COL749: Computational Social Choice

Lecture 10

Towards Stronger Fairness Guarantees

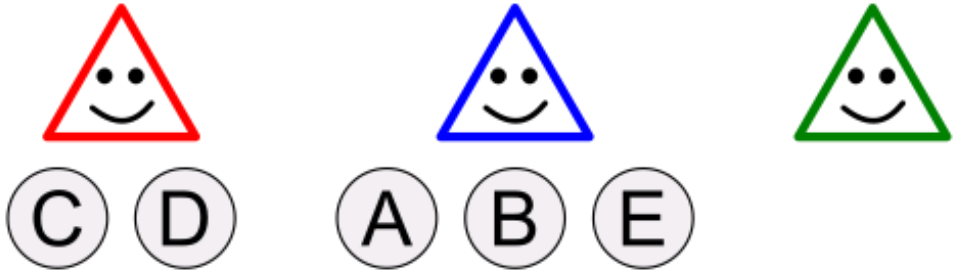
Feb 06, 2025 | Rohit Vaish

The Model

Set of agents

Set of indivisible items

Allocation



Envy-Freeness

[Gamow and Stern, 1958; Foley, 1967]

Each agent prefers its own bundle over that of any other agent.

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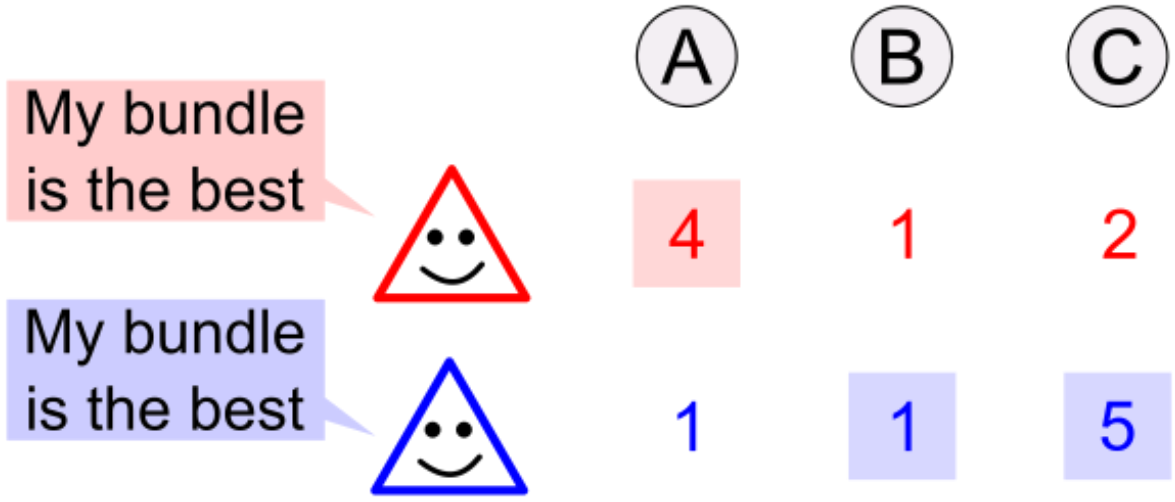

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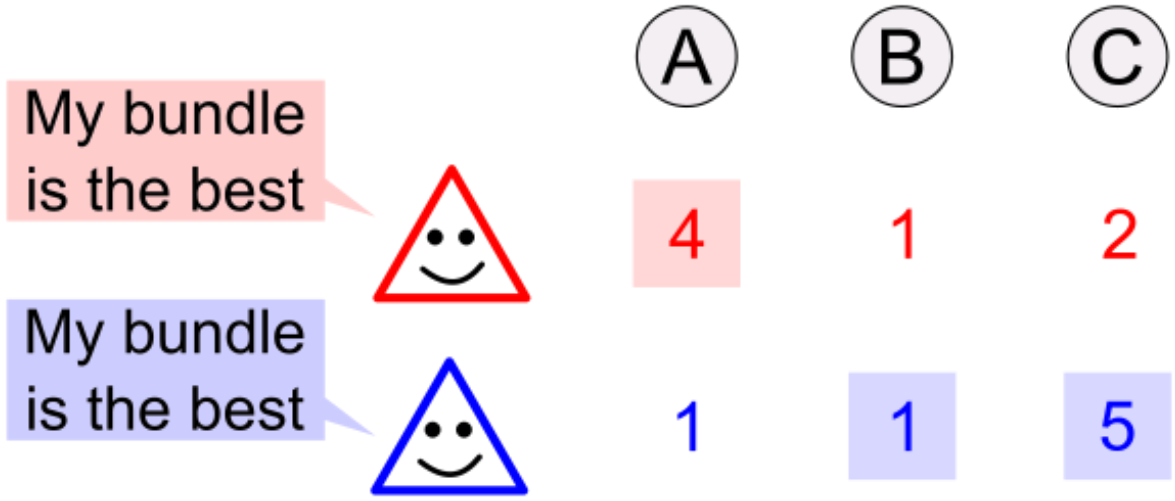

☹ Not guaranteed to exist (two agents, one good)

☹ Checking whether an EF allocation exists is NP-complete

Envy-Freeness Up To One Good

[Budish, 2011]

Envy can be eliminated by removing some good in the envied bundle.

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My bundle is better if Ⓐ is removed

My bundle is better if Ⓒ is removed

Ⓐ	Ⓑ	Ⓒ
4	1	2
1	1	5

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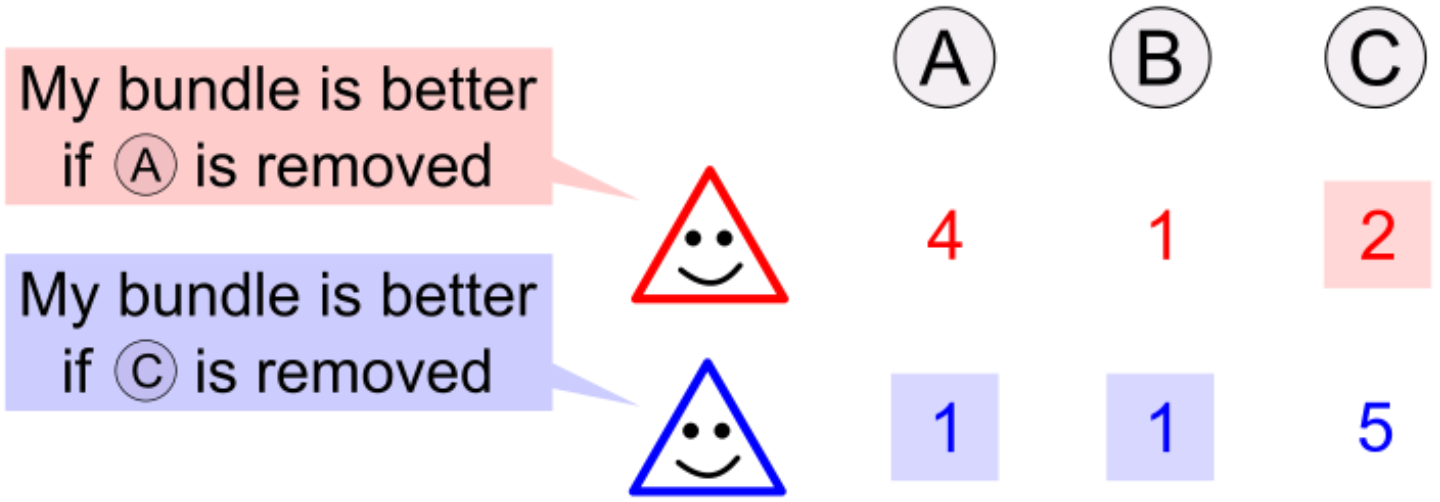
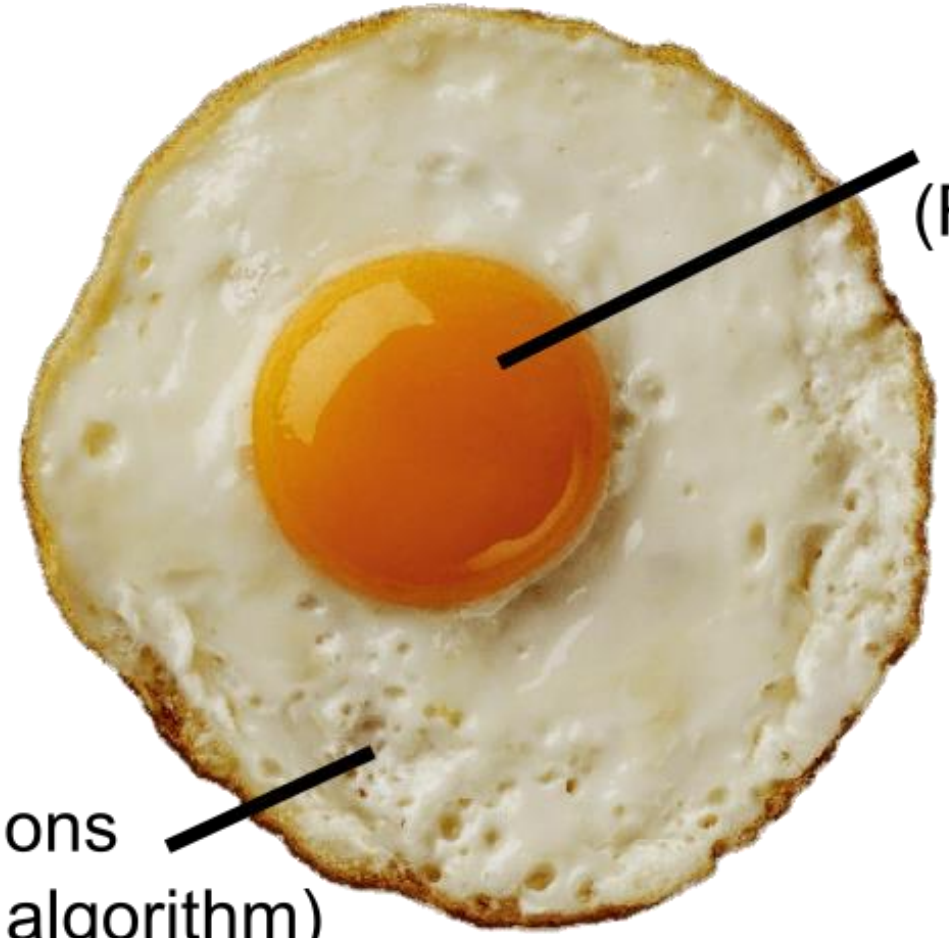


Guaranteed to exist and efficiently computable

Algorithms for finding an EF1 allocation



Additive valuations
(Round-robin algorithm)

Monotone valuations
(Envy-cycle elimination algorithm)

A Limitation of EF1

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I donut think this is fair!

Why are you sad?
Aren't you envy-free up to a car?

A Limitation of EF1

Envy-Freeness Up To Any Good

[Caragiannis, Kurokawa, Moulin, Procaccia, Shah, and Wang, *EC* 2016, *TEAC* 2019]

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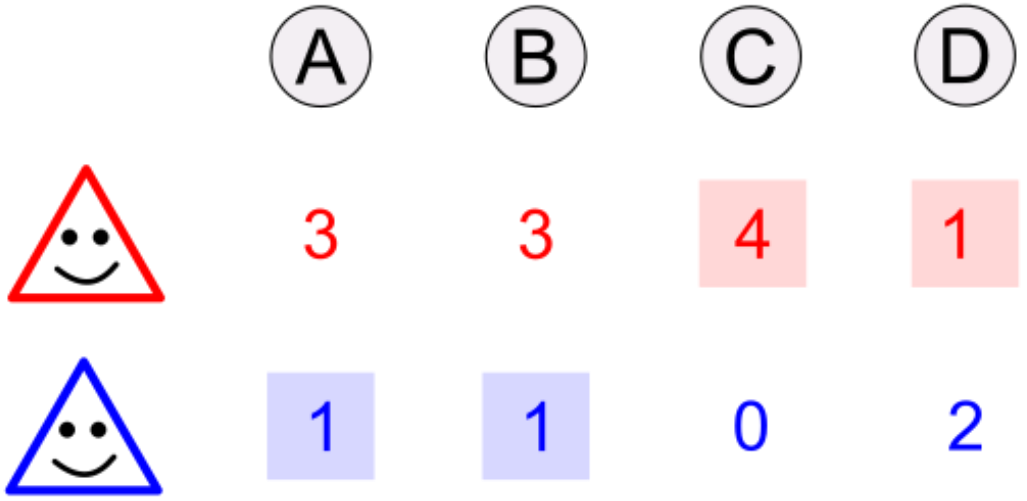

Allocation $A = (A_1, \ldots, A_n)$ is EFX if for every pair of agents i, k and for every good $j \in A_k$, we have $v_i(A_i) \geq v_i(A_k \setminus \{j\})$.

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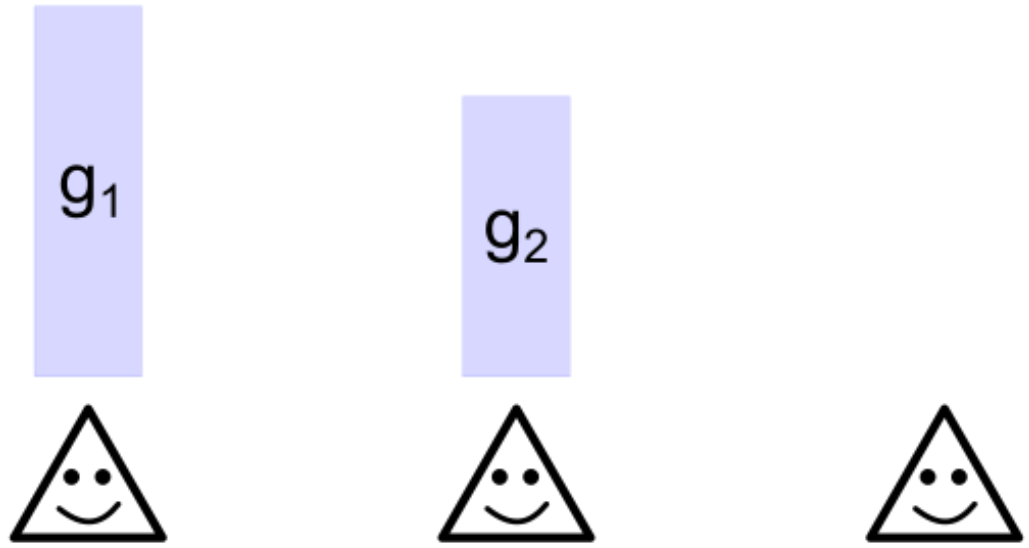

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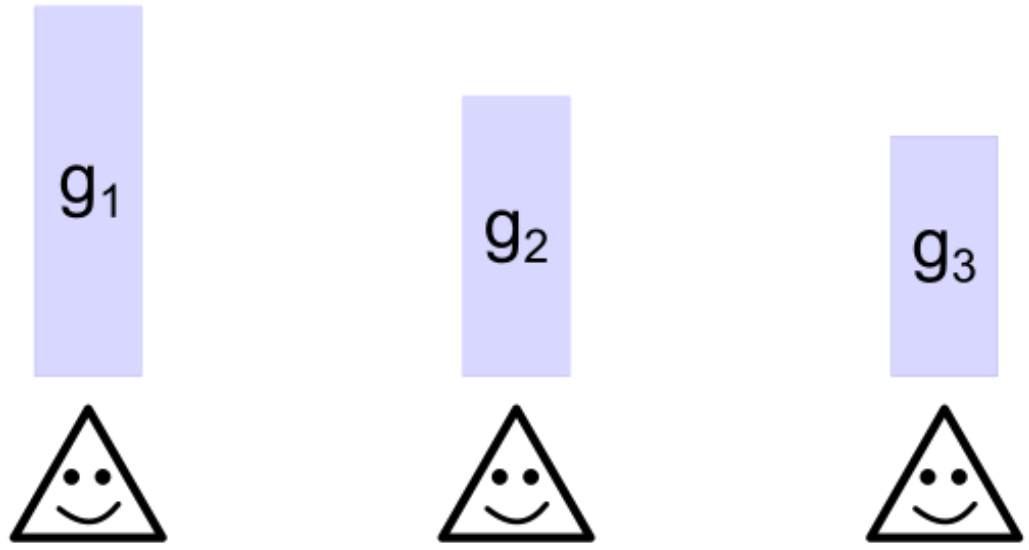

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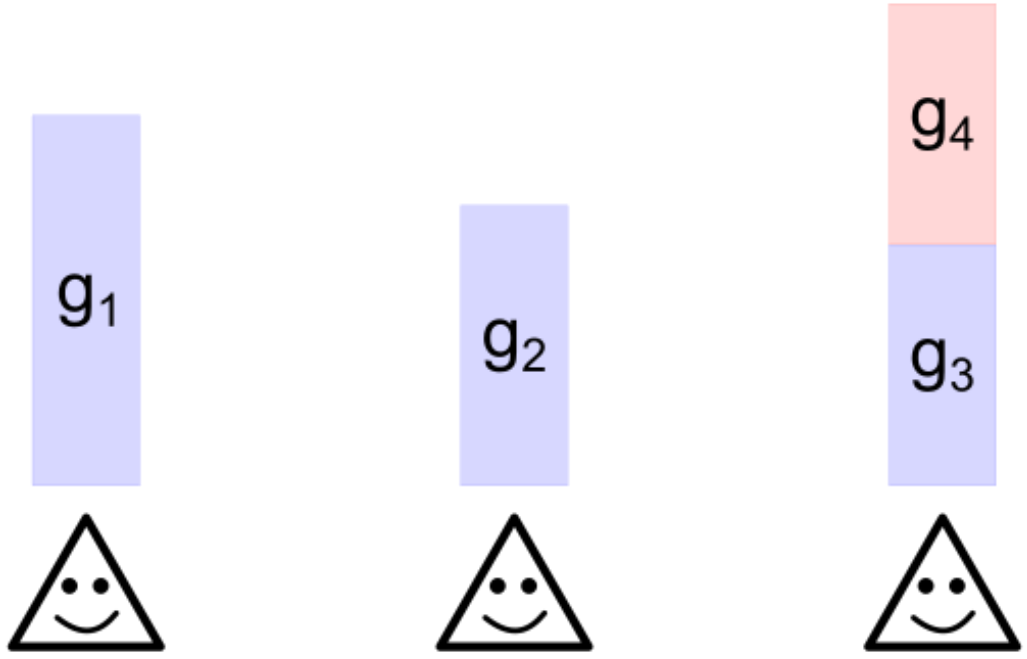

