COL 202: DISCRETE MATHEMATICAL STRUCTURES

LECTURE 37

CONDITIONAL PROBABILITY & PARADOXES

APR 19, 2024 | ROHIT VAISH

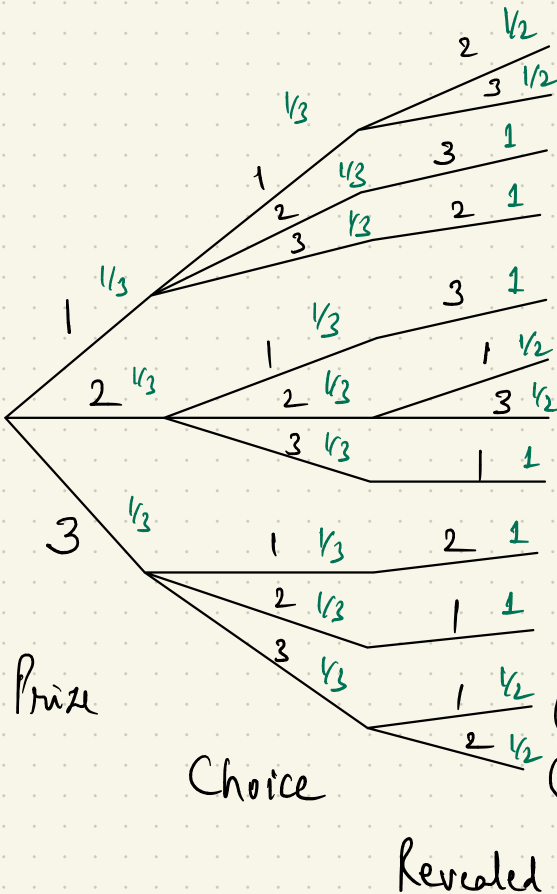
An event is a set of outcomes.

The probability that an event $E \subseteq S$ occurs is

$$\sum_{w \in E} \Pr(w).$$

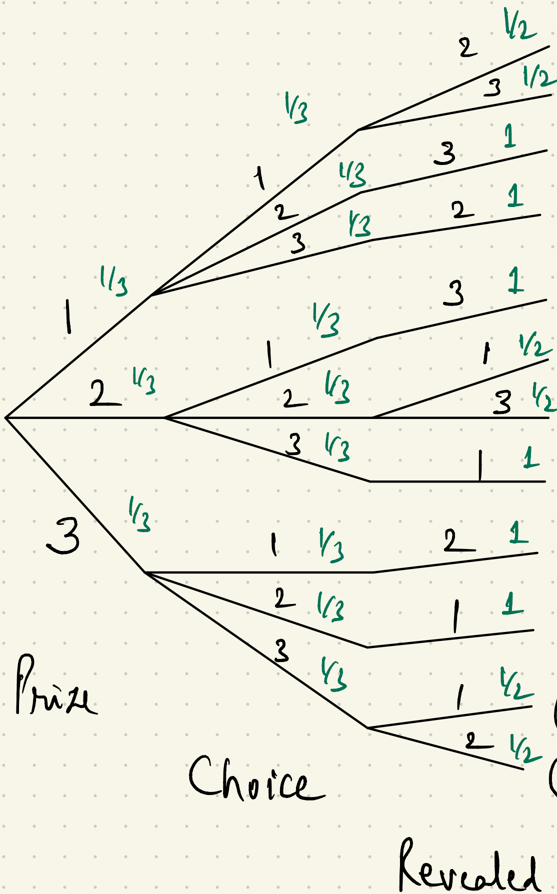
Monty Hall Tree

Strategy : Switch



(1, 1, 2)	L	1/18
(1, 1, 3)	L	1/18
(1, 2, 3)	W	1/9
(1, 3, 2)	W	1/9
(2, 1, 3)	W	1/9
(2, 2, 1)	L	1/18
(2, 2, 3)	L	1/18
(2, 3, 1)	W	1/9
(3, 1, 2)	W	1/9
(3, 2, 1)	W	1/9
(3, 3, 1)	L	1/18
(3, 3, 2)	L	1/18

