COL202: Discrete Mathematical Structures **Spring 2024**

Tutorial Sheet 3

Announced on: Jan 13 (Sat)

1. Show that ordinary induction implies well ordering principle.

2. (Based on Problem 2.5 in [LLM17].)

   Use the Well Ordering Principle to prove that there is no solution over the positive integers for the following equation:
   $$4a^3 + 2b^3 = c^3.$$

3. (Based on Problem 2.7 in [LLM17].)

   Use the Well Ordering Principle to prove that any integer greater than or equal to 8 can be represented as the sum of nonnegative integer multiples of 3 and 5.

4. Consider the *selection sort* algorithm (you may have seen this in COL106). The input to this algorithm is an array of n integers. The desired output is a sorted list of these integers arranged in ascending order.

   The algorithm runs for n rounds. In the i^{th} round (where $i \in \{1, 2, \ldots, n\}$), the algorithm finds the smallest element between (and including) the positions i and n in the array, and swaps it with the element at position i.

   a) What property is satisfied by the array maintained by selection sort at the end of the i^{th} round? (The property may depend on i.)

   b) Use the property from part (a) to prove that selection sort returns a sorted array after n rounds.

5. Consider the *insertion sort* algorithm (you may have seen this in COL106). The input to this algorithm is an array of n integers. The desired output is a sorted list of these integers arranged in ascending order.

   The algorithm runs for n rounds. In the i^{th} round (where $i \in \{1, 2, \ldots, n\}$), the algorithm inserts the element at position i into the subarray between (and including) the positions 1 and $i - 1$ at the correct location, say position j, and shifts all elements between (and including) the positions j and $i - 1$ by one position each to their right side.

   a) What property is satisfied by the array maintained by insertion sort at the end of the i^{th} round? (The property may depend on i.)

   b) Use the property from part (a) to prove that insertion sort returns a sorted array after n rounds.

6. [**Bonus problem: Not to be included in tutorial quiz**]
   Source: https://jdh.hamkins.org/buckets-of-fish/

   Consider a two-player game called *Buckets of Fish* played with finitely many buckets in a line on the beach, each containing a finite number of fish. There is also a large supply of additional fish available nearby, fresh off the boats.

   Taking turns, each player selects a bucket and removes exactly one fish from it and then, if desired, adds any finite number of fish from the nearby supply to the buckets to the left.

   For example, if we label the buckets from the left as 1, 2, 3 and so on, then a legal move would be to take one fish from bucket 4 and then add ten fish to bucket 1, no fish to bucket 2, and ninety-four fish to bucket 3. The winner is whoever takes the very last fish from the buckets, leaving them empty.

   Since huge numbers of fish can often be added to the buckets during play, thereby prolonging the length of play, a skeptical reader may wonder whether the game will necessarily come to an end.

   Prove that every play of the game Buckets of fish necessarily come to an end. Show a proof by induction and another proof by contradiction.

References

[LLM17] Eric Lehman, Tom Leighton, and Albert R Meyer. *Mathematics for Computer Science*. 2017. URL: https://courses.csail.mit.edu/6.042/spring18/mcs.pdf.