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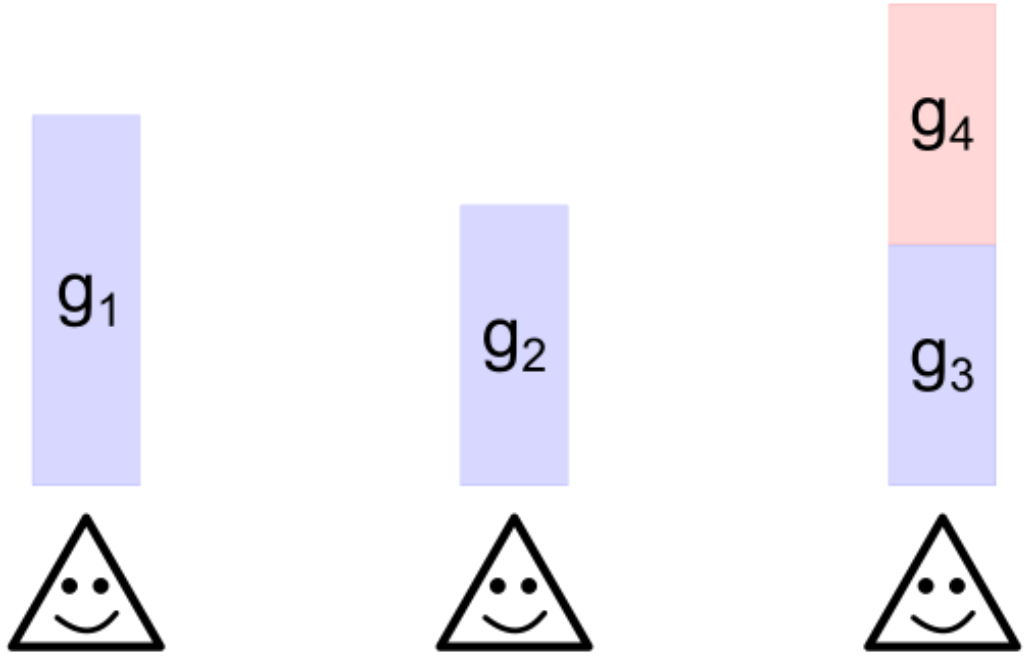

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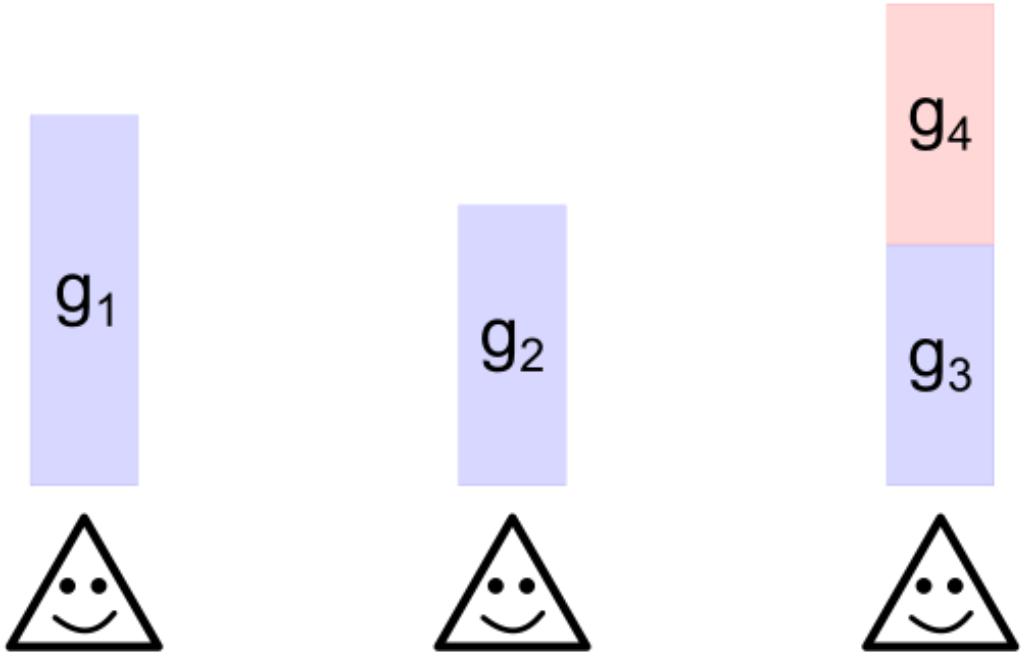

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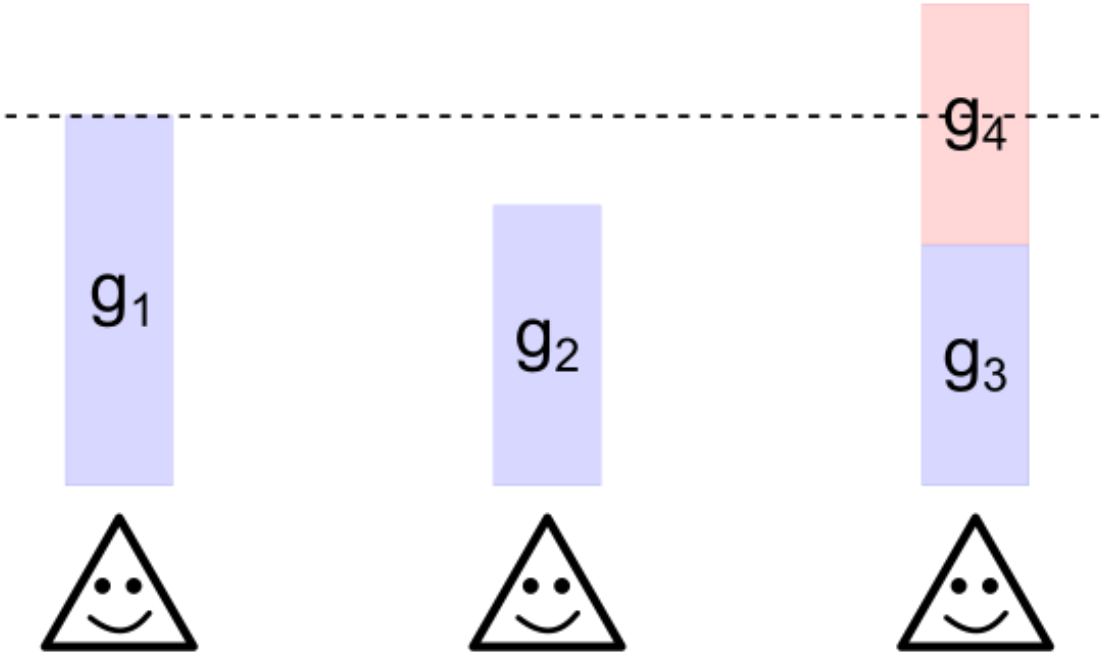

Why EFX?

Most recent good = least valued.

Envy-free up to most recent good.

[Plaut and Roughgarden; *SODA* 2018; *SIDMA* 2020]

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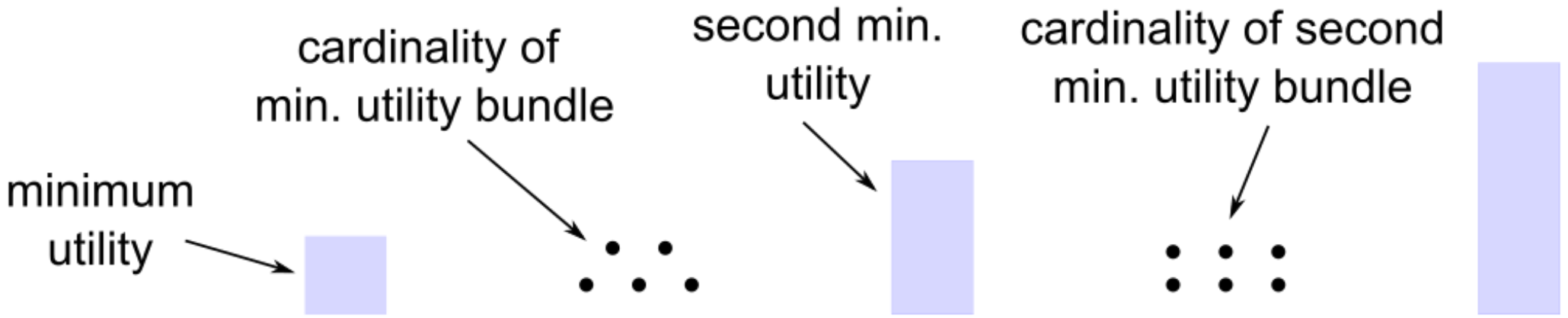

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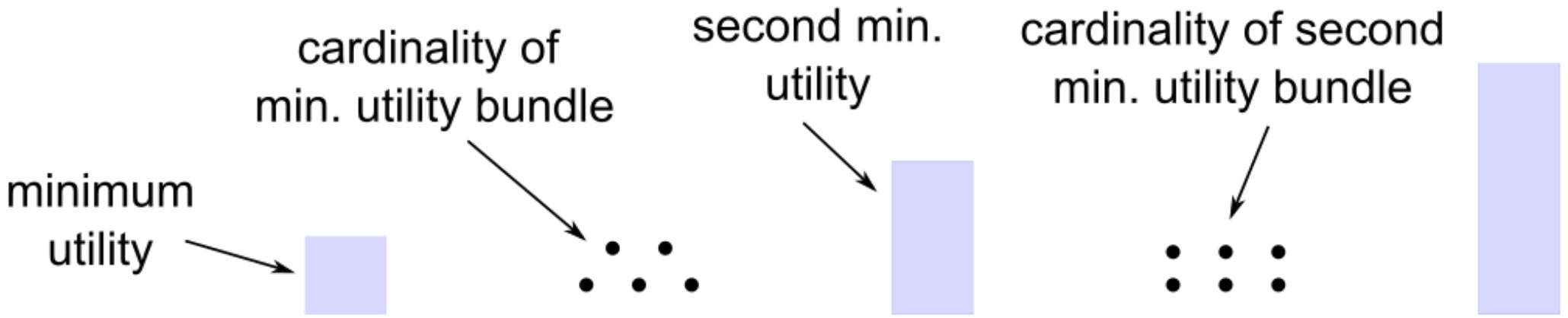

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Finding an EFX allocation can take exponential-in-#goods value queries even for two identical agents with submodular valuations.

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identical valuations result + "cut and choose"

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Exists for "2 value" additive instances

[Amanatidis, Birmpas, Filos-Ratsikas, Hollender, and Voudouris, *IJCAI* 2020, *TCS* 2021; Garg and Murhekar, *SAGT* 2021]

Fairness via Charity

EFX-with-charity

[Caragiannis, Gravin, and Huang, *EC* 2019; Chaudhury, Kavitha, Mehlhorn, and Sgouritsa, *SODA* 2020, *SICOMP* 2021]

A partition $(A_1, \ldots, A_n, P)$ into $n+1$ bundles satisfies EFX-with-charity if

- the partial allocation $(A_1, \ldots, A_n)$ is EFX,
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EFX-with-charity

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For monotone valuations, an EFX-with-charity allocation always exists.

Minimal Envied Subset

[Chaudhury, Kavitha, Mehlhorn, and Sgouritsa, *SODA* 2020, *SICOMP* 2021]

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S = {a,b,c,d}

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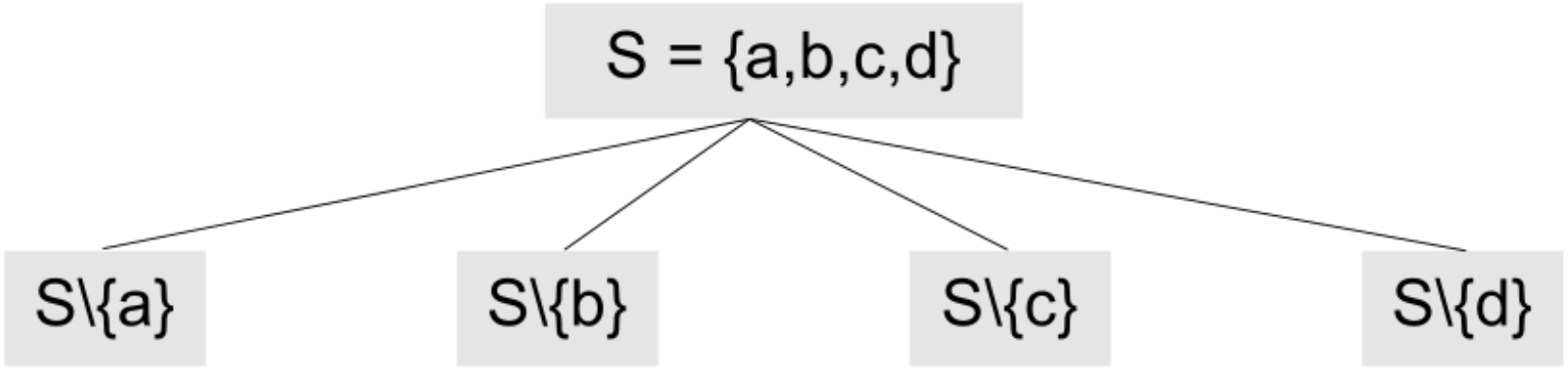

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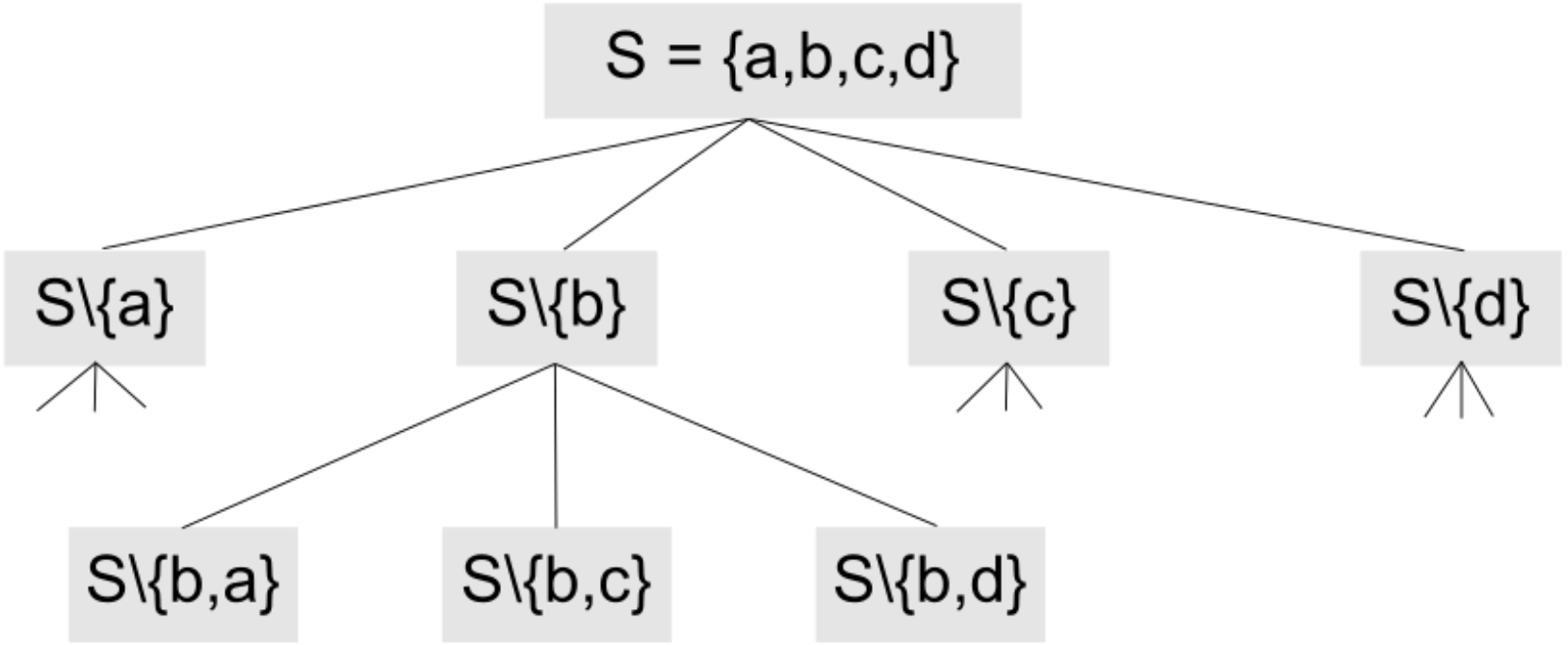

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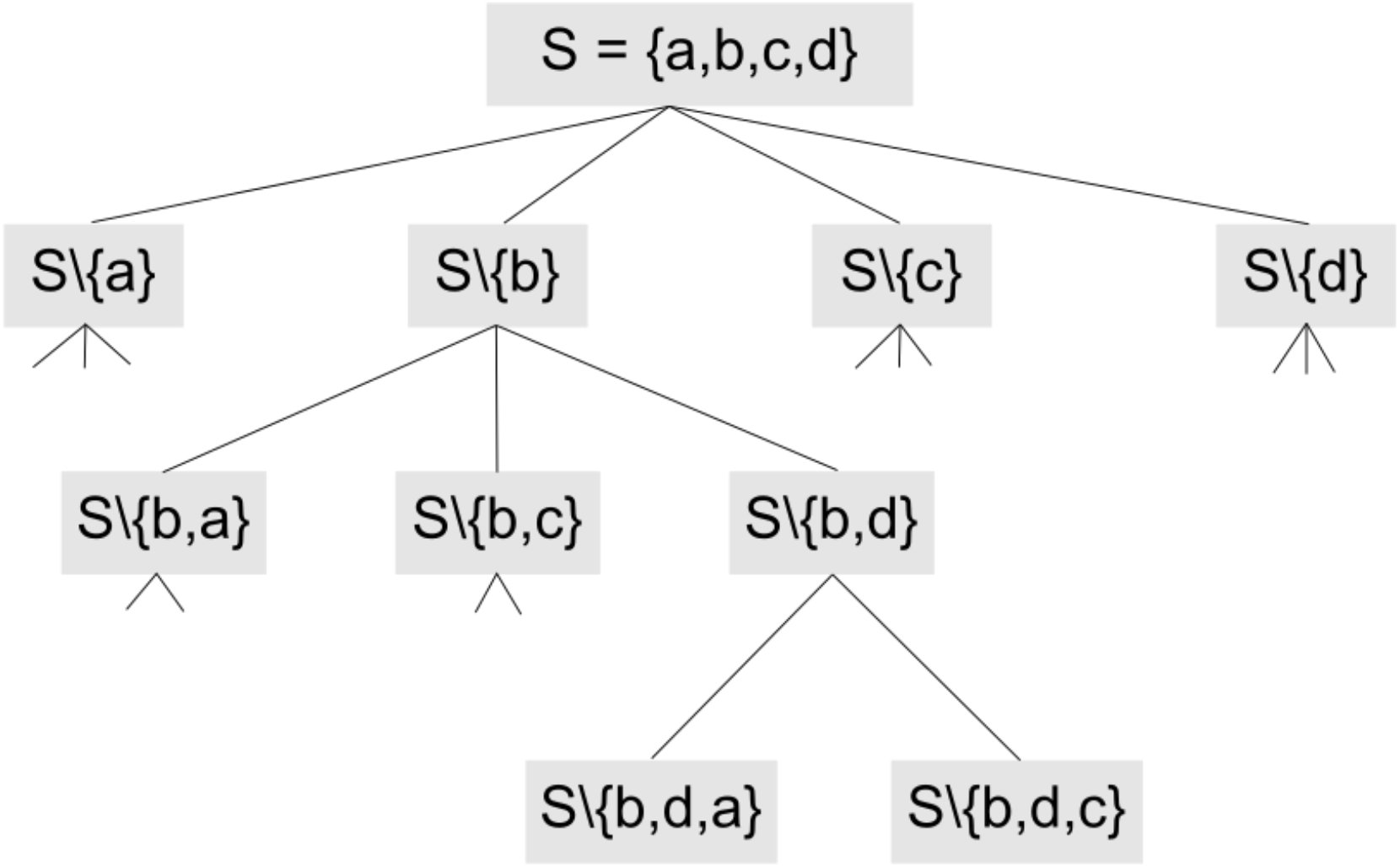