MONTY HALL TREE

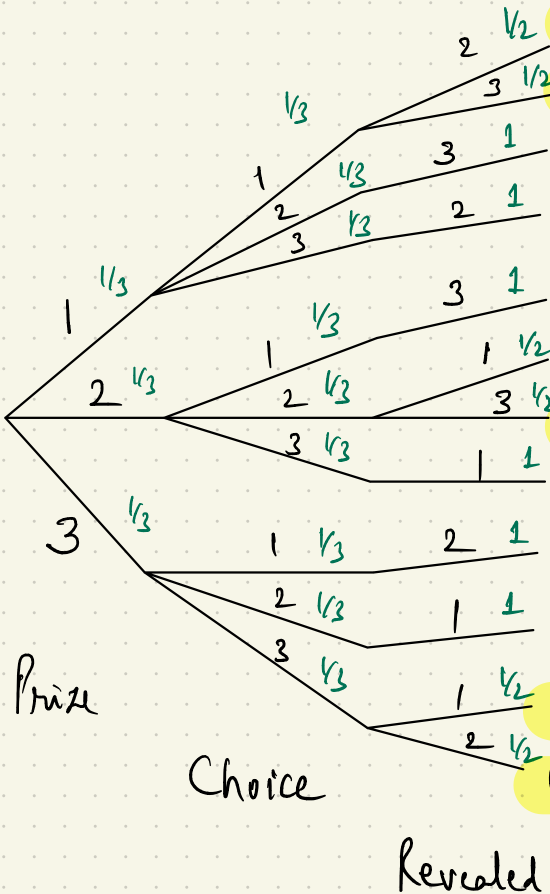


(1, 1, 2)	L	1/18
(1, 1, 3)	L	1/18
(1, 2, 3)	W	1/9
(1, 3, 2)	W	1/9
(2, 1, 3)	W	1/9
(2, 2, 1)	L	1/18
(2, 2, 3)	L	1/18
(2, 3, 1)	W	1/9
(3, 1, 2)	W	1/9
(3, 2, 1)	W	1/9
(3, 3, 1)	L	1/18
(3, 3, 2)	L	1/18

