

Tutorial Sheet 8

Announced on: Mar 06 (Wed)

1. Based on Problem 12.53 in [LLM17].

Procedure *Mark* starts with a connected, unweighted, and simple graph with all edges unmarked and then marks some edges. At any point in the procedure a path that includes only marked edges is called a *fully marked* path, and an edge that has no fully marked path between its endpoints is called *eligible*.

Procedure Mark simply keeps marking eligible edges, and terminates when there are none. Prove that Mark terminates, and that when it does, the set of marked edges forms a spanning tree of the original graph.

2. This problem describes the *cut property* of minimum spanning trees (MSTs).

Let $G = (V, E)$ be a connected and weighted graph. For any subset $S \subseteq V$ of the vertices, let $\bar{S} := V \setminus S$ denote the vertices not in S . Let $E(S, \bar{S})$ denote the set of all edges with one endpoint each in S and \bar{S} . Show that if there exists a unique edge $e \in E(S, \bar{S})$ with the smallest weight among the edges in $E(S, \bar{S})$, then e must belong to every MST of graph G .

3. This problem describes the well-known *Prim's algorithm* for constructing an MST.

Consider a procedure that, given a connected and weighted graph $G = (V, E)$, grows a tree as follows: Start with a vertex $u \in V$, and then successively add a minimum weight edge with exactly one endpoint in the tree. The procedure stops when the tree spans the vertices in V .

Prove that the above procedure terminates and returns an MST.

Hint: You may use the cut property in Problem 2.

4. Given a connected and weighted graph, show that if each edge has a distinct weight, then there is a unique minimum spanning tree.
5. Prove or disprove: A graph is a tree if and only if there is a unique path between any pair of vertices.
6. Prove or disprove: Let $G = (V, E)$ be a connected graph with $|V| = n$ vertices. Then, G is a tree if and only if $|E| = n - 1$.
7. Prove or disprove: A graph is bipartite if and only if it has no cycle of odd length.

References

- [LLM17] Eric Lehman, Tom Leighton, and Albert R Meyer. *Mathematics for Computer Science*. 2017. URL: <https://courses.csail.mit.edu/6.042/spring18/mcs.pdf>.