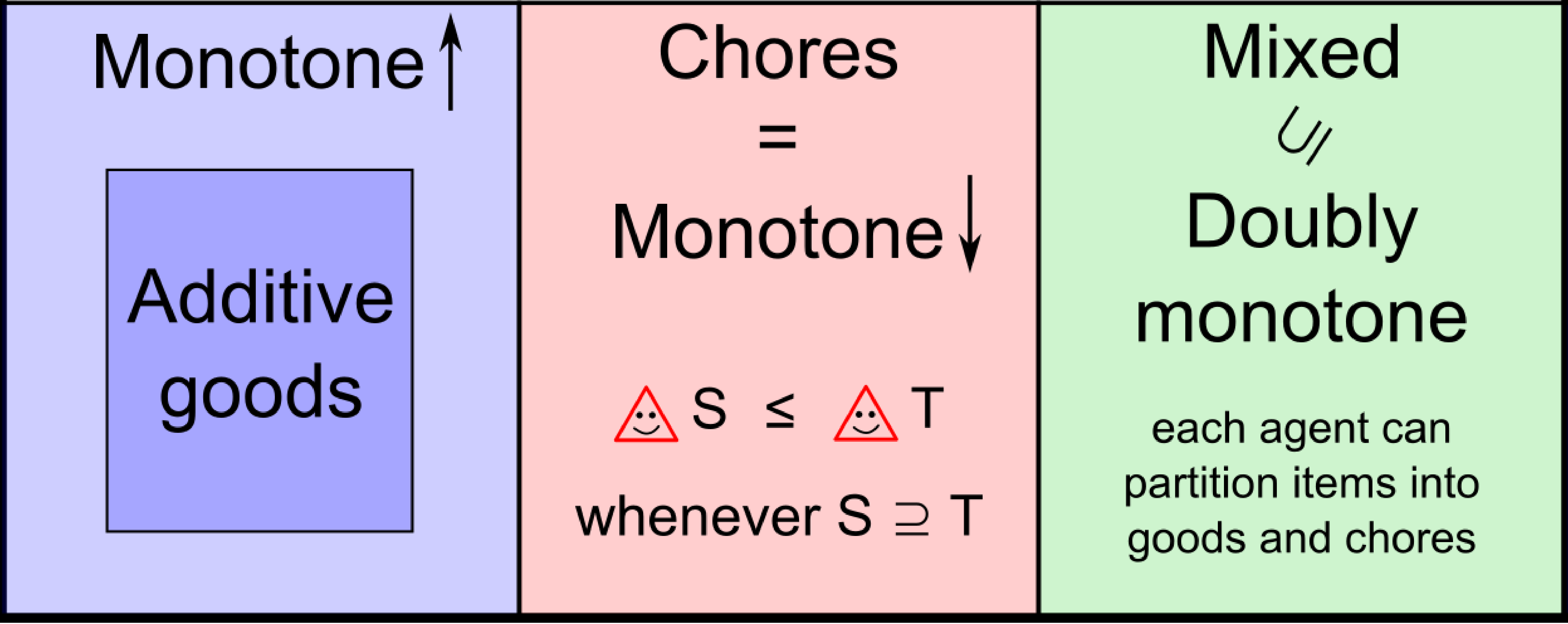
## Mixed

⊇

Doubly monotone

each agent can partition items into goods and chores

# Types of Valuation Functions



Monotone↑

Additive goods

Chores = Monotone↓

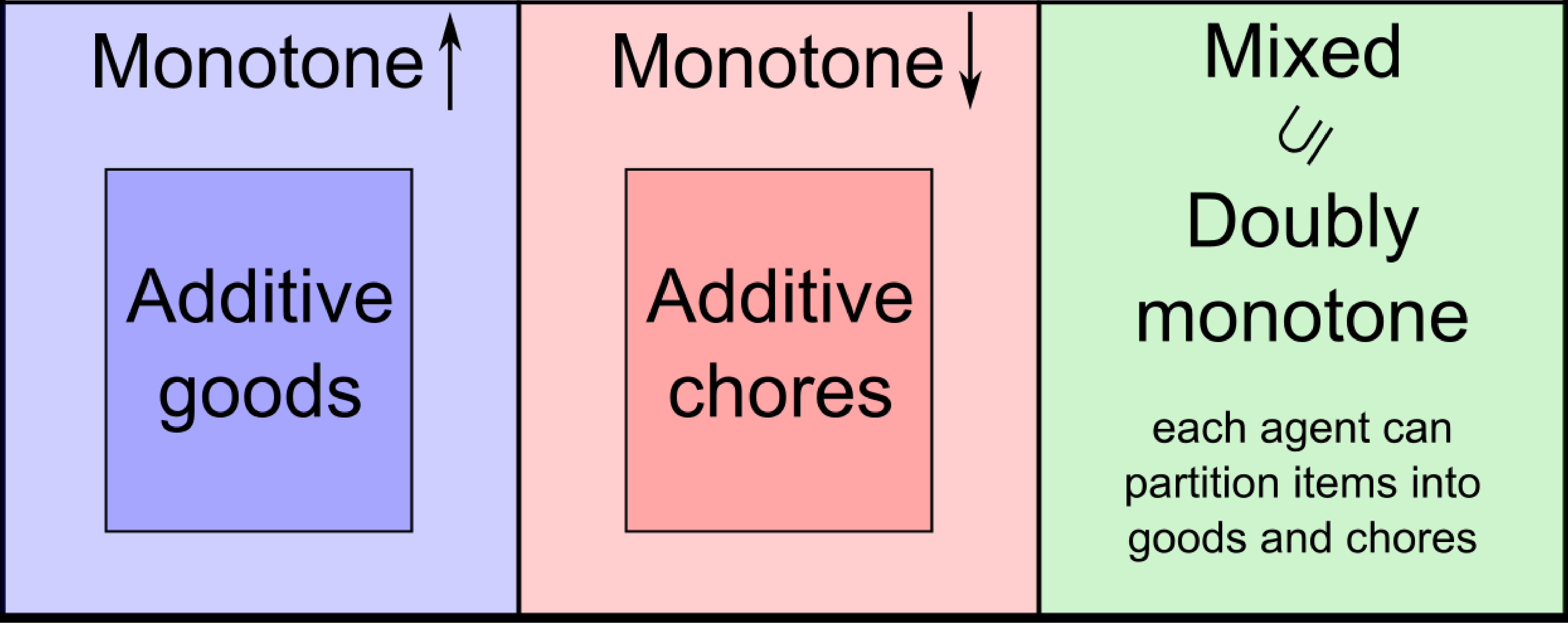
△S ≤ △T

whenever S ⊇ T

Mixed ⊇ Doubly monotone

each agent can partition items into goods and chores

# Types of Valuation Functions



Monotone↑

Additive goods

Monotone↓

Additive chores

Mixed

$\cup\!/$

Doubly monotone

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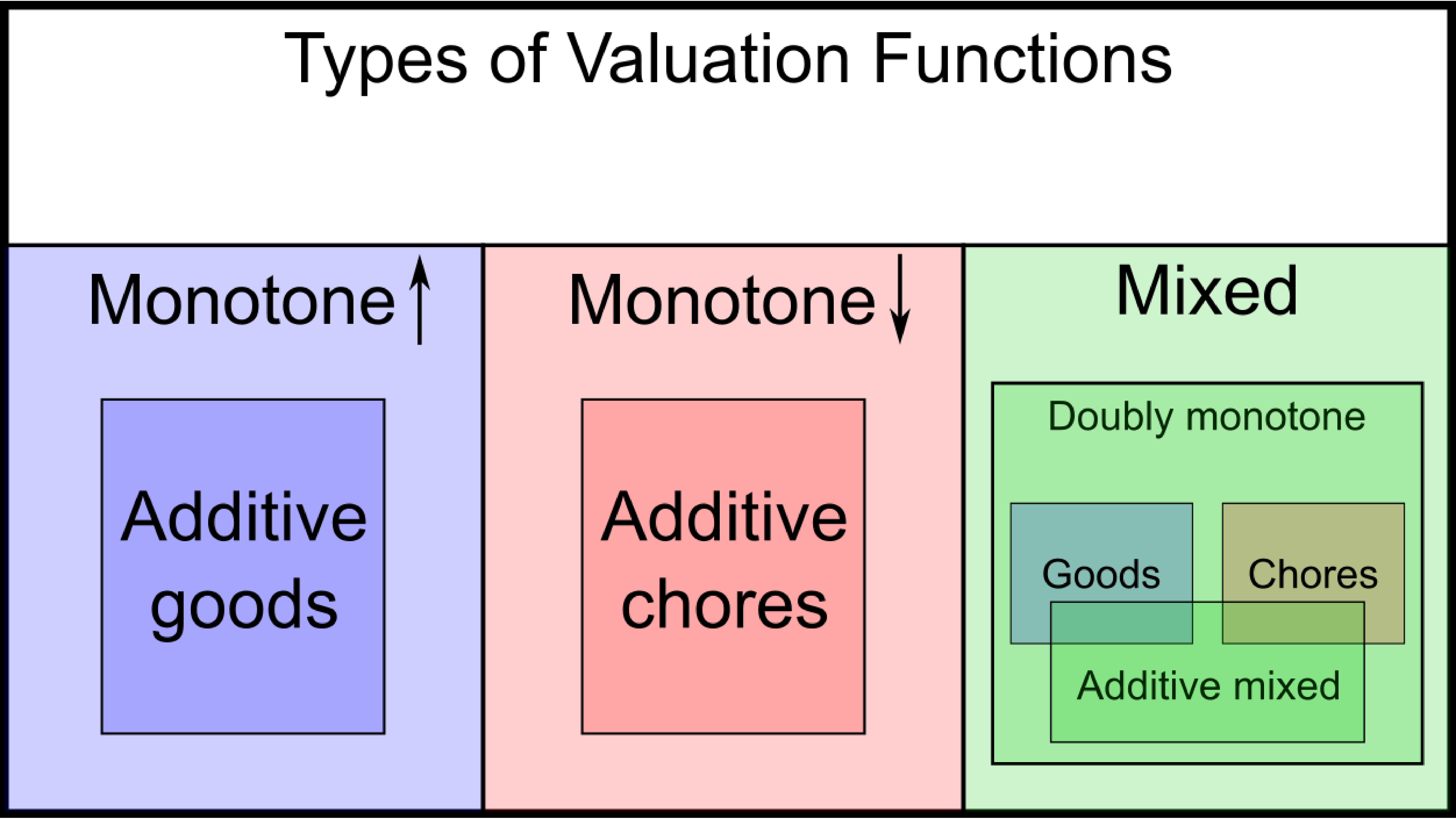


Types of Valuation Functions

Monotone↑

Additive goods

Monotone↓

Additive chores

Mixed

Doubly monotone

Goods

Chores

Additive mixed

# Types of Valuation Functions

Under additive valuations

## Goods

|   | A | B | C |
|---|---|---|---|
| △ | 4 | 1 | 2 |
| △ | 1 | 0 | 5 |

## Chores

|   | A | B | C |
|---|---|---|---|
| △ | -1 | 0 | -2 |
| △ | -5 | -1 | -1 |

## Mixed

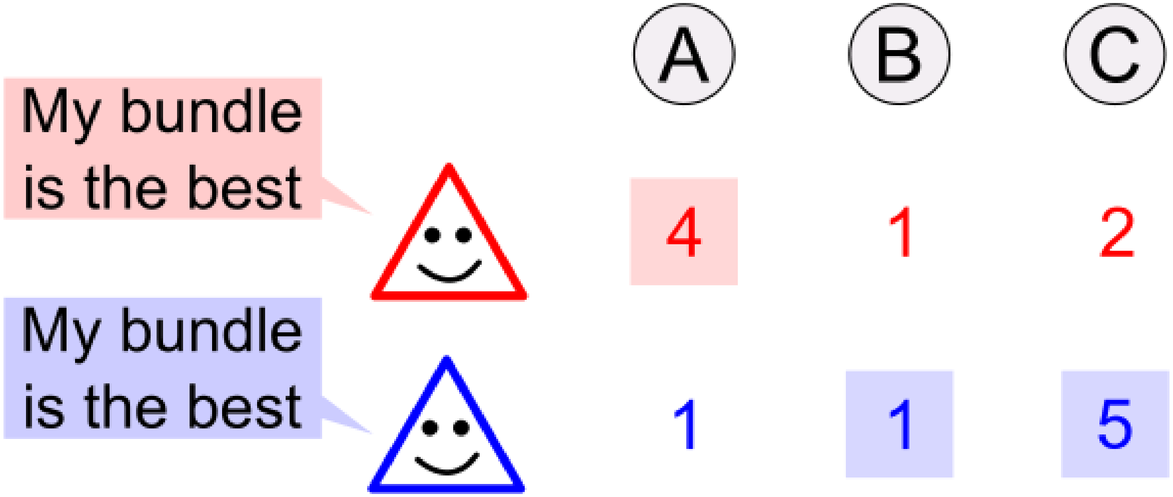
|   | A | B | C |
|---|---|---|---|
| △ | 1 | 1 | -1 |
| △ | -2 | 0 | -2 |

# Fairness Notions

# Envy-Freeness

[Gamow and Stern, 1958; Foley, 1967]

Each agent prefers its own bundle over that of any other agent.



My bundle is the best

My bundle is the best

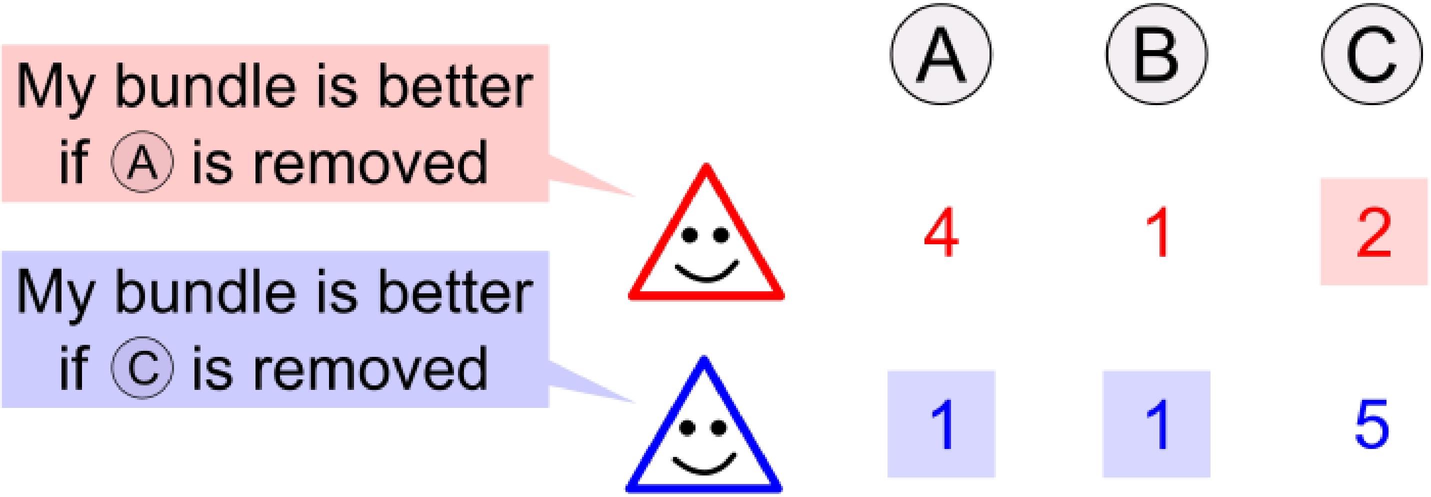
|   | A | B | C |
|---|---|---|---|
| △ | 4 | 1 | 2 |
| △ | 1 | 1 | 5 |

- ☹ Not guaranteed to exist (two agents, one good)
- ☹ Checking whether an EF allocation exists is NP-complete

# Envy-Freeness Up To One Good

[Budish, 2011]

Envy can be eliminated by removing some good in the envied bundle.



My bundle is better if Ⓐ is removed

My bundle is better if Ⓒ is removed

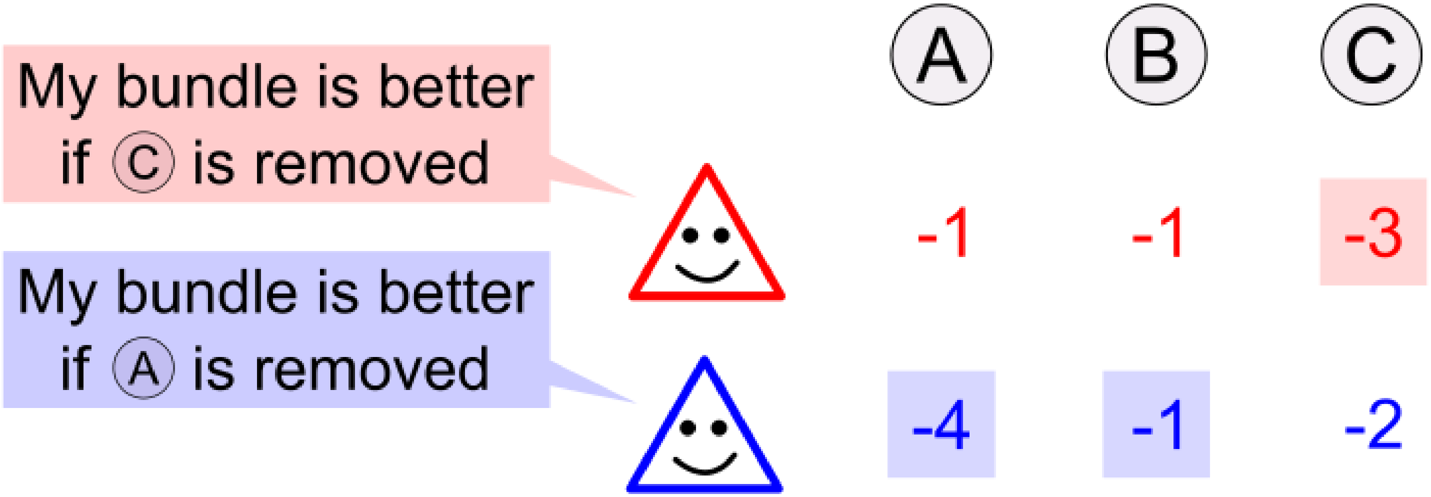
|  | Ⓐ | Ⓑ | Ⓒ |
|---|---|---|---|
| △ | 4 | 1 | 2 |
| △ | 1 | 1 | 5 |

Guaranteed to exist and efficiently computable

# Envy-Freeness Up To One Chore

[Aziz, Caragiannis, Igarashi, and Walsh; *IJCAI* 2019; *JAAMAS* 2022]

Eliminate envy by removing some chore in the envious agent's bundle.



My bundle is better if Ⓒ is removed

My bundle is better if Ⓐ is removed

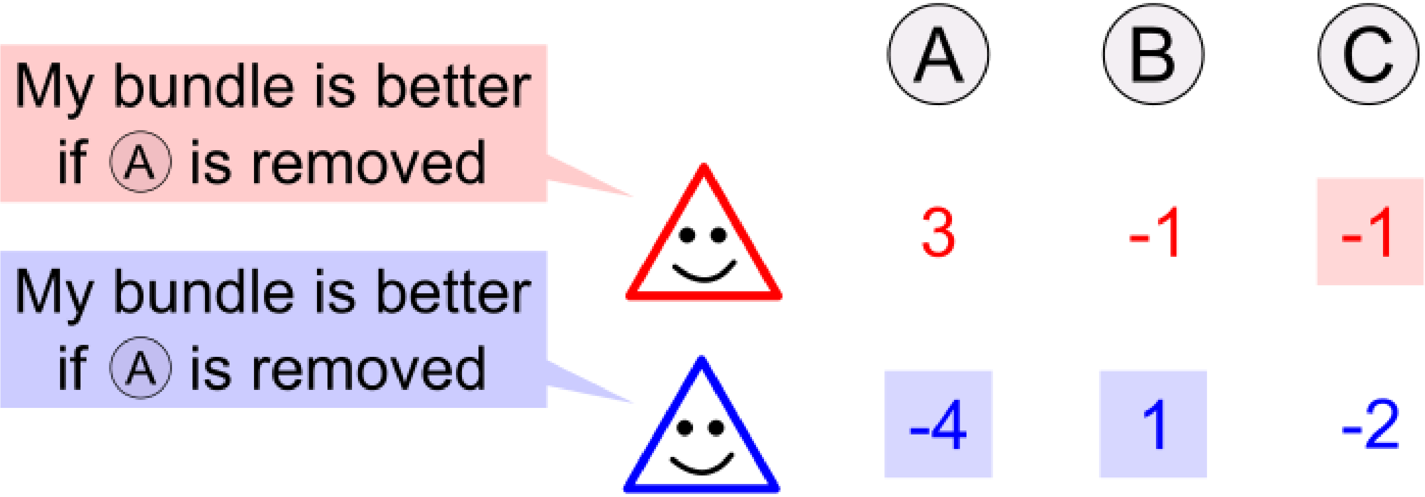
|   | Ⓐ | Ⓑ | Ⓒ |
|---|---|---|---|
| △ | -1 | -1 | -3 |
| △ | -4 | -1 | -2 |

Allocation $A = (A_1, \ldots, A_n)$ is EF1 if for every pair of agents $i, k$, there exists a chore $j \in A_i$ such that $v_i(A_i \setminus \{j\}) \geq v_i(A_k)$.

# Envy-Freeness Up To One Item

[Aziz, Caragiannis, Igarashi, and Walsh; *IJCAI* 2019; *JAAMAS* 2022]

Eliminate envy by removing some "good" in the envied bundle
or some "chore" in the envious agent's bundle.



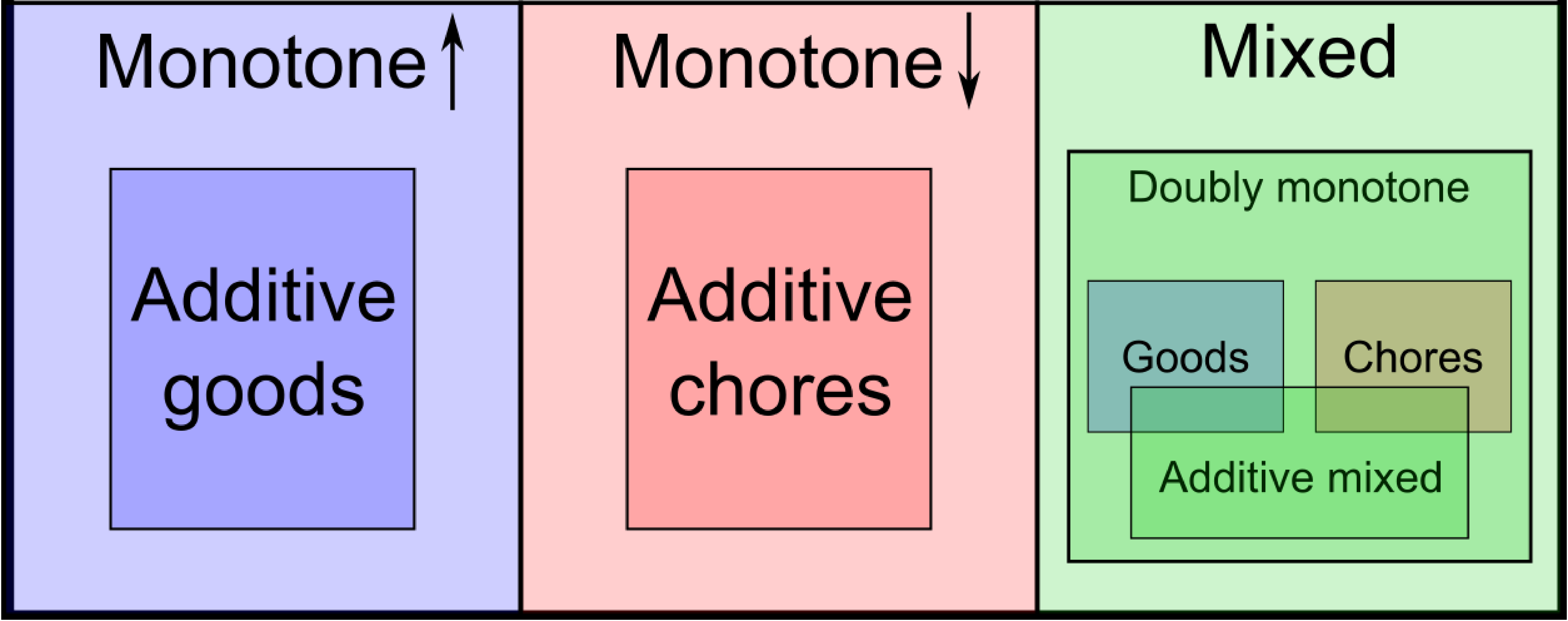
My bundle is better if Ⓐ is removed

My bundle is better if Ⓐ is removed

|  | Ⓐ | Ⓑ | Ⓒ |
|---|---|---|---|
| △ | 3 | -1 | -1 |
| △ | -4 | 1 | -2 |

Allocation $A = (A_1, \ldots, A_n)$ is EF1 if for every pair of agents $i, k$, there exists an item $j \in A_i \cup A_k$ s.t. $v_i(A_i \setminus \{j\}) \geq v_i(A_k \setminus \{j\})$.

