COL202: Discrete Mathematical Structures **Spring 2024**

Tutorial Sheet 7

Announced on: Feb 14 (Wed)

1. Based on Problem 6.25 in [LLM17].

   In the stable matching problem, call a person (a man or a woman) *lucky* if they are matched with someone in the top half of their preference list under some stable matching. Prove that there must be at least one lucky person.

2. Show that under the men-proposing deferred acceptance algorithm, there is always at least one woman who receives exactly one proposal during the execution of the algorithm.

3. Consider any input to the DA algorithm consisting of n men and n women, where n is an arbitrary natural number. As a function of n, what is the maximum number of rounds for which the DA algorithm can run before it terminates?

   Construct an instance of the stable matching problem with n men and n women (again, for a general n) where the DA algorithm runs for the number of rounds specified in your answer above. Explain the *correctness* of your answer—specifically, why does your instance satisfy the stated bound and why is it the optimal bound.

4. Prove that an instance has a unique stable matching if and only if the men-optimal and women-optimal stable matchings are identical. Can you give an algorithm for quickly determining if a given instance has a unique stable matching?

5. Based on Problem 12.47 in [LLM17].

   a) Prove that the average degree of a tree is less than 2.

   b) Suppose every vertex in a graph has degree at least k. Explain why the graph has a path of length at least k. Does such a graph also always have a path of length *exactly* k?

References

[LLM17] Eric Lehman, Tom Leighton, and Albert R Meyer. *Mathematics for Computer Science*. 2017. URL: https://courses.csail.mit.edu/6.042/spring18/mcs.pdf.