COL202: Discrete Mathematical Structures **Spring 2024**

Tutorial Sheet 2

Announced on: Jan 09 (Tue)

1. A *graph* (or a network) is a structure consisting of a set of objects (also known as *vertices* or *nodes*) some pairs of which are connected via *edges*. Assume that there are no self-edges (or loops). See Figure 1 for an example.

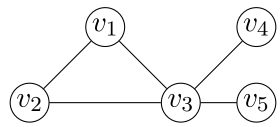

Figure 1: A graph with five vertices v_1, v_2, v_3, v_4, v_5, five edges $\{v_1, v_2\}$, $\{v_2, v_3\}$, $\{v_1, v_3\}$, $\{v_3, v_4\}$, and $\{v_3, v_5\}$, and maximum degree four.

The *degree* of a vertex is the number of other vertices that share an edge with it.

Say we are given a set of k colors. A graph is said to be k*-colorable* if each vertex can be assigned one of the k colors in a way that all neighboring vertices (i.e., any pair of vertices that share an edge) have different colors. (Some of the k colors may be left unused.)

Prove via induction that any graph with maximum degree d is $(d+1)$-colorable.

2. The sequence of Fibonacci numbers $\{F_n\}_{n \in \mathbb{N} \cup \{0\}}$ is defined as follows:

$$F_0 = 0$$
$$F_1 = 1$$
$$\forall n \geqslant 2, F_n = F_{n-1} + F_{n-2}$$

Let r be a positive real number satisfying $r^2 = r + 1$. Using induction, show that for all $n \in \mathbb{N}$, $F_n \geqslant r^{n-2}$.

3. Consider two collections of numbers,

$$a_1 \leqslant a_2 \leqslant \ldots \leqslant a_n \text{ and } b_1 \leqslant b_2 \leqslant \ldots \leqslant b_n,$$

and suppose that you have to make n disjoint pairs of numbers, each pair consisting of one a and one b.

Prove that the sum of the products of the members of each pair is maximized when, for each $i \in \{1, 2, \ldots, n\}$, a_i is paired with b_i.

For example, consider the collection of numbers $1 \leqslant 3 \leqslant 10$ and $2 \leqslant 4 \leqslant 7$. One way to construct a pairing is $\{(1,4), (3,2), (10,7)\}$, which gives a sum of products equal to $1 \times 4 + 3 \times 2 + 10 \times 7 = 80$. However, a better pairing is $\{(1,2), (3,4), (10,7)\}$, which gives a sum of products equal to $1 \times 2 + 3 \times 4 + 10 \times 7 = 84$.

4. Prove the following statements using induction:

   a) The number of subsets of an n-element set is 2^n.

   b) The number of ways of ranking n different objects is $n!$.

5. Prove the following using induction:

   a) For all $n \in \mathbb{N}$ such that $n \geqslant 3$, the sum of internal angles of a regular n-gon is $180(n-2)$.

   b) For all $n \in \mathbb{N}$, the last digit in the decimal expansion of 6^n is 6.

6. [**Bonus problem: Not to be included in tutorial quiz**] Imagine a board with the numbers $1, 2, \ldots, 2n$ written on it, where n is an odd natural number. Pick any two numbers i and j written on the board and erase them, and then write $|i - j|$ on the board. Continue until only one integer remains on the board. Show that this integer must be odd.

   For example, if $n = 3$, then you start with $\langle 1, 2, 3, 4, 5, 6\rangle$. Say you erase 1 and 3 first and add $|1 - 3| = 2$ on the board to get $\langle 2, 2, 4, 5, 6\rangle$. Next, say you erase 2 and 4 to get $\langle 2, 2, 5, 6\rangle$. Next, if you erase 2 and 5, then you get $\langle 2, 3, 6\rangle$, and then suppose you erase 3 and 6 to be left with $\langle 2, 3\rangle$. Finally, you must erase 2 and 3 to end up with 1, which is odd.