# The Story of EF1



Monotone↑

Additive goods

Monotone↓

Additive chores

Mixed

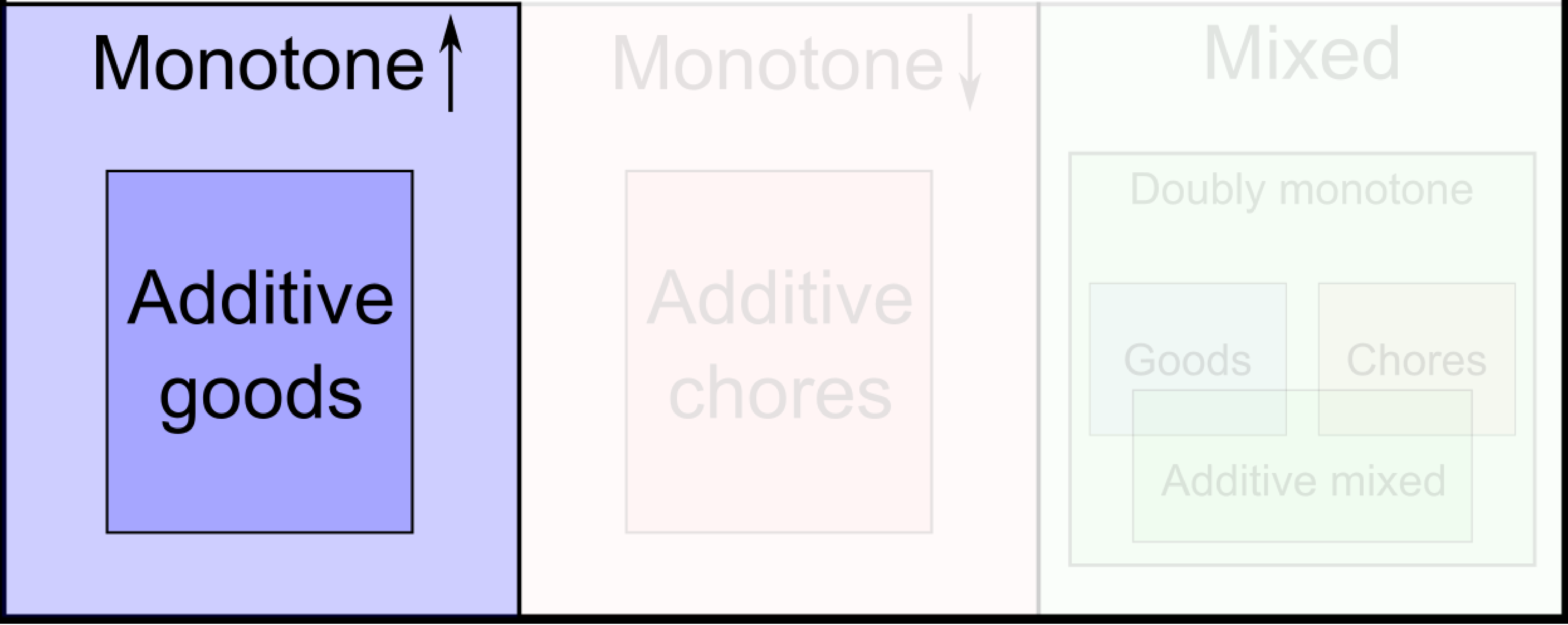
Doubly monotone

Goods

Chores

Additive mixed

# The Story of EF1



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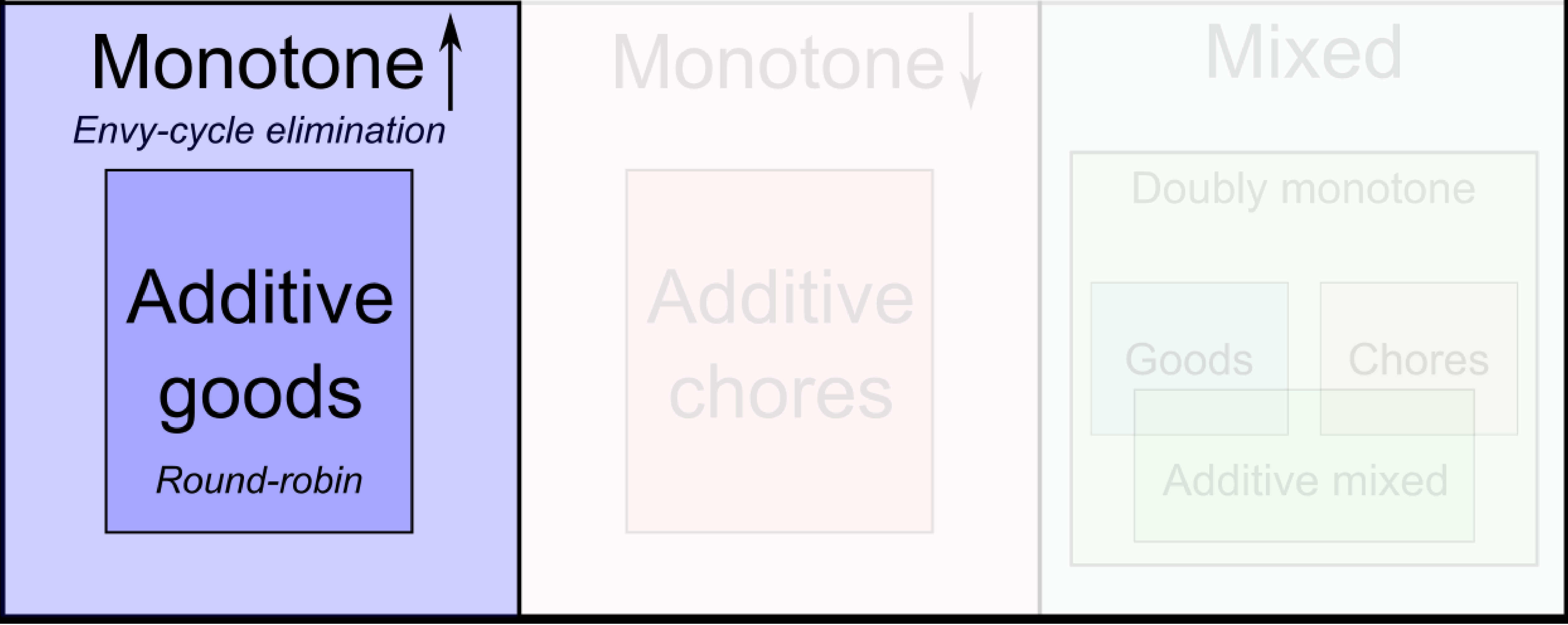

Monotone↑

*Envy-cycle elimination*

Additive goods

*Round-robin*

Monotone↓

Additive chores

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Doubly monotone

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Chores

Additive mixed

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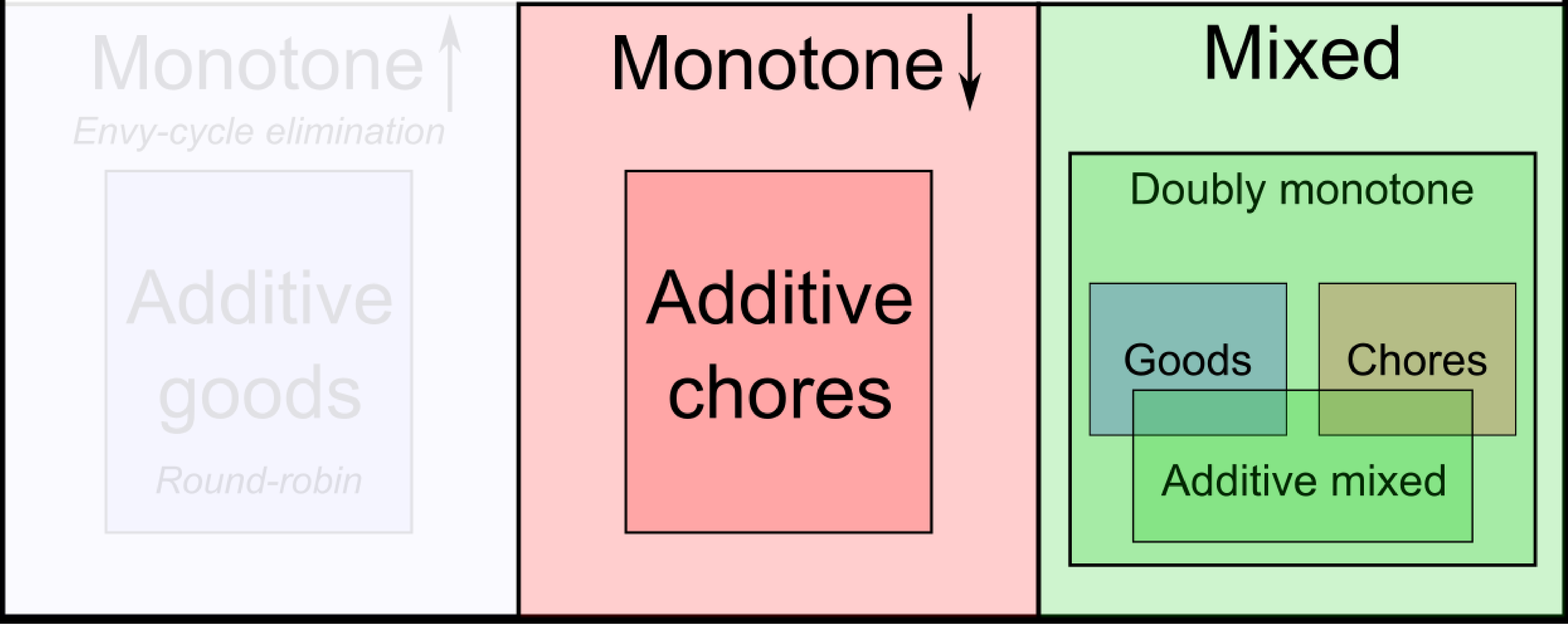

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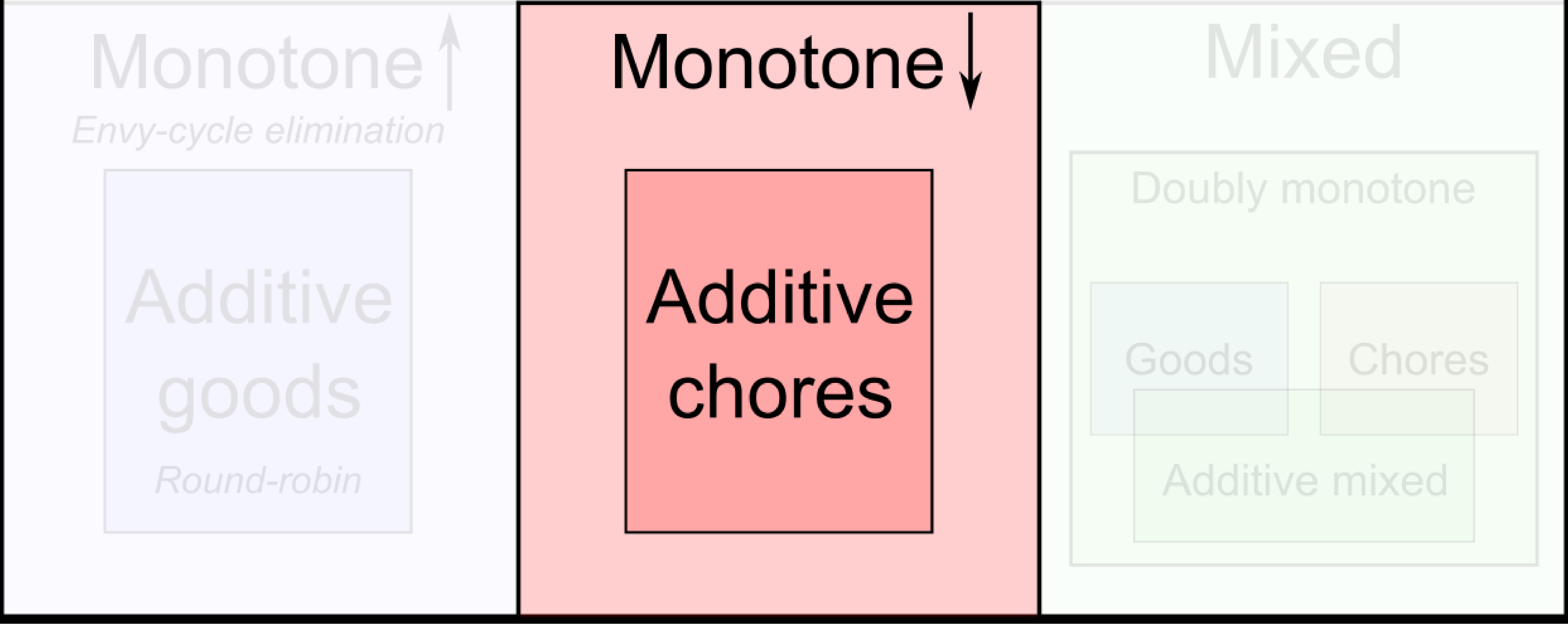

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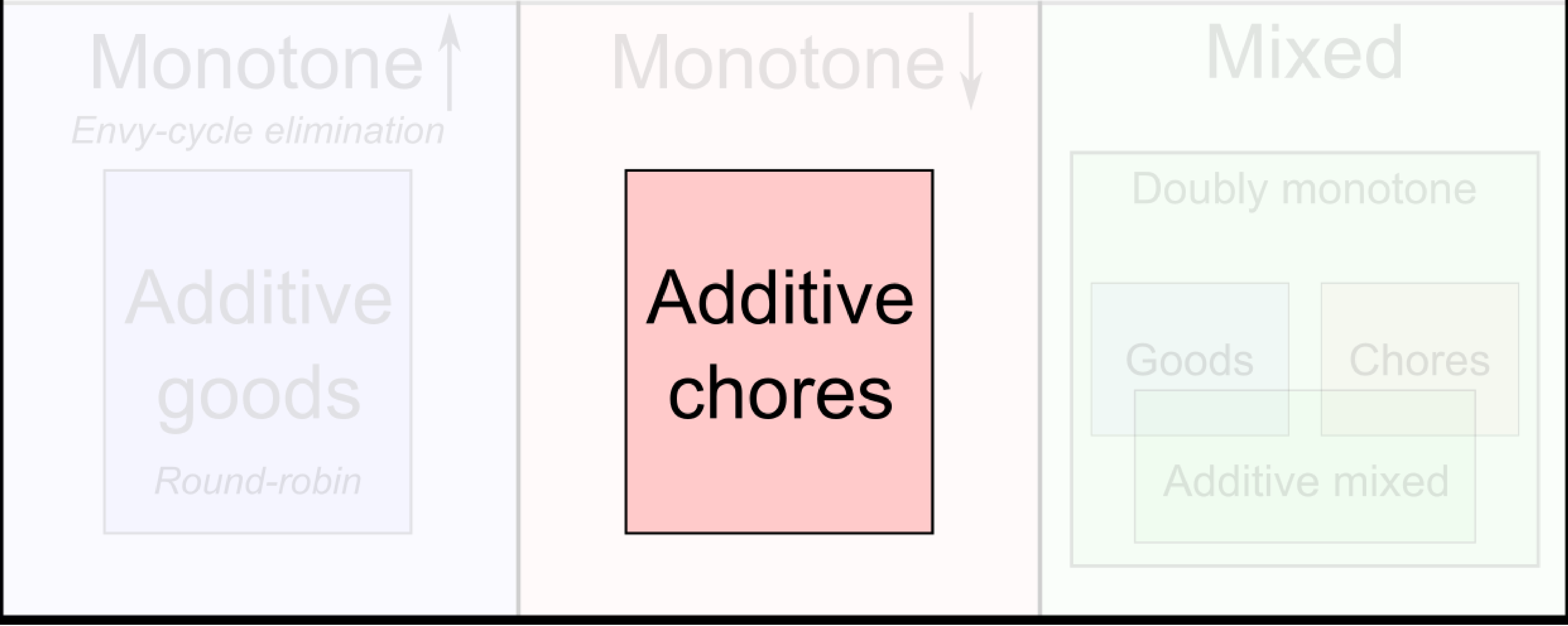

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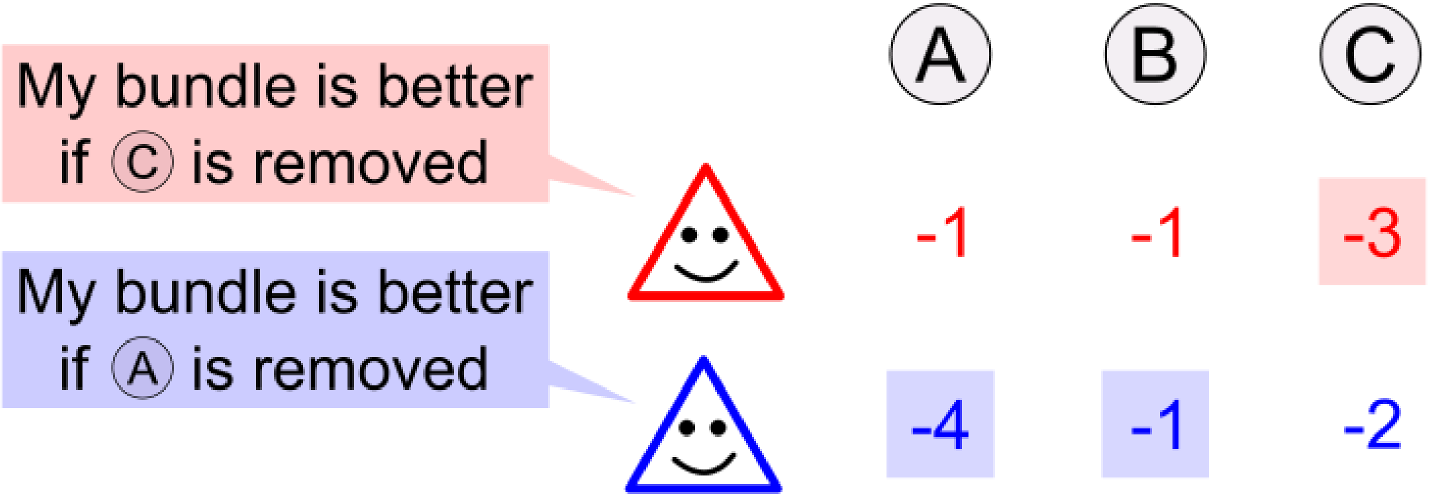

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| | Ⓐ | Ⓑ | Ⓒ |
|---|---|---|---|
| △ | -1 | -1 | -3 |
| △ | -4 | -1 | -2 |

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For additive chores, the allocation computed by round-robin algorithm satisfies EF1.

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| | A | B | C | D | E |
|---|---|---|---|---|---|
| △ | -4 | -1 | -3 | -2 | -4 |
| △ | 0 | -1 | -5 | -2 | -1 |
| △ | -4 | -2 | -5 | -3 | -1 |

For additive chores, the allocation computed by round-robin algorithm satisfies EF1.

|   | A | B | C | D | E |
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|---|---|---|---|---|---|
| 🟥 | -4 | -1 | -3 | -2 | -4 |
| 🟦 | 0 | -1 | -5 | -2 | -1 |
| 🟩 | -4 | -2 | -5 | -3 | -1 |

For additive chores, the allocation computed by round-robin algorithm satisfies EF1.

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|---|---|---|---|---|---|
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|   | A | B | C | D | E |
|---|---|---|---|---|---|
| △ | -4 | -1 | -3 | -2 | -4 |
| △ | 0 | -1 | -5 | -2 | -1 |
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| | Ⓐ | Ⓑ | Ⓒ | Ⓓ | Ⓔ |
|---|---|---|---|---|---|
| △ | -4 | **-1** | -3 | **-2** | -4 |
| △ | **0** | -1 | **-5** | -2 | -1 |
| △ | -4 | -2 | -5 | -3 | **-1** |

For additive chores, the allocation computed by round-robin algorithm satisfies EF1.

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Fix a pair of agents (r,b). Analyze envy of r towards b.

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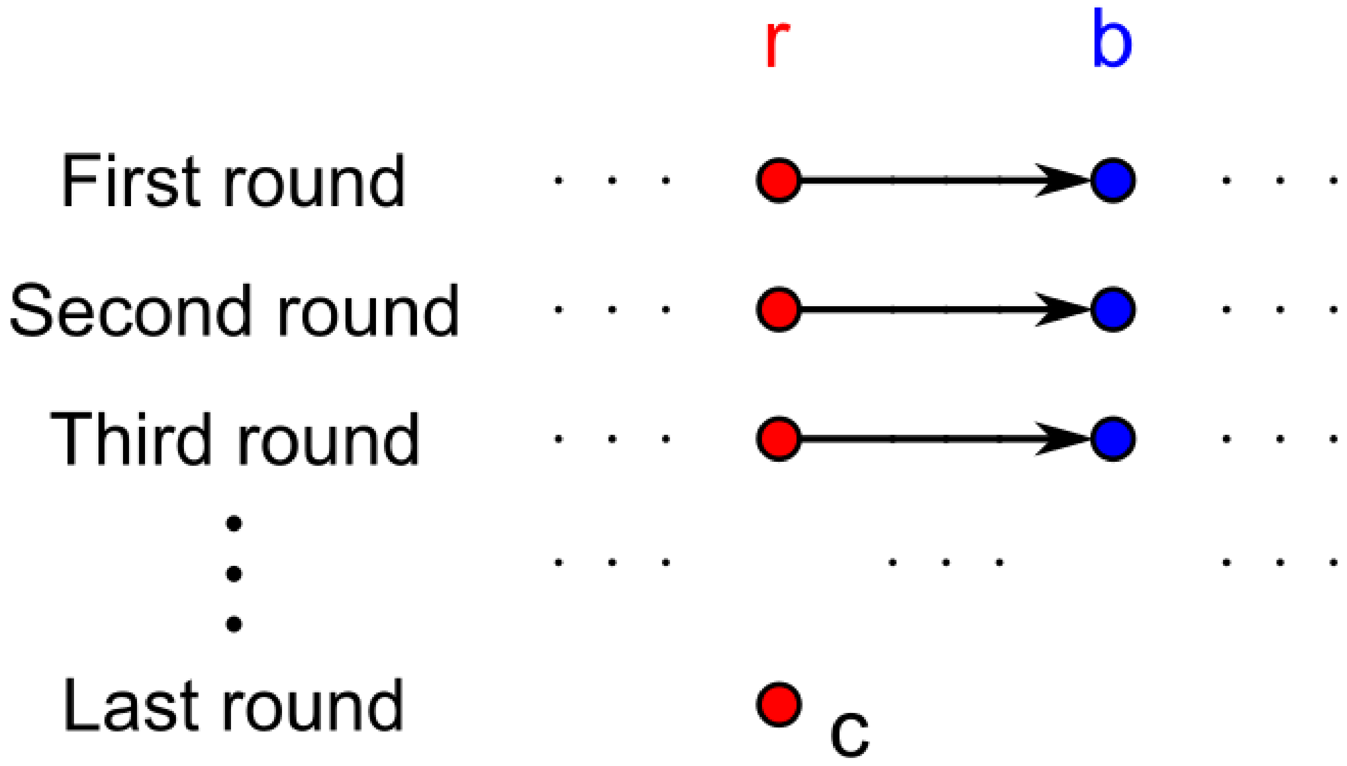

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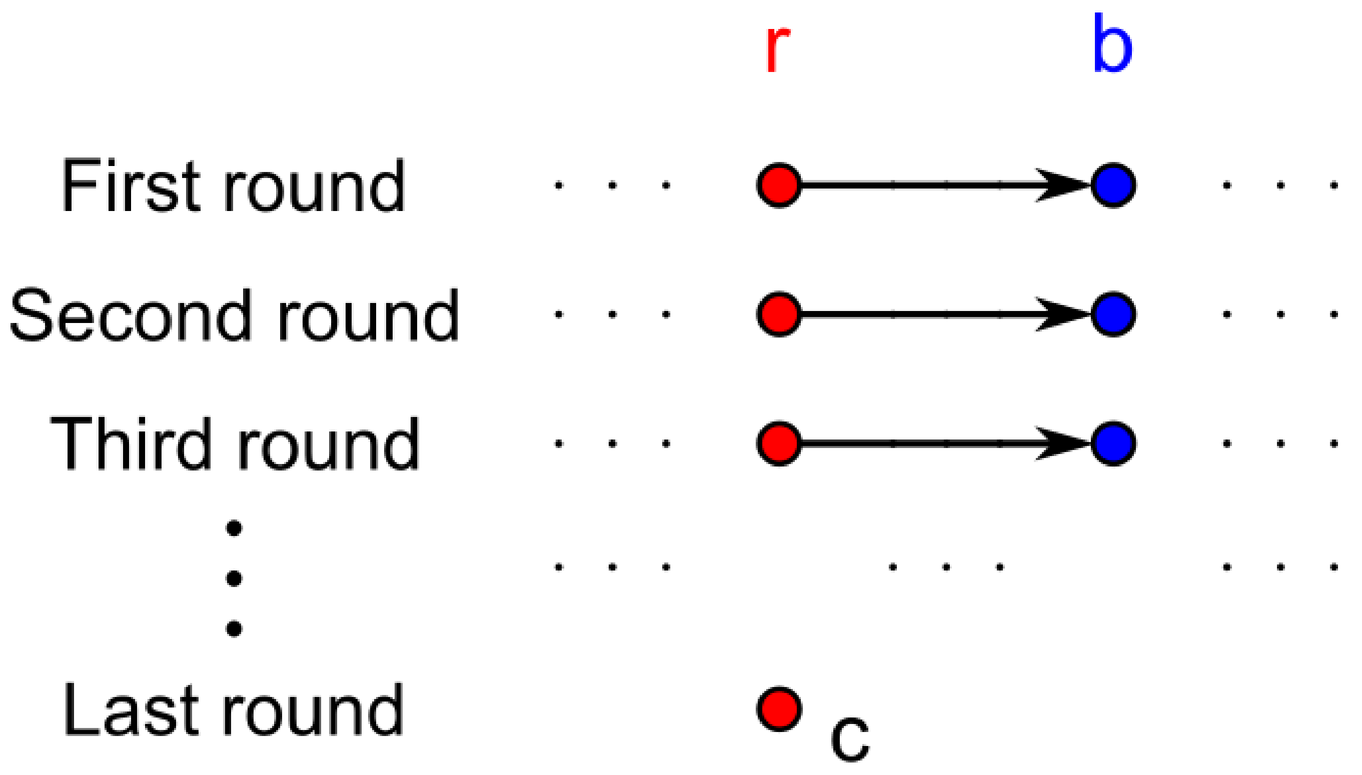

If r precedes b: Then, by additivity, $v_r(A_r \backslash \{c\}) \geq v_r(A_b)$.

