COL 202: DISCRETE MATHEMATICAL STRUCTURES

LECTURE 4

INDUCTION

JAN 09, 2024 | ROHIT VAISH

SO FAR

PROOF

PROPOSITION

A statement that is either TRUE or FALSE

AXIOMS

Assumptions / Propositions that are "accepted" as TRUE

LOGICAL DEDUCTIONS

A collection of rules for proving new propositions using previously known ones.

SO FAR

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A collection of rules for proving new propositions using previously known ones.

Propositional Calculus

$\neg p$	Negation
$p \vee q$	Disjunction / Or
$p \wedge q$	Conjunction / And
$p \Rightarrow q$	Implication / If-Then

SO FAR

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Consistency and completeness

Gödel's Incompleteness Theorem

SO FAR

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A collection of rules for proving new propositions using previously known ones.

Modus Ponens

$$\frac{p,\ p \Rightarrow q}{q}$$

Modus Tollens

$$\frac{p \Rightarrow q,\ \neg q}{\neg p}$$

Chain Rule

$$\frac{p \Rightarrow q,\ q \Rightarrow r}{p \Rightarrow r}$$

Contrapositive

$$\frac{p \Rightarrow q}{\neg q \Rightarrow \neg p}$$

SO FAR

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PROPOSITION

A statement that is either TRUE or FALSE

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Assumptions / Propositions that are "accepted" as TRUE

LOGICAL DEDUCTIONS

A collection of rules for proving new propositions using previously known ones.

PROOF TEMPLATES

By case analysis

By contradiction

By picture

By induction (Today)

INDUCTION AXIOM

Let $P(n)$ be a predicate.

If $P(0)$ is TRUE and

$\forall n \in \mathbb{N} \cup \{0\}$, $P(n) \Rightarrow P(n+1)$ is TRUE

then $\forall n \in \mathbb{N} \cup \{0\}$ $P(n)$ is TRUE.

INDUCTION AXIOM

Suppose $P(0)$ is TRUE. and

$P(0) \Rightarrow P(1)$ is TRUE and

$P(1) \Rightarrow P(2)$ is TRUE and

$\vdots$

Then, $\forall n \in \mathbb{N} \cup \{0\}$ $P(n)$ is TRUE.

INDUCTION AXIOM

Suppose $P(0)$ is TRUE. and

$P(0) \Rightarrow P(1)$ is TRUE and

$P(1) \Rightarrow P(2)$ is TRUE and

$\vdots$

P(1) is TRUE
(modus ponens)

Then, $\forall n \in \mathbb{N} \cup \{0\}$ $P(n)$ is TRUE.

Induction Axiom

Suppose $P(0)$ is TRUE. and

$P(0) \Rightarrow P(1)$ is TRUE and

$P(1) \Rightarrow P(2)$ is TRUE and

$\vdots$

$\rightarrow$ $P(1)$ is TRUE (modus ponens)

$\downarrow$

$\rightarrow$ $P(2)$ is TRUE (modus ponens)

Then, $\forall n \in \mathbb{N} \cup \{0\}$ $P(n)$ is TRUE.

INDUCTION AXIOM

Suppose $P(0)$ is TRUE and

$P(0) \Rightarrow P(1)$ is TRUE and

$P(1) \Rightarrow P(2)$ is TRUE and

$\vdots$

$\rightarrow$ $P(1)$ is TRUE (modus ponens)

$\downarrow$

$\rightarrow$ $P(2)$ is TRUE (modus ponens)

$\downarrow$

$\rightarrow$ $P(3)$ is TRUE

$\vdots$

Then, $\forall n \in \mathbb{N} \cup \{0\}$ $P(n)$ is TRUE.

Theorem: For all $n \in \mathbb{N} \cup \{0\}$

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

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$$\sum_{i=1}^{n} i \quad \text{or} \quad \sum_{1 \le i \le n} i$$

Theorem: For all $n \in \mathbb{N} \cup \{0\}$

$$1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}.$$

Proof: By induction. ← Always mention this.

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Let $P(n)$ be the proposition that

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

Theorem: For all $n \in \mathbb{N} \cup \{0\}$

$$1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}.$$

Proof: By induction. ← Always mention this.

Let $P(n)$ be the proposition that

$$1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}.$$

We want to show that:

* $P(0)$ is TRUE, and
* $\forall n \in \mathbb{N} \cup \{0\}$, $P(n) \Rightarrow P(n+1)$.

