COL202: Discrete Mathematical Structures **Spring 2024**

Tutorial Sheet 4

Announced on: Jan 24 (Wed)

1. Based on Problem 9.10 in [LLM17].

   Prove or disprove the following:

   a) For any triple of integers a, b, c, if $\gcd(a, b) \neq 1$ and $\gcd(b, c) \neq 1$, then $\gcd(a, c) \neq 1$.

   b) For any triple of integers a, b, c, $\gcd(ab, ac) = a \cdot \gcd(b, c)$.

   c) For any pair of integers a and b and any natural number n, $\gcd(a^n, b^n) = \gcd(a, b)^n$.

2. Recall the water-filling puzzle discussed in Lecture 10. Imagine that now you are given *three* jugs of capacities a, b, and c litres, where $a, b, c \in \mathbb{N}$. Further, as with the two-jug puzzle discussed in class, there is a faucet with an unlimited water supply and a drain with an unlimited capacity.

   What water levels can you create in a jug using a sequence of standard moves and why? (As with the two-jug puzzle, a standard move includes drawing water from the faucet, discarding water into the drain, and pouring water from one jug into the other.)

3. Based on Problems 9.18, 9.21, and 9.26 in [LLM17].

   Prove or disprove the following:

   a) For any triple of integers a,b, and c, $\gcd(a, \gcd(b, c)) = \gcd(\gcd(a, b), c)$.

   b) Let
   $$a = 2^9 \cdot 5^{24} \cdot 7^4 \cdot 11^7,$$
   $$b = 2^3 \cdot 7^{22} \cdot 11^{211} \cdot 19^7, \text{ and}$$
   $$c = 2^5 \cdot 3^4 \cdot 7^{6042} \cdot 19^{30}.$$
   Then, $\gcd(a, b, c) = 2 \cdot 7 \cdot 19$.

   c) For any integers a and b, if $a \equiv b \pmod 5$ and $a \equiv b \pmod{14}$, then $a \equiv b \pmod{70}$.

4. Based on Problems 9.7 and 9.31 in [LLM17].

   Find:

   a) $\gcd(3^{101}, 21)$

   b) $\gcd(13a + 8b, 5a + 3b)$ - $\gcd(a, b)$ for arbitrary integers a and b

   c) $\gcd(m, n)/\gcd(m/2, n/2)$ for arbitrary even natural numbers m and n.

   In each case, provide the reasoning behind your answer.

References

[LLM17] Eric Lehman, Tom Leighton, and Albert R Meyer. *Mathematics for Computer Science*. 2017. URL: https://courses.csail.mit.edu/6.042/spring18/mcs.pdf.