For additive chores, the allocation computed by round-robin algorithm satisfies EF1.

Fix a pair of agents (r,b). Analyze envy of r towards b.

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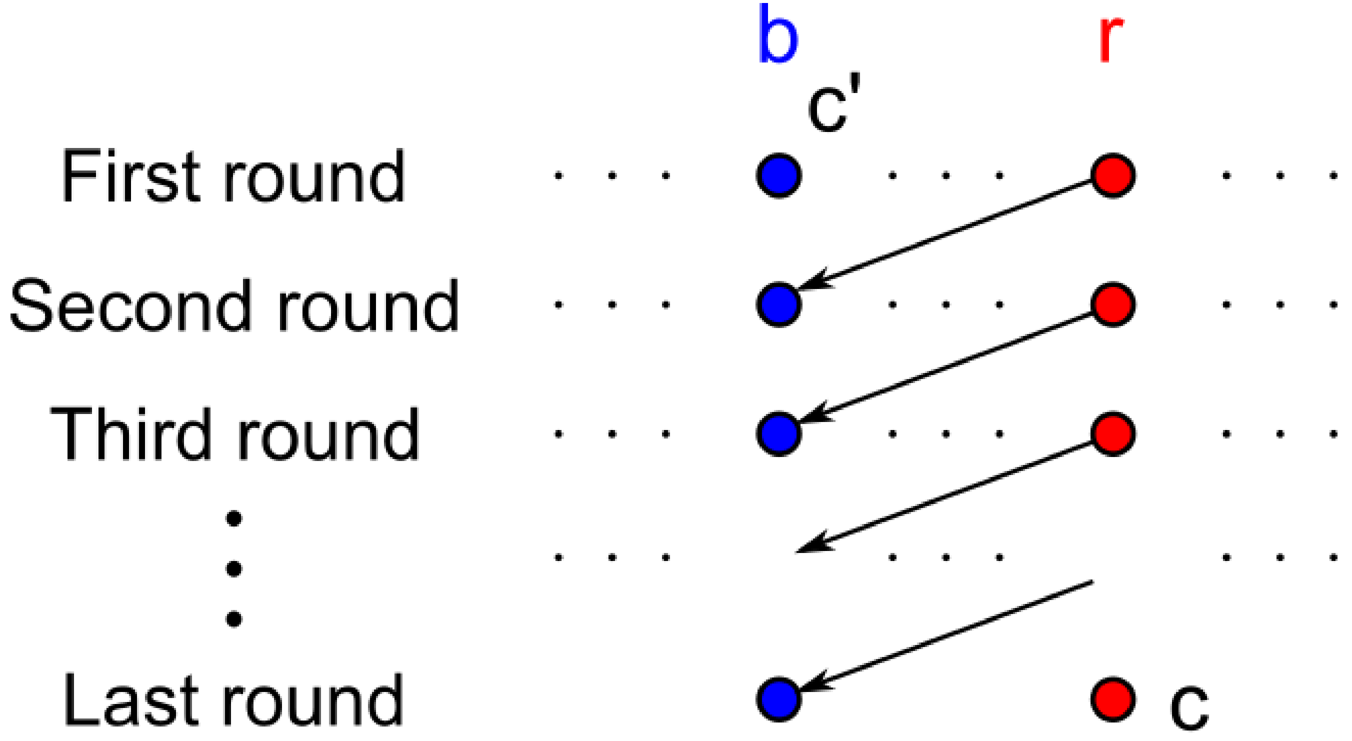

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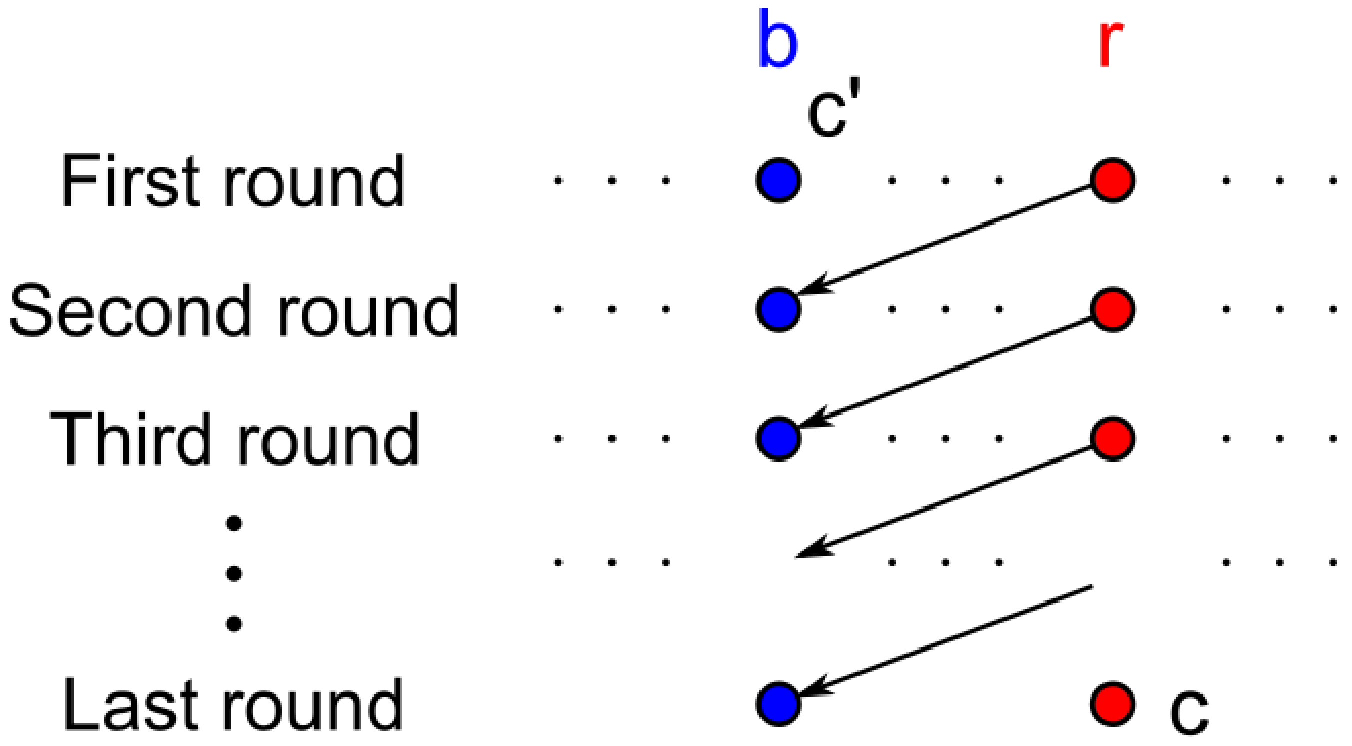

If b precedes r: Again, by additivity, $v_r(A_r \setminus \{c\}) \geq v_r(A_b \setminus \{c'\}) \geq v_r(A_b)$.

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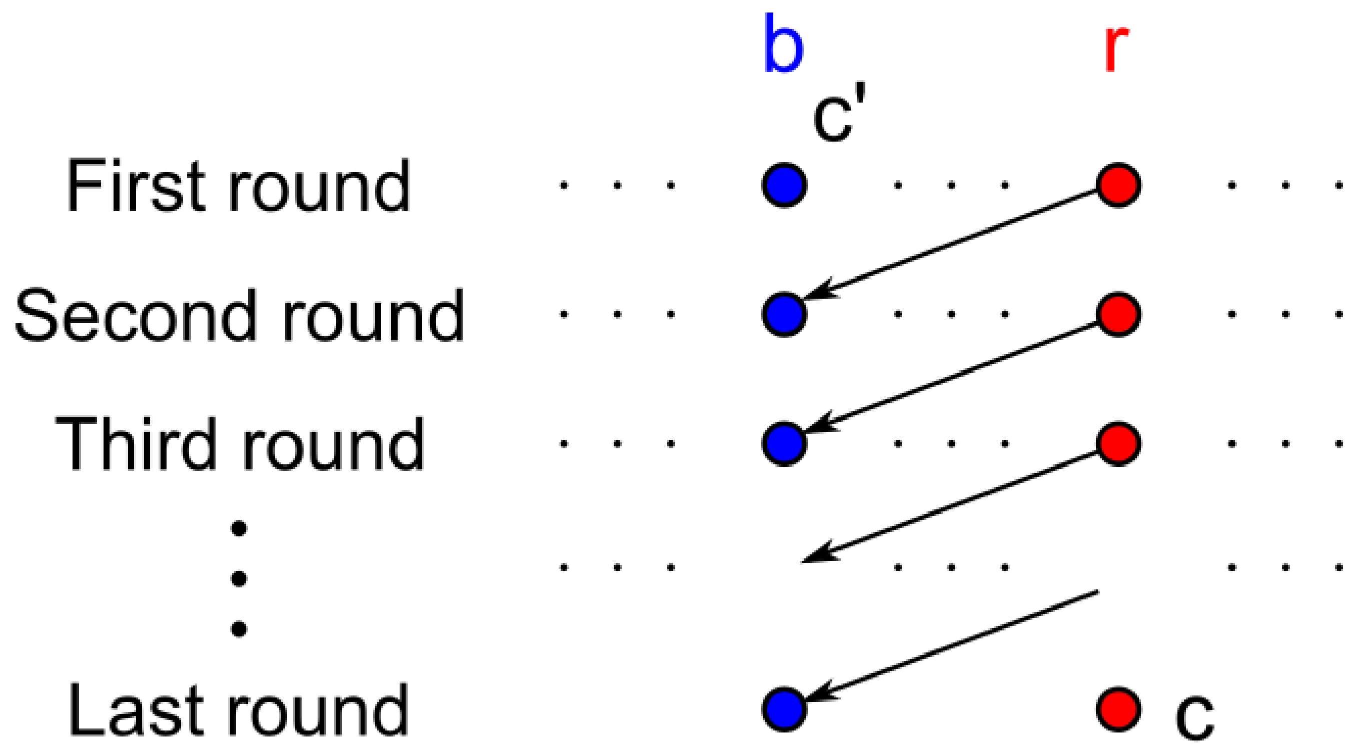

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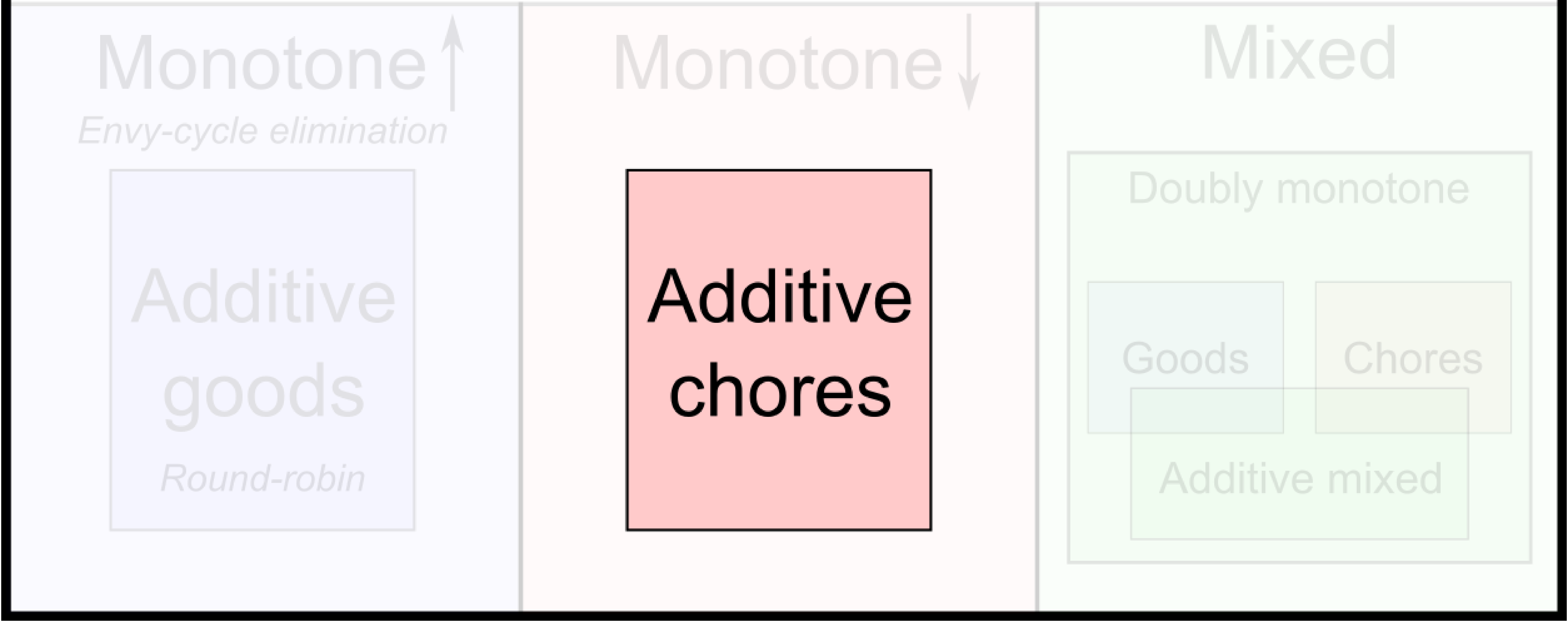

Monotone↑

*Envy-cycle elimination*

Additive goods

*Round-robin*

Monotone↓

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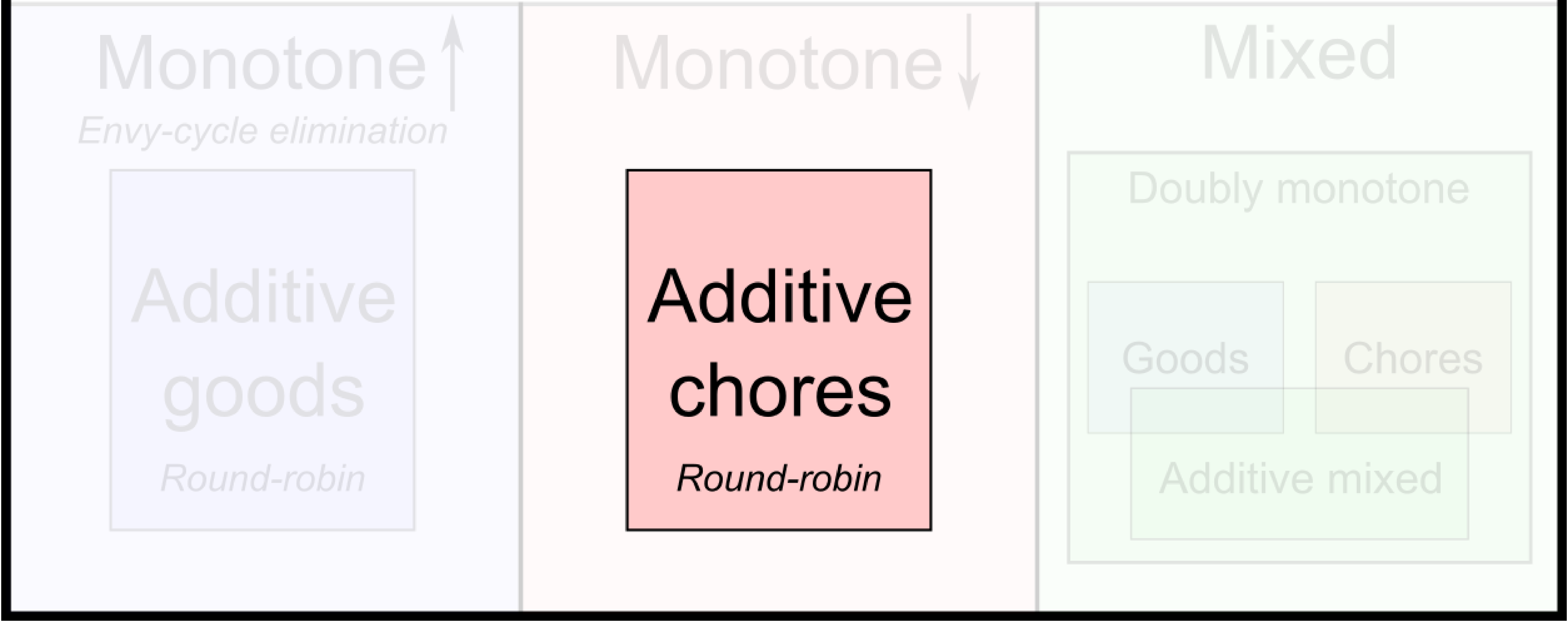

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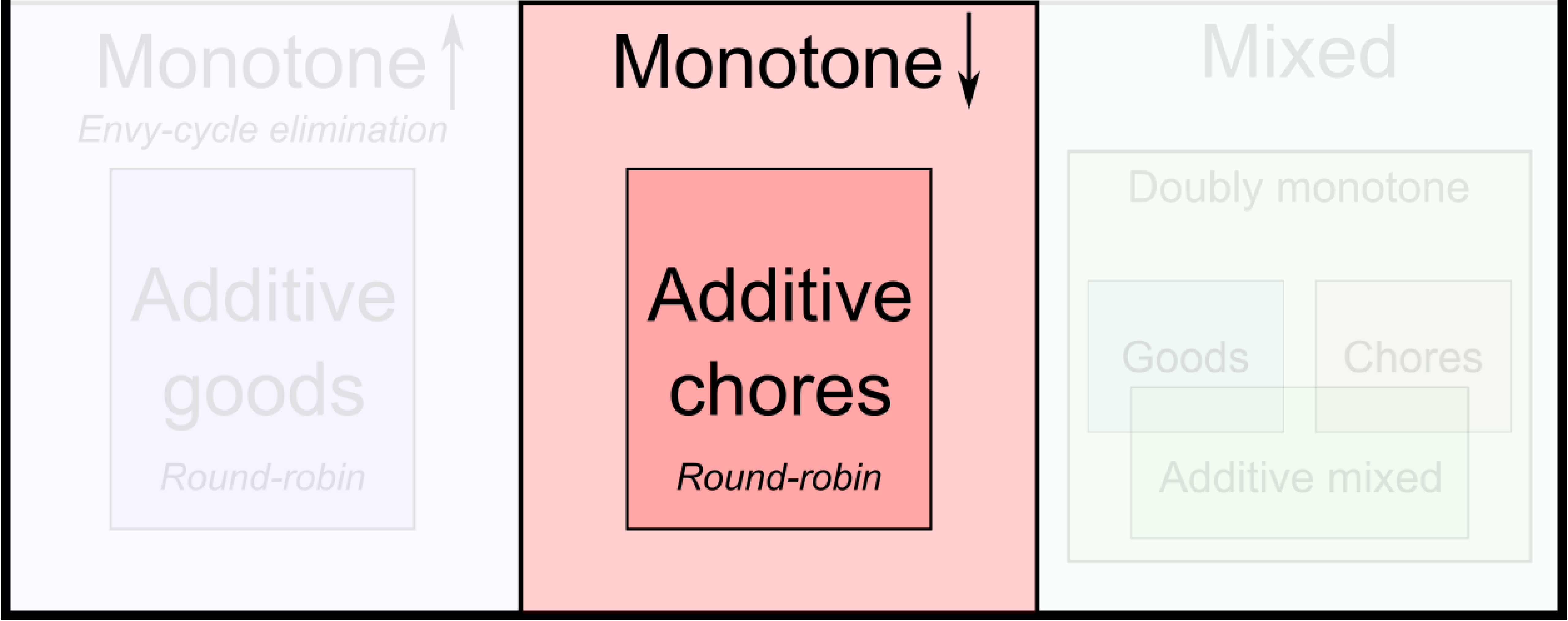

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# Adapting envy-cycle elimination to chores

While there is an unallocated good

- If the envy graph has a source vertex, assign the good to that agent.
- Otherwise, resolve envy cycles until a source vertex shows up, and then assign the good to it.

# Adapting envy-cycle elimination to chores

While there is an unallocated ~~good~~ chore

- If the envy graph has a ~~source~~ sink vertex, assign the ~~good~~ chore to that agent.
- Otherwise, resolve envy cycles until a ~~source~~ sink vertex shows up, and then assign the ~~good~~ chore to it.