Theorem: For all $n \in \mathbb{N} \cup \{0\}$

$$1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}.$$

Proof: Base Case: $P(0)$ is TRUE

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Proof: Base Case: $P(0)$ is TRUE

For $n = 0$, $1 + 2 + \dots + n = 0$ (Convention)

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Proof: Base Case: $P(0)$ is TRUE

For $n=0$, $1+2+\dots+n = 0$ (Convention)

$$\frac{n(n+1)}{2} = \frac{0(0+1)}{2} = 0$$

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Proof: Base Case: $P(0)$ is TRUE

For $n=0$, $1+2+\cdots+n = 0$ (Convention)

$$\frac{n(n+1)}{2} = \frac{0(0+1)}{2} = 0$$

Therefore, $P(0)$ is TRUE.

Theorem: For all $n \in \mathbb{N} \cup \{0\}$

$$1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}.$$

Proof: Inductive Step: $\forall n \in \mathbb{N} \cup \{0\}$ $P(n) \Rightarrow P(n+1)$ is TRUE.

Theorem: For all $n \in \mathbb{N} \cup \{0\}$

$$1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}.$$

Proof: Inductive Step: $\forall n \in \mathbb{N} \cup \{0\}$ $P(n) \Rightarrow P(n+1)$ is TRUE.

Recall:

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Theorem: For all $n \in \mathbb{N} \cup \{0\}$

$$1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}.$$

Proof: Inductive Step: $\forall n \in \mathbb{N} \cup \{0\}$ $P(n) \Rightarrow P(n+1)$ is TRUE.

Recall:

p	q	$p \Rightarrow q$
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If $P(n)$ is FALSE, we are done.

Theorem: For all $n \in \mathbb{N} \cup \{0\}$

$$1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}.$$

Proof: Inductive Step: $\forall n \in \mathbb{N} \cup \{0\}$ $P(n) \Rightarrow P(n+1)$ is TRUE.

Recall:

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
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If $P(n)$ is FALSE, we are done.

So, let's assume $P(n)$ is TRUE and show $P(n+1)$ is TRUE.

Theorem: For all $n \in \mathbb{N} \cup \{0\}$

$$1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}.$$

Proof: Inductive Step: $\forall n \in \mathbb{N} \cup \{0\}$ $P(n) \Rightarrow P(n+1)$ is TRUE.

Assume $P(n)$ is TRUE for the purpose of induction,

i.e., $1 + 2 + \dots + n = \frac{n(n+1)}{2}$

Theorem: For all $n \in \mathbb{N} \cup \{0\}$

$$1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}.$$

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Assume $P(n)$ is TRUE for the purpose of induction,

i.e., $1 + 2 + \dots + n = \frac{n(n+1)}{2}$

Need to show that $P(n+1)$ is TRUE, i.e.,

$$1 + 2 + \dots + (n+1) = \frac{(n+1)(n+2)}{2}.$$

Theorem: For all $n \in \mathbb{N} \cup \{0\}$

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

Proof: Inductive Step: $\forall n \in \mathbb{N} \cup \{0\}$ $P(n) \Rightarrow P(n+1)$ is TRUE.

Assume $P(n)$ is TRUE for the purpose of induction,

i.e., $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$ ← Does NOT mean we are assuming what we want to prove.

Need to show that $P(n+1)$ is TRUE, i.e.,

$$1 + 2 + \cdots + (n+1) = \frac{(n+1)(n+2)}{2}.$$

Theorem: For all $n \in \mathbb{N} \cup \{0\}$

$$1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}.$$

Proof: Inductive Step: $\forall n \in \mathbb{N} \cup \{0\}$ $P(n) \Rightarrow P(n+1)$ is TRUE.

$$1 + 2 + \; -- \; + (n+1)$$

Theorem: For all $n \in \mathbb{N} \cup \{0\}$

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

Proof: Inductive Step: $\forall n \in \mathbb{N} \cup \{0\}$ $P(n) \Rightarrow P(n+1)$ is TRUE.

$$1 + 2 + \cdots + n + (n+1)$$

$$= \frac{n(n+1)}{2} + (n+1) \qquad \text{(since } P(n) \text{ is TRUE)}$$

$$= \frac{(n+1)(n+2)}{2}.$$

Theorem: For all $n \in \mathbb{N} \cup \{0\}$

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

Proof: Inductive Step: $\forall n \in \mathbb{N} \cup \{0\}$ $P(n) \Rightarrow P(n+1)$ is TRUE.