Strategy : Switch

Probability that switch strategy lose

MONTY HALL TREE

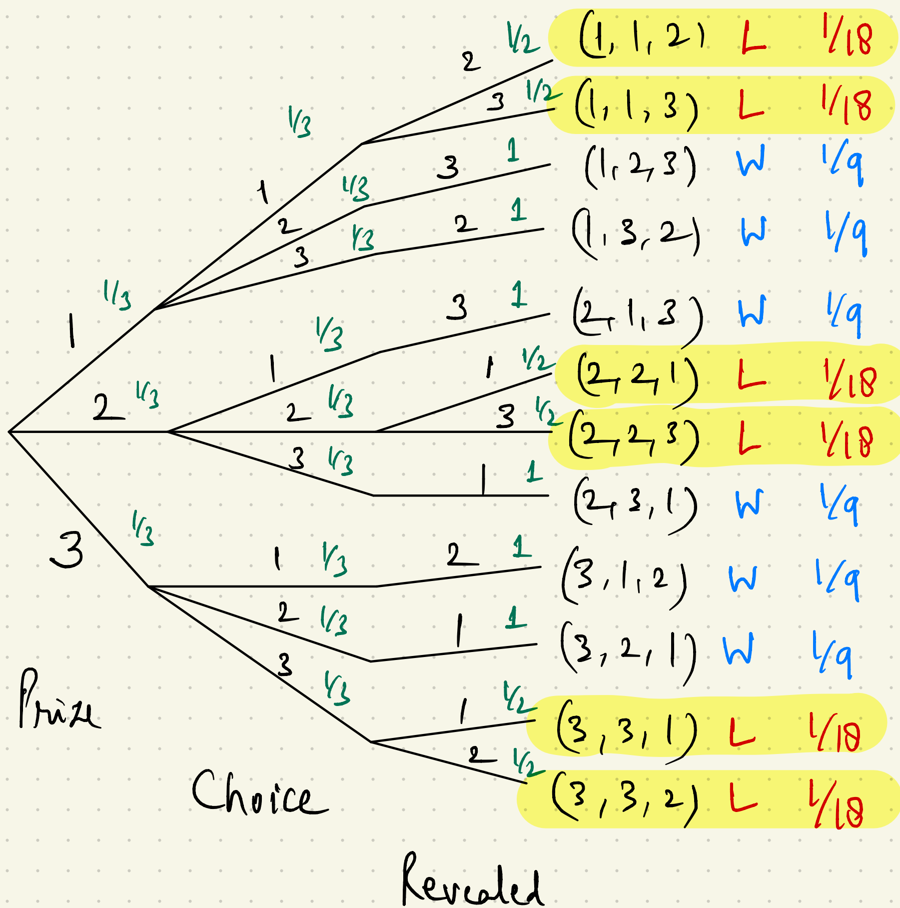


(1, 1, 2)	L	1/18
(1, 1, 3)	L	1/18
(1, 2, 3)	W	1/9
(1, 3, 2)	W	1/9
(2, 1, 3)	W	1/9
(2, 2, 1)	L	1/18
(2, 2, 3)	L	1/18
(2, 3, 1)	W	1/9
(3, 1, 2)	W	1/9
(3, 2, 1)	W	1/9
(3, 3, 1)	L	1/18
(3, 3, 2)	L	1/18

Strategy : Switch

Probability that switch strategy loses

Monty Hall Tree



Strategy : Switch

Probability that switch strategy lose

$$= 6 \times \frac{1}{18} = \frac{1}{3}$$

CONDITIONAL PROBABILITY

$$\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$

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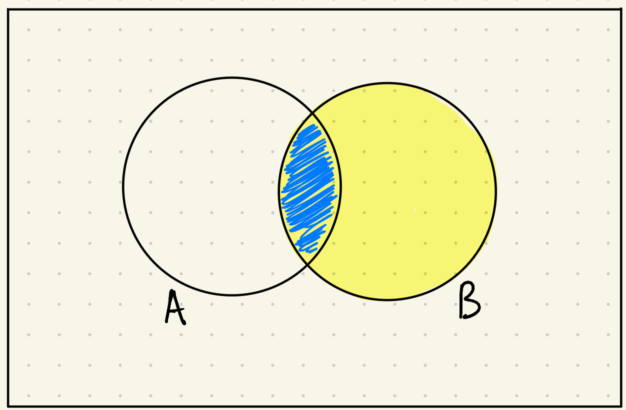


sample space S

CONDITIONAL PROBABILITY

$$\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$

Product rule : $\Pr(A \cap B) = \Pr(B) \cdot \Pr(A \mid B)$.

CONDITIONAL PROBABILITY

$$Pr(A \mid B) = \frac{Pr(A \cap B)}{Pr(B)}$$

Product rule : $Pr(A \cap B) = Pr(B) \cdot Pr(A \mid B)$.

General Product rule : $Pr(A_1 \cap A_2 \cap \cdots \cap A_n) = Pr(A_1) \cdot Pr(A_2 \mid A_1) \cdot Pr(A_3 \mid A_1 \cap A_2) \cdot \ \vdots \ Pr(A_n \mid A_1 \cap A_2 \cap \cdots \cap A_{n-1})$

BEST 2-OUT-OF-3 SERIES

2 teams play three games

First to win two games wins the series

BEST 2-OUT-OF-3 SERIES

2 teams play three games

First to win two games wins the series

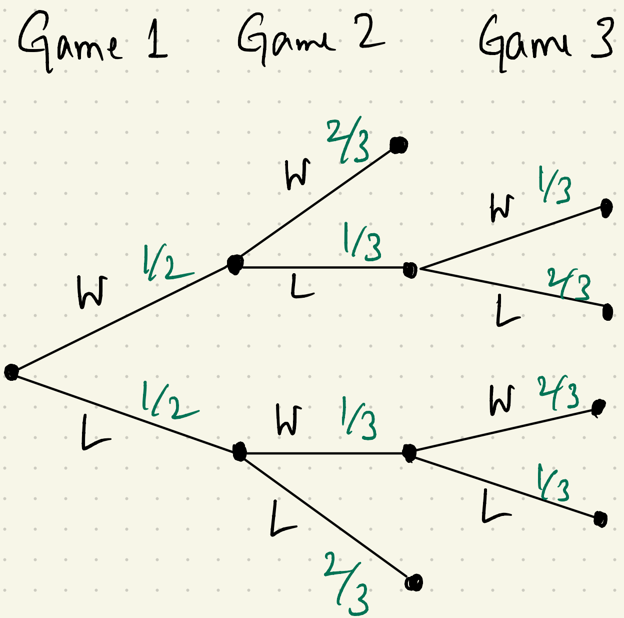
For any team :

* Prob of winning the 1^{st} game $= 1/2$.