# Adapting envy-cycle elimination to chores

While there is an unallocated chore

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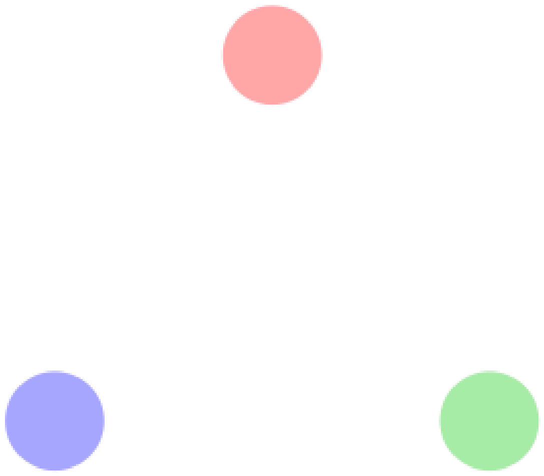
While there is an unallocated chore

- If the envy graph has a sink vertex, assign the chore to that agent.
- Otherwise, resolve envy cycles until a sink vertex shows up, and then assign the chore to it.

| | A | B | C | D | E | F |
|---|---|---|---|---|---|---|
| △ | -1 | -4 | -2 | -3 | 0 | -1 |
| △ | -2 | -1 | -2 | -2 | -3 | -1 |
| △ | -1 | -3 | -1 | -1 | -3 | -10 |



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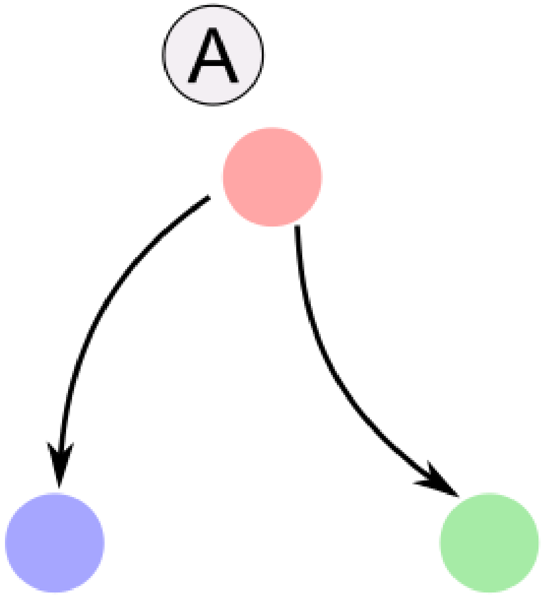

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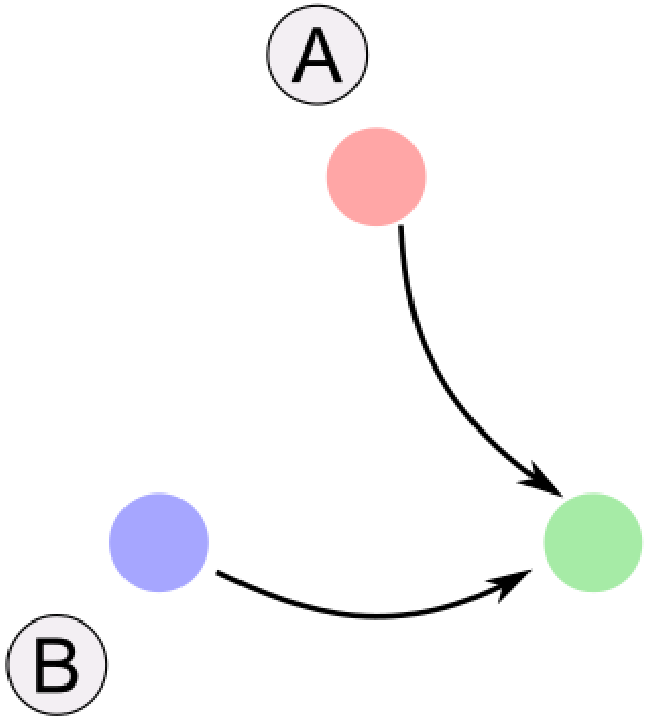

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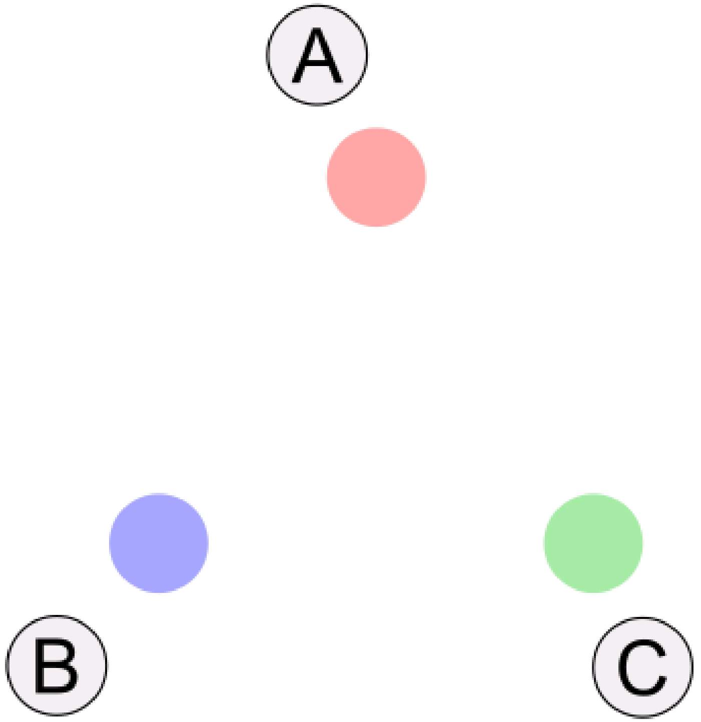

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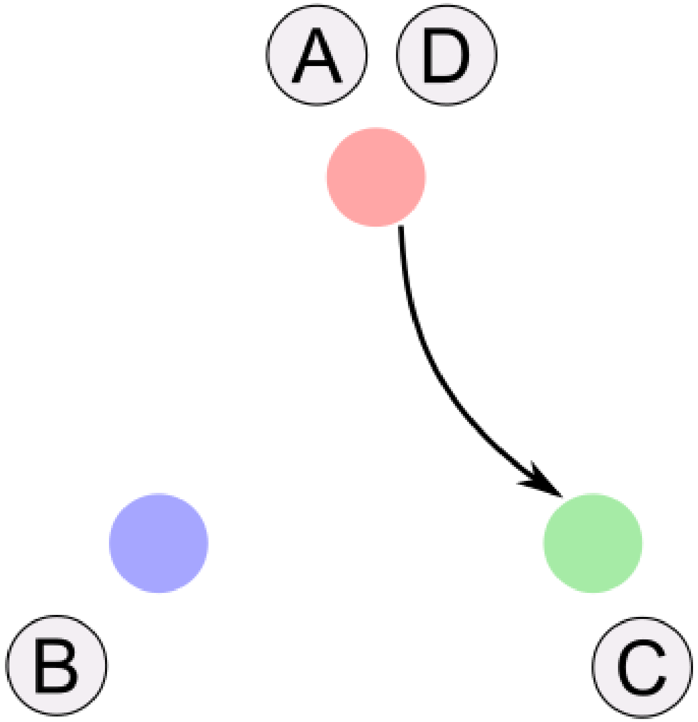
| △ | -1 | -3 | -1 | -1 | -3 | -10 |



# Adapting envy-cycle elimination to chores

While there is an unallocated chore

- If the envy graph has a sink vertex, assign the chore to that agent.
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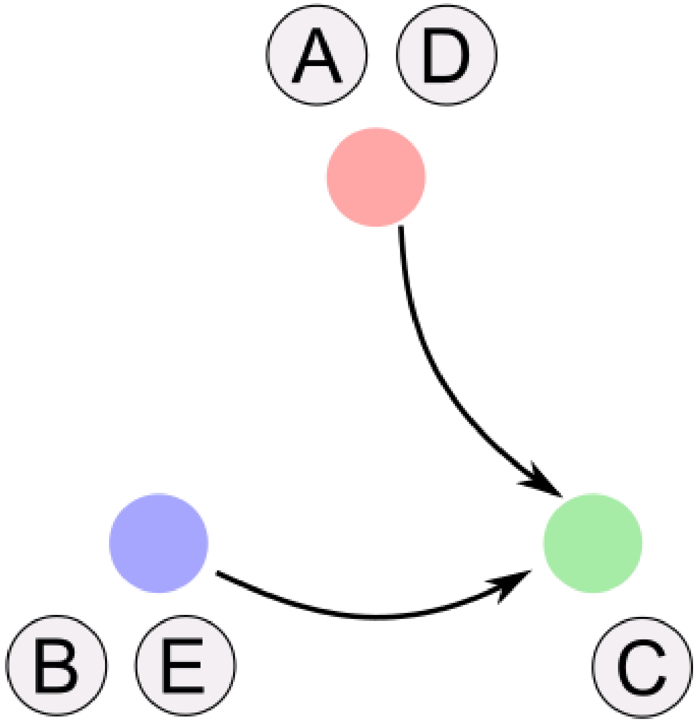
| | A | B | C | D | E | F |
|---|---|---|---|---|---|---|
| △ | -1 | -4 | -2 | -3 | 0 | -1 |
| △ | -2 | -1 | -2 | -2 | -3 | -1 |
| △ | -1 | -3 | -1 | -1 | -3 | -10 |



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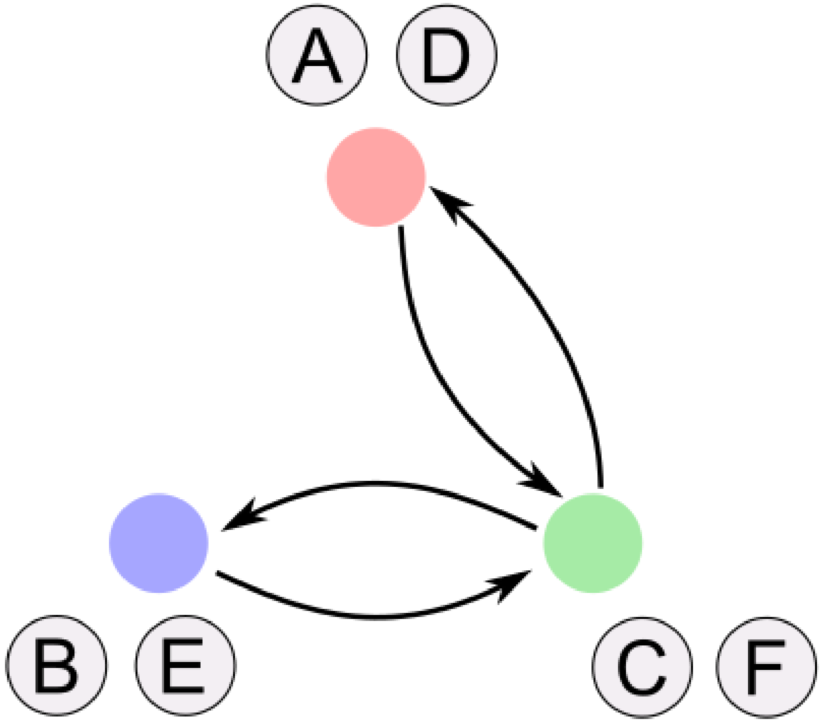