$$1 + 2 + \cdots + n + (n+1)$$

$$= \frac{n(n+1)}{2} + (n+1) \qquad (\text{since } P(n) \text{ is TRUE})$$

$$= \frac{(n+1)(n+2)}{2}.$$

Thus, $\forall n \in \mathbb{N} \cup \{0\}$ $P(n) \Rightarrow P(n+1)$. $\blacksquare$

Theorem: In any set of n non-negative integers,
(Absurd)
all elements are equal to each other.

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Proof: By induction.

Let $P(n)$ be the statement of the theorem.

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Base case: $P(0)$ is TRUE because

Theorem: In any set of n non-negative integers,
(Absurd) all elements are equal to each other.

Proof: By induction.

Let $P(n)$ be the statement of the theorem.

Base case: $P(0)$ is TRUE because in a set with zero elements, all elements are equal to each other.

Theorem: In any set of n non-negative integers,
(Absurd)
all elements are equal to each other.

Proof: Inductive step. $\forall n \in \mathbb{N} \cup \{0\}\ P(n) \Rightarrow P(n+1)$.

Theorem: In any set of n non-negative integers,
(Absurd)
all elements are equal to each other.

Proof: Inductive step. $\forall n \in \mathbb{N} \cup \{0\}$ $P(n) \Rightarrow P(n+1)$.

Consider any set S of $n+1$ non-negative integers.

$$S = \{a_1, a_2, a_3, \dots a_{n+1}\}$$

Theorem: In any set of n non-negative integers,
(Absurd) all elements are equal to each other.

Proof: Inductive step. $\forall n \in \mathbb{N} \cup \{0\}\ P(n) \Rightarrow P(n+1)$.

Consider any set S of $n+1$ non-negative integers.

$$S = \{a_1, a_2, a_3, \ldots a_{n+1}\}$$

Then, by induction hypothesis:

for $S \setminus \{a_{n+1}\}$: $a_1 = a_2 = a_3 = \ldots = a_n$

Theorem: (Absurd) In any set of n non-negative integers, all elements are equal to each other.

Proof: Inductive step. $\forall n \in \mathbb{N} \cup \{0\} \quad P(n) \Rightarrow P(n+1)$.

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$\boxtimes$

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Then, by induction hypothesis:

PROBLEM: We never proved $P(1) \Rightarrow P(2)$.

for $S \setminus \{a_{n+1}\}$: $a_1 = a_2 = a_3 = \ldots = a_n$

for $S \setminus \{a_1\}$: $a_2 = a_3 = \ldots = a_{n+1}$

$\blacksquare$

Theorem: In any set of n non-negative integers,
(Absurd) all elements are equal to each other.

Proof: Inductive step. $\forall n \in \mathbb{N} \cup \{0\}$ $P(n) \Rightarrow P(n+1)$.

Consider any set S of $n+1$ non-negative integers.

$$S = \{a_1, a_2\}$$

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for $S \setminus \{a_{n+1}\}$: $a_1 = a_2 = a_3 = \cdots = a_n$

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for $S \setminus \{a_{n+1}\}$: a_1

for $S \setminus \{a_1\}$: $a_2 = a_3 = \ - - - = a_{n+1}$

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Theorem: In any set of n non-negative integers,
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Consider any set S of $n+1$ non-negative integers.

$$S = \{a_1, a_2\}$$

Then, by induction hypothesis:

PROBLEM: We never proved $P(1) \Rightarrow P(2)$.

for $S \setminus \{a_{n+1}\}$: a_1

No reason to infer $a_1 = a_2$

for $S \setminus \{a_1\}$: a_2

45

Theorem: In any set of n non-negative integers, all elements are equal to each other.
(Absurd)

Bug: We proved $P(0)$ is TRUE and

$$P(0) \Rightarrow P(1)$$

but not $P(1) \Rightarrow P(2)$

$$P(2) \Rightarrow P(3)$$

$$P(3) \Rightarrow P(4)$$

$$\vdots$$

Theorem: In any set of n non-negative integers,
(Absurd) all elements are equal to each other.