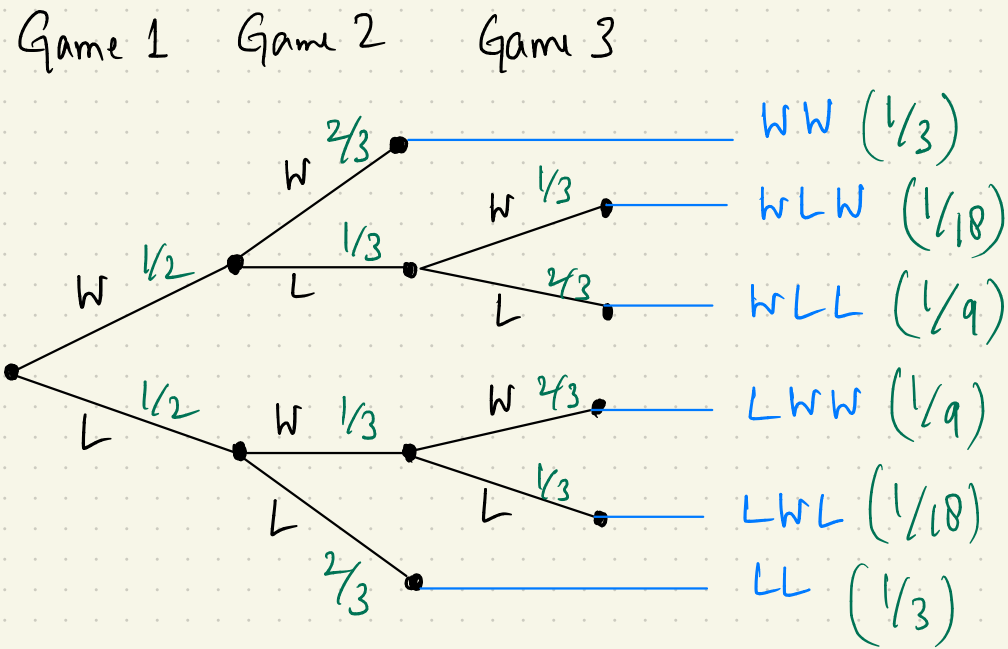
* Prob of winning a game after a win $= 2/3$.

* Prob of winning a game after a loss $= 1/3$.

BEST 2-OUT-OF-3 SERIES



BEST 2-OUT-OF-3 SERIES



BEST 2-OUT-OF-3 SERIES

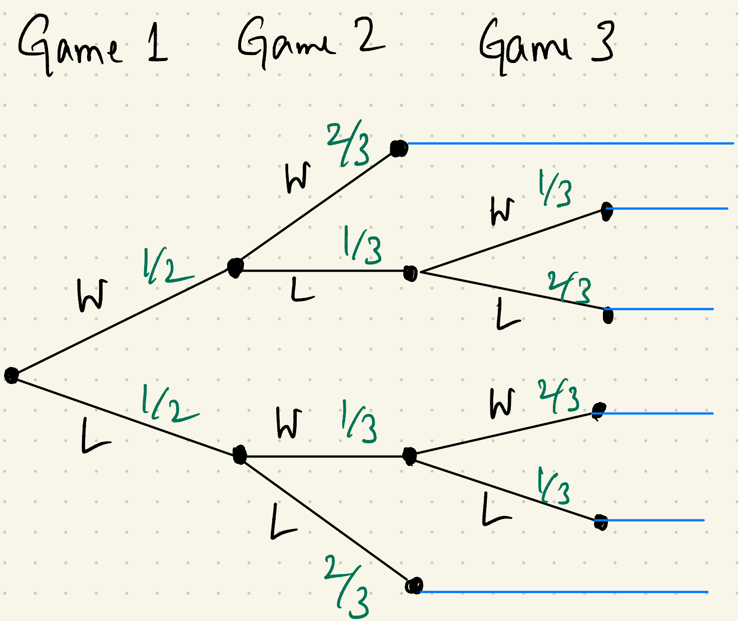
$$\Pr(\text{winning the series} \mid \text{won first game}) = ?$$