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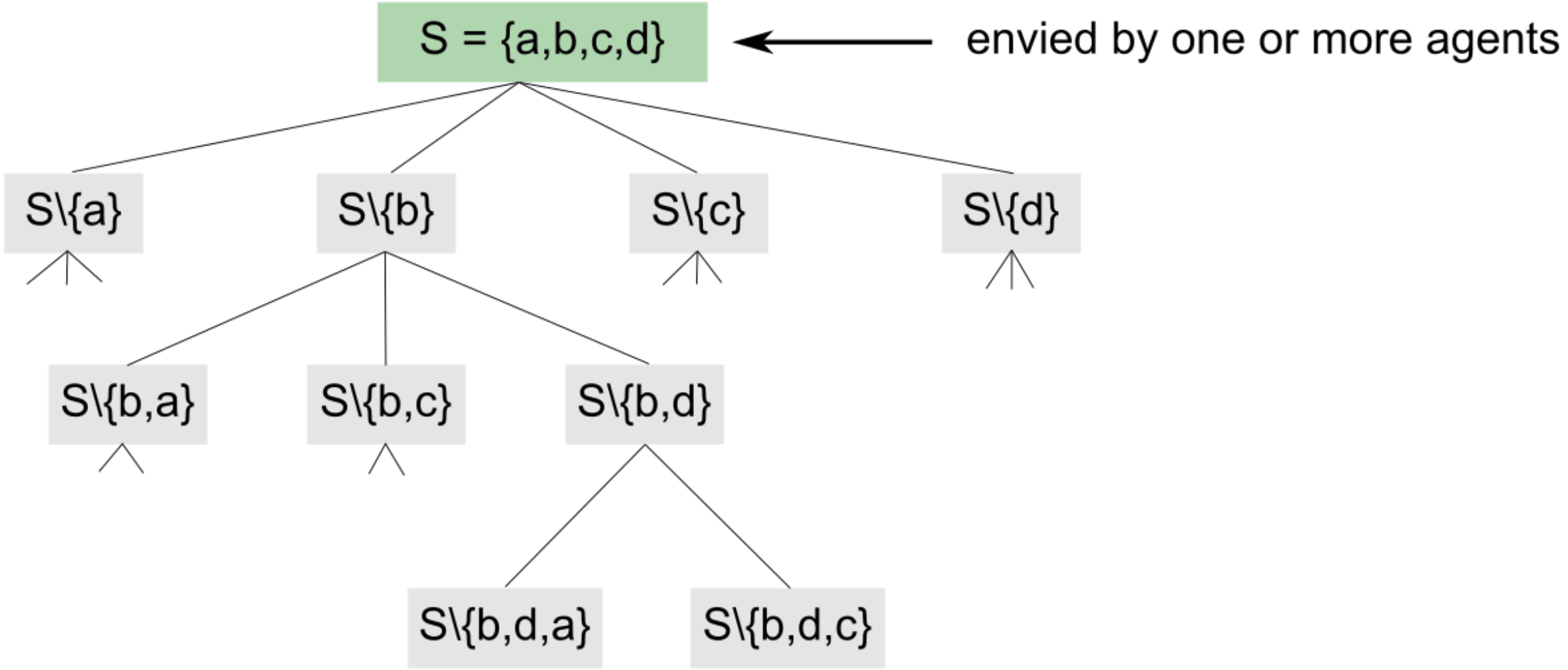

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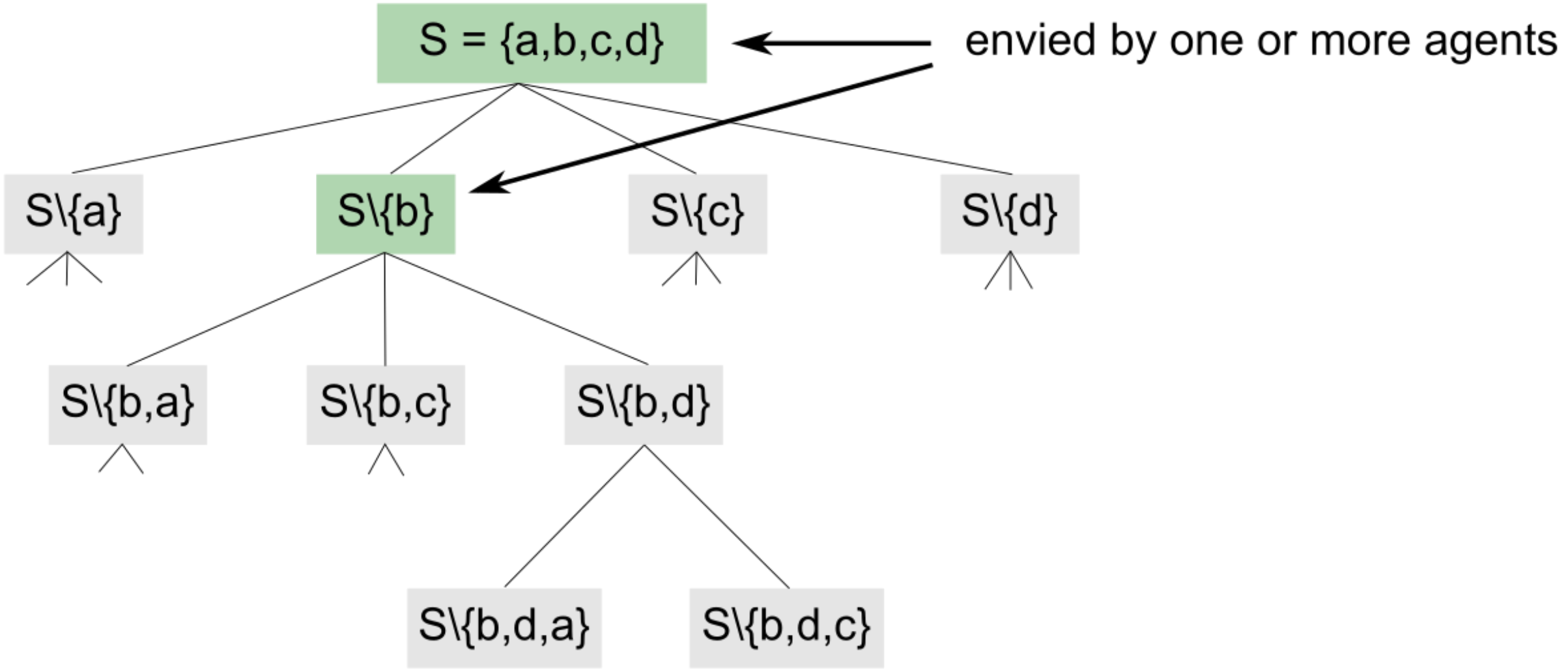

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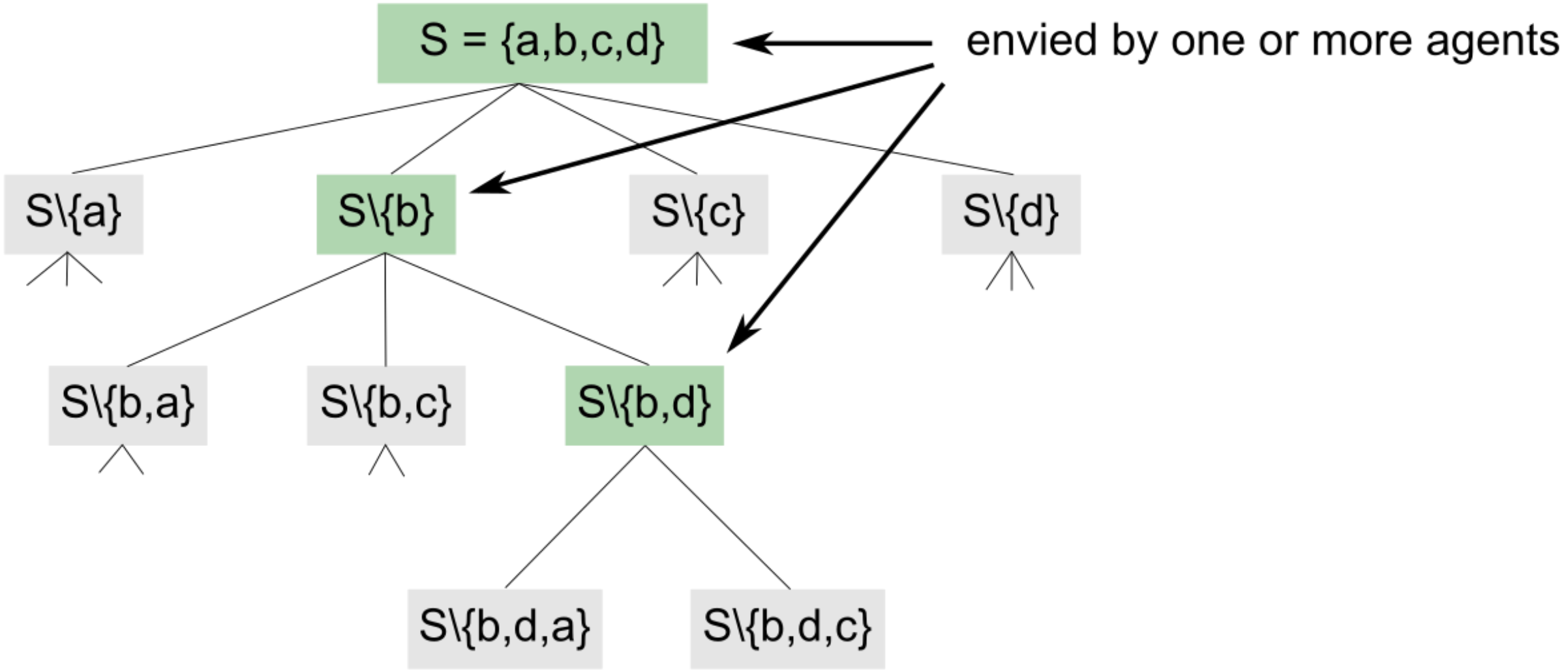

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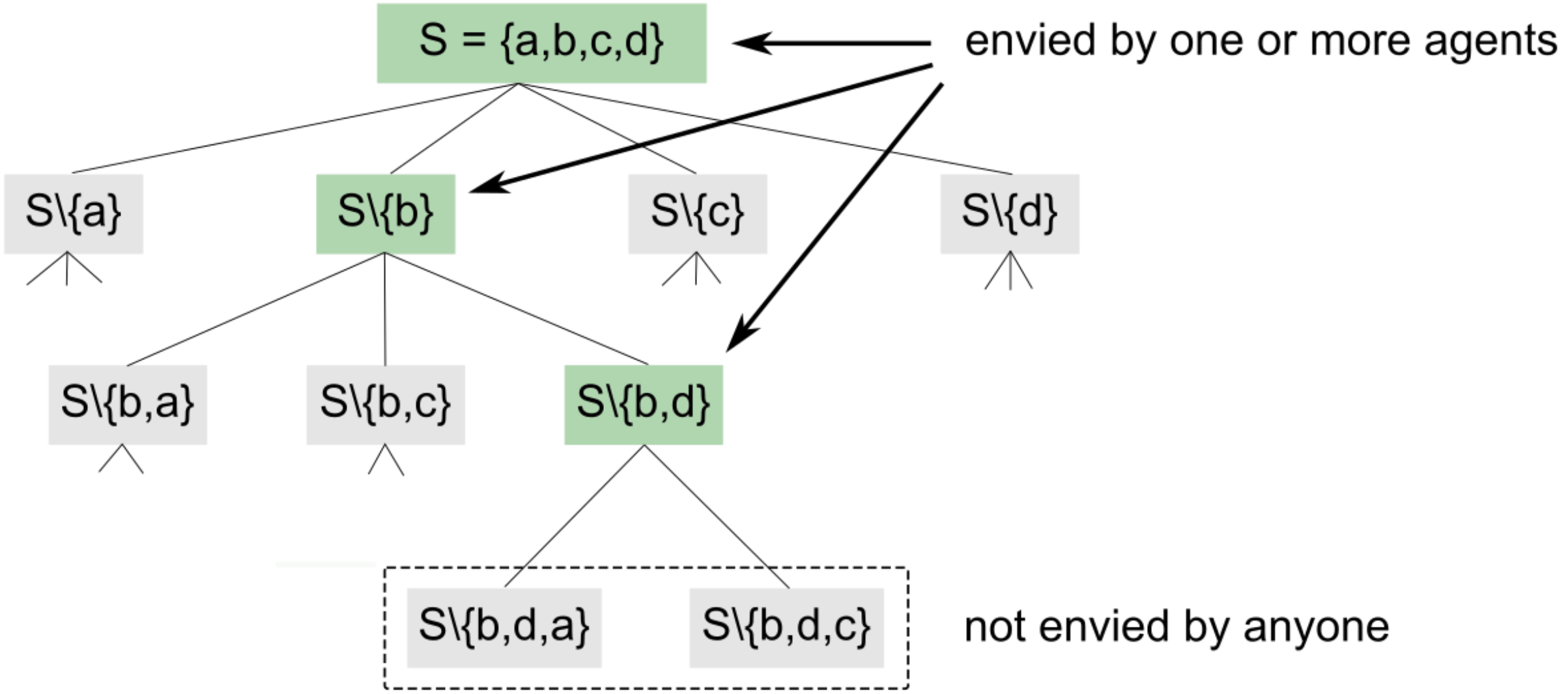

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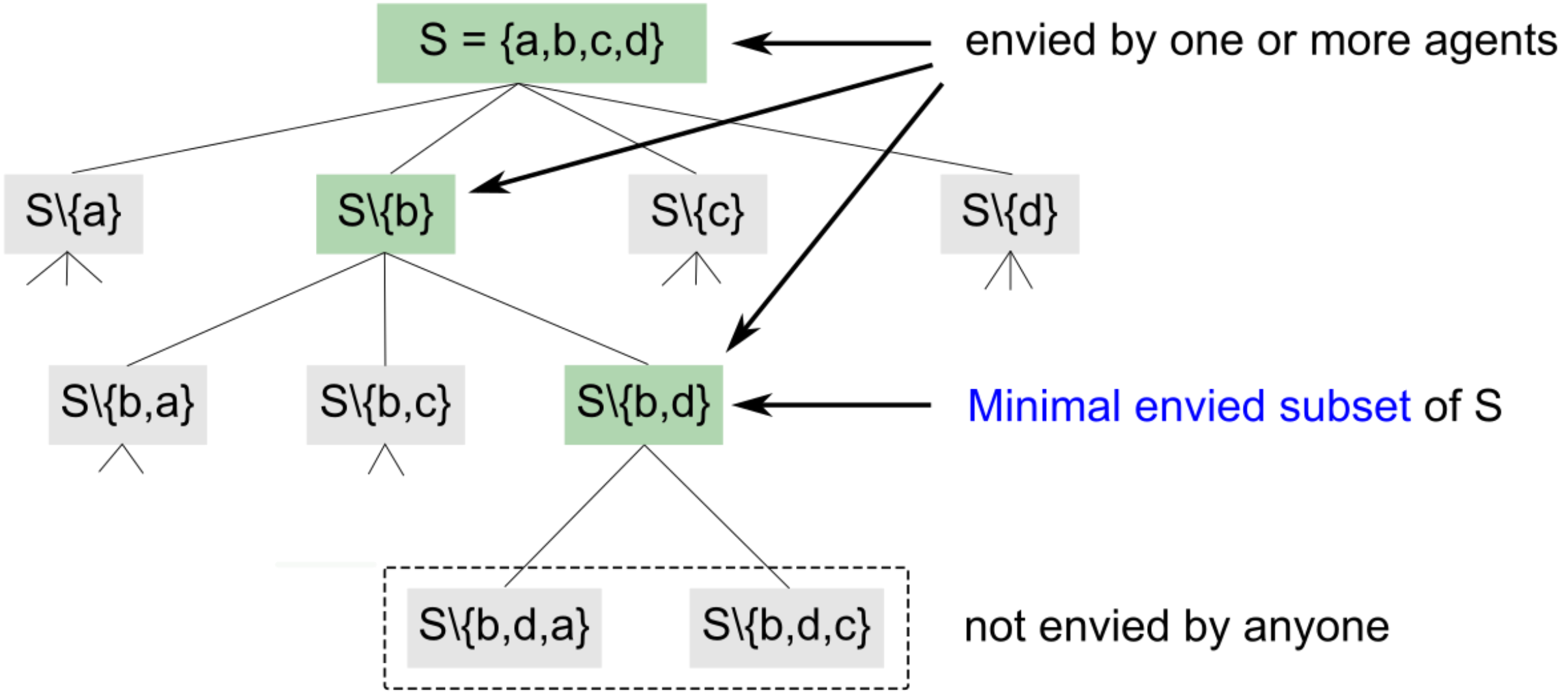