# Adapting envy-cycle elimination to chores

```
While there is an unallocated chore
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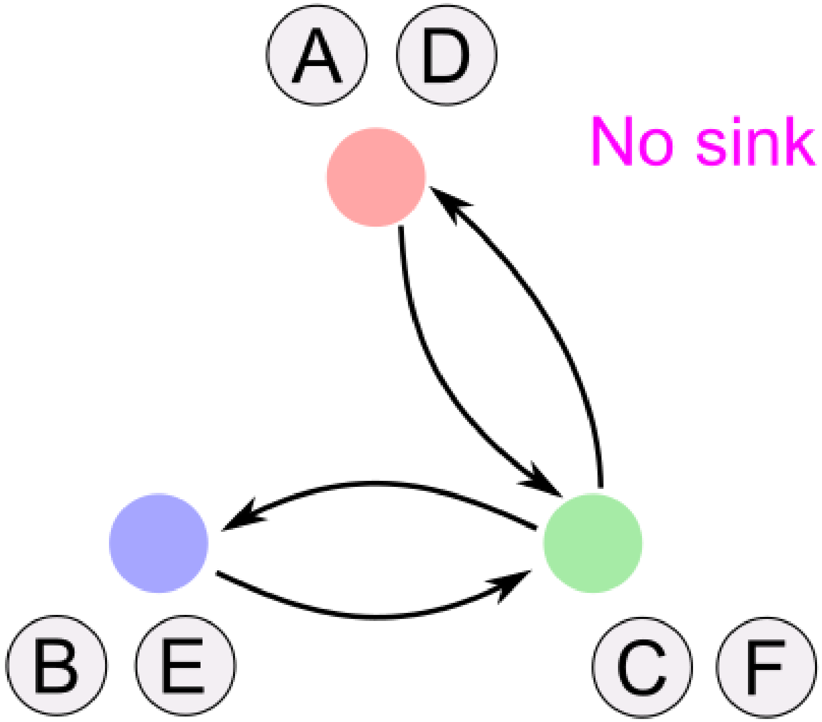

No sink

# Adapting envy-cycle elimination to chores

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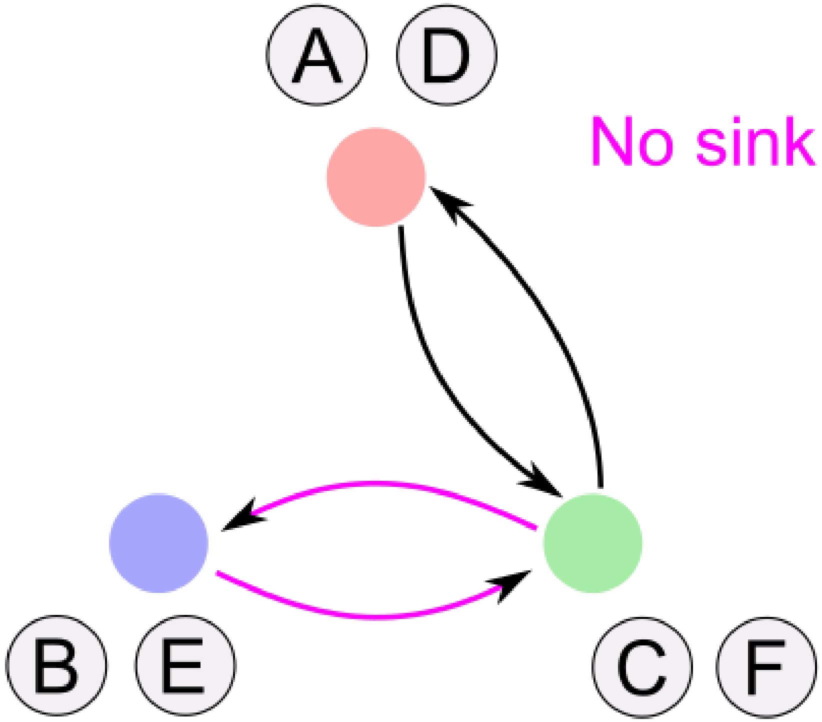

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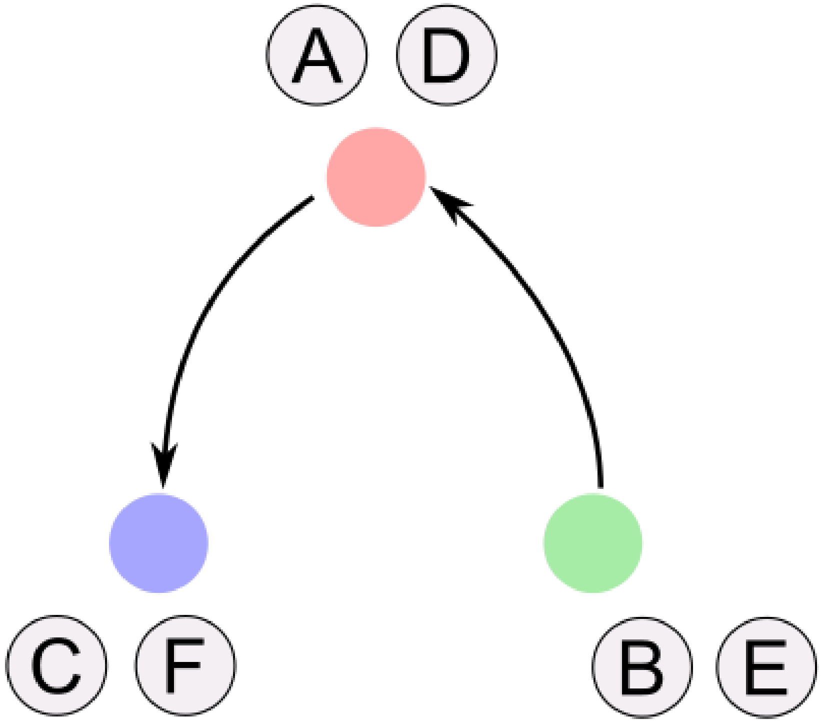

# Adapting envy-cycle elimination to chores

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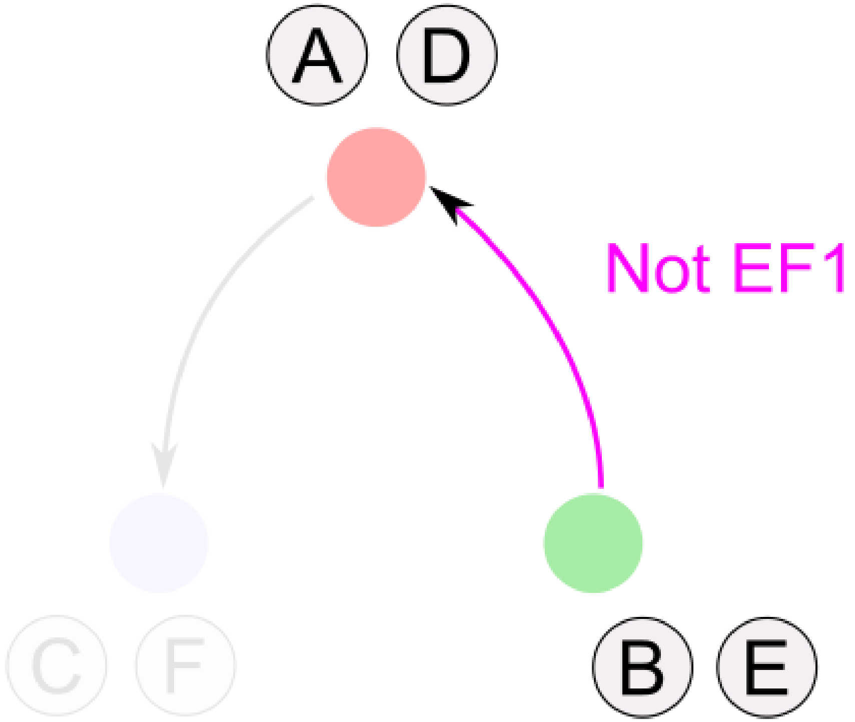

# Adapting envy-cycle elimination to chores

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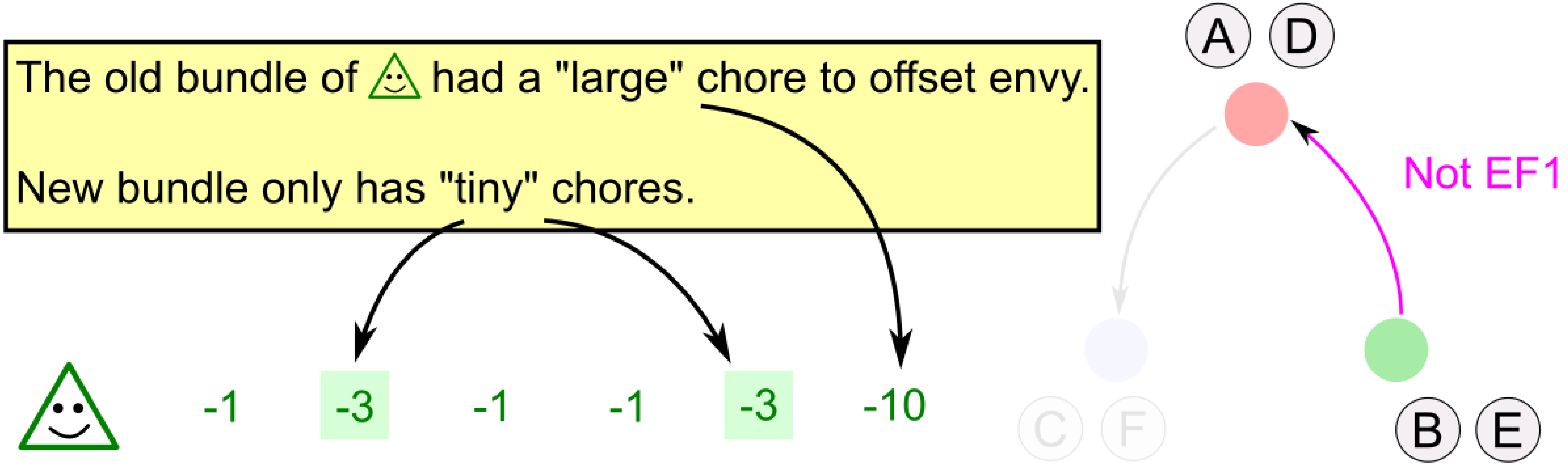

The old bundle of △ had a "large" chore to offset envy.

New bundle only has "tiny" chores.

# Adapting envy-cycle elimination to chores

While there is an unallocated chore

- If the envy graph has a sink vertex, assign the chore to that agent.
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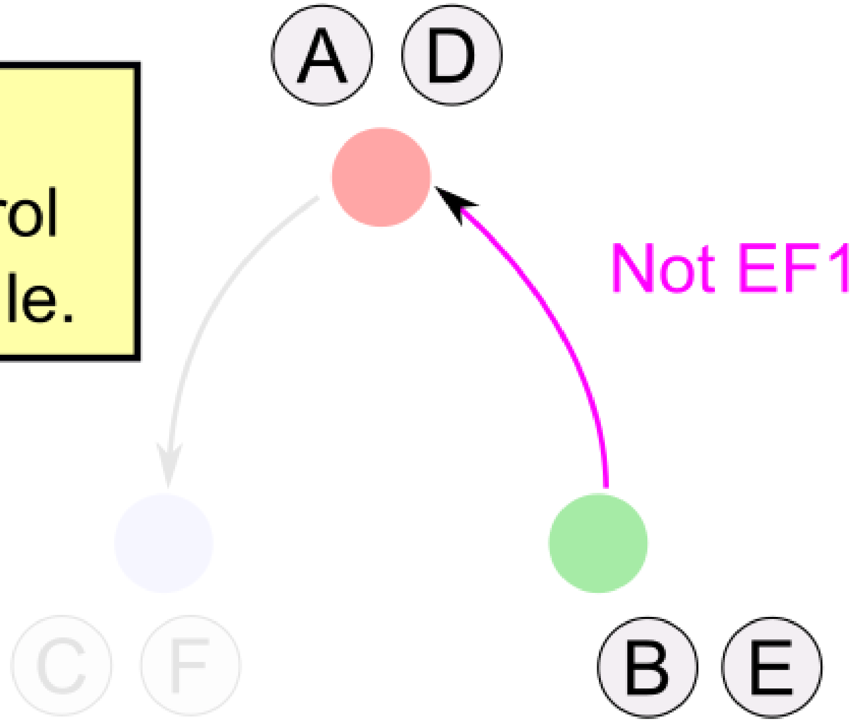
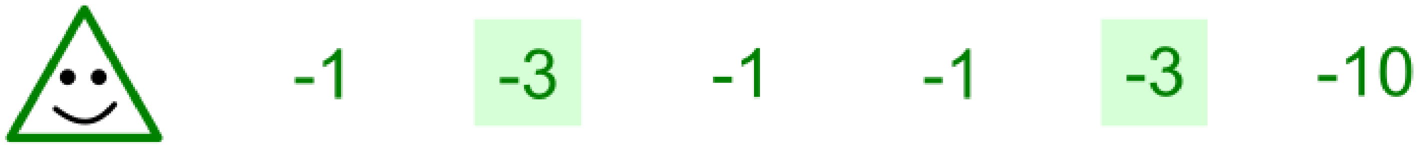
**Source of the problem**

Resolving arbitrary envy cycles gives us no control over the size of individual chores in the new bundle.





# Adapting envy-cycle elimination to chores

While there is an unallocated chore

- If the envy graph has a sink vertex, assign the chore to that agent.
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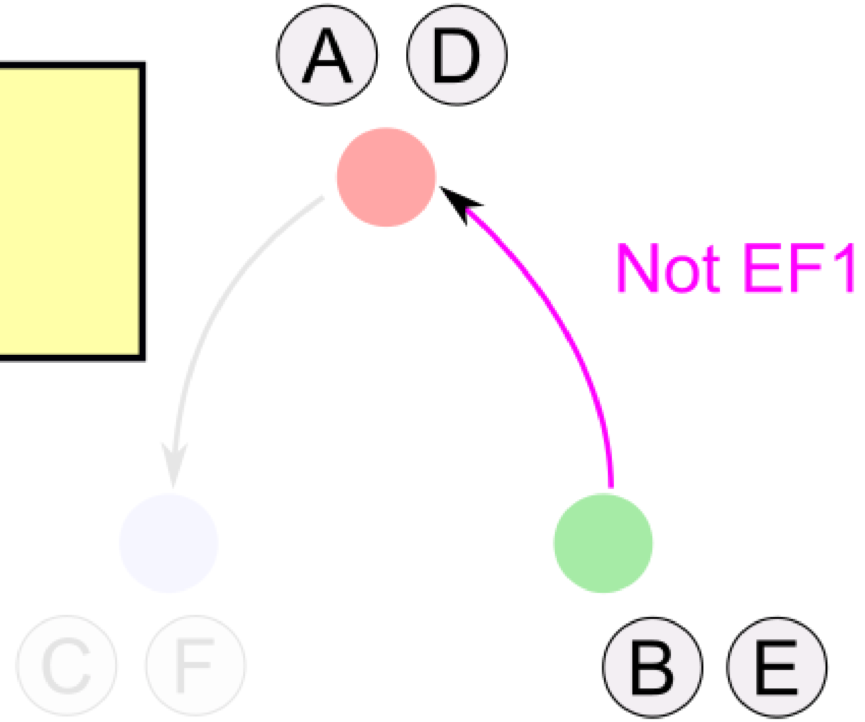
**Solution**

Resolve top-trading envy cycle

Each agent points to its *favorite* envied bundle



△ -1 -3 -1 -1 -3 -10

# Top-trading envy-cycle elimination

[Bhaskar, Sricharan, and Vaish, *APPROX* 2021]

While there is an unallocated chore

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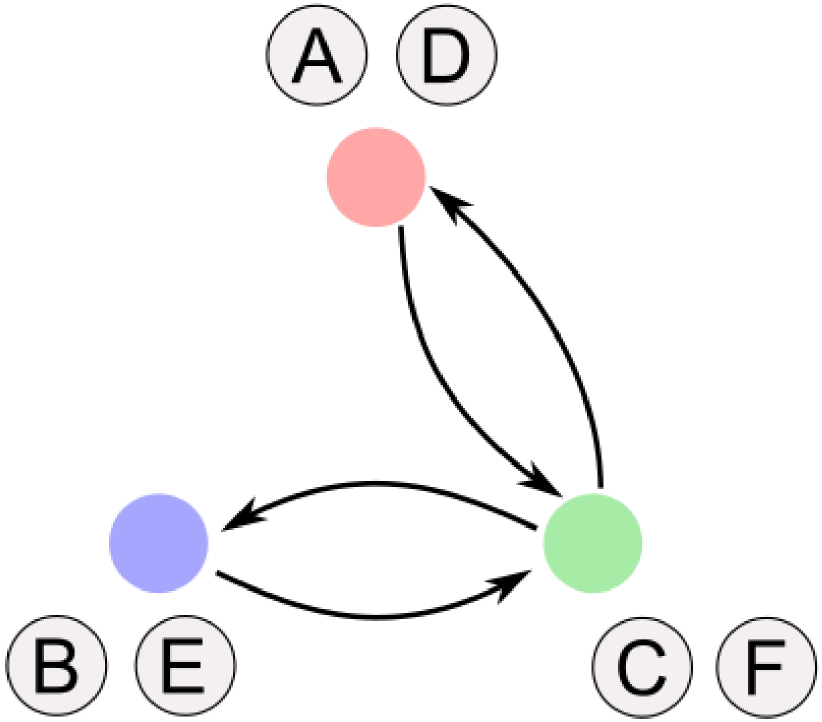

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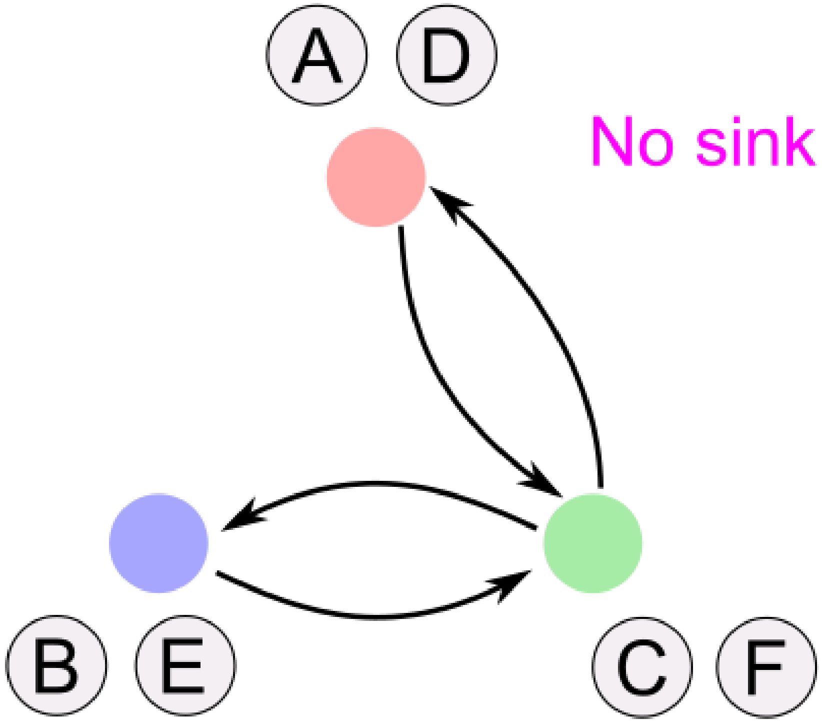

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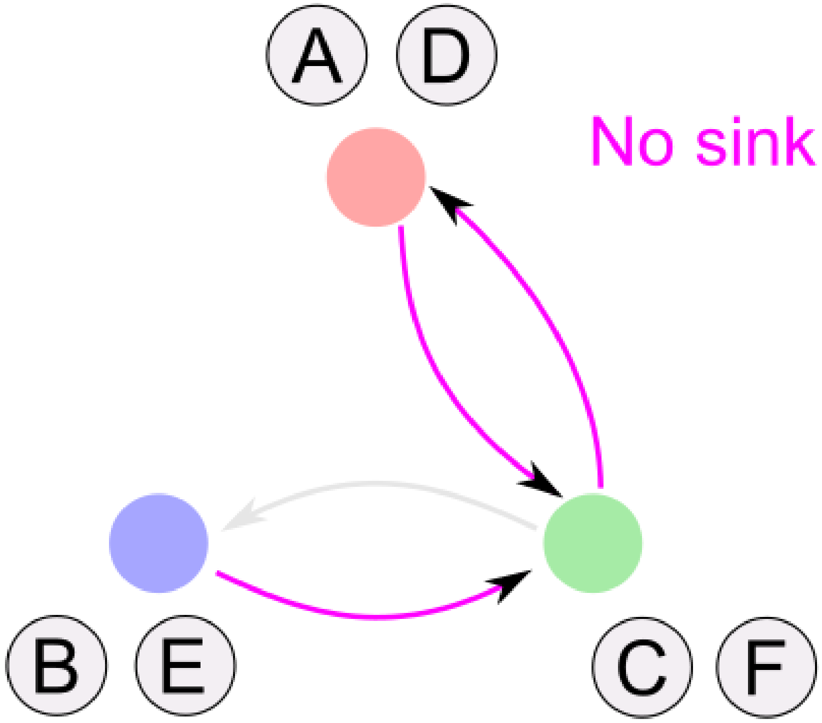

No sink

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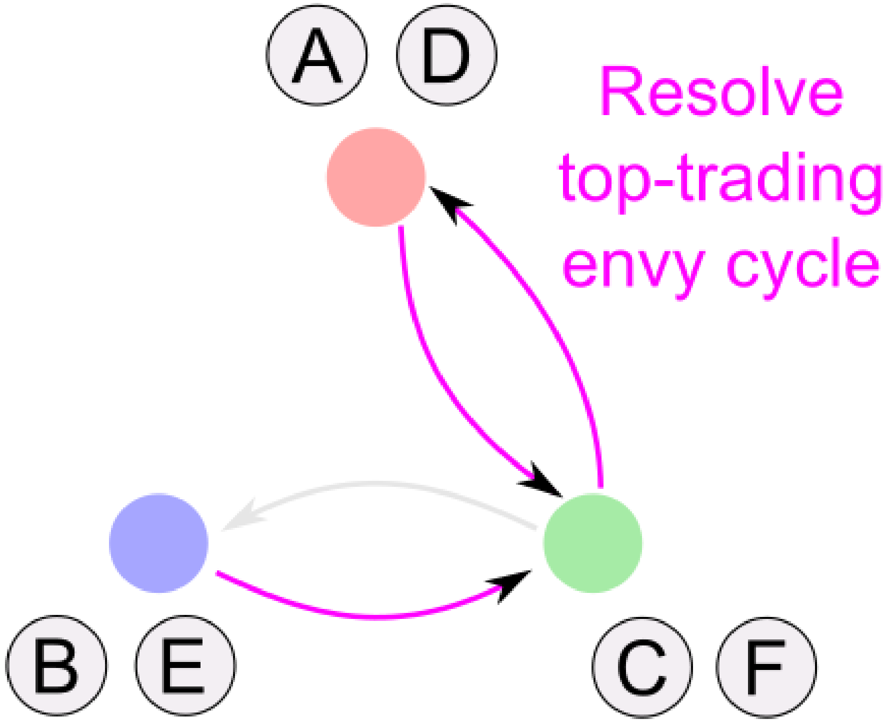

Resolve top-trading envy cycle

# Top-trading envy-cycle elimination

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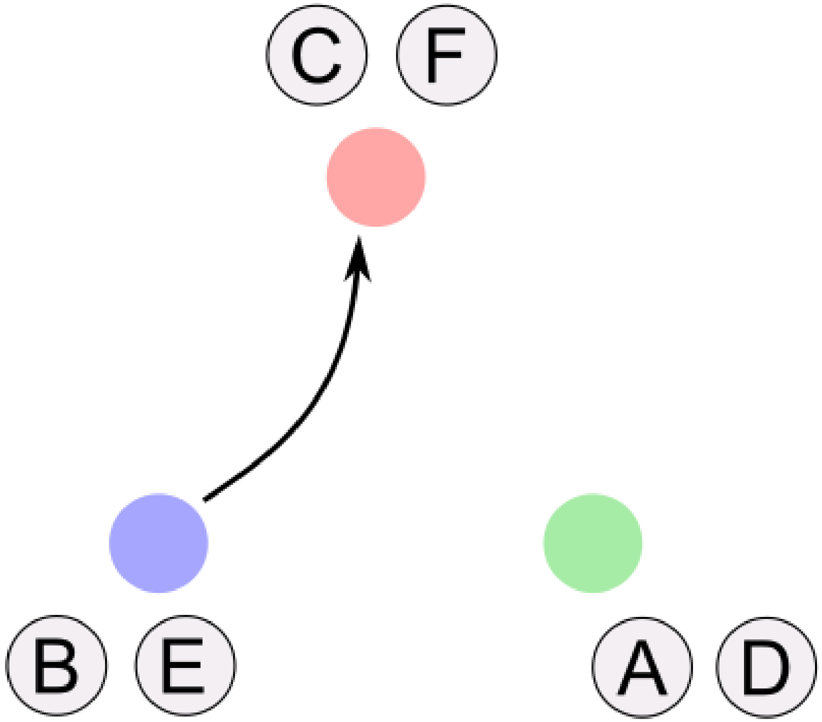

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Why does a top-trading envy cycle exist when there is no sink?

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No sink ⇒ Every vertex has an outgoing envy edge

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Why does a top-trading envy cycle exist when there is no sink?

No sink ⇒ Every vertex has ~~an~~ a favorite outgoing envy edge

⇒ There is a cycle of "most envied" edges

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Why does top-trading envy cycle algorithm satisfy EF1?

# Top-trading envy-cycle elimination

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- If the envy graph has a sink vertex, assign the chore to that agent.
- Otherwise, resolve a top-trading envy cycle until a sink vertex shows up, and then assign the chore to it.