BEST 2-OUT-OF-3 SERIES

$$Pr\left(\underbrace{\text{winning the series}}_{A} \mid \underbrace{\text{won first game}}_{B}\right) = ?$$

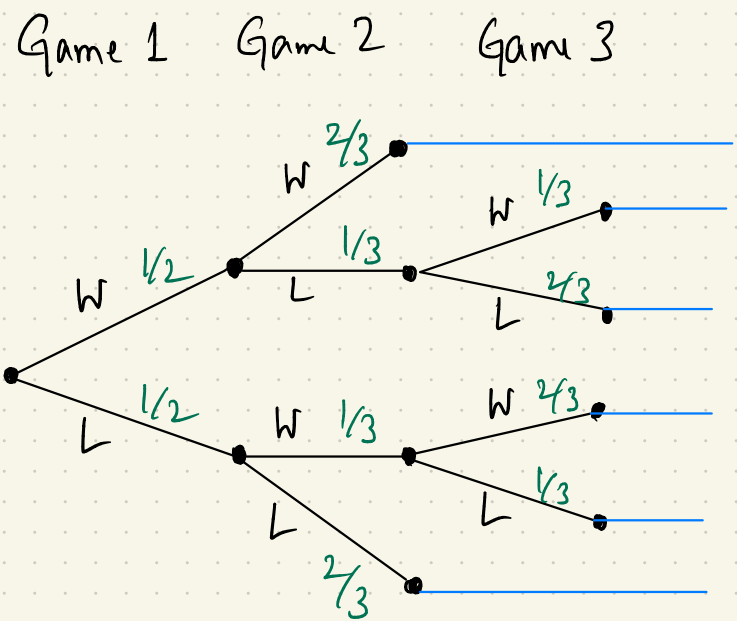
$$Pr(A \mid B) = \frac{Pr(A \cap B)}{Pr(B)} = ?$$

BEST 2-OUT-OF-3 SERIES



		A (win series)	B (win 1st game)	$A \cap B$
WW	(1/3)	✓	✓	✓
WLW	(1/18)	✓	✓	✓
WLL	(1/9)		✓	
LWW	(1/9)	✓		
LWL	(1/18)			
LL	(1/3)			

BEST 2-OUT-OF-3 SERIES



		A (win series)	B (win 1st game)	$A \cap B$
WW	$(1/3)$	✓	✓	✓
WLW	$(1/18)$	✓	✓	✓
WLL	$(1/9)$		✓	
LWW	$(1/9)$	✓		
LWL	$(1/18)$			
LL	$(1/3)$			