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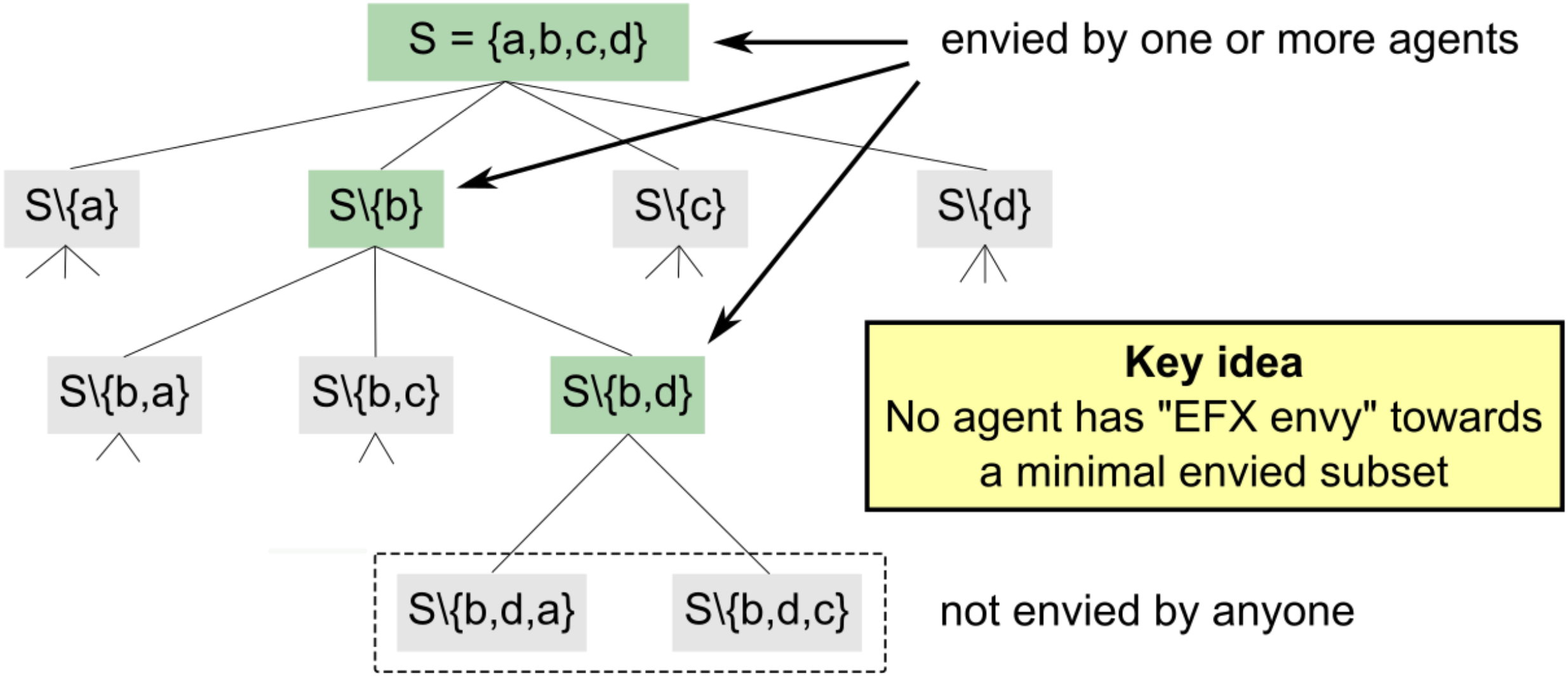

Key idea

No agent has "EFX envy" towards a minimal envied subset

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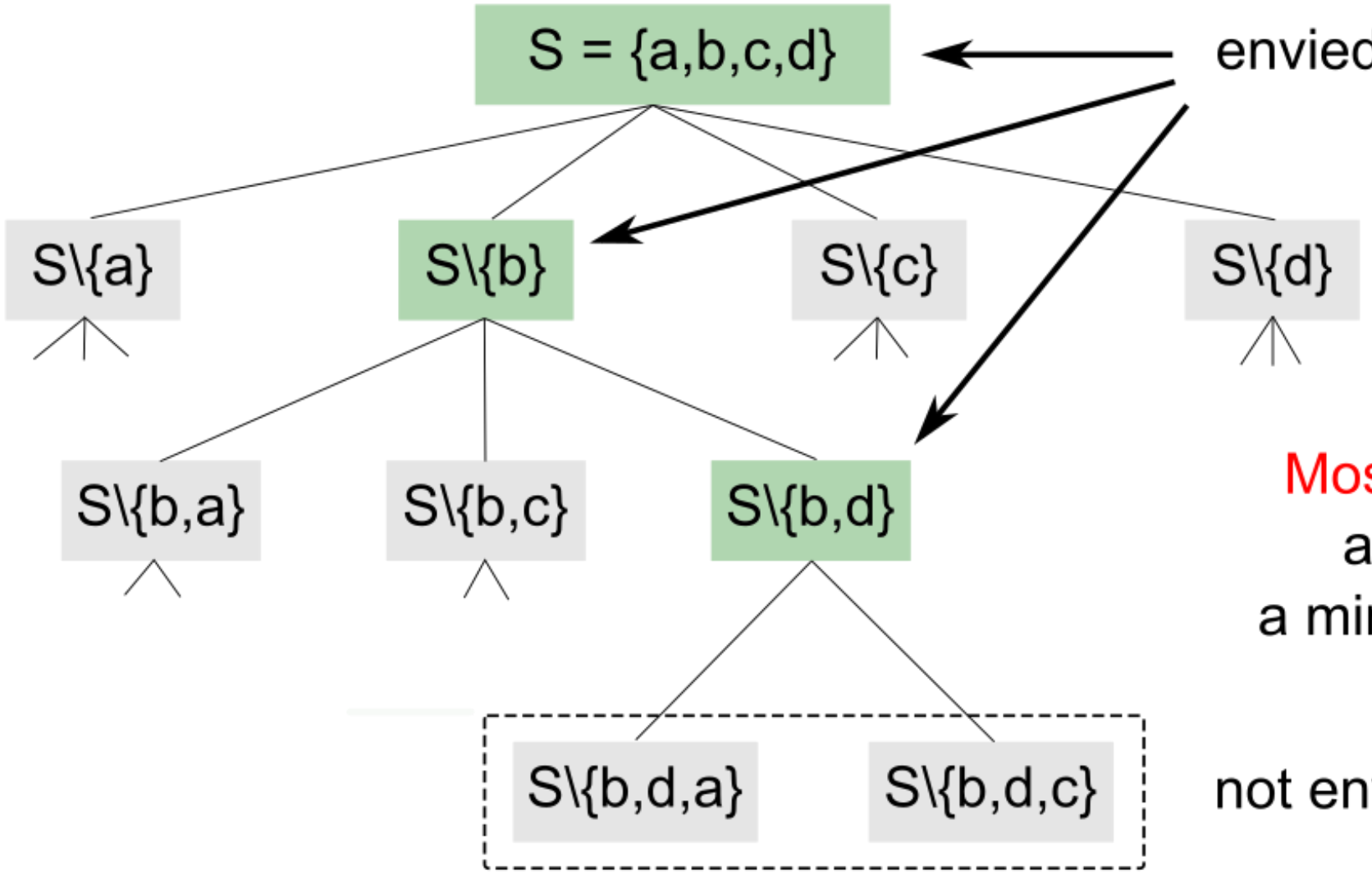

envied by one or more agents

Most envious agent of S =
any agent that envies
a minimal envied subset of S

not envied by anyone

Achieving EFX-with-charity

[Chaudhury, Kavitha, Mehlhorn, and Sgouritsa, *SODA* 2020, *SICOMP* 2021]

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make progress towards these

Achieving EFX-with-charity

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Start with everything unallocated (i.e., all goods in the pool P).

Trivially EFX

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Otherwise, return the current allocation.

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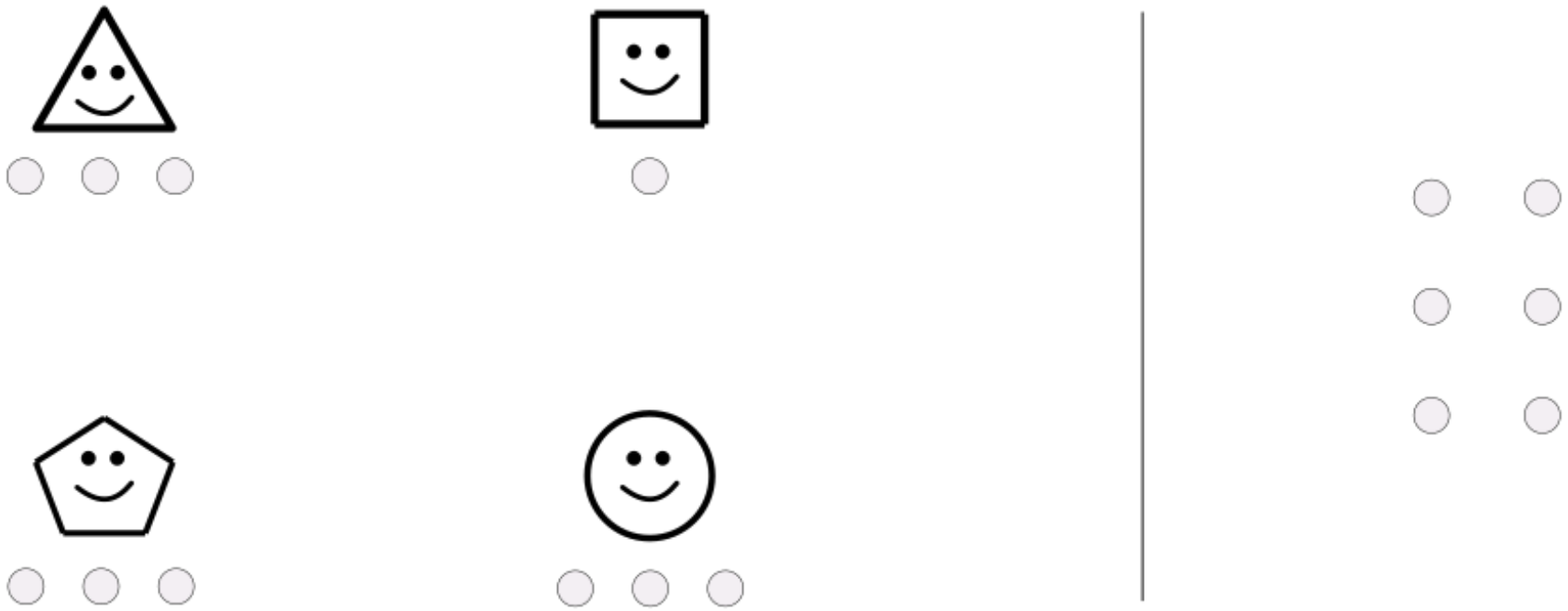

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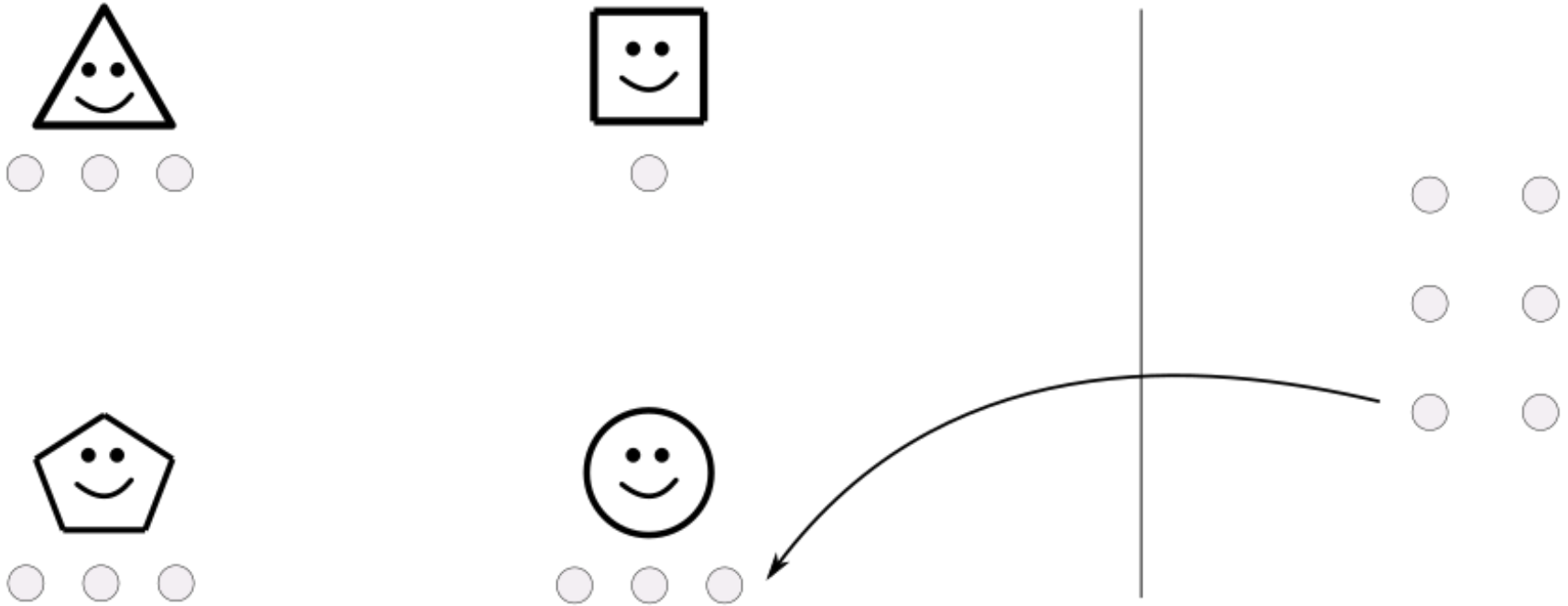

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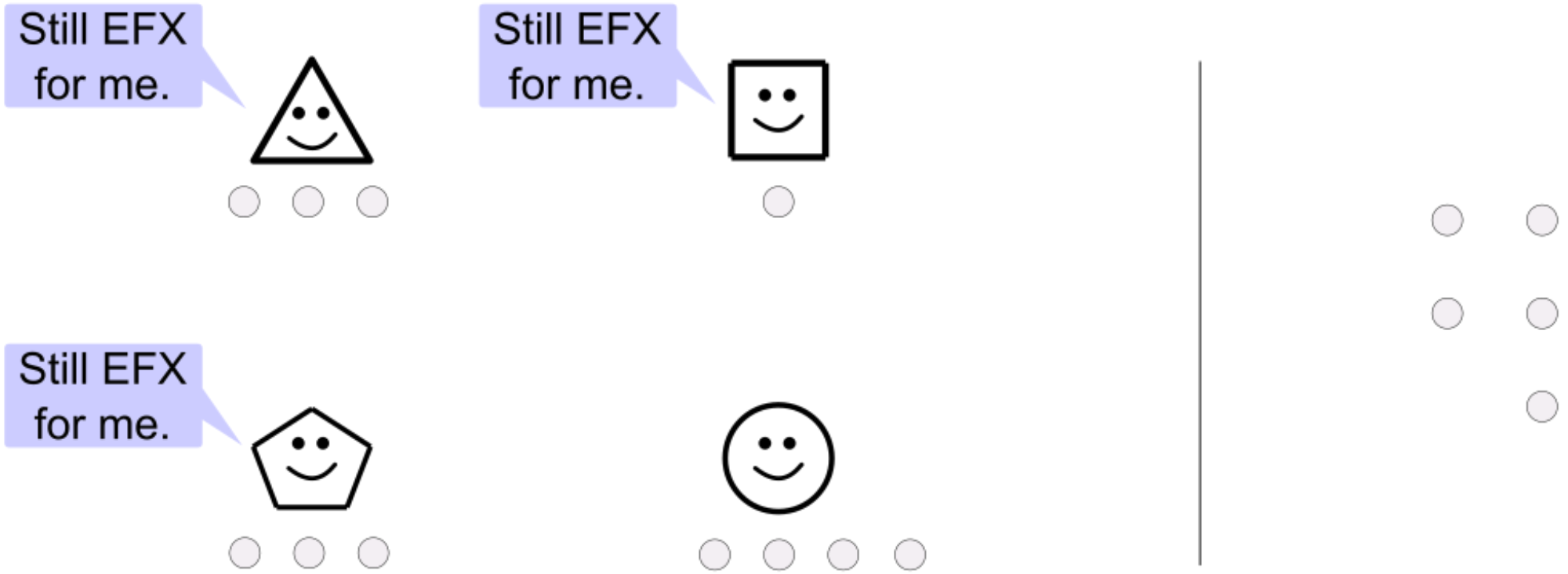

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Rule 2: If any agent envies the pool P, then give a minimal envied subset of P to the most envious agent (and return old bundle to P).