Why does top-trading envy cycle algorithm satisfy EF1?

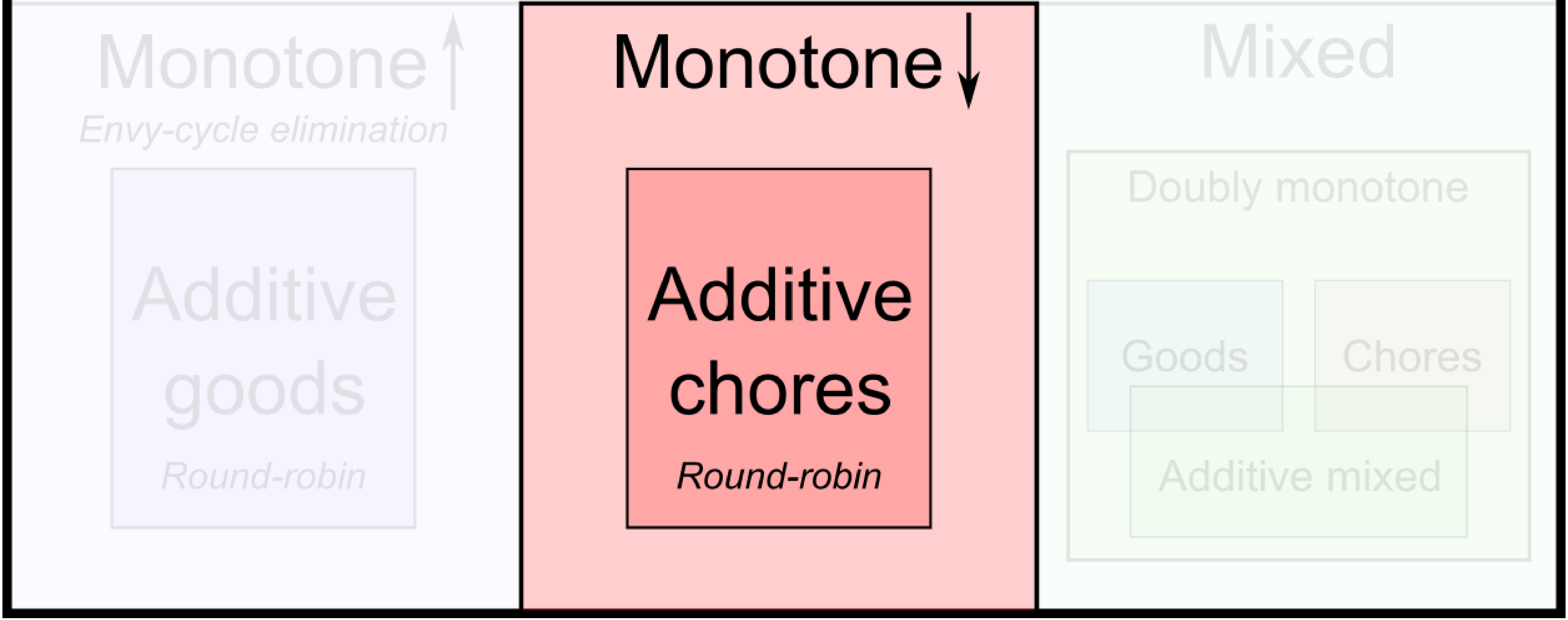
Every vertex in the top-trading cycle becomes envy-free.

The problem of "new bundle with tiny chores" does not arise.

[Bhaskar, Sricharan, and Vaish, *APPROX* 2021]

For monotone chores, the allocation computed by the top-trading envy-cycle elimination algorithm satisfies EF1.

# The Story of EF1



Monotone↑

*Envy-cycle elimination*

Additive goods

*Round-robin*

Monotone↓

Additive chores

*Round-robin*

Mixed

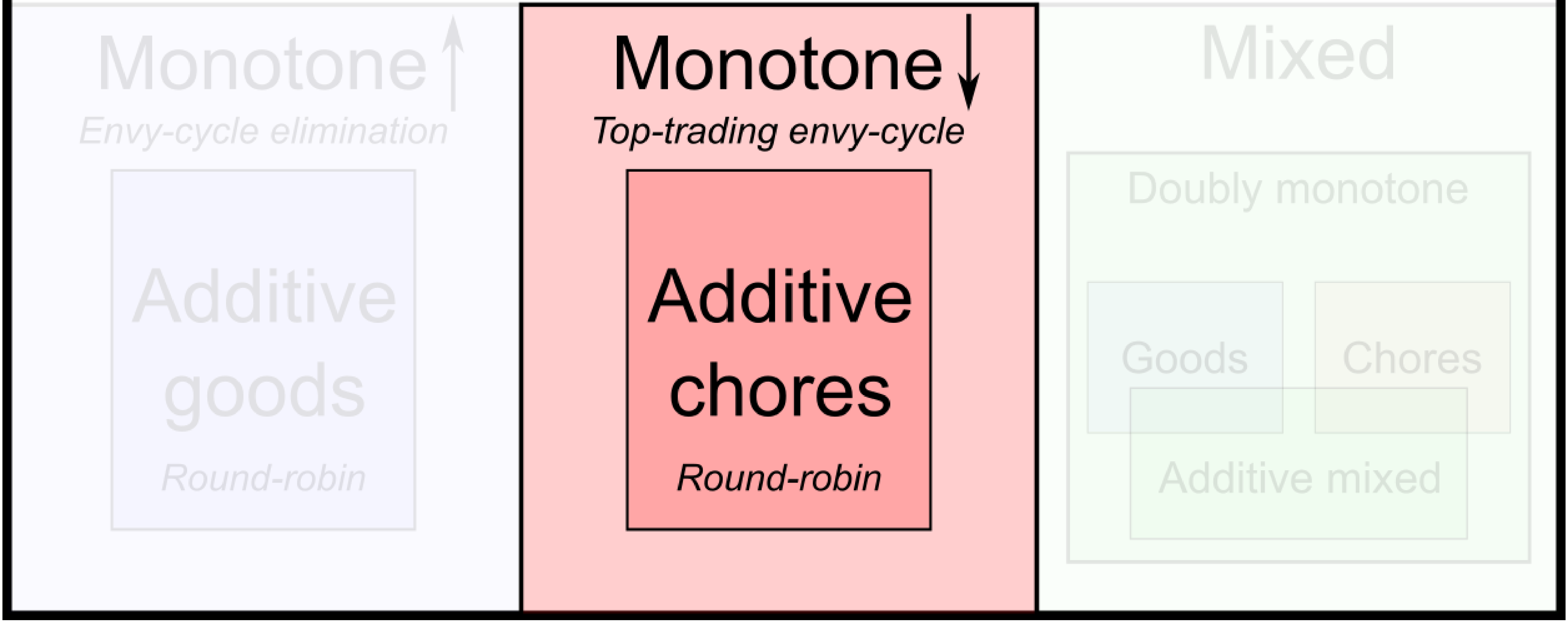
Doubly monotone

Goods

Chores

Additive mixed

# The Story of EF1



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Monotone↓

*Top-trading envy-cycle*

Additive chores

*Round-robin*

Mixed

Doubly monotone

Goods

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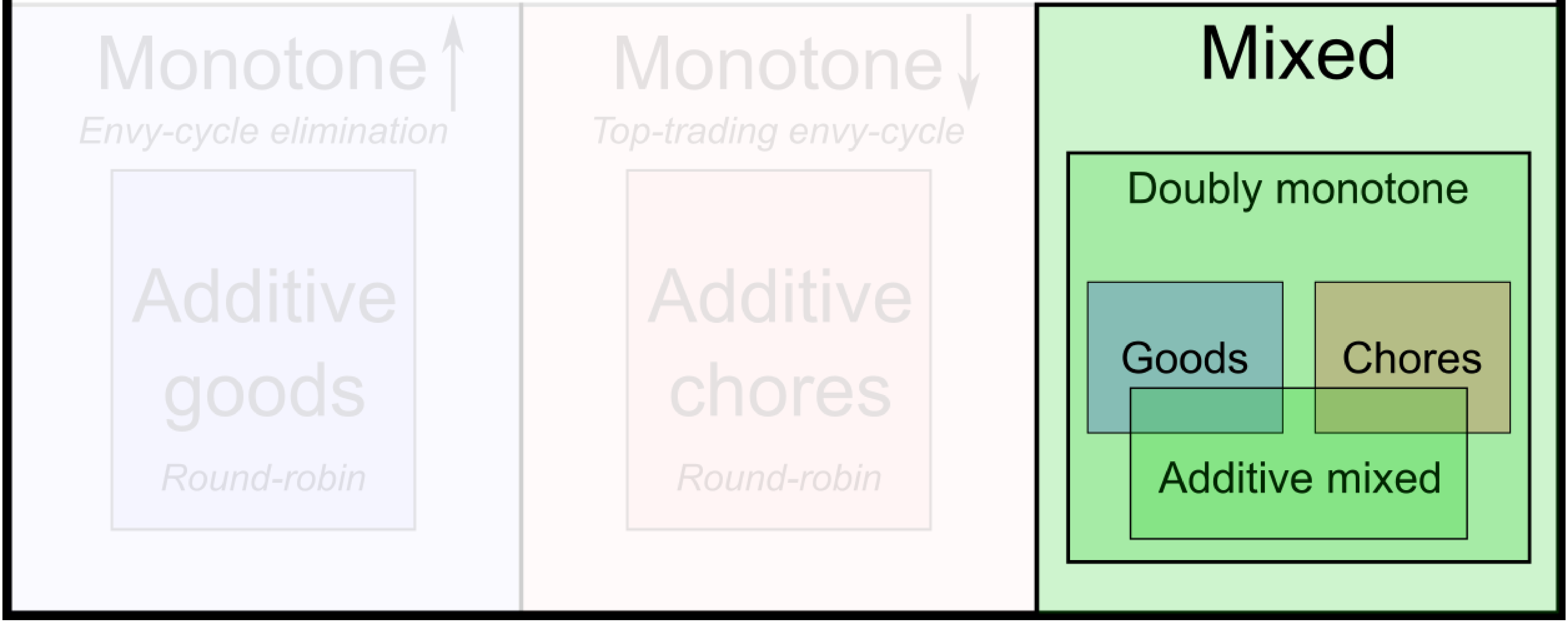
Additive mixed

# The Story of EF1



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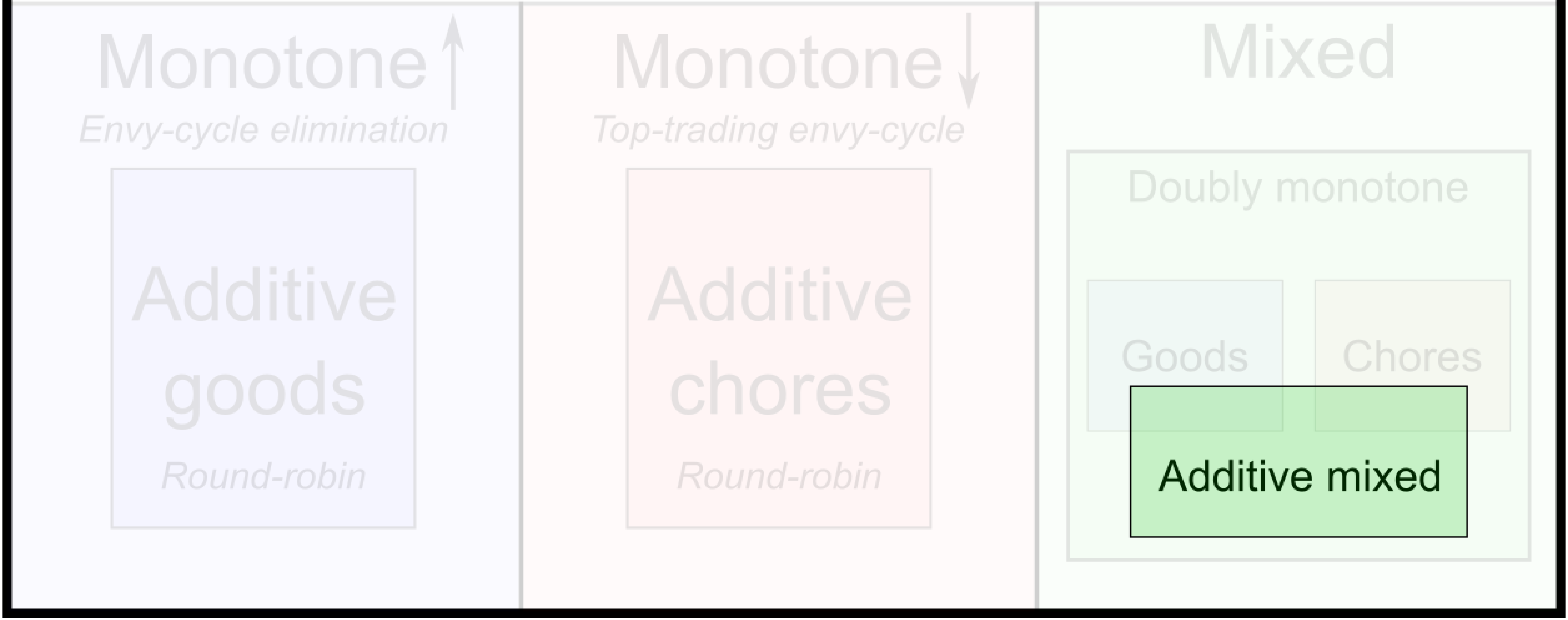

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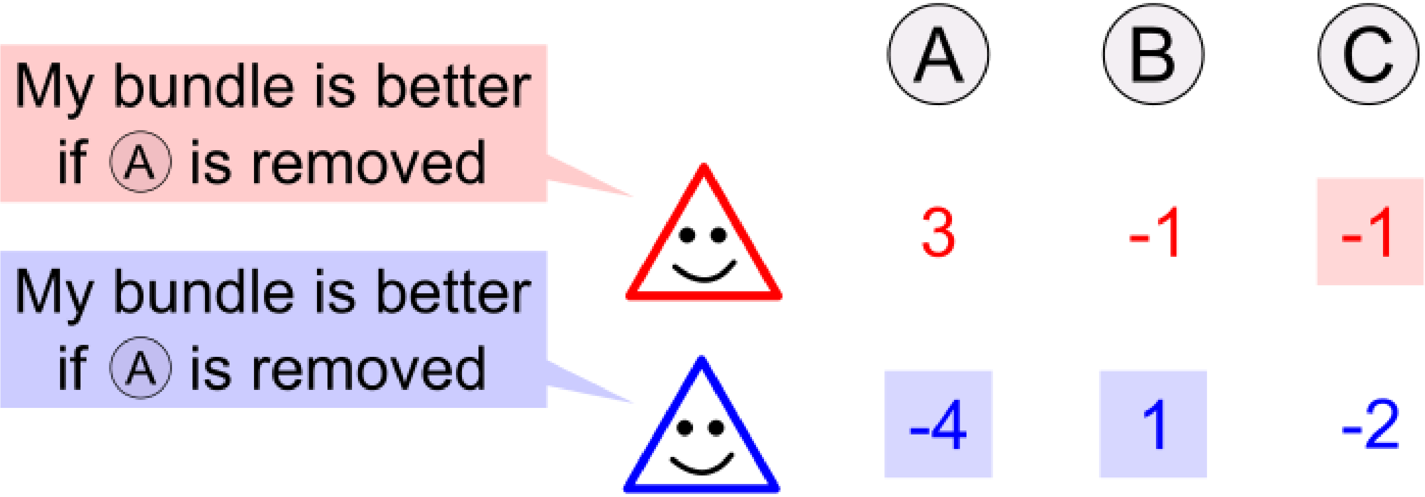
# Envy-Freeness Up To One Item

[Aziz, Caragiannis, Igarashi, and Walsh; *IJCAI* 2019; *JAAMAS* 2022]

Eliminate envy by removing some "good" in the envied bundle or some "chore" in the envious agent's bundle.



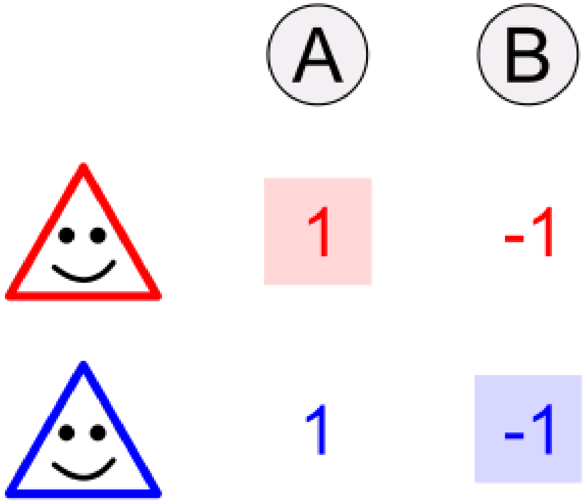
My bundle is better if Ⓐ is removed

My bundle is better if Ⓐ is removed

|   | Ⓐ | Ⓑ | Ⓒ |
|---|---|---|---|
| △ | 3 | -1 | -1 |
| △ | -4 | 1 | -2 |

Allocation $A = (A_1, \ldots, A_n)$ is EF1 if for every pair of agents $i, k$, there exists an item $j \in A_i \cup A_k$ s.t. $v_i(A_i \setminus \{j\}) \geq v_i(A_k \setminus \{j\})$.

# For goods+chores, naive round-robin fails EF1.



|  | A | B |
|---|---|---|
| △ | 1 | -1 |
| △ | 1 | -1 |

# Double round-robin algorithm

[Aziz, Caragiannis, Igarashi, and Walsh; *IJCAI* 2019; *JAAMAS* 2022]

# Double round-robin algorithm

[Aziz, Caragiannis, Igarashi, and Walsh; *IJCAI* 2019; *JAAMAS* 2022]

Partition the items into two sets: positive and negative

Positive: items with strictly positive value for at least one agent

(considered to be a "good" by at least one agent)

Negative: all other items

(considered a "chore" by all agents)

# Double round-robin algorithm

[Aziz, Caragiannis, Igarashi, and Walsh; *IJCAI* 2019; *JAAMAS* 2022]

| | Ⓐ | Ⓑ | Ⓒ | Ⓓ | Ⓔ |
|---|---|---|---|---|---|
| △ | -4 | -1 | -2 | 2 | -4 |
| △ | 0 | -1 | -5 | -2 | -1 |
| △ | -4 | -2 | -5 | 0 | 2 |

# Double round-robin algorithm

[Aziz, Caragiannis, Igarashi, and Walsh; *IJCAI* 2019; *JAAMAS* 2022]

|  | - | - | - | + | + |
|---|---|---|---|---|---|
|  | A | B | C | D | E |
| △ | -4 | -1 | -2 | 2 | -4 |
| △ | 0 | -1 | -5 | -2 | -1 |
| △ | -4 | -2 | -5 | 0 | 2 |

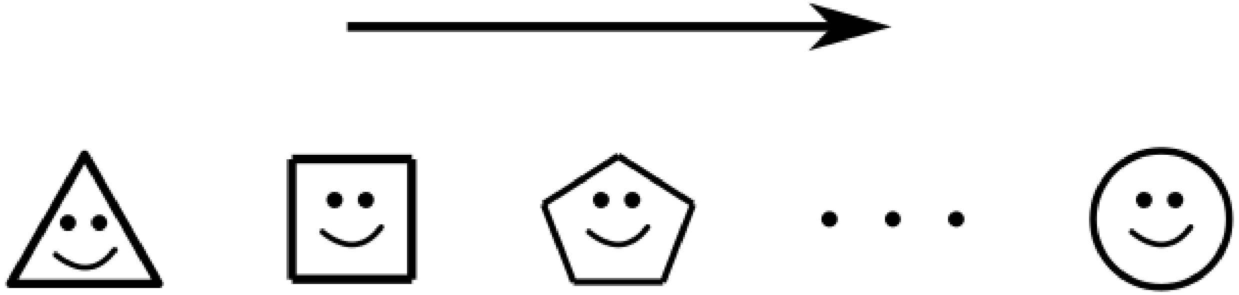
# Double round-robin algorithm

[Aziz, Caragiannis, Igarashi, and Walsh; *IJCAI* 2019; *JAAMAS* 2022]

# Double round-robin algorithm

[Aziz, Caragiannis, Igarashi, and Walsh; *IJCAI* 2019; *JAAMAS* 2022]

allocate negative items in this order