$$\Pr(B) = \frac{1}{3} + \frac{1}{18} + \frac{1}{9} = \frac{9}{18}$$

$$\Pr(A \cap B) = \frac{1}{3} + \frac{1}{18} = \frac{7}{18}$$

$$\Pr(A \mid B) = \frac{7}{9}$$

BEST 2-OUT-OF-3 SERIES

$$\Pr(\text{winning the series} \mid \text{won first game}) = 7/9$$

$$\Pr(\text{won first game} \mid \text{winning the series}) = ?$$

BEST 2-OUT-OF-3 SERIES

$$\Pr(\text{winning the series} \mid \text{won first game}) = 7/9$$

$$\Pr(\text{won first game} \mid \text{winning the series}) = ?$$

earlier in time

later in time

BEST 2-OUT-OF-3 SERIES

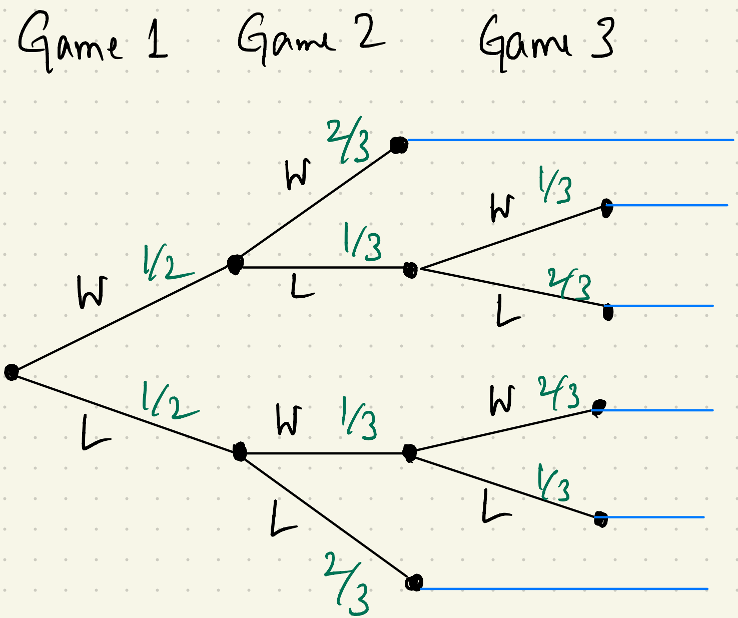
Pr (winning the series | won first game) = 7/9

Pr (won first game | winning the series) = ?

$$\Pr(\underbrace{\text{won first game}}_{B} \mid \underbrace{\text{winning the series}}_{A}) = ?$$

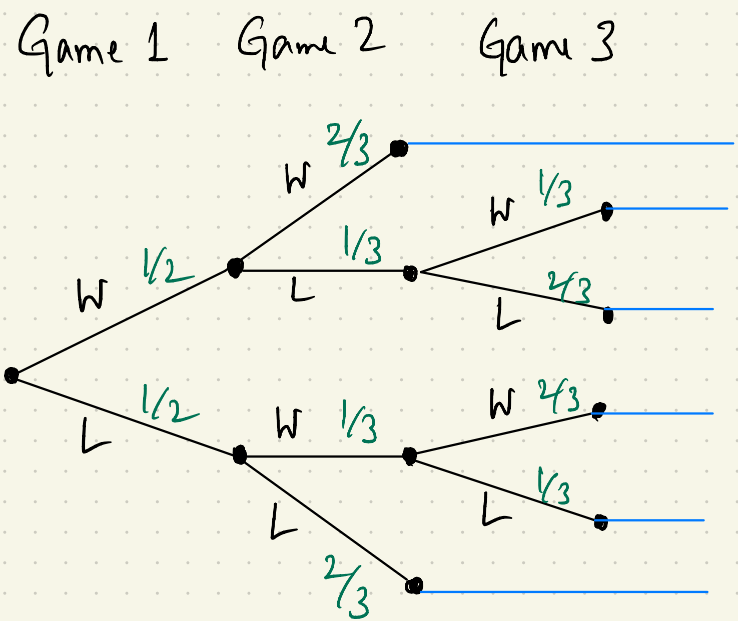
$$\Pr(B|A) = \frac{\Pr(A \cap B)}{\Pr(A)} =$$

Best 2-out-of-3 Series



	A (win series)	B (win 1st game)	$A \cap B$
WW $(1/3)$	✓	✓	✓
WLW $(1/18)$	✓	✓	✓
WLL $(1/9)$		✓	
LWW $(1/9)$	✓		
LWL $(1/18)$			
LL $(1/3)$			

Best 2-out-of-3 Series



A: win series

B: win 1st game

Outcome	A	B	$A \cap B$
WW $(1/3)$	✓	✓	✓
WLW $(1/18)$	✓	✓	✓
WLL $(1/9)$		✓	
LWW $(1/9)$	✓		
LWL $(1/18)$			
LL $(1/3)$			

$$\Pr(A \cap B) = 1/3 + 1/18$$

$$\Pr(A) = 1/3 + 1/18 + 1/9$$

BEST 2-OUT-OF-3 SERIES

$$\Pr(\text{winning the series} \mid \text{won first game}) = 7/9$$

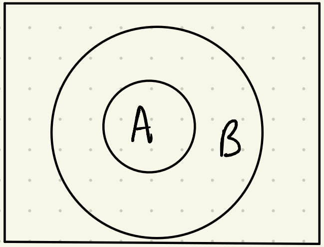
$$\Pr(\text{won first game} \mid \text{winning the series}) = 7/9$$