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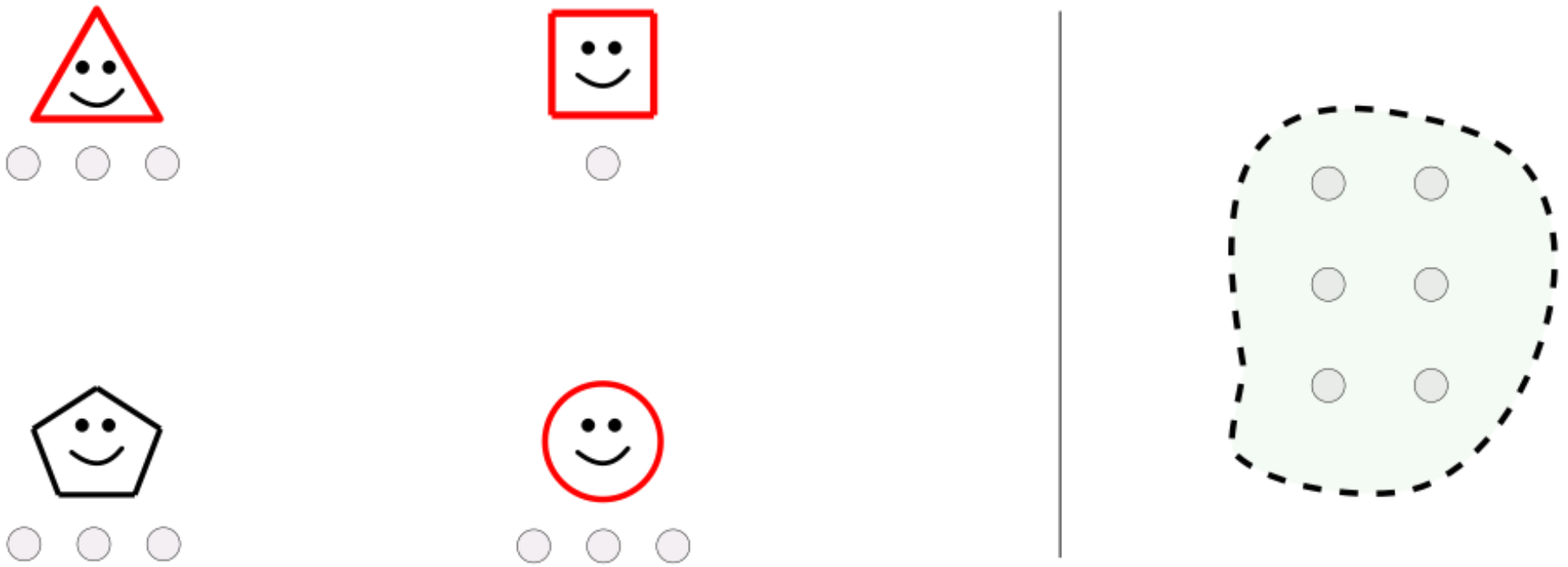

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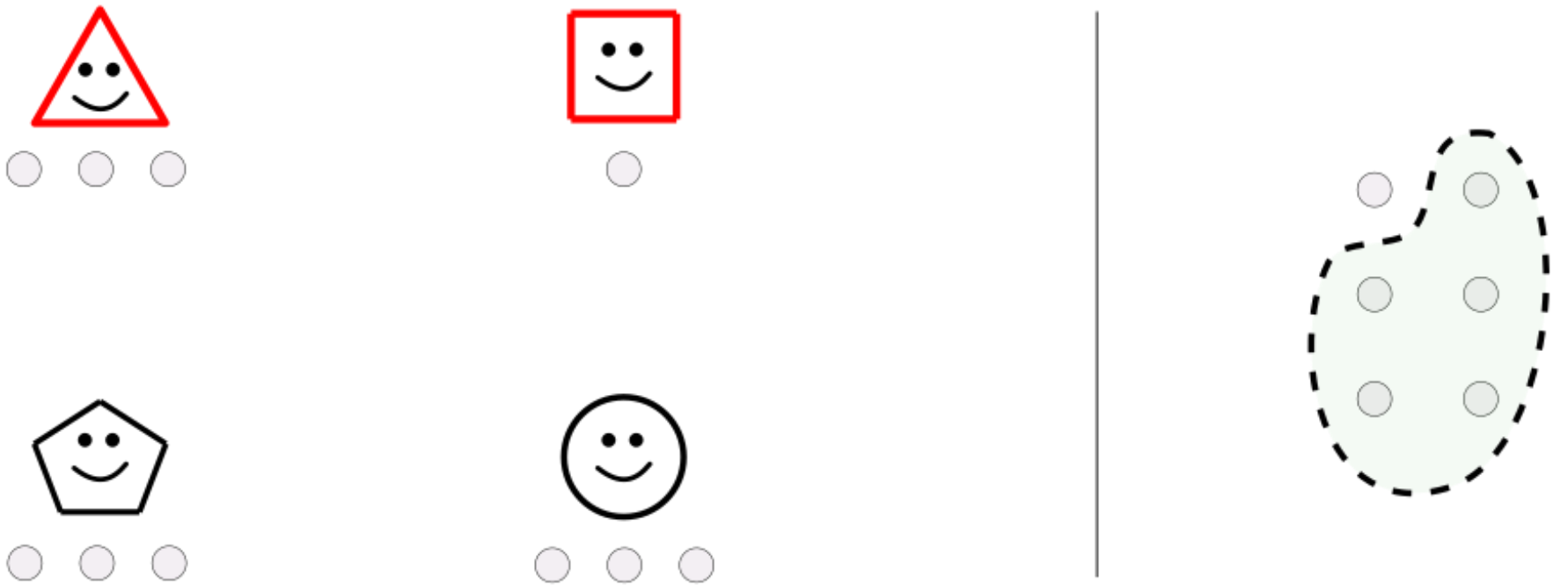

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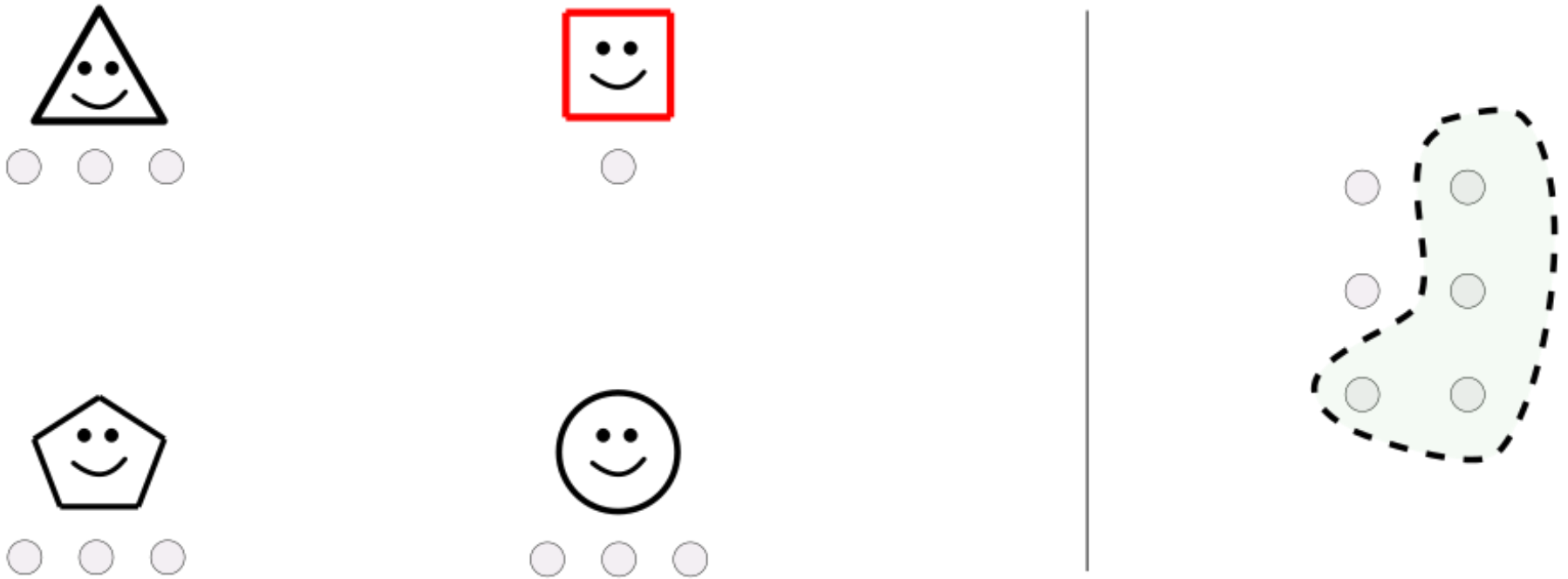

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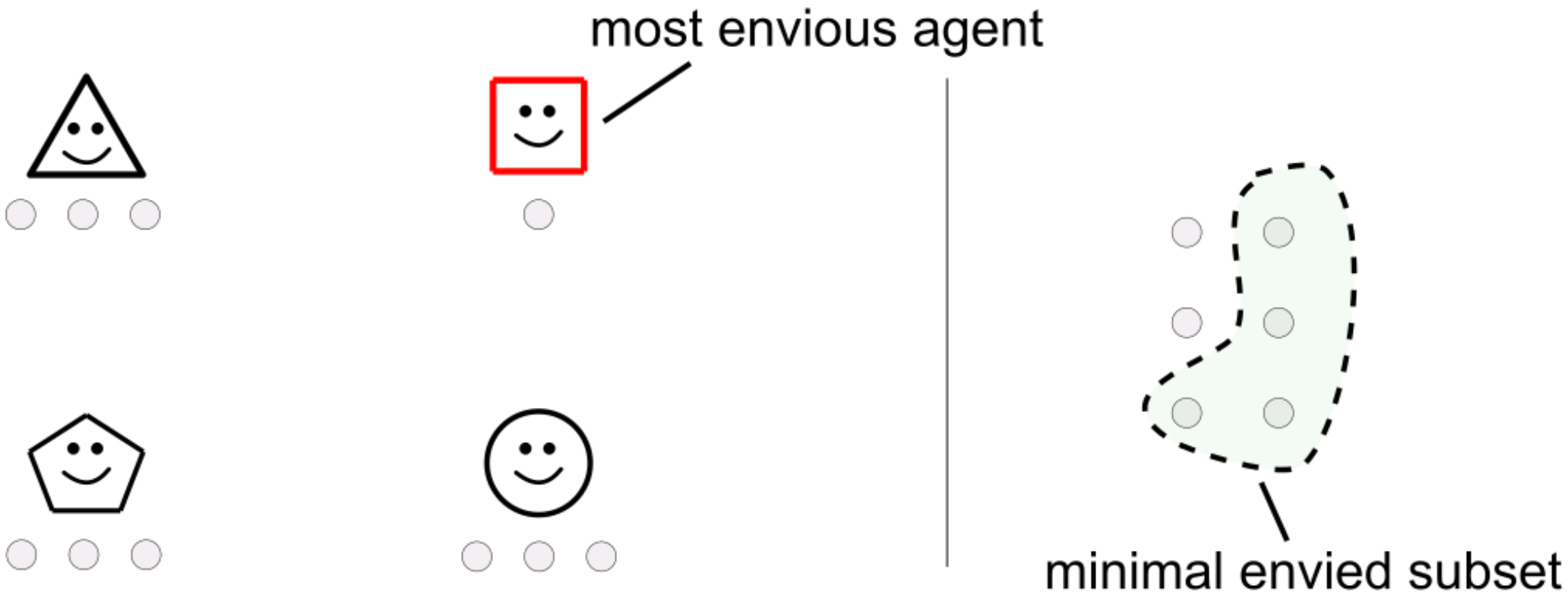

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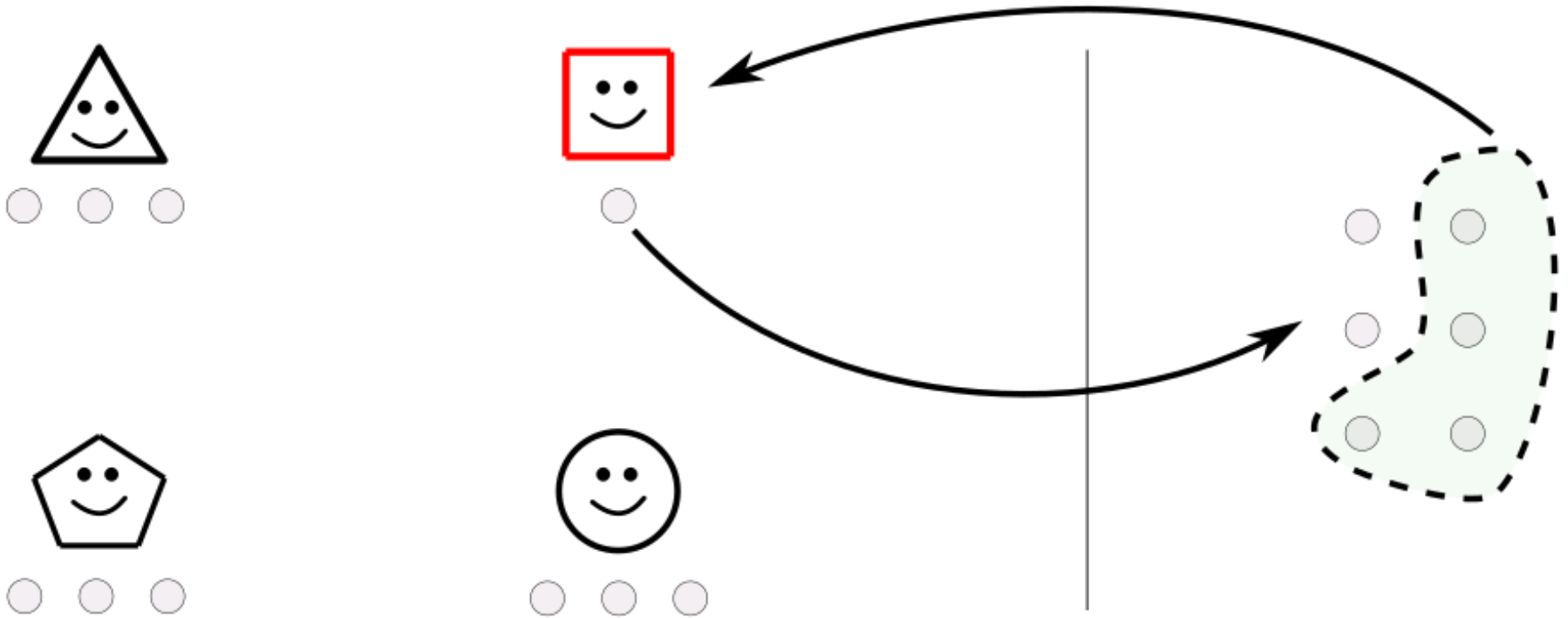

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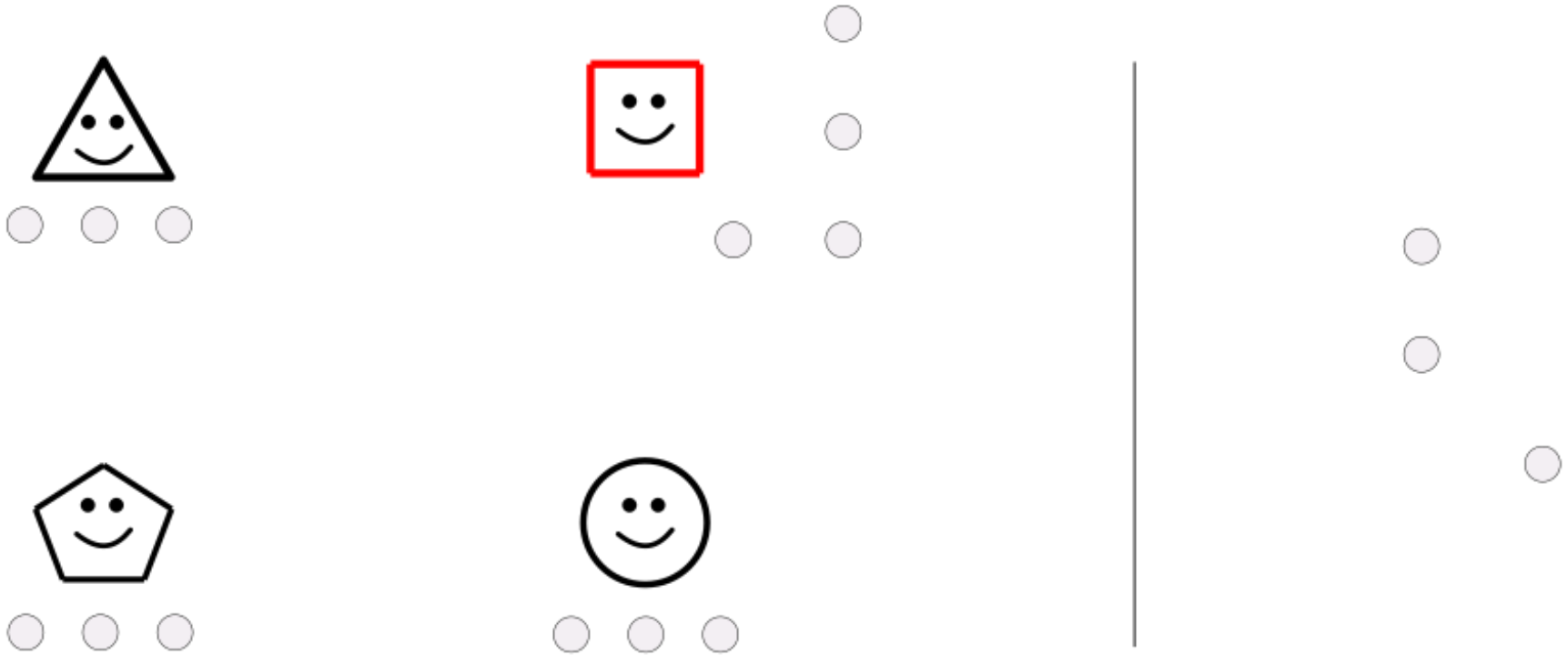

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Otherwise, return the current allocation.

Achieving EFX-with-charity

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Achieving EFX-with-charity

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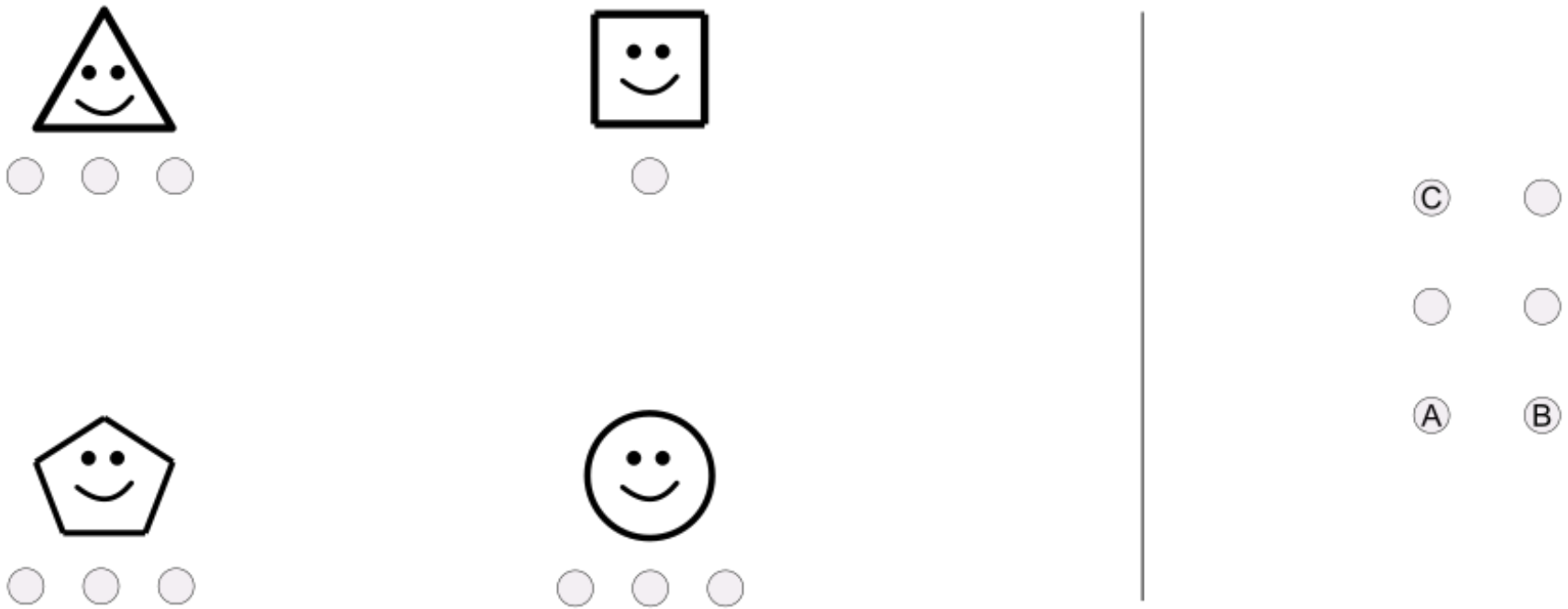