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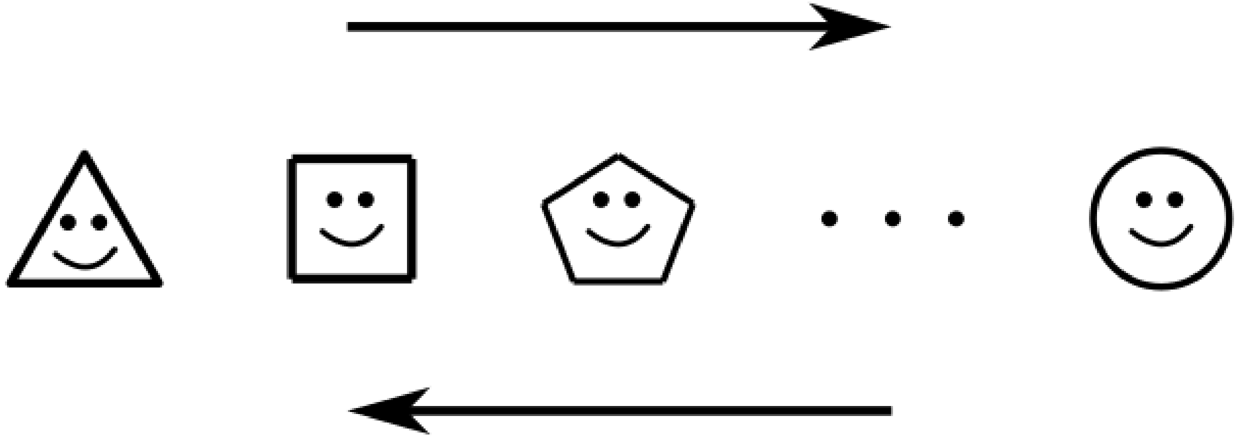


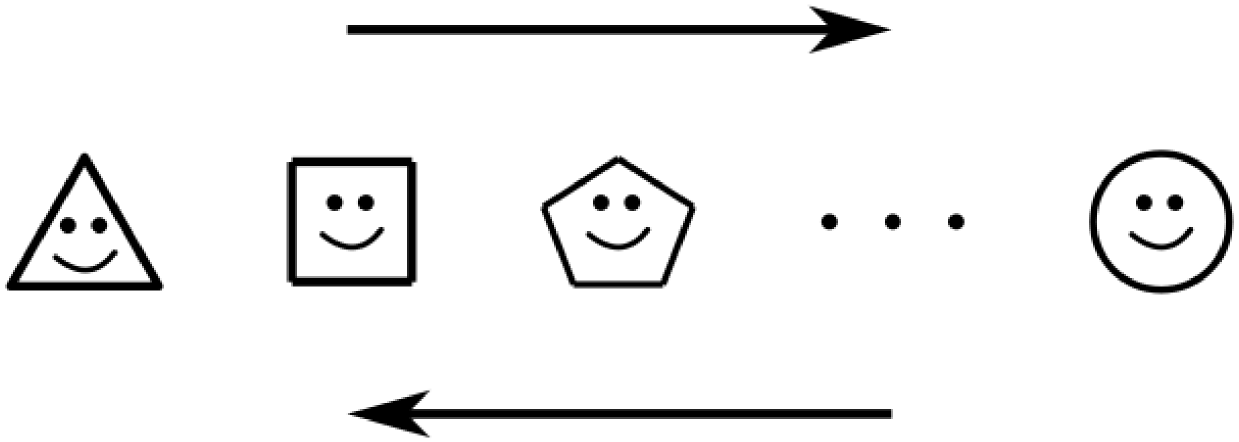
and positive items in the opposite order

# Double round-robin algorithm

[Aziz, Caragiannis, Igarashi, and Walsh; *IJCAI* 2019; *JAAMAS* 2022]

allocate negative items in this order

No. of negative items
is an integer multiple of n
(add zero valued items)



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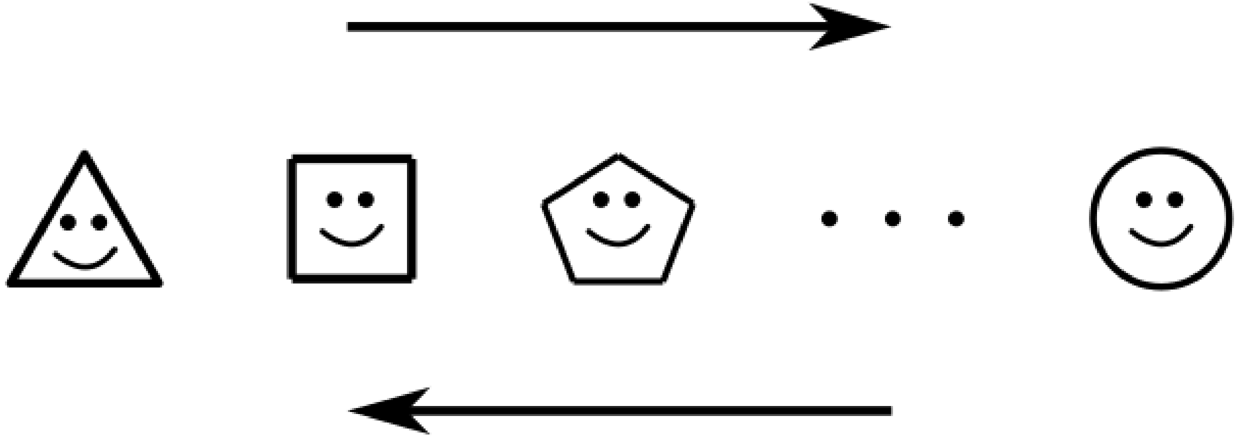


and positive items in the opposite order

Picking with skipping

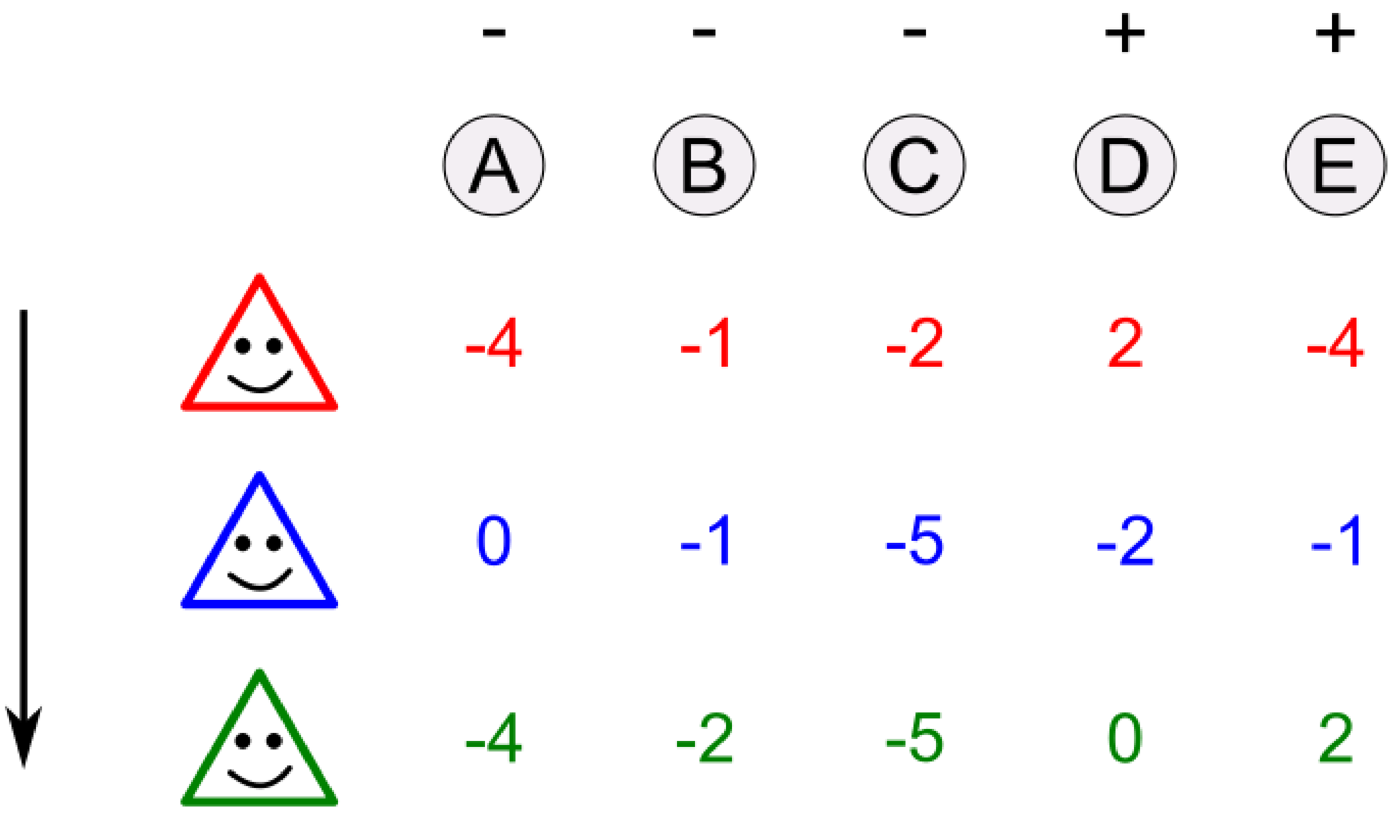
# Double round-robin algorithm

[Aziz, Caragiannis, Igarashi, and Walsh; *IJCAI* 2019; *JAAMAS* 2022]

|   | - | - | - | + | + |
|---|---|---|---|---|---|
|   | A | B | C | D | E |
| △ | -4 | -1 | -2 | 2 | -4 |
| △ | 0 | -1 | -5 | -2 | -1 |
| △ | -4 | -2 | -5 | 0 | 2 |

# Double round-robin algorithm

[Aziz, Caragiannis, Igarashi, and Walsh; *IJCAI* 2019; *JAAMAS* 2022]



| | - | - | - | + | + |
|---|---|---|---|---|---|
| | A | B | C | D | E |
| △ | -4 | -1 | -2 | 2 | -4 |
| △ | 0 | -1 | -5 | -2 | -1 |
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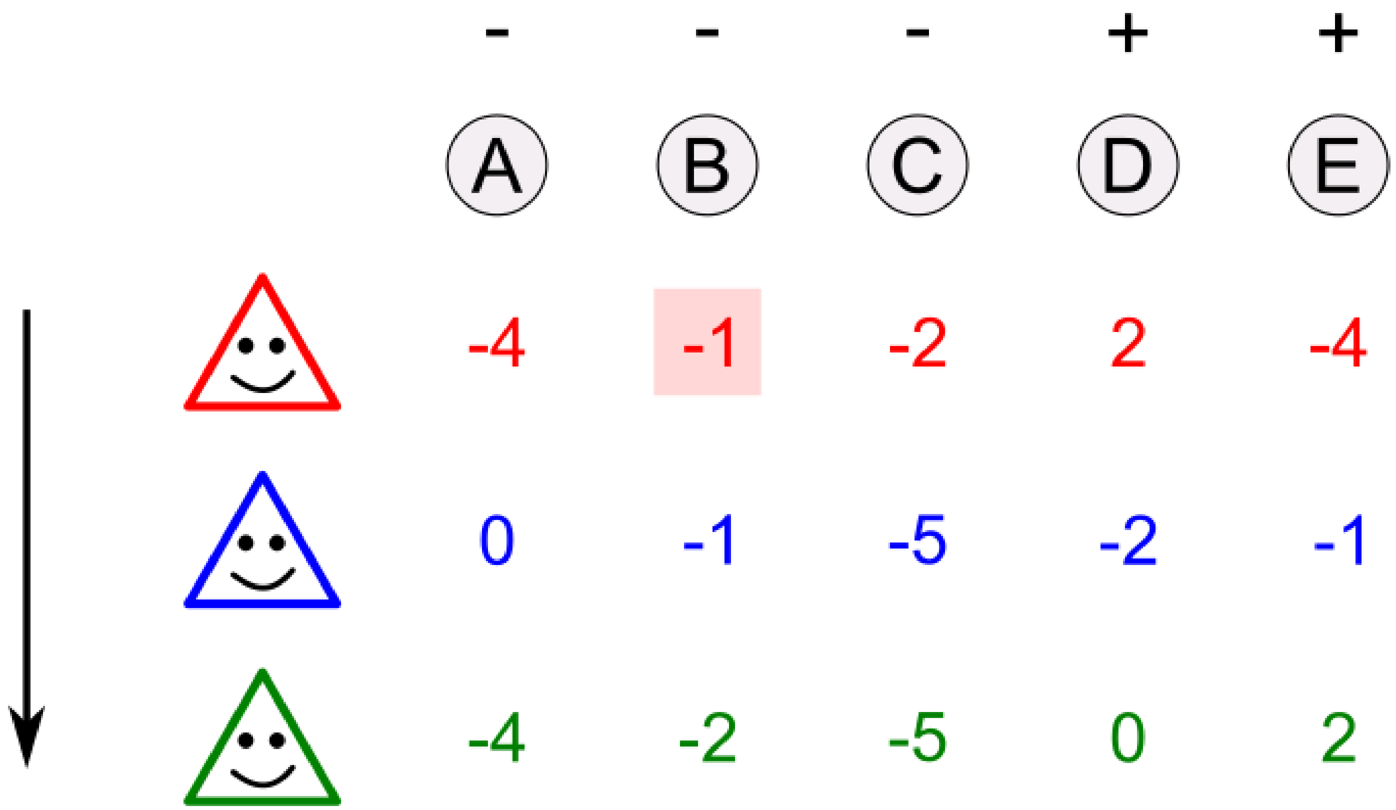

|  | - | - | - | + | + |
|---|---|---|---|---|---|
|  | A | B | C | D | E |
| △ | -4 | -1 | -2 | 2 | -4 |
| △ | 0 | -1 | -5 | -2 | -1 |
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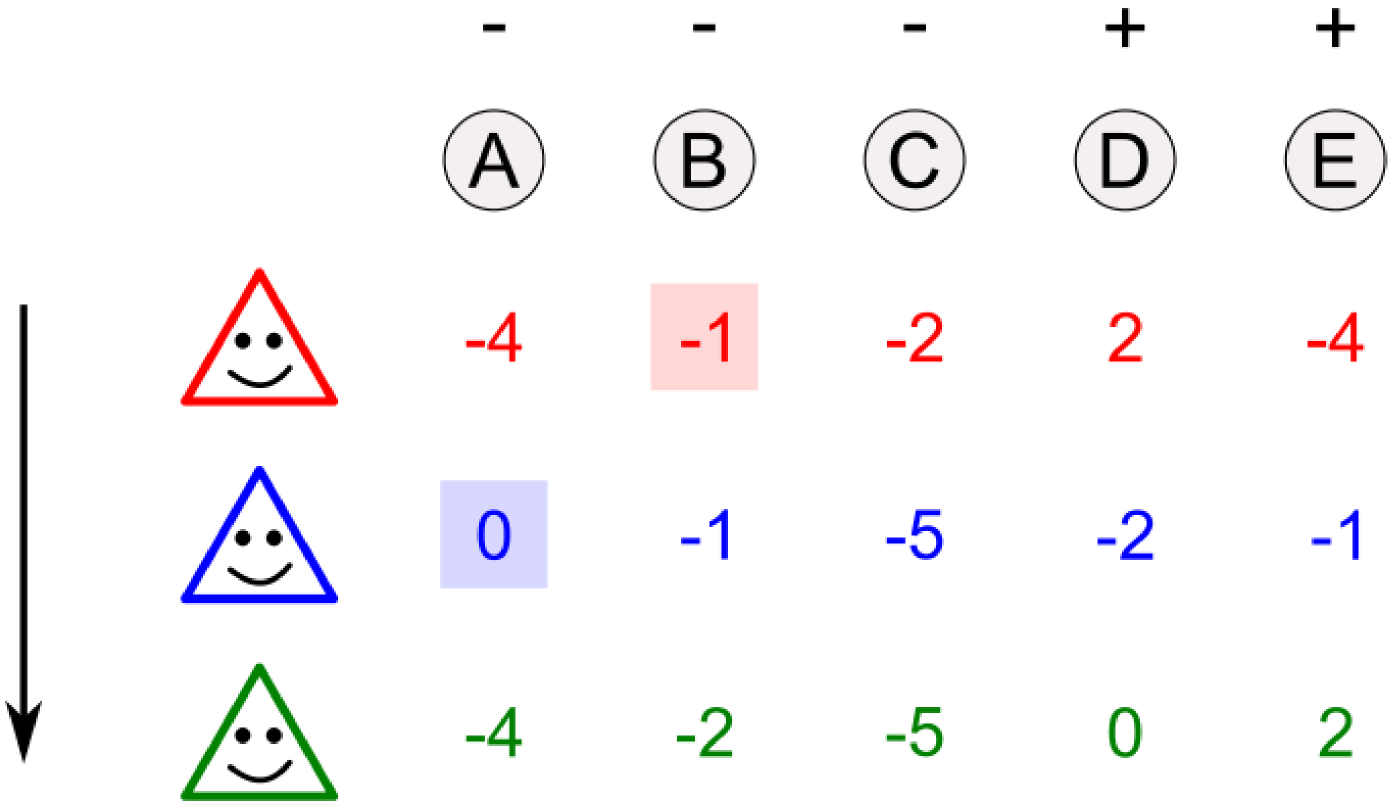
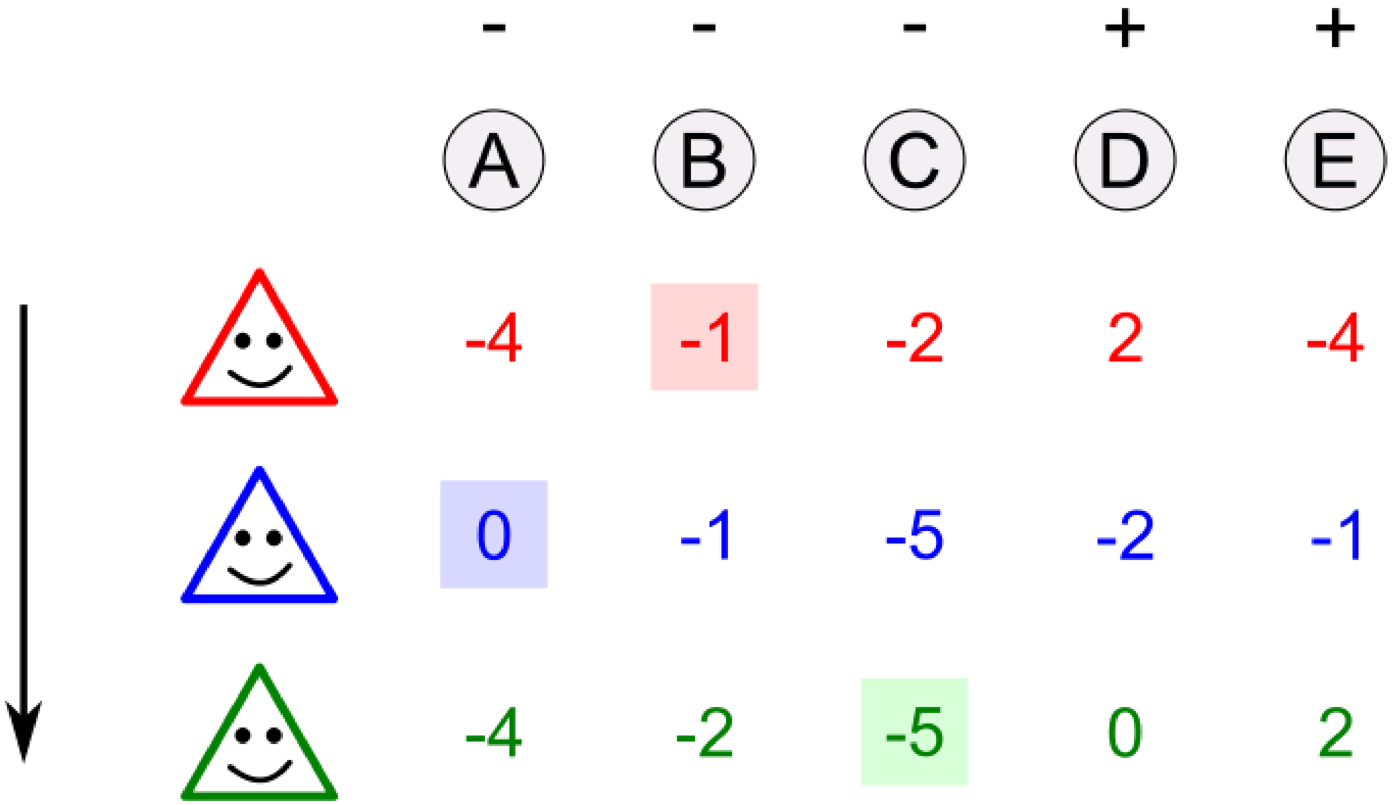


|   | - | - | - | + | + |
|---|---|---|---|---|---|
|   | A | B | C | D | E |
| △ | -4 | -1 | -2 | 2 | -4 |
| △ | 0 | -1 | -5 | -2 | -1 |
| △ | -4 | -2 | -5 | 0 | 2 |

# Double round-robin algorithm

[Aziz, Caragiannis, Igarashi, and Walsh; *IJCAI* 2019; *JAAMAS* 2022]



|   | - | - | - | + | + |
|---|---|---|---|---|---|
|   | A | B | C | D | E |
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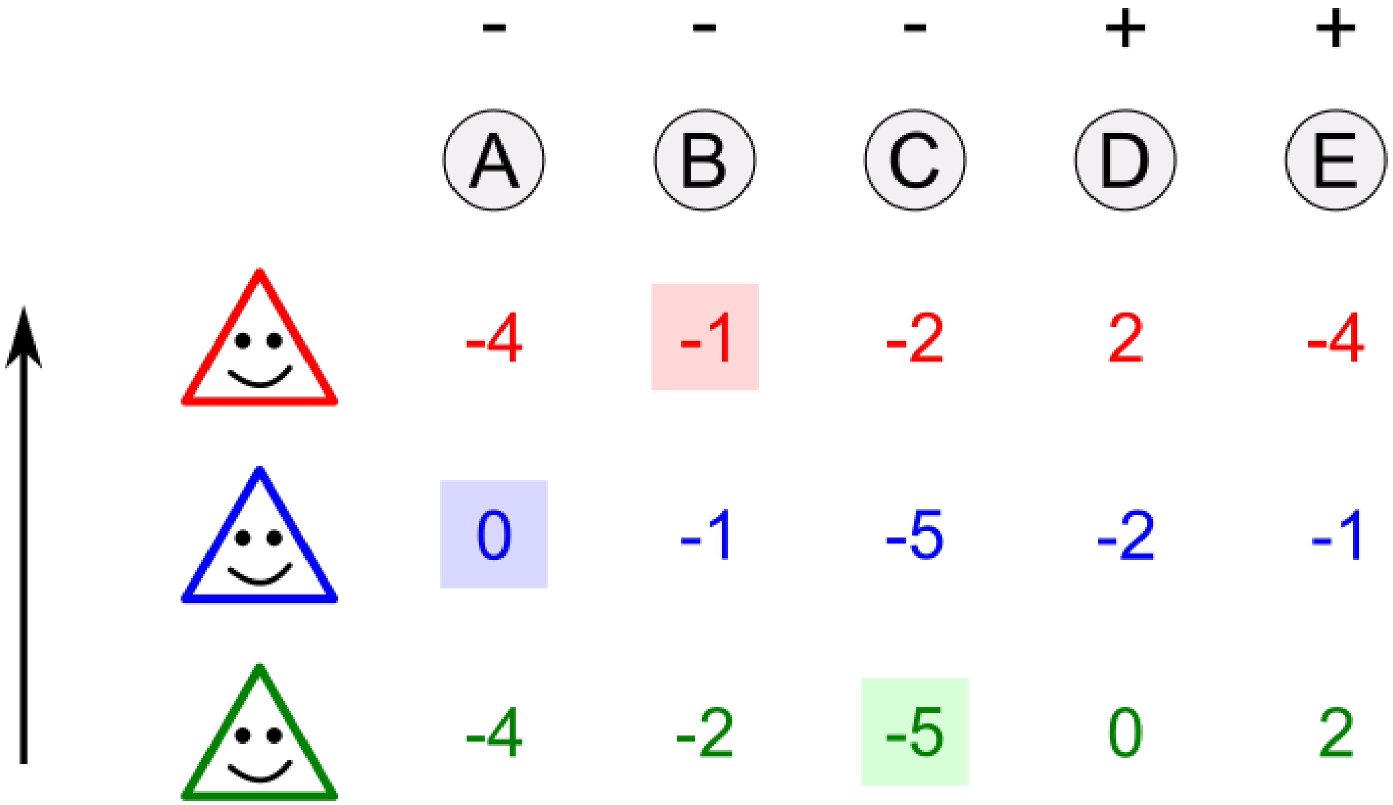

|  | - | - | - | + | + |
|---|---|---|---|---|---|
|  | A | B | C | D | E |
| 🟥 | -4 | **-1** | -2 | 2 | -4 |
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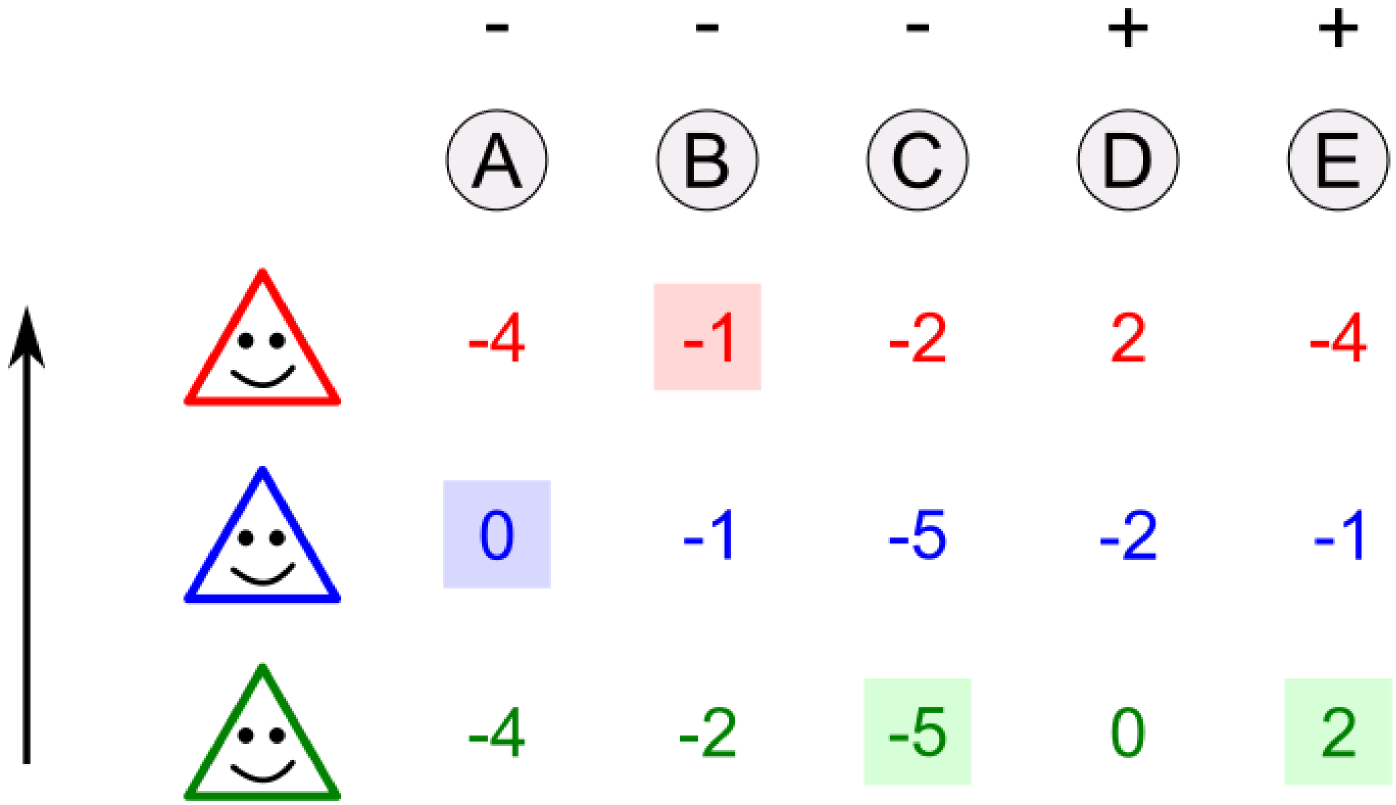

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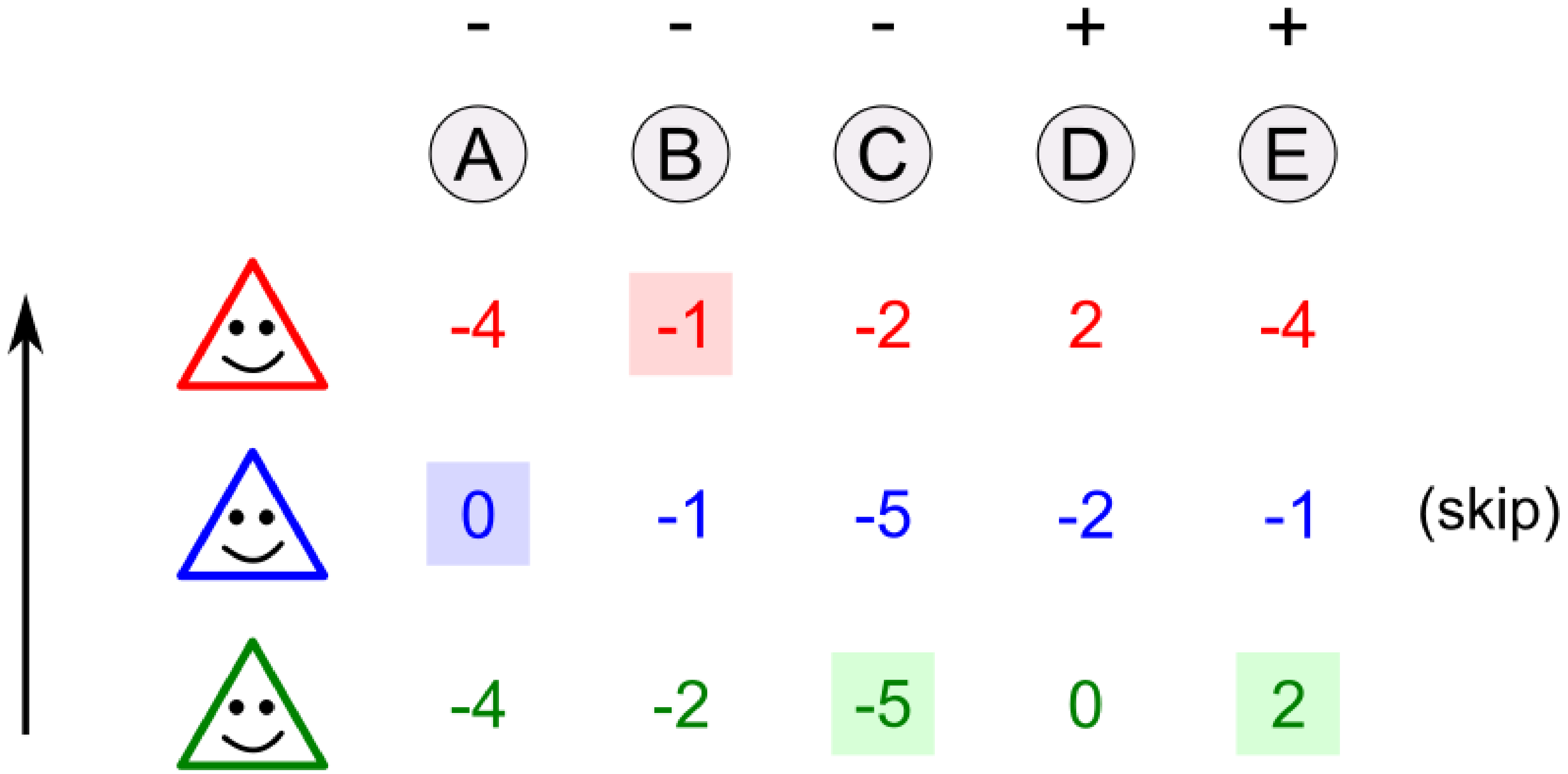

| | - | - | - | + | + | |
|---|---|---|---|---|---|---|
| | A | B | C | D | E | |
| | -4 | -1 | -2 | 2 | -4 | |
| | 0 | -1 | -5 | -2 | -1 | (skip) |
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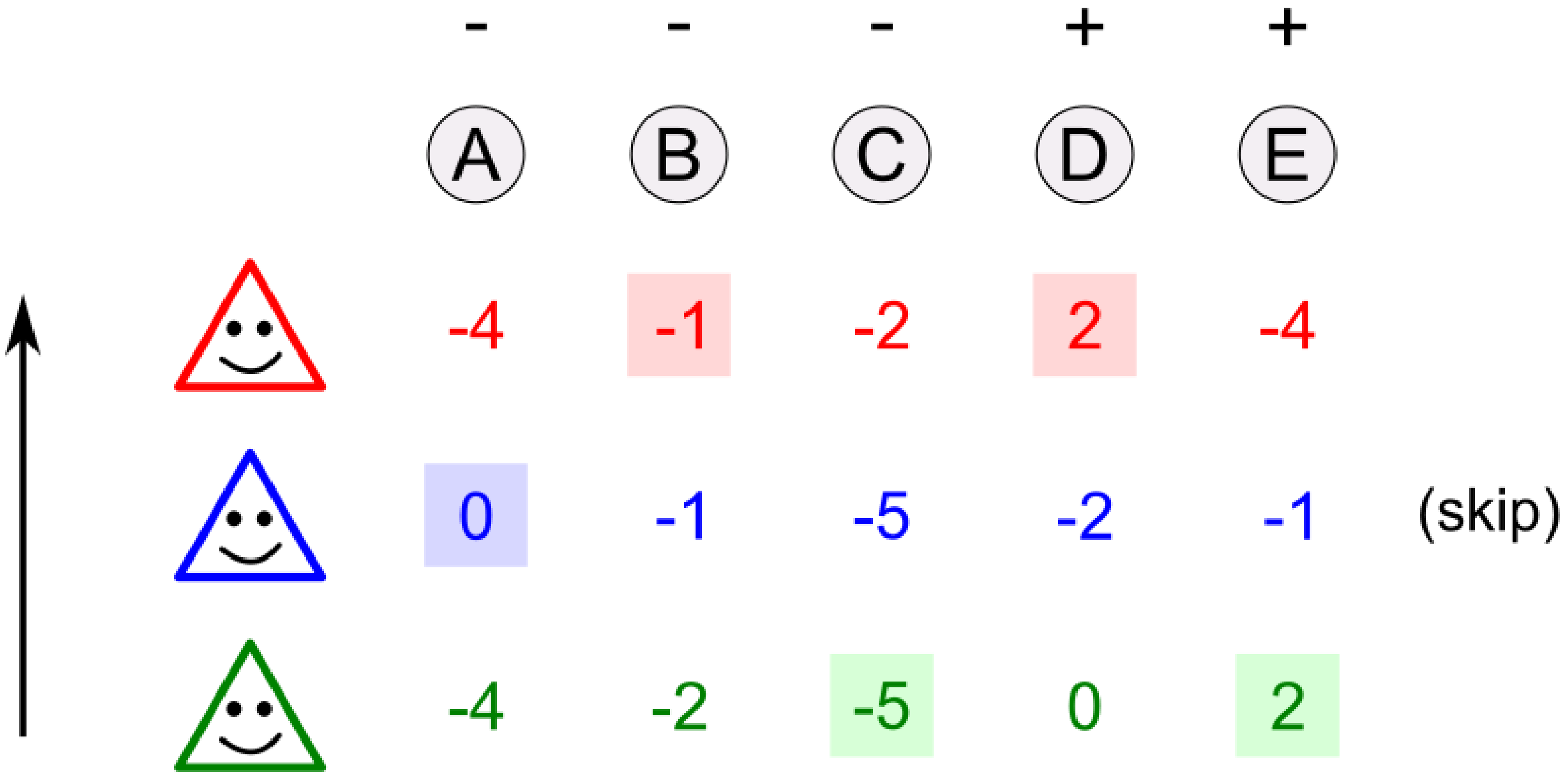

|  | - | - | - | + | + |  |
|---|---|---|---|---|---|---|
|  | A | B | C | D | E |  |
|  | -4 | **-1** | -2 | **2** | -4 |  |
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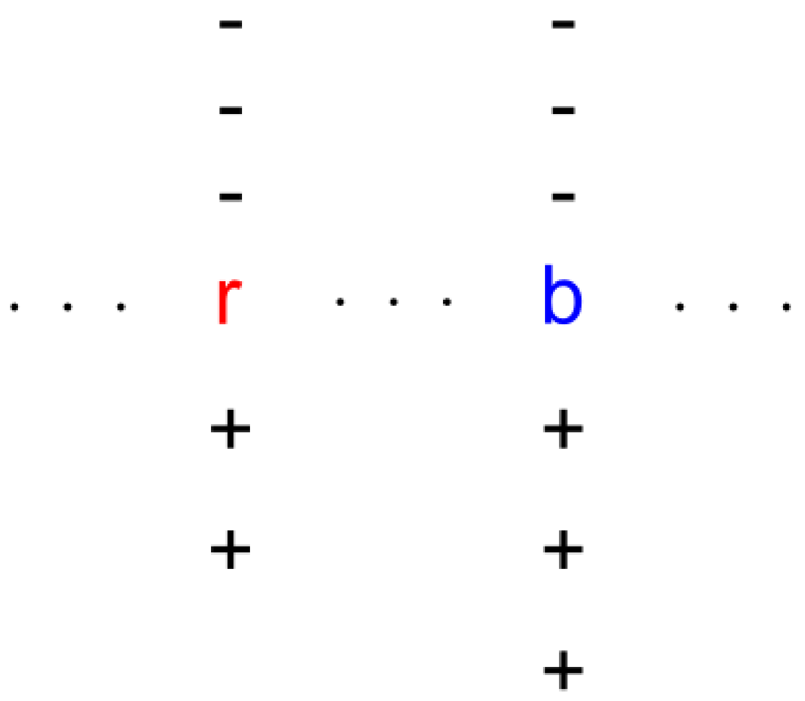

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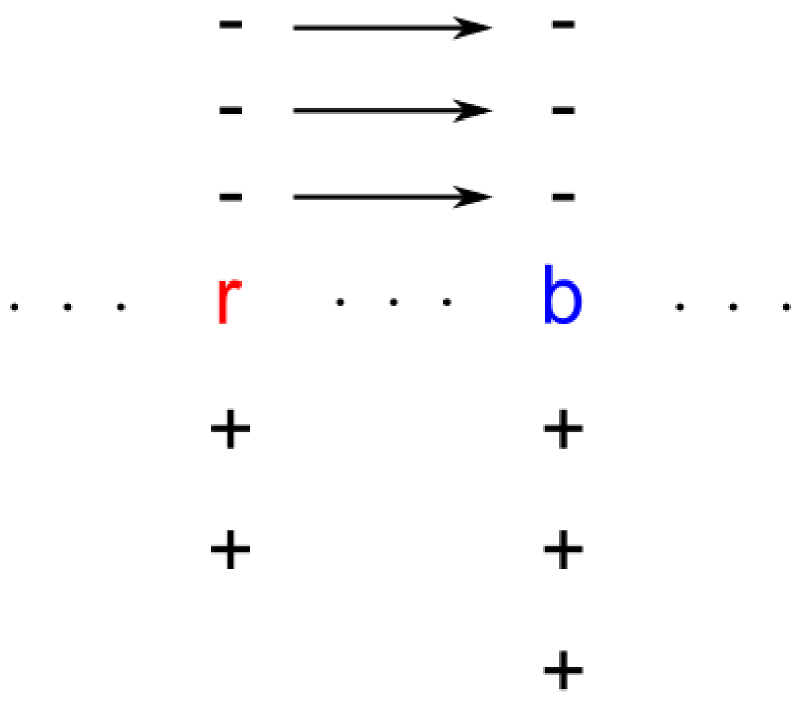

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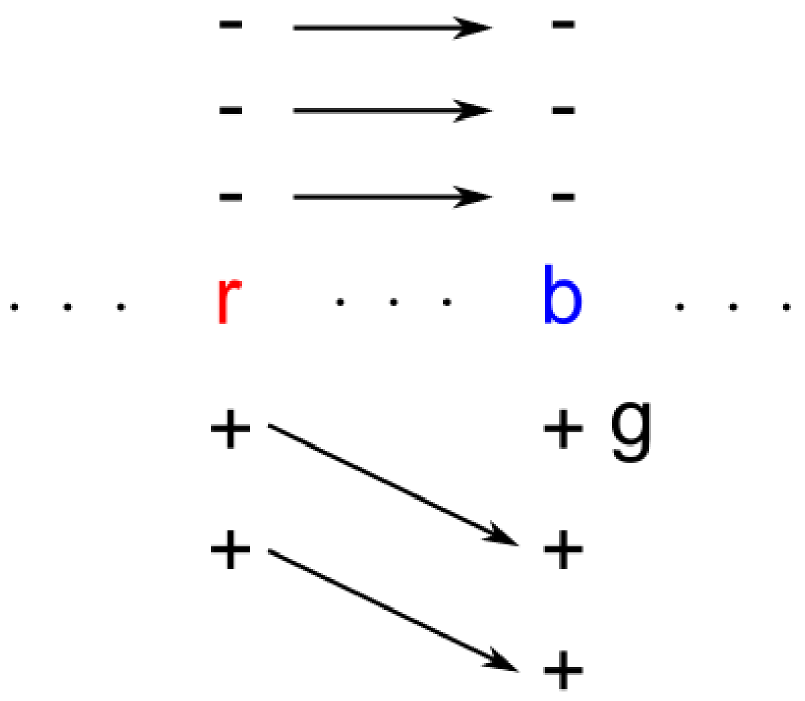

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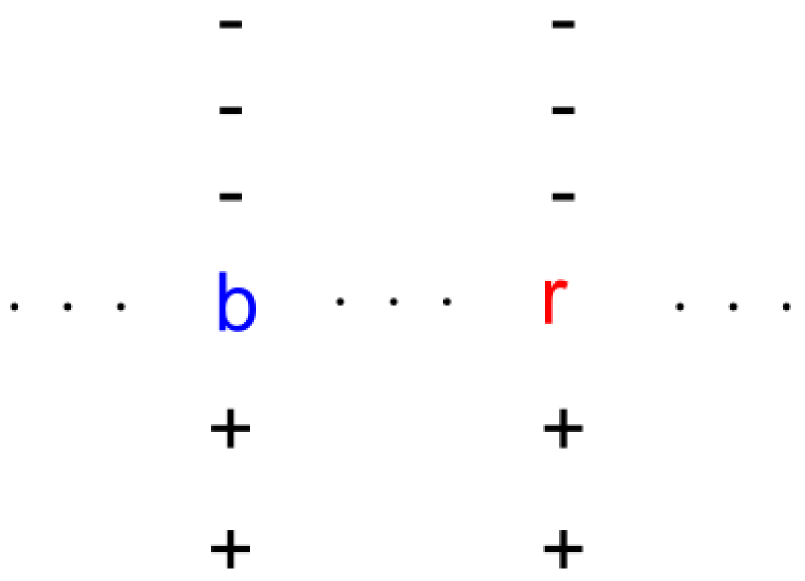

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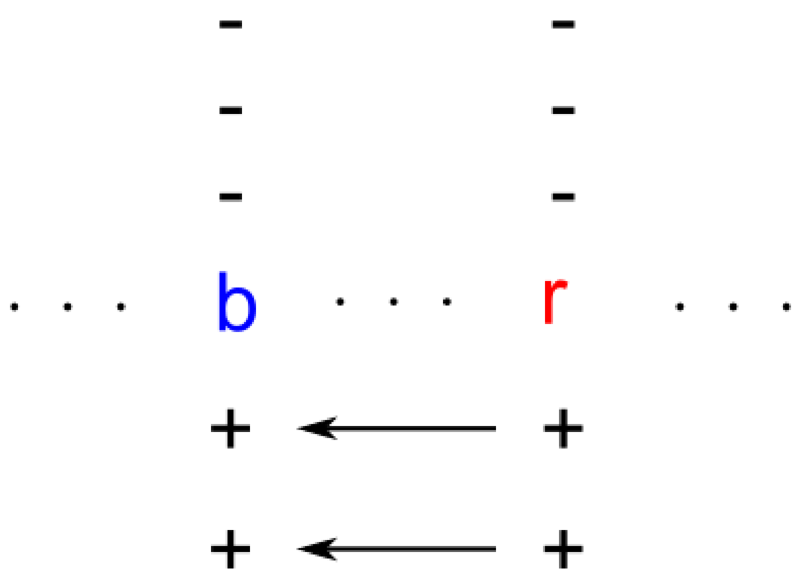