BEST 2-OUT-OF-3 SERIES

$$\Pr(\text{winning the series} \mid \text{won first game}) = 7/9$$

$$\Pr(\text{won first game} \mid \text{winning the series}) = 7/9$$

In general, $\Pr(A \mid B) \neq \Pr(B \mid A)$.



$$\Pr(\text{Received } 1^{st} \text{ vaccine shot} \mid \text{Received a booster shot}) = 1$$

$$\Pr(\text{Received a booster shot} \mid \text{Received } 1^{st} \text{ vaccine shot}) < 1$$

Fair and Unfair Coins

FAIR AND UNFAIR COINS

Coin 1 : 50% H , 50% T

Coin 2 : 100% H

FAIR AND UNFAIR COINS

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Suppose a coin is chosen uniformly at random.

FAIR AND UNFAIR COINS

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Suppose a coin is chosen uniformly at random.

When this coin is tossed, say it shows H.

FAIR AND UNFAIR COINS

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Suppose a coin is chosen uniformly at random.

When this coin is tossed, say it shows H.

What is the probability that we picked the fair coin?

FAIR AND UNFAIR COINS

Coin 1 : 50% H , 50% T

Coin 2 : 100% H

Suppose a coin is chosen uniformly at random.

When this coin is tossed, say it shows H.

What is the probability that we picked the fair coin?

Similarity with Monty Hall : Prob that car is behind door #3 given that door #1 chosen and door #2 revealed?

FAIR AND UNFAIR COINS

1. Draw the tree

FAIR AND UNFAIR COINS

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Coin

FAIR AND UNFAIR COINS

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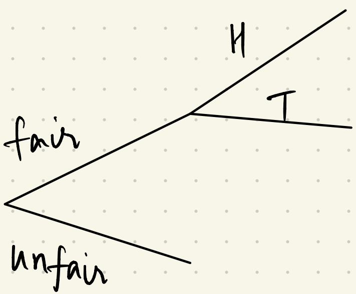
Coin Flip

FAIR AND UNFAIR COINS

1. Draw the tree

Coin Flip



FAIR AND UNFAIR COINS

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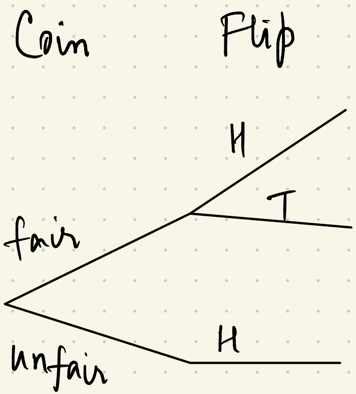

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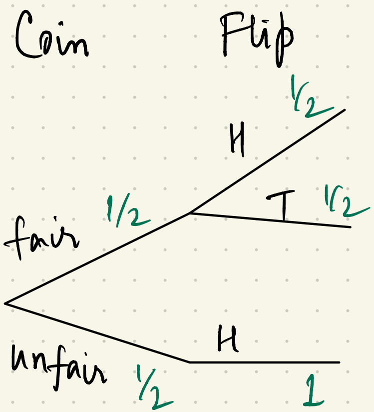

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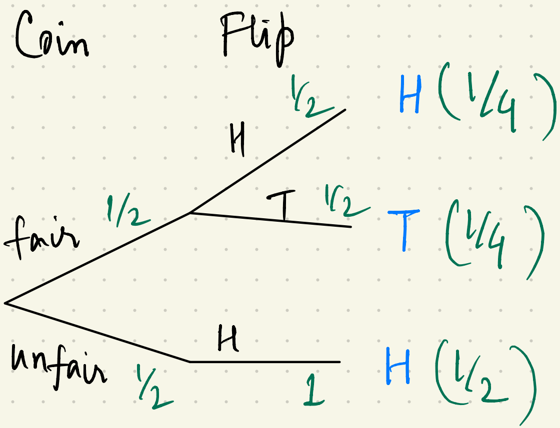