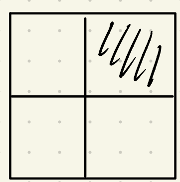
Theorem: If every pair of non-negative integers are equal,
(Not absurd) then for any $n \geq 2$, any n-sized set of non-negative integers has all elements equal.

A Tiling Puzzle

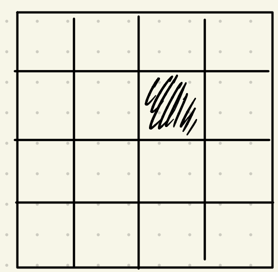
A TILING PUZZLE

Given : A $2^n \times 2^n$ grid with missing square

$n=1$



$n=2$

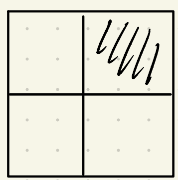


A TILING PUZZLE

Given : A $2^n \times 2^n$ grid with missing square

Want : Tile it using "L" shaped trominos/tiles

$n=1$



$n=2$

A TILING PUZZLE

Given : A $2^n \times 2^n$ grid with missing square

Want : Tile it using "L" shaped trominos/tiles

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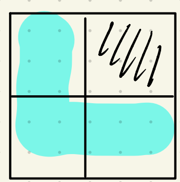
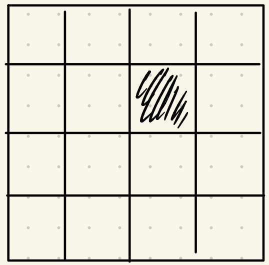


$n=2$



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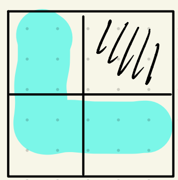
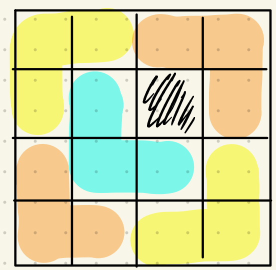


$n=2$



Theorem: Any $2^n \times 2^n$ grid with a missing center square can be tiled using trominos.

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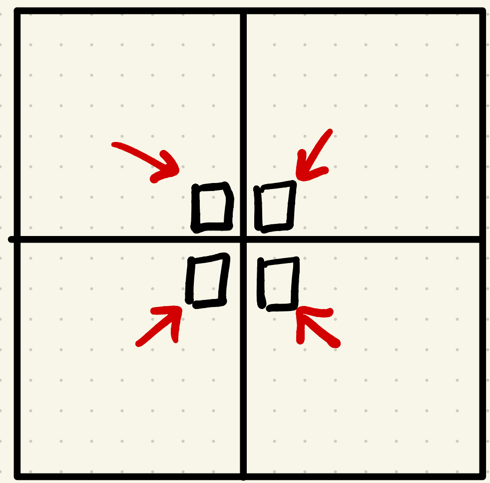


All of these are center squares

Theorem: Any $2^n \times 2^n$ grid with a missing center square can be tiled using trominos.

Proof: By induction.

$P(n)$: Any $2^n \times 2^n$ grid with a missing center square can be tiled using trominos.

Theorem: Any $2^n \times 2^n$ grid with a missing center square can be tiled using trominos.

Proof: By induction.

$P(n)$: Any $2^n \times 2^n$ grid with a missing center square can be tiled using trominos.

Base case:  Zero tiles suffice!

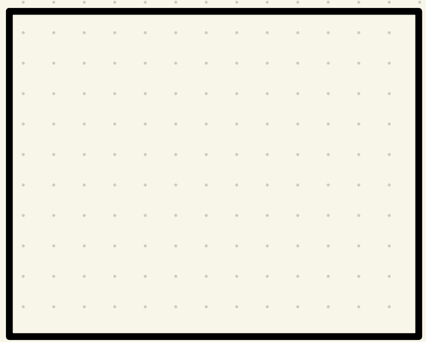
Theorem: Any $2^n \times 2^n$ grid with a missing center square can be tiled using trominos.

Proof: By induction.

$P(n)$: Any $2^n \times 2^n$ grid with a missing center square can be tiled using trominos.