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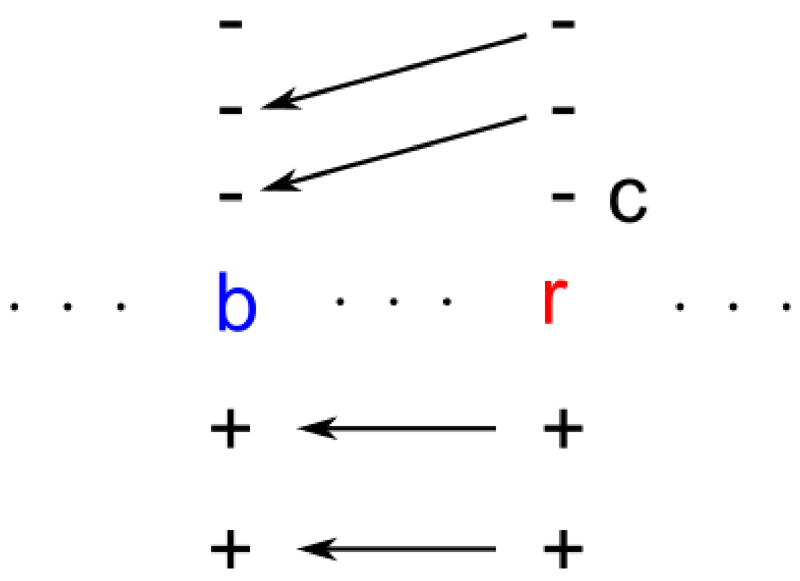

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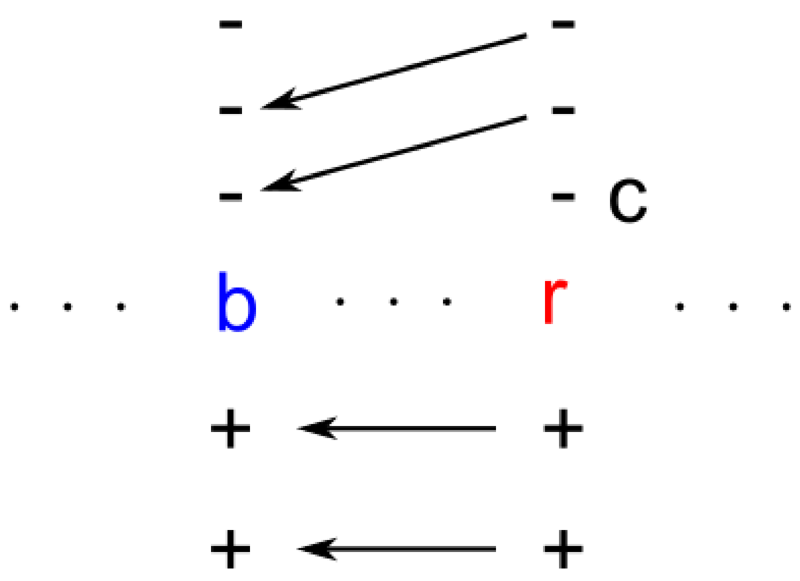
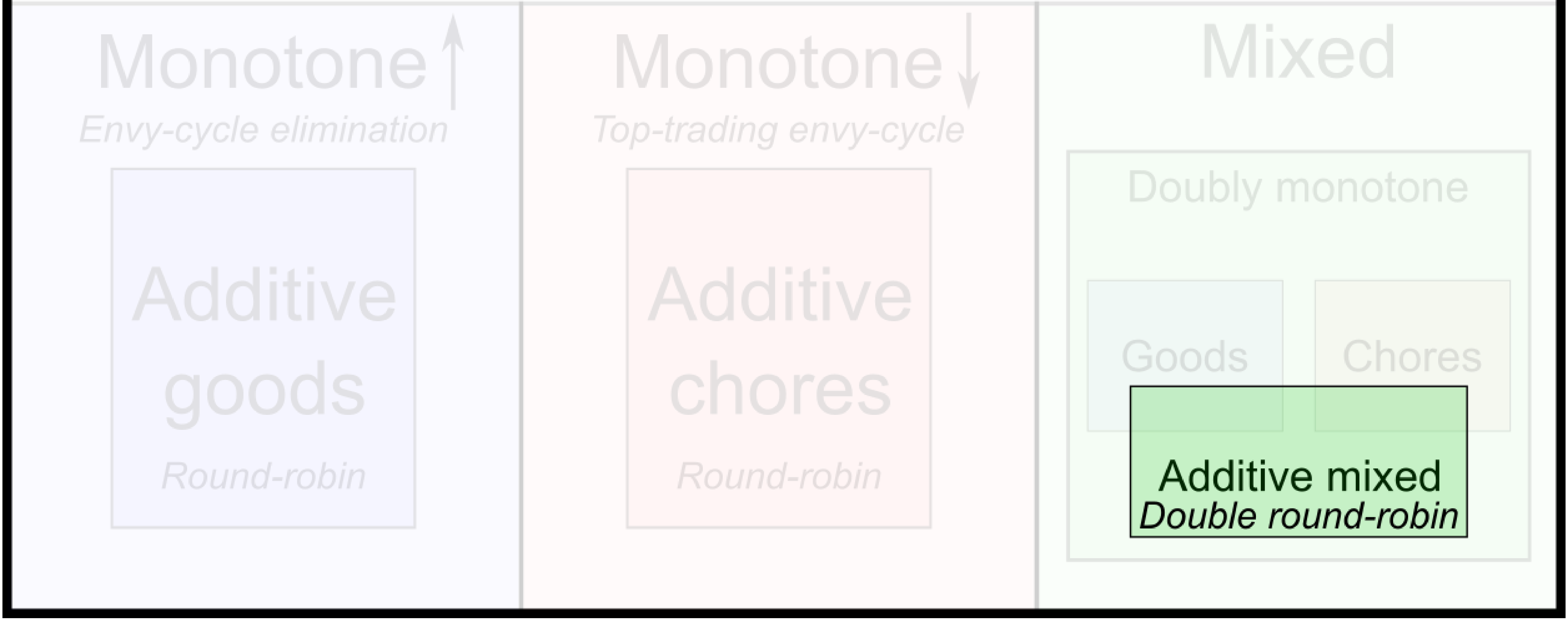


# The Story of EF1



Monotone↑

*Envy-cycle elimination*

Additive goods

*Round-robin*

Monotone↓

*Top-trading envy-cycle*

Additive chores

*Round-robin*

Mixed

Doubly monotone

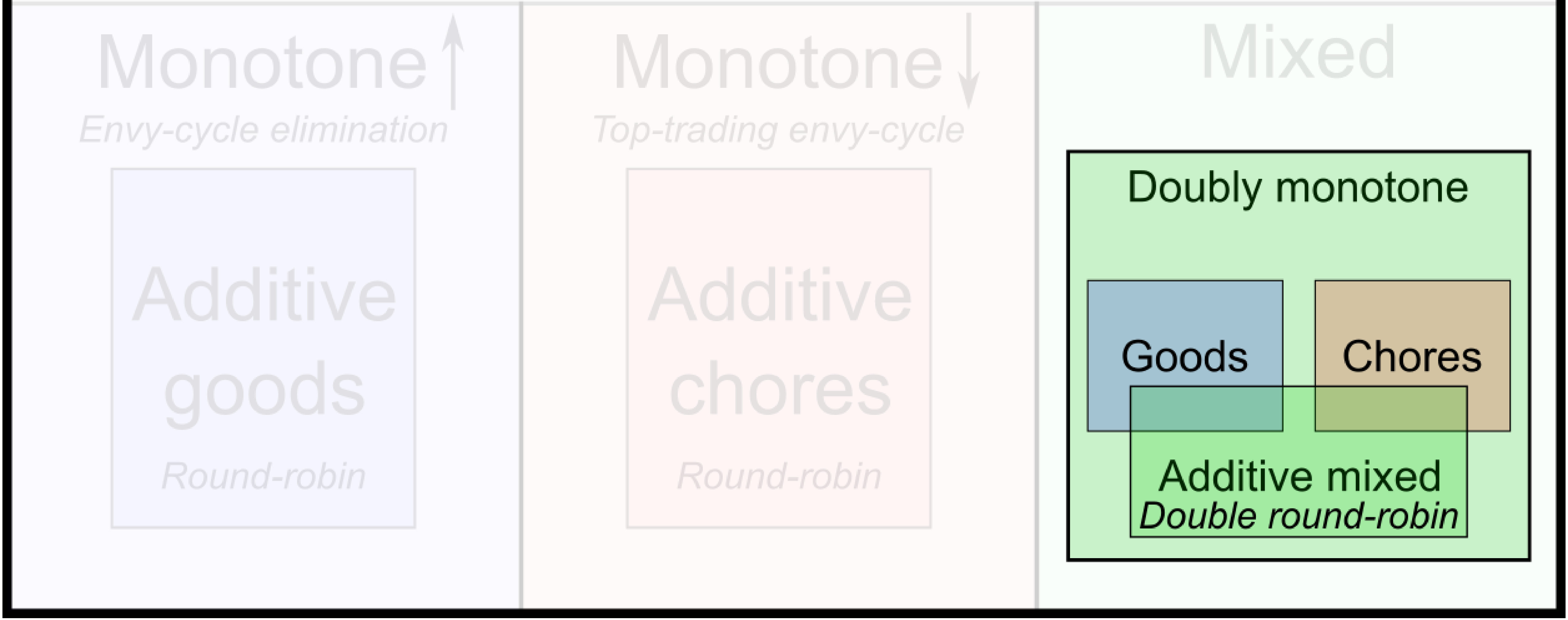
Goods

Chores

Additive mixed

*Double round-robin*

# The Story of EF1



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Each agent can partition the items into "goods" and "chores".

marginal ≥ 0

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|  | Ⓐ | Ⓑ | Ⓒ | Ⓓ | Ⓔ |
|---|---|---|---|---|---|
| △ | - | - | - | + | + |
| △ | - | + | + | + | + |
| △ | - | + | - | + | - |

# EF1 for Doubly Monotone Valuations

Partition the items into two sets: positive and negative

Positive: items considered "good" by at least one agent

Negative: items considered "chore" by everyone

# EF1 for Doubly Monotone Valuations

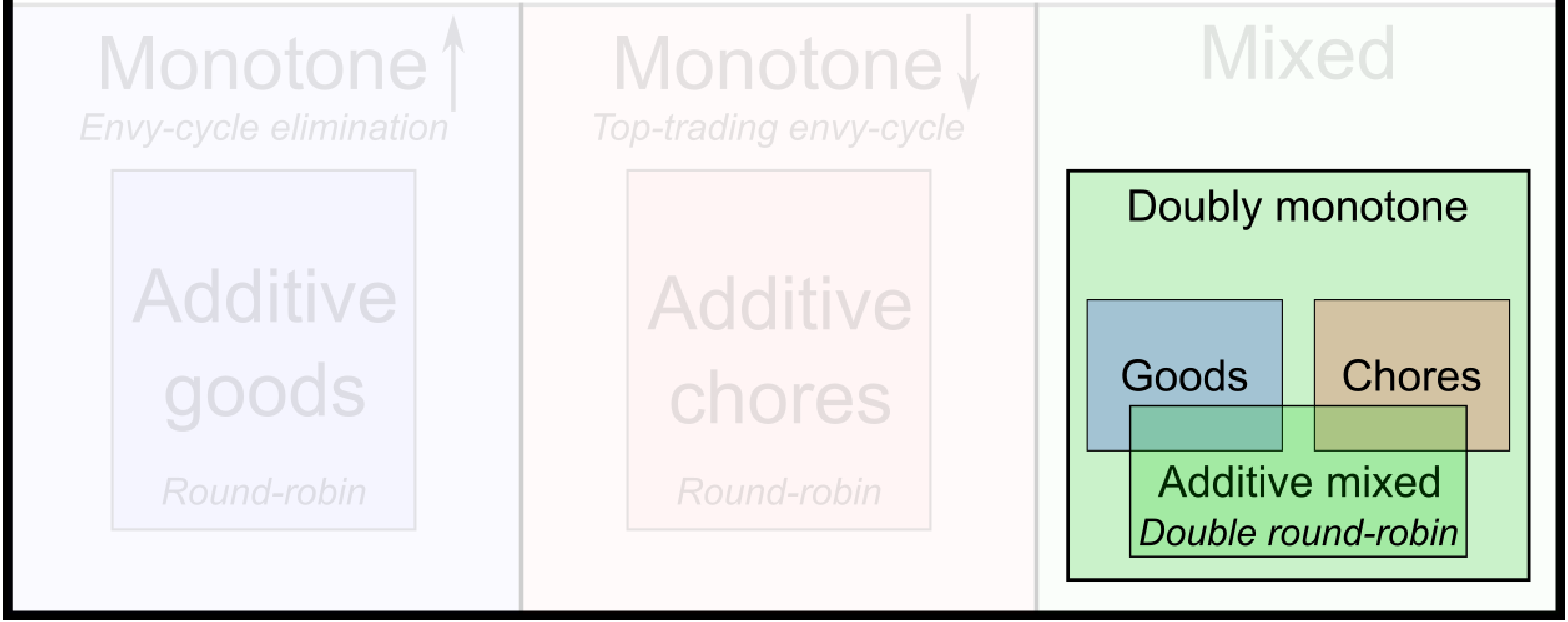
- Assign positive items via envy-cycle elimination
(envy graph defined w.r.t. agents who consider the item a "good")

- Assign negative items via top-trading envy-cycle elimination

[Bhaskar, Sricharan, and Vaish, *APPROX* 2021]

For doubly monotone items, the above algorithm returns an EF1 allocation.

# The Story of EF1



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*Envy-cycle elimination*

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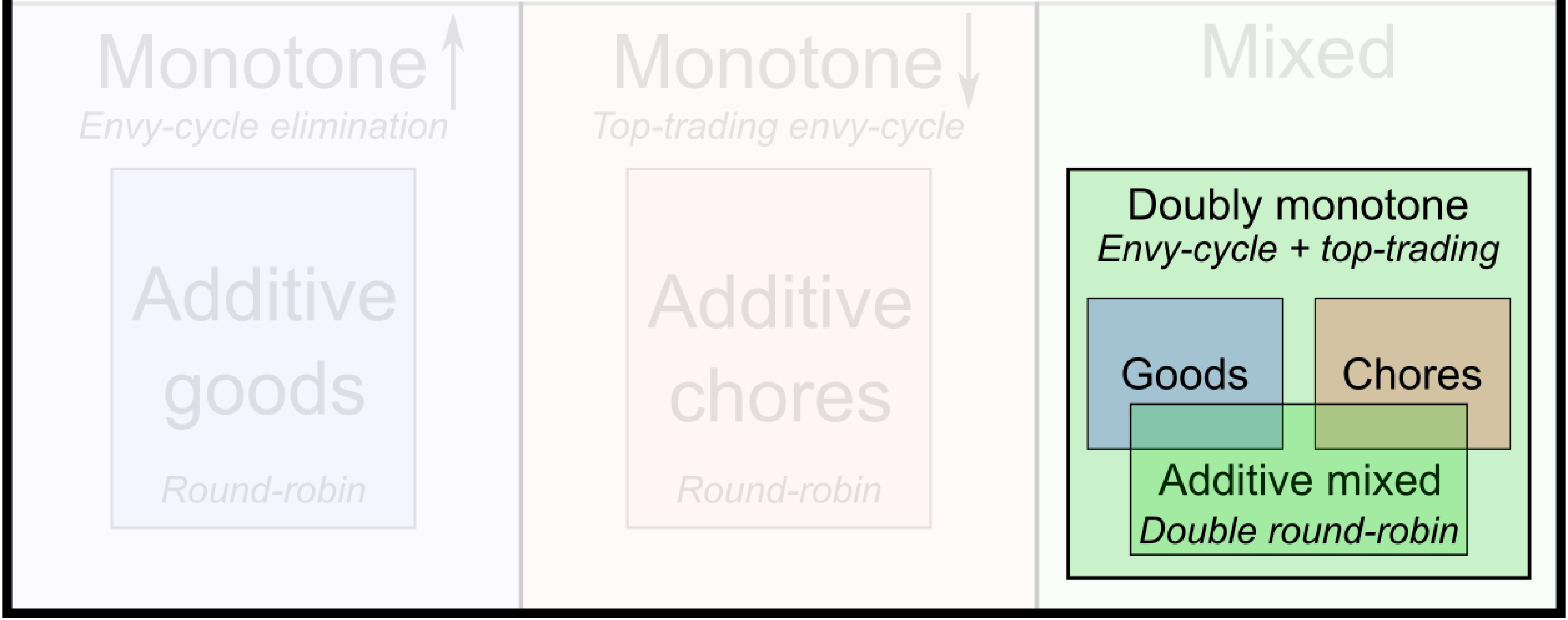

Monotone↑

*Envy-cycle elimination*

Additive goods

*Round-robin*

Monotone↓

*Top-trading envy-cycle*

Additive chores

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Mixed

Doubly monotone

*Envy-cycle + top-trading*

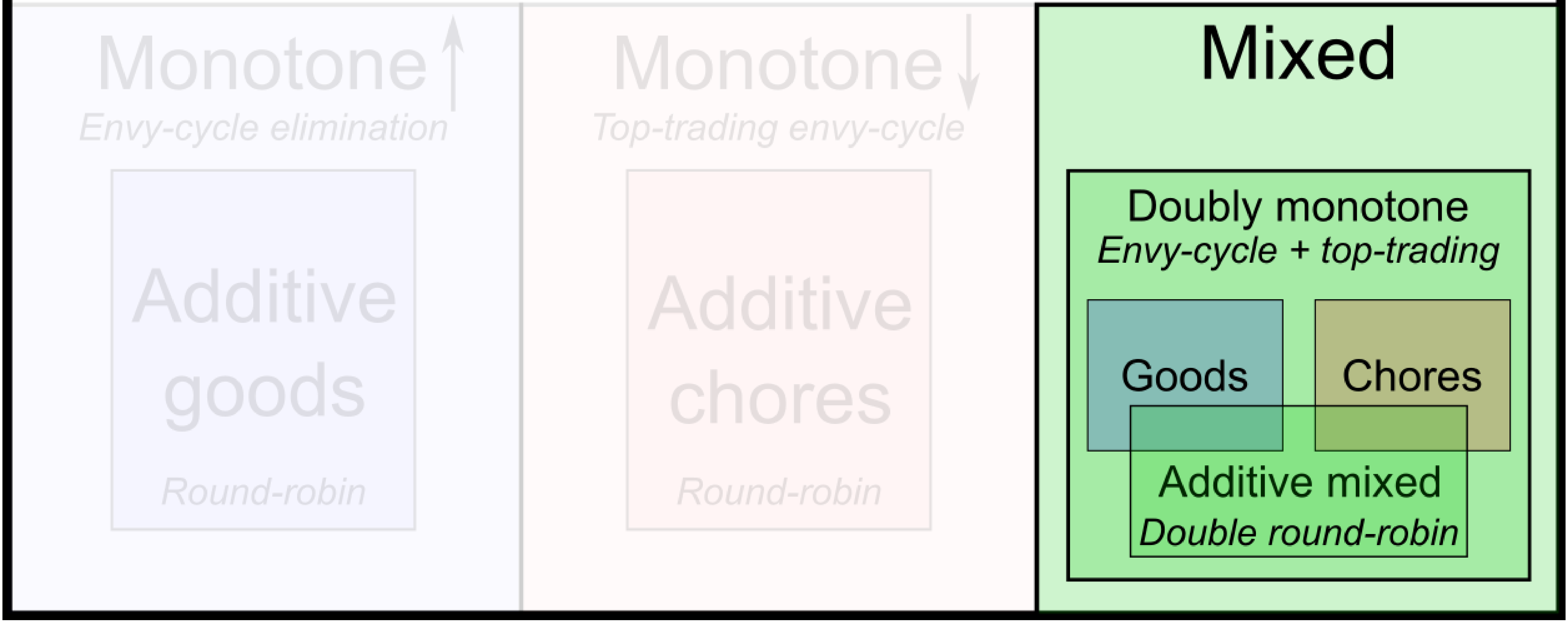
Goods

Chores

Additive mixed

*Double round-robin*

# The Story of EF1



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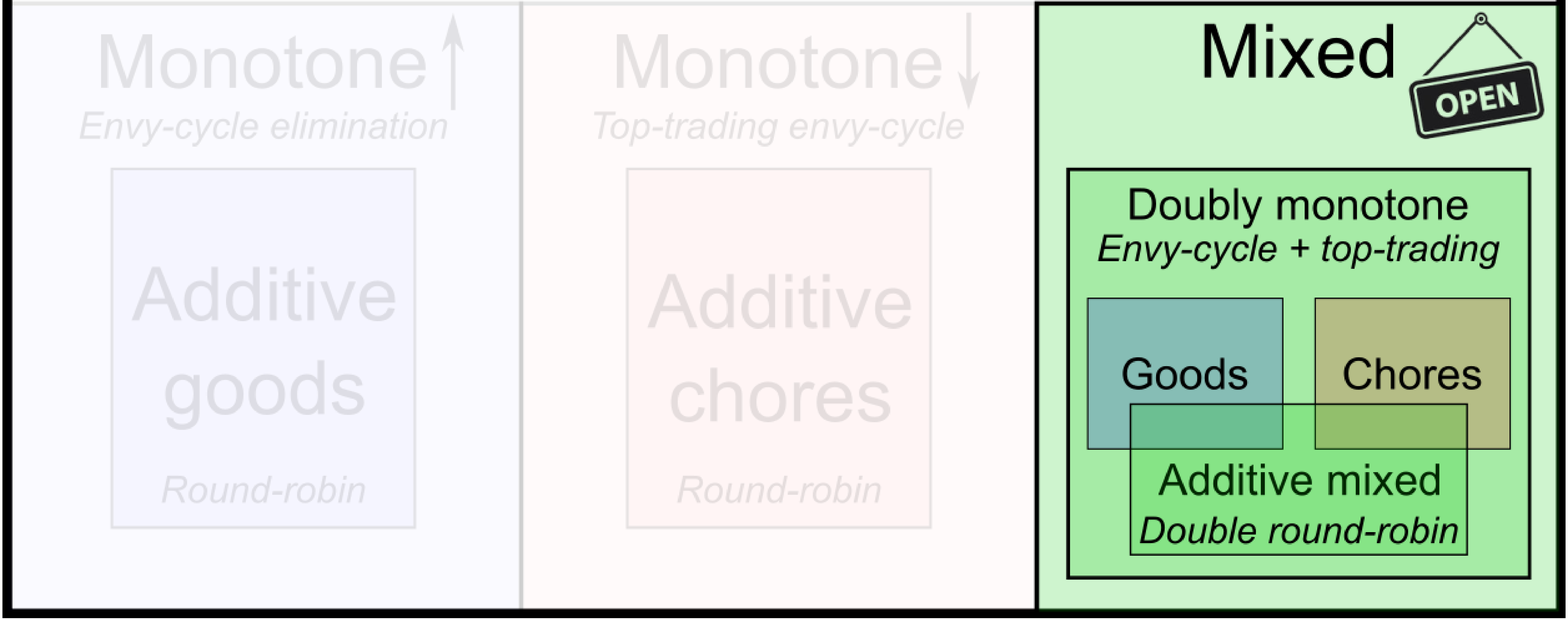

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*Envy-cycle elimination*

Additive goods

*Round-robin*

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*Top-trading envy-cycle*

Additive chores

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Mixed

Doubly monotone

*Envy-cycle + top-trading*

Goods

Chores

Additive mixed

*Double round-robin*

# Next Time

Fair Rent Division

# Quiz

# Quiz

Prove or disprove:

For n identical agents with additive valuations over mixed items, an EFX allocation always exists.

# References

- Double round-robin algorithm

  Haris Aziz, Ioannis Caragiannis, Ayumi Igarashi and Toby Walsh
  *“Fair allocation of indivisible goods and chores”*
  Autonomous Agents and Multiagent Systems, 36(3), 2022 pg 1-21
  https://link.springer.com/article/10.1007/s10458-021-09532-8

- Top-trading envy-cycle elimination

  Umang Bhaskar, A R Sricharan, and Rohit Vaish
  “*On approximate envy-freeness for indivisible chores and mixed resources*”
  APPROX 2021
  https://drops.dagstuhl.de/opus/volltexte/2021/14694/