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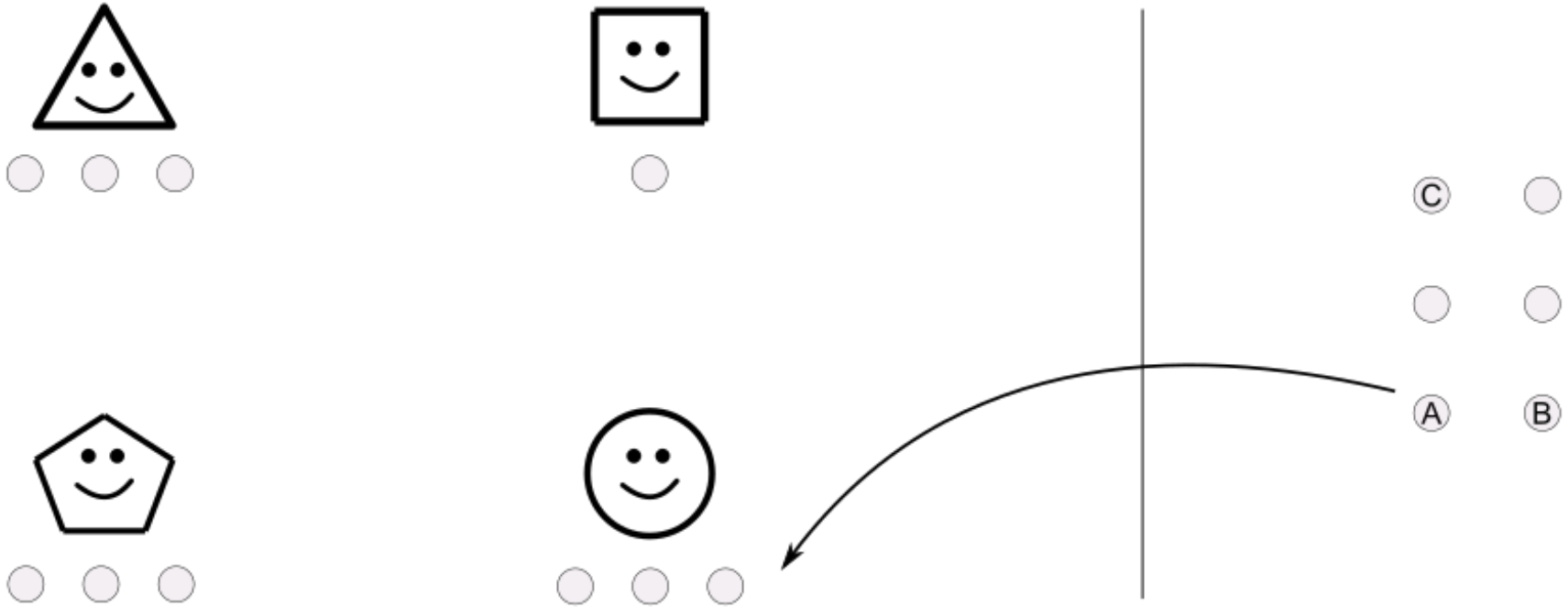

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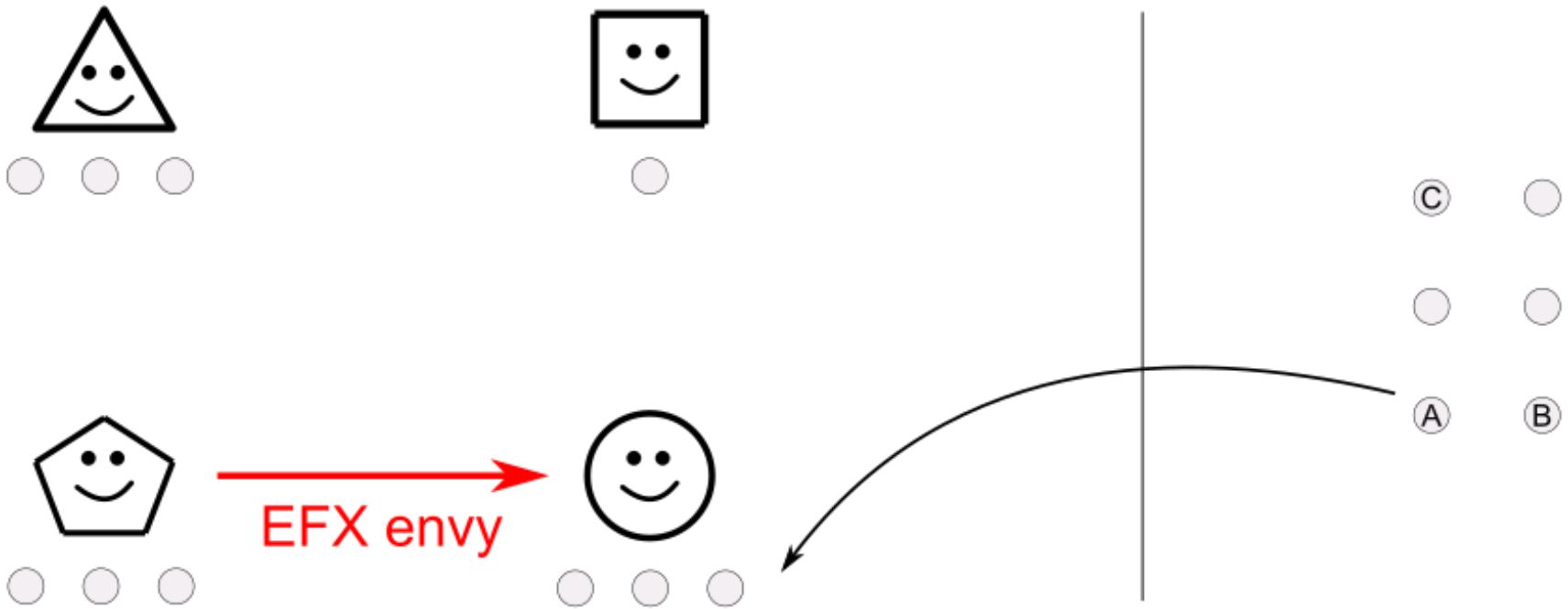

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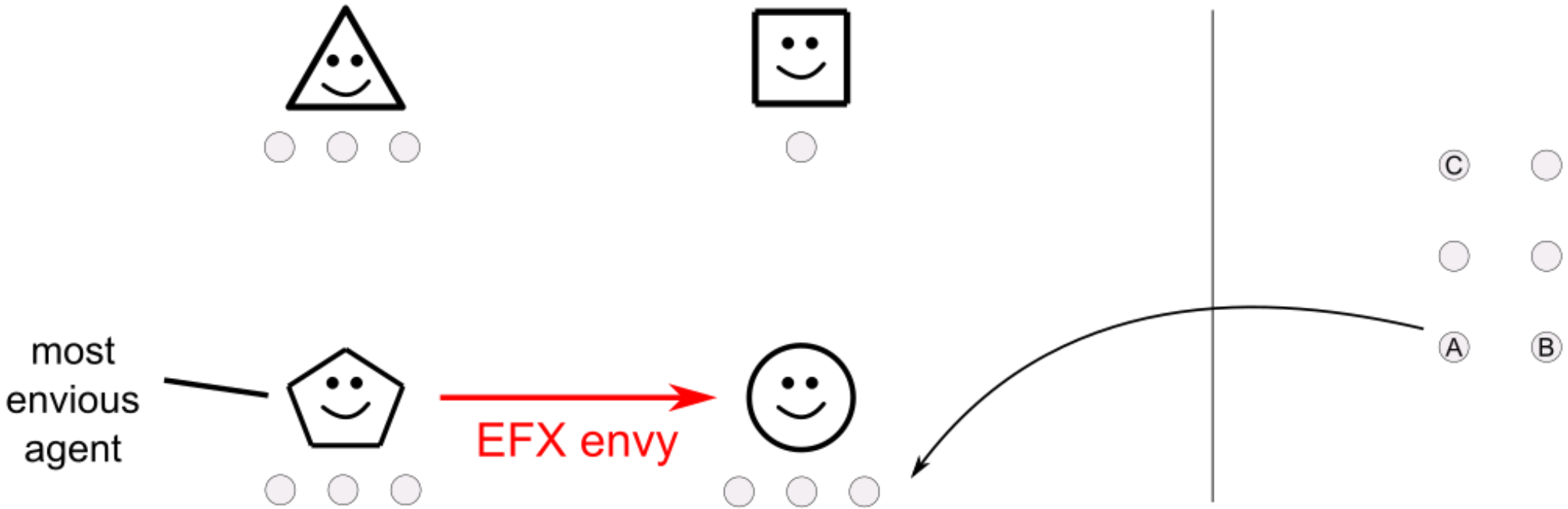

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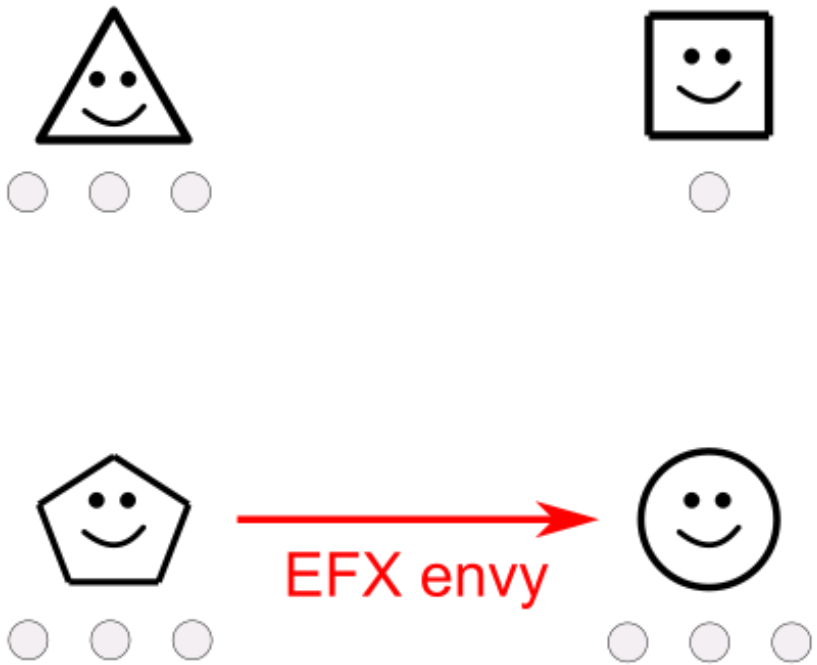

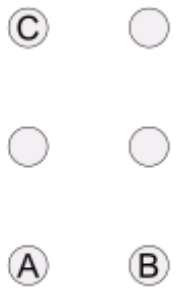

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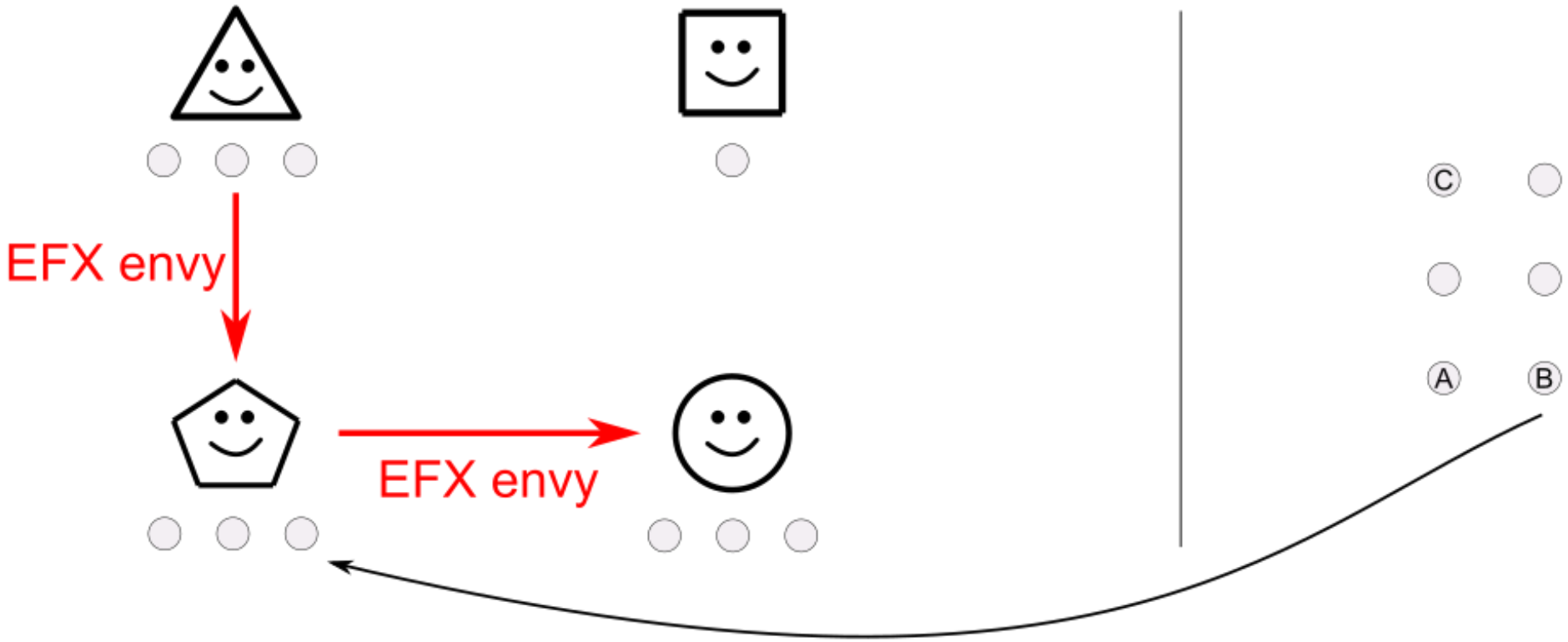

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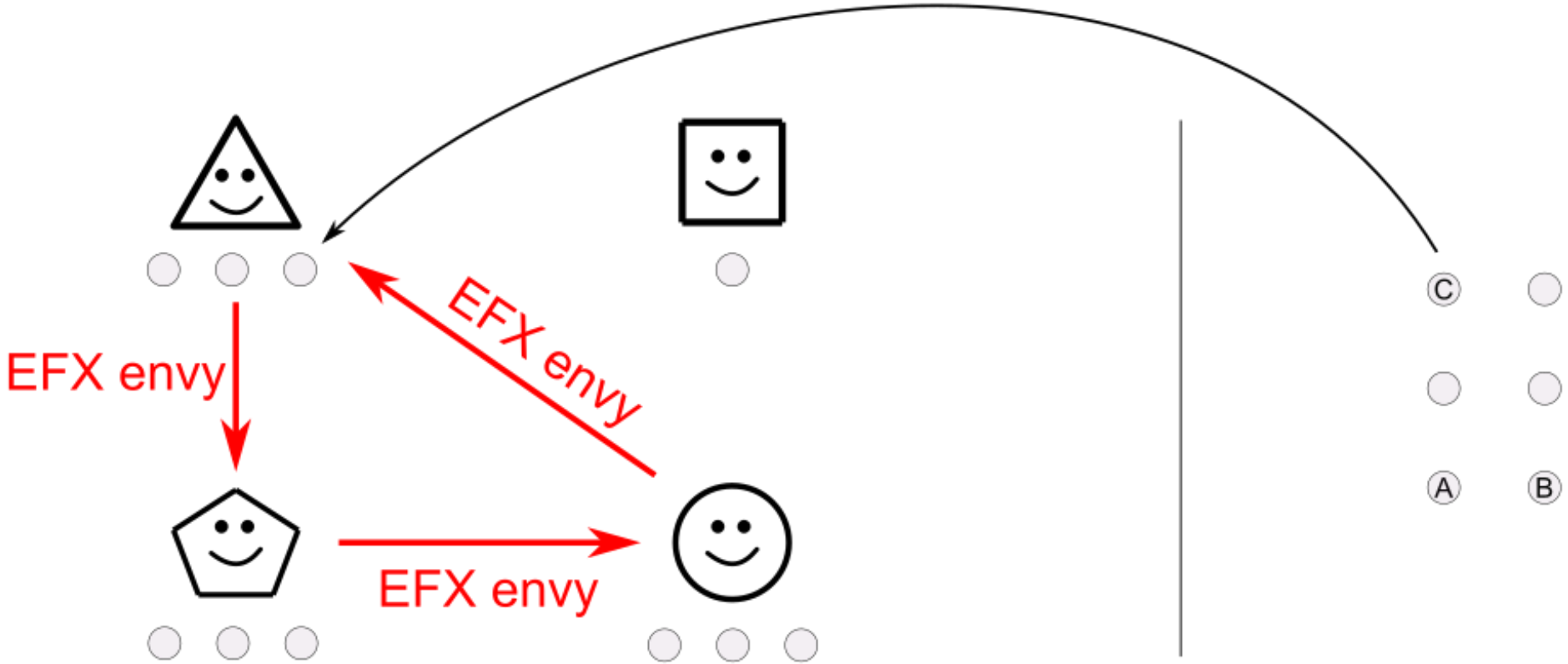

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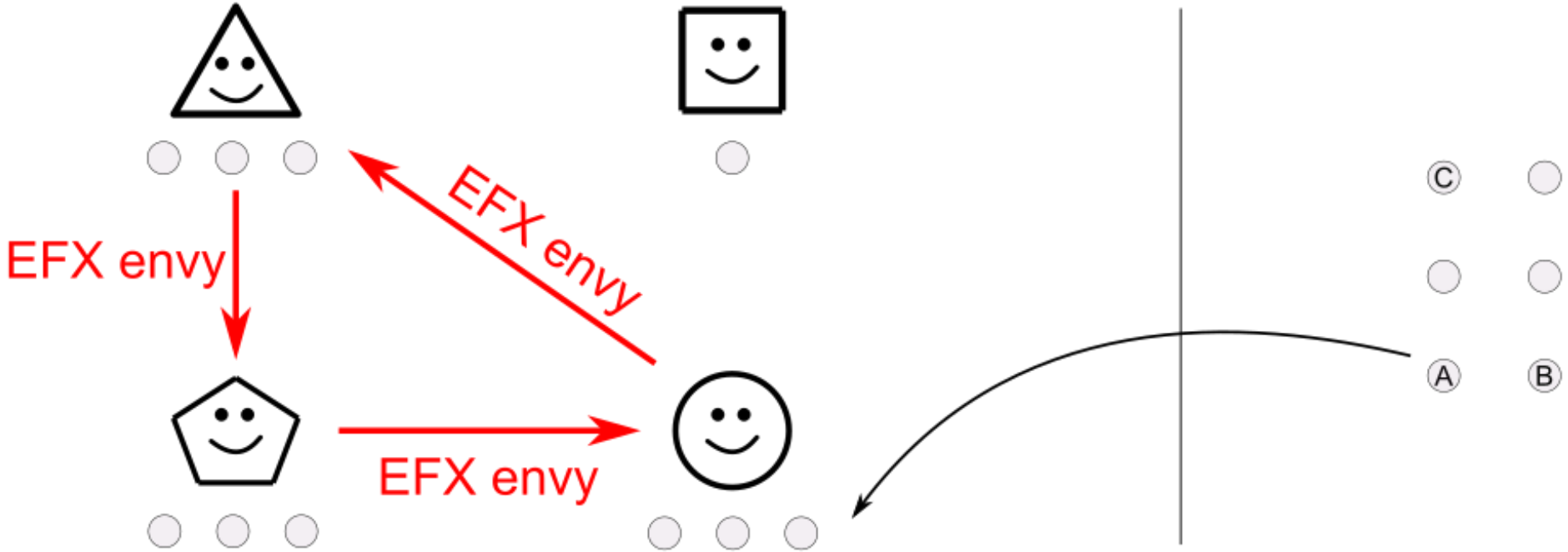