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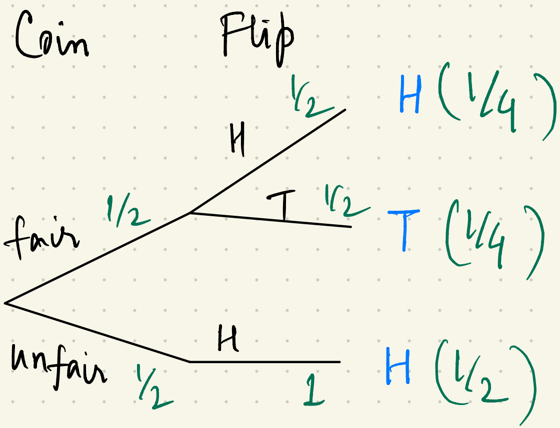

2. Define events of interest

FAIR AND UNFAIR COINS

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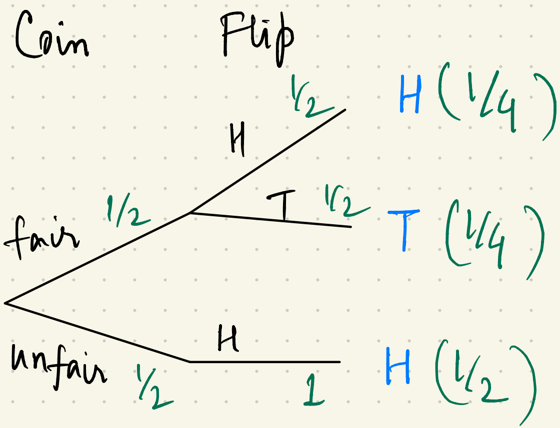

2. Define events of interest

A = choose fair coin

B = result is H

FAIR AND UNFAIR COINS

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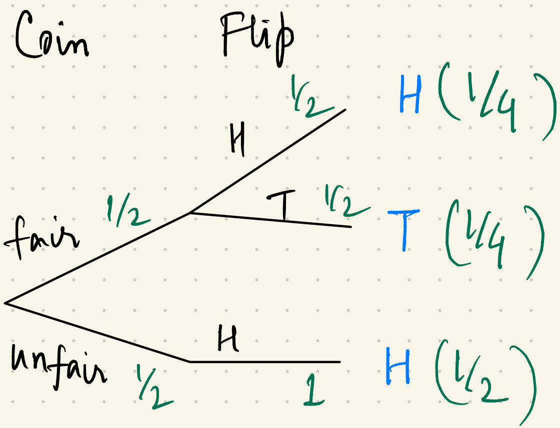

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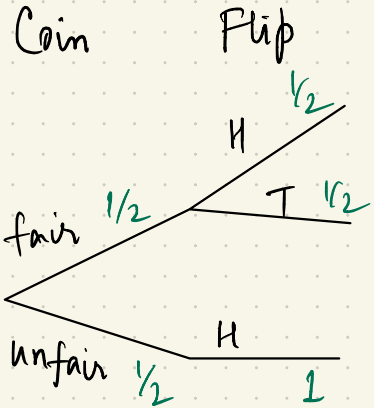

	A	B	$A \cap B$
H (1/4)	✓	✓	✓
T (1/4)	✓		
H (1/2)		✓	

2. Define events of interest

A = choose fair coin

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FAIR AND UNFAIR COINS

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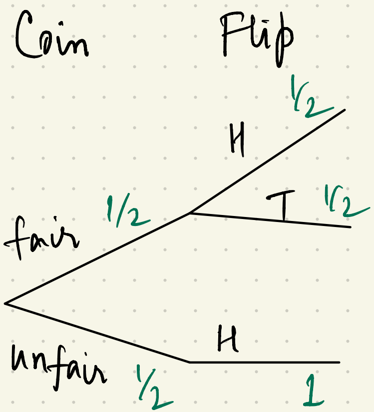

	A	B	$A \cap B$
H (1/4)	✓	✓	✓
T (1/4)	✓		
H (1/2)		✓	

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3. Analyze the probability

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