Induction step:

$2^{n+1} \times 2^{n+1}$ grid



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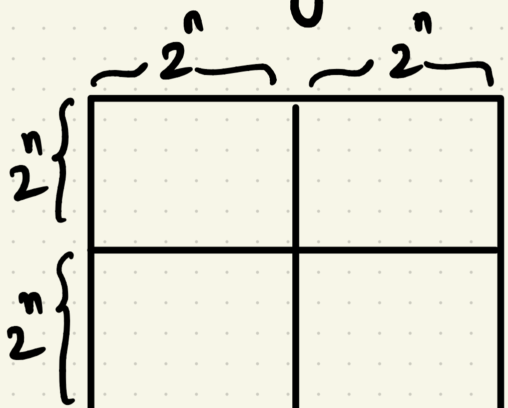

Theorem: Any $2^n \times 2^n$ grid with a missing center square can be tiled using triominos.

Proof: By induction.

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Induction step:

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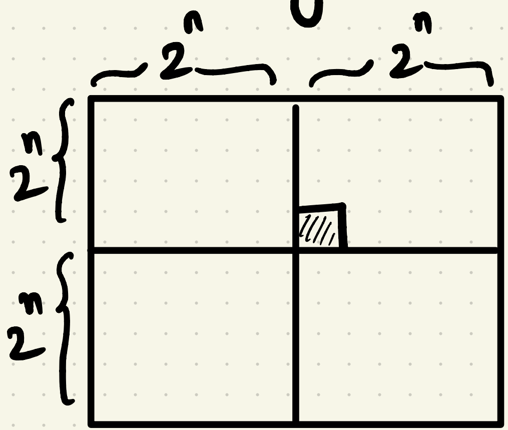

Theorem: Any $2^n \times 2^n$ grid with a missing center square can be tiled using trominos.

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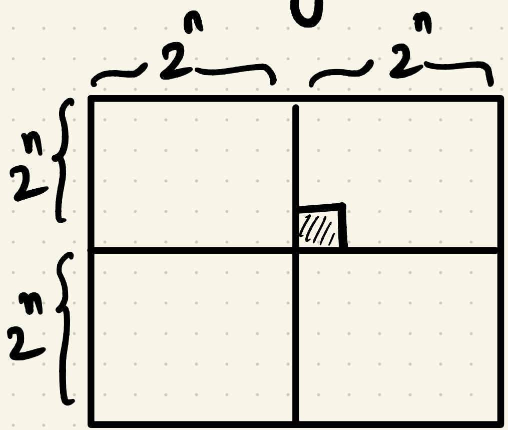

How to apply the induction hypothesis?

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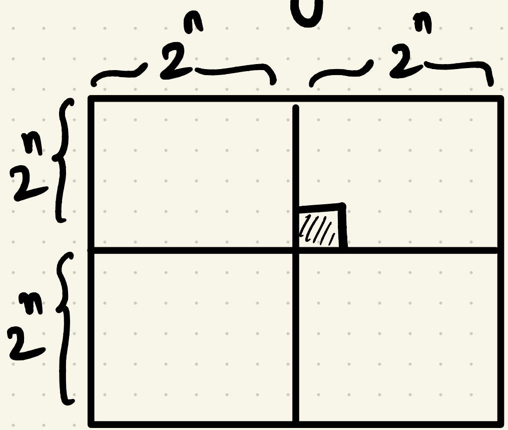


How to apply the induction hypothesis?

Modify it.

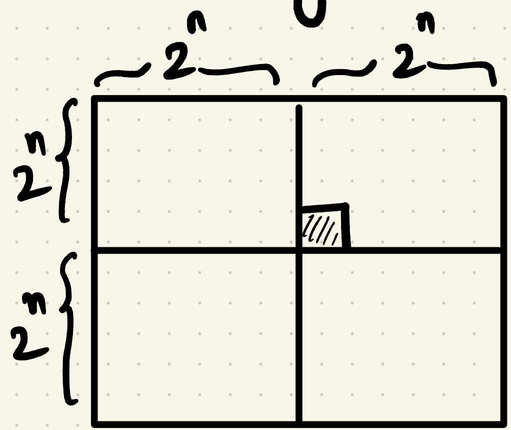
Theorem: Any $2^n \times 2^n$ grid with a missing center square can be tiled using trominos.

Proof: By induction.

$P(n)$: Any $2^n \times 2^n$ grid with ~~a~~ any missing ~~center~~ square can be tiled using trominos.