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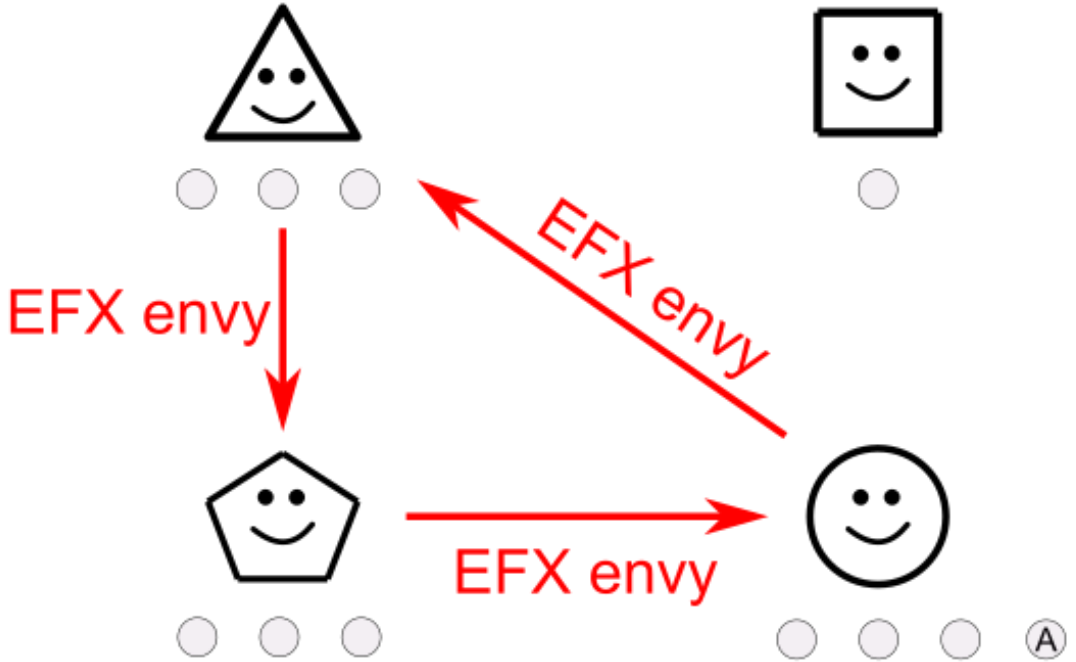

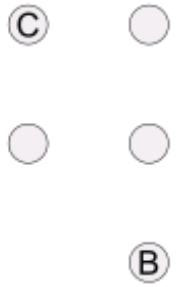

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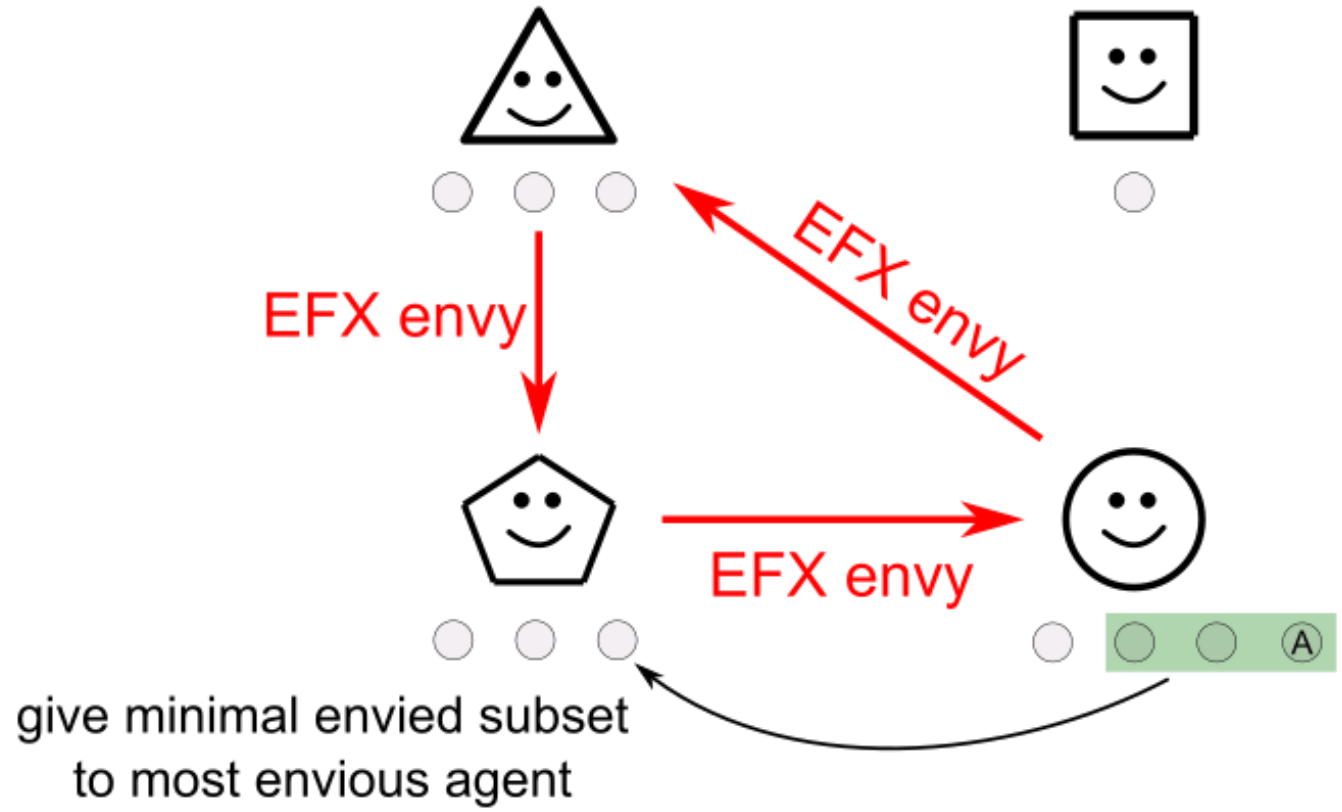

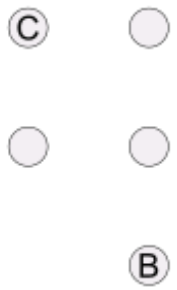

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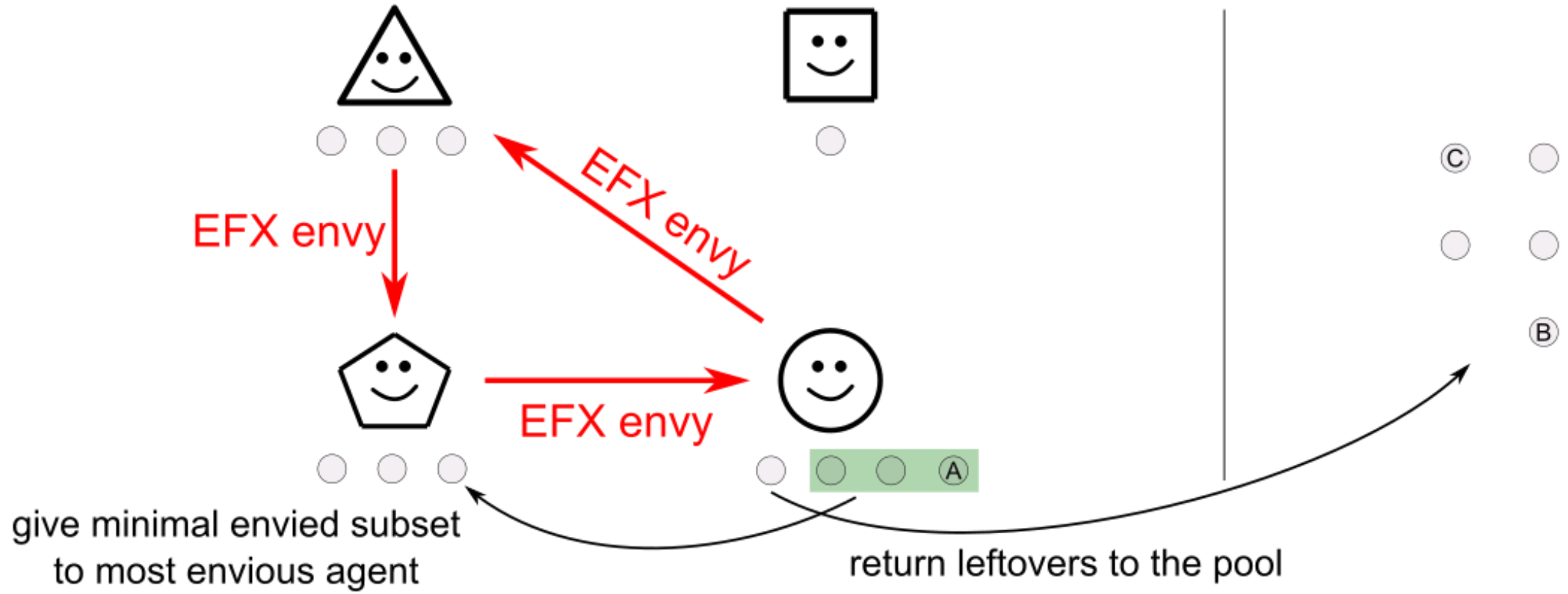

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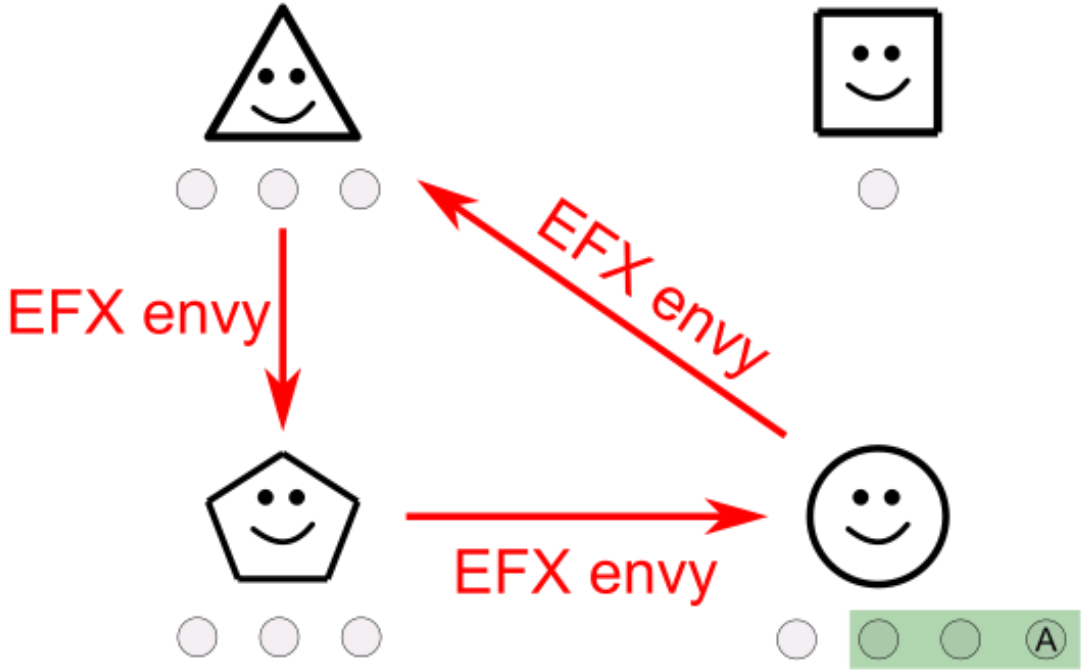

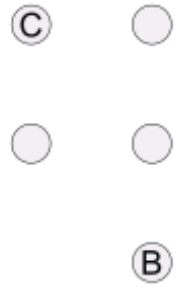

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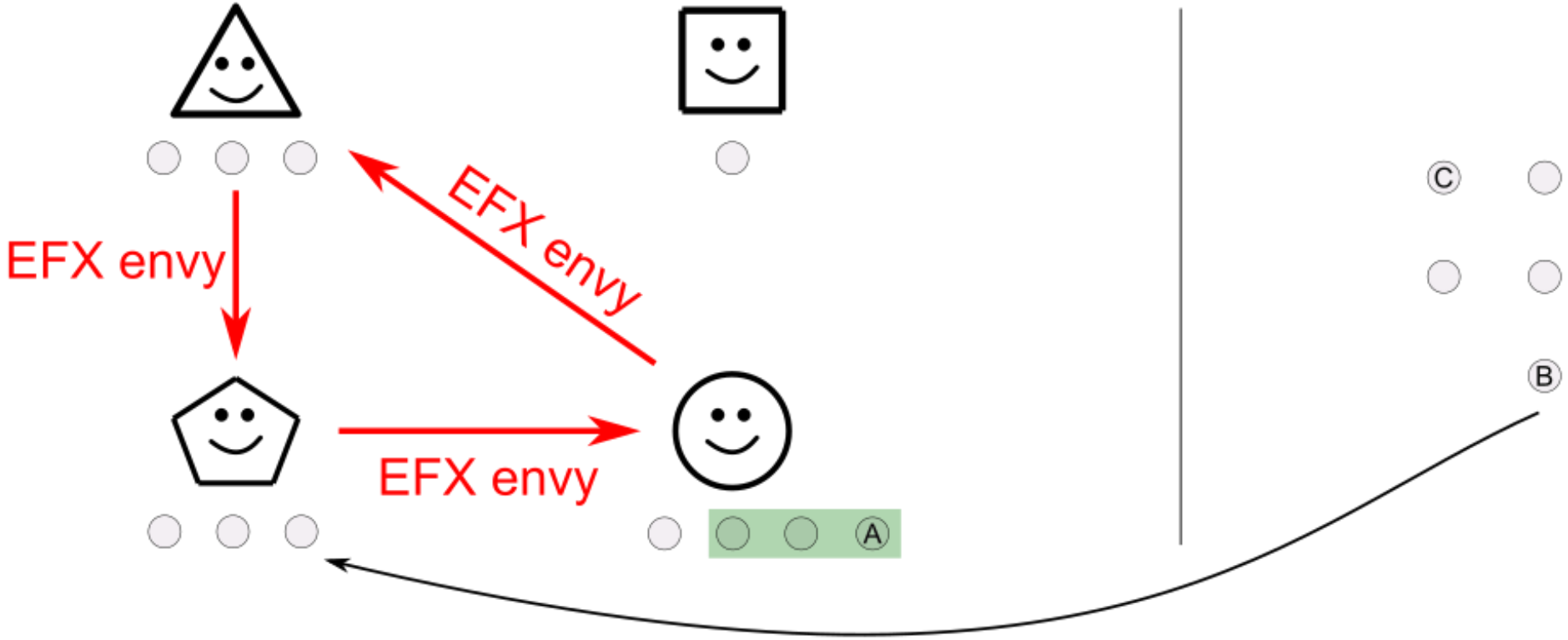

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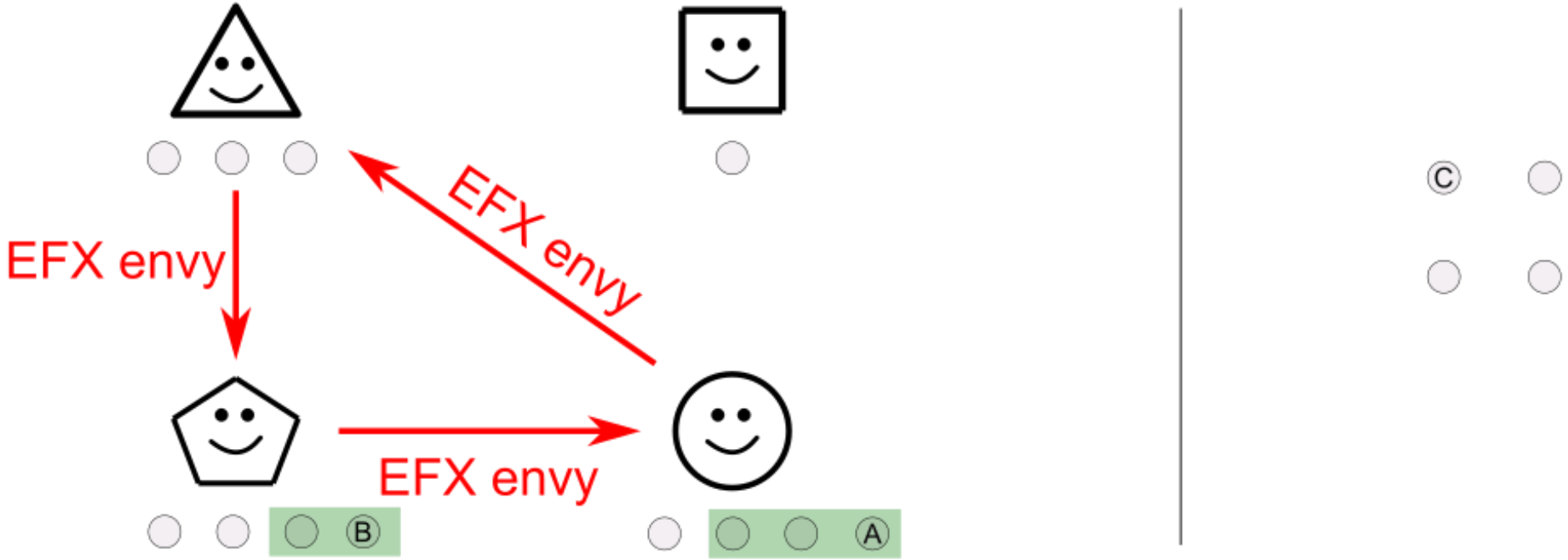

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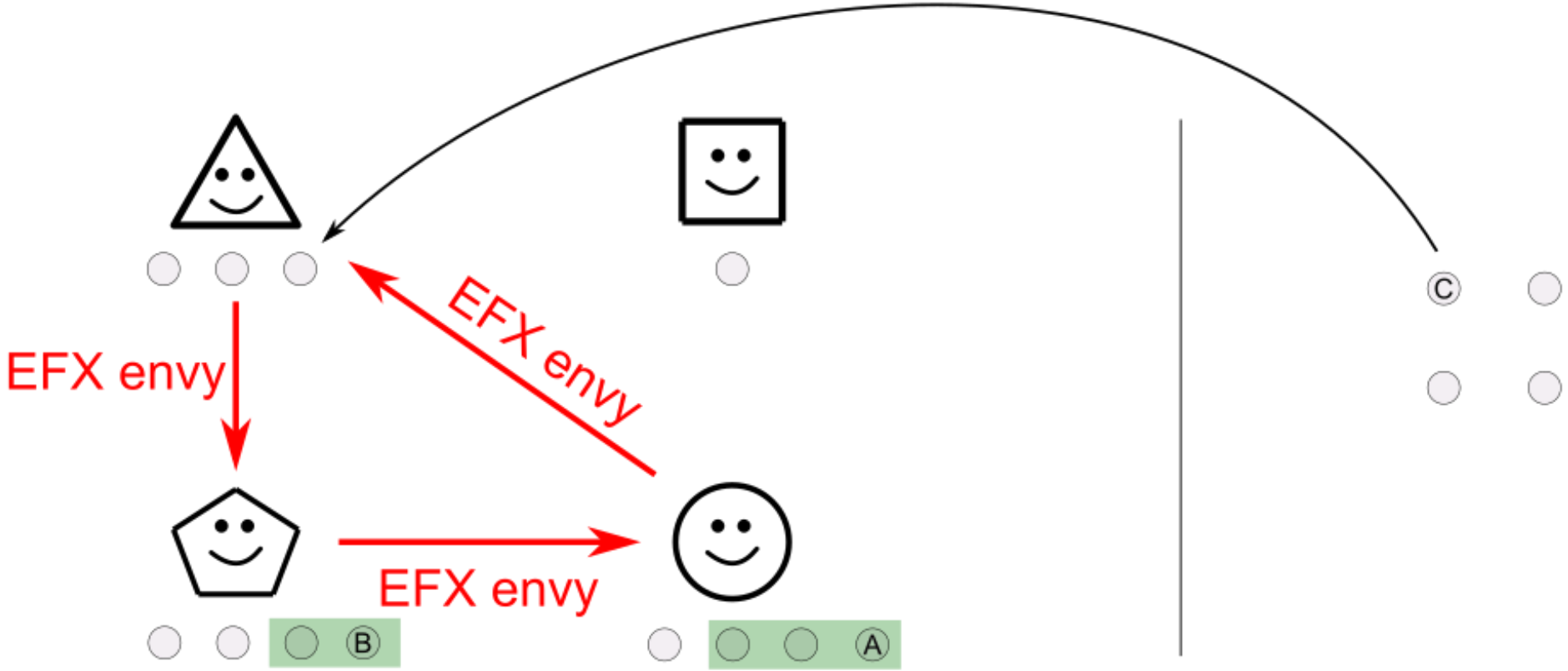

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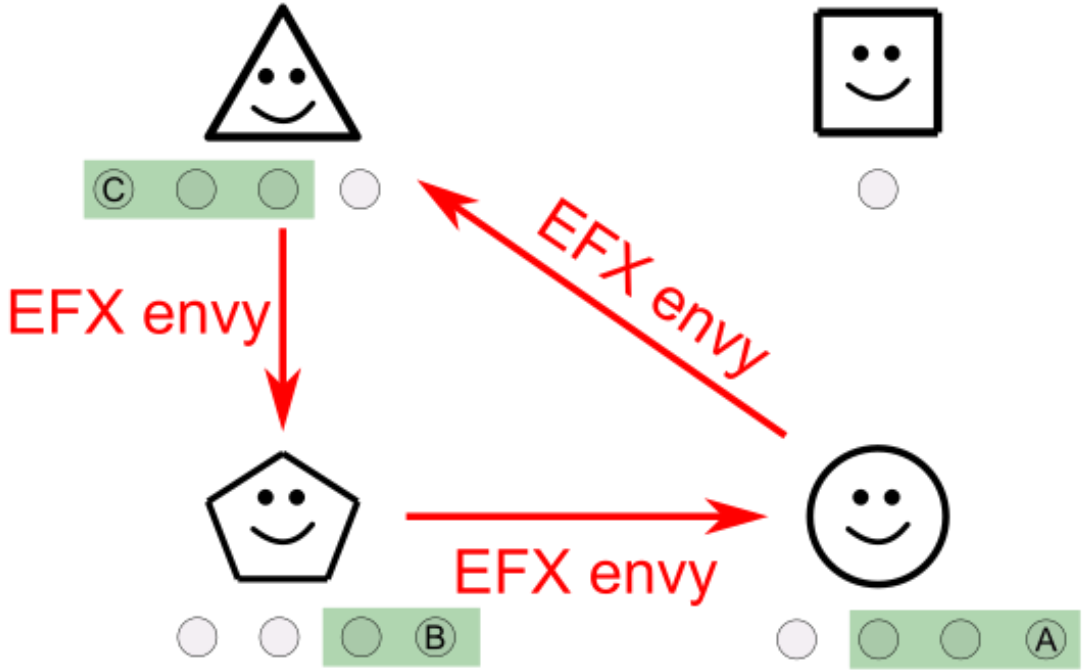

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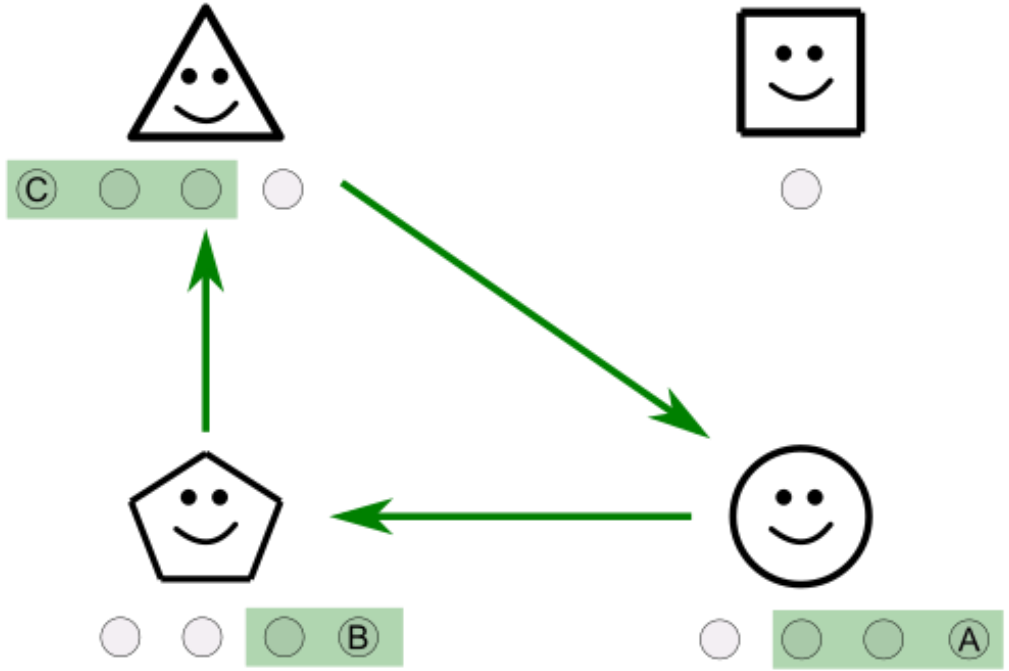

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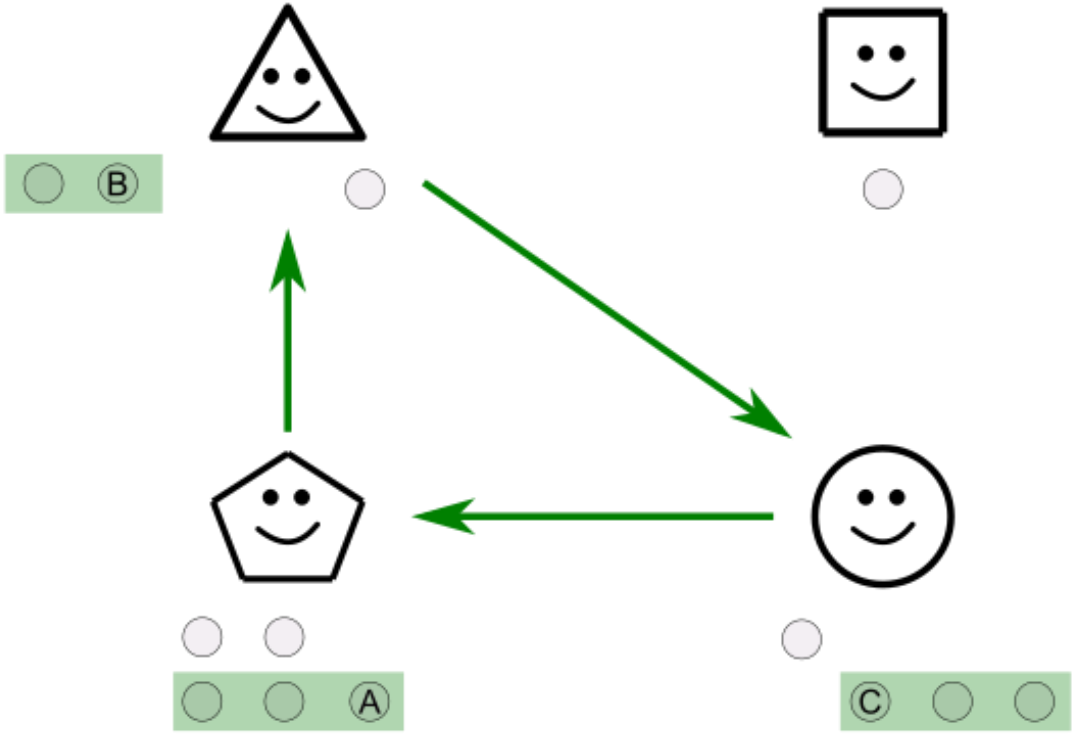

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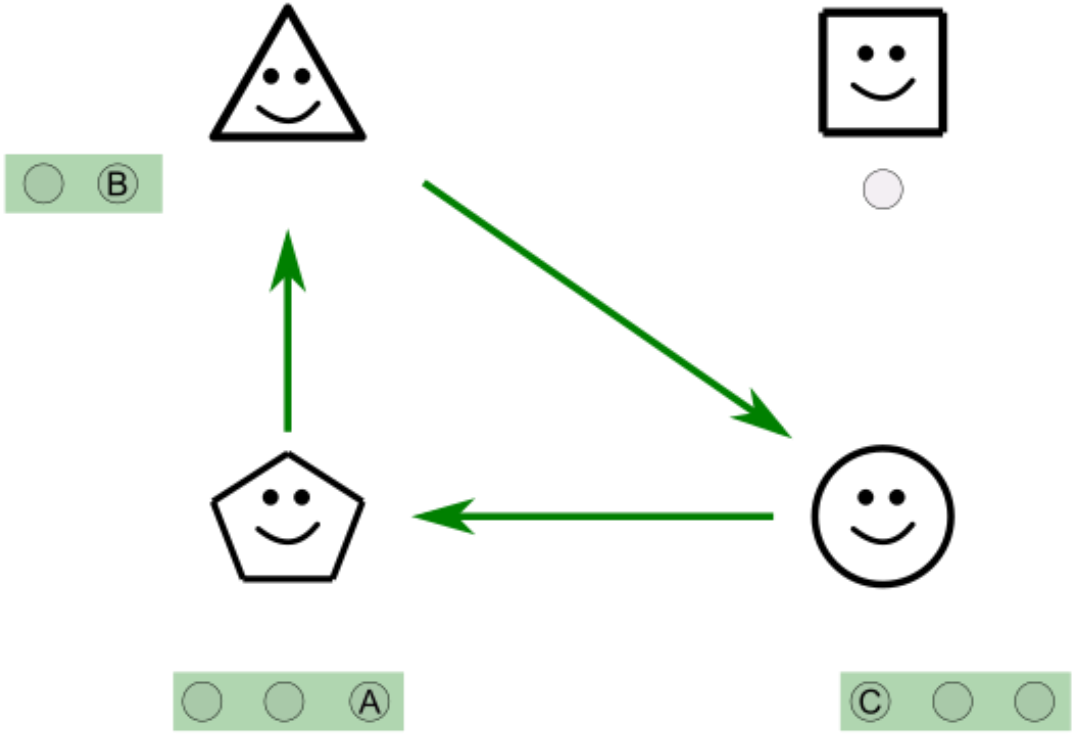

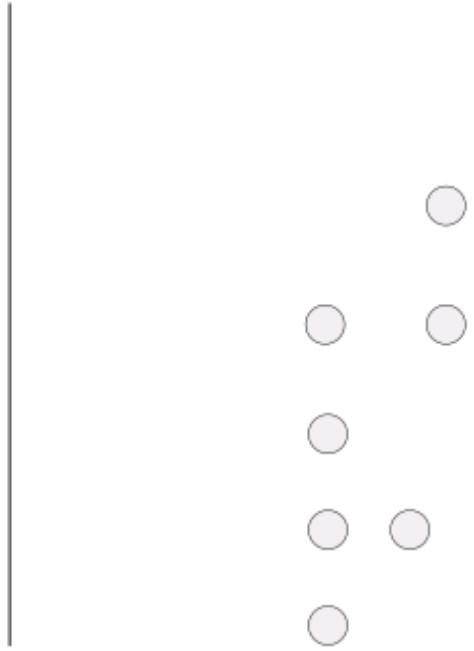

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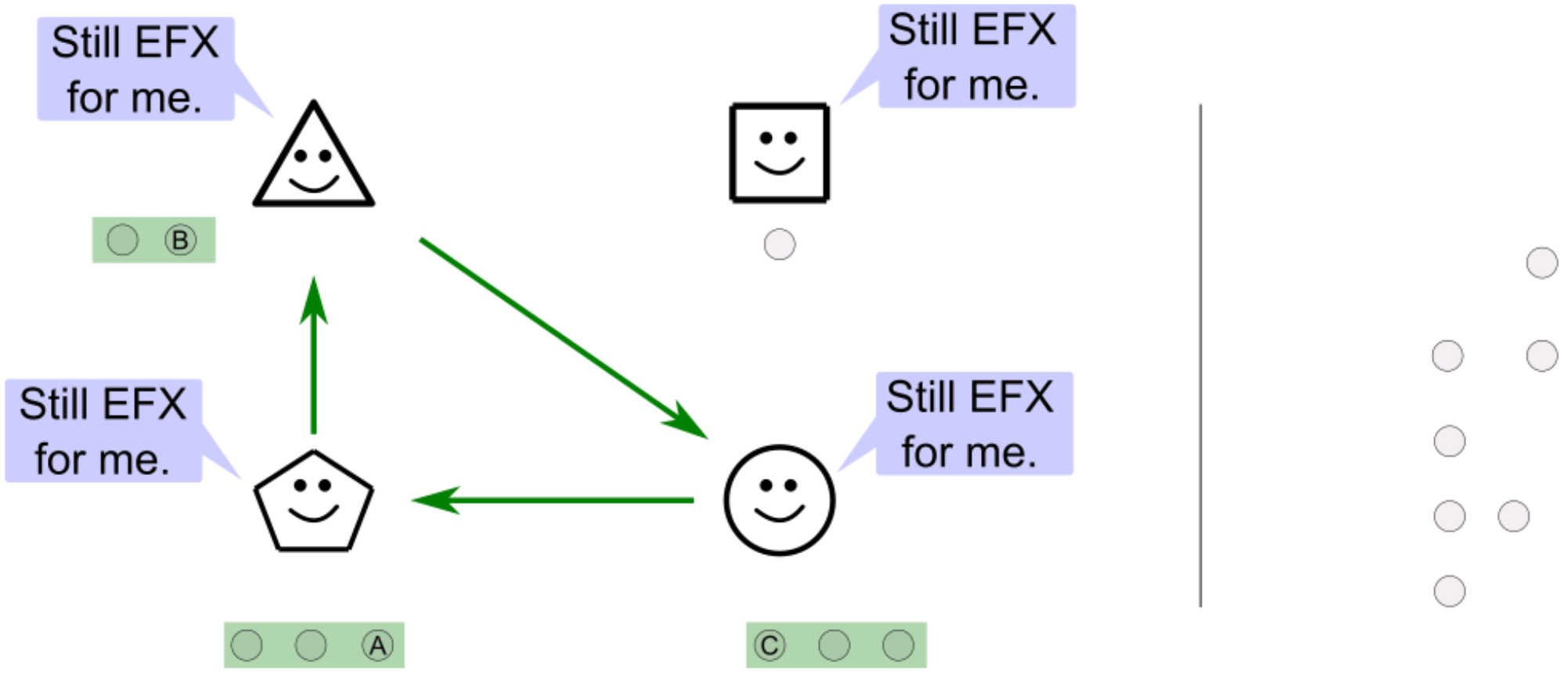

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As long as one of the following rules is applicable, apply it.

- Rule 1
- Rule 2
- Rule 3

Otherwise, return the current allocation.

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If the algorithm terminates:

- EFX among main agents (invariant)
- No envy towards pool (Rule 2)
- $|P|<n$ (Rule 3)

$\Rightarrow$ EFX-with-charity

Achieving EFX-with-charity

[Chaudhury, Kavitha, Mehlhorn, and Sgouritsa, *SODA* 2020, *SICOMP* 2021]

Why does the algorithm terminate?

Achieving EFX-with-charity

[Chaudhury, Kavitha, Mehlhorn, and Sgouritsa, *SODA* 2020, *SICOMP* 2021]

Why does the algorithm terminate?

Rule 1: If an unallocated good can be given to an agent while maintaining EFX, then do so.

Rule 2: If any agent envies the pool P, then give a minimal envied subset of P to the most envious agent (and return old bundle to P).

Rule 3: If $|P| \geq n$, then resolve a cycle of most envious agents.

Achieving EFX-with-charity

[Chaudhury, Kavitha, Mehlhorn, and Sgouritsa, *SODA* 2020, *SICOMP* 2021]

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Social welfare ↕ , $|P| \downarrow$

Rule 2: If any agent envies the pool P, then give a minimal envied subset of P to the most envious agent (and return old bundle to P).

Social welfare $\uparrow$, $|P| \updownarrow$

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Social welfare $\uparrow$, $|P| \updownarrow$

Achieving EFX-with-charity

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Why does the algorithm terminate?

Potential function

$$\phi(A) = m \cdot \sum_i v_i(A_i) - |P|$$

$$m = \text{no. of goods}$$

EFX-with-charity in polynomial time

Reminders

Mid-semester feedback (due Feb 9)

Assignment 1 is out (due Feb 15)

Next Time

Mid-term presentations

Feb 10 (Monday)

Quiz

Quiz

Construct an instance with two agents where no EFX allocation is Pareto optimal.