Induction step:

$2^{n+1} \times 2^{n+1}$ grid



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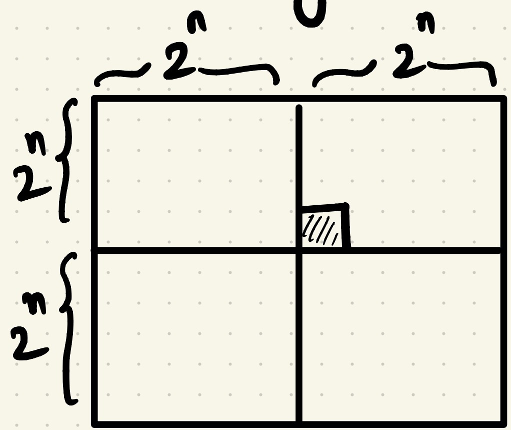
Theorem: Any $2^n \times 2^n$ grid with a missing center square can be tiled using trominos.

Proof: By induction.

$P(n)$: Any $2^n \times 2^n$ grid with ~~a~~ any missing ~~center~~ square can be tiled using trominos.

Induction step:

$2^{n+1} \times 2^{n+1}$ grid



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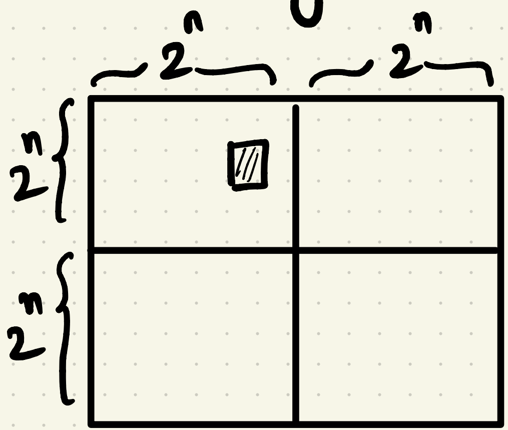

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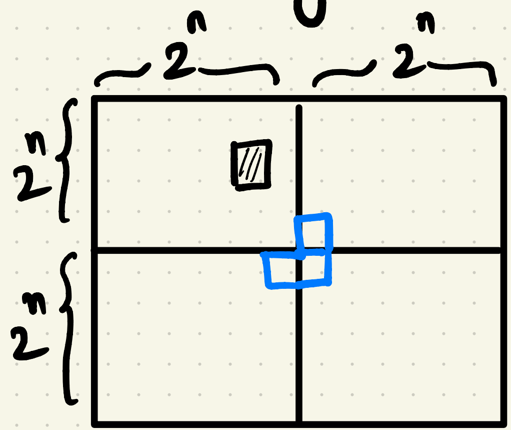

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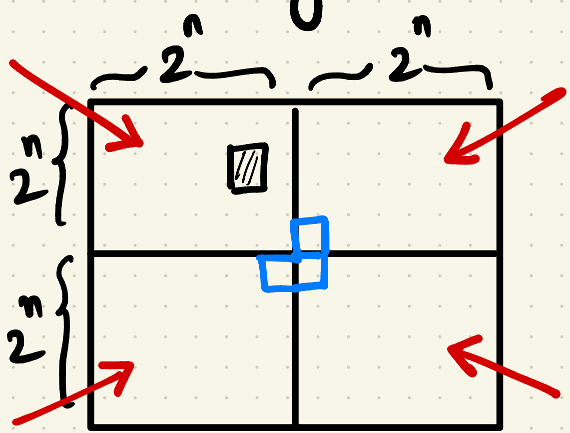

Apply $P(n)$ to each of these.

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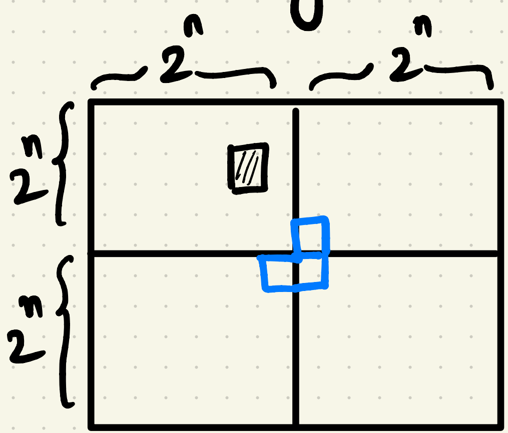

Takeaway:

$P(n) \Longrightarrow P(n+1)$

assume something stronger

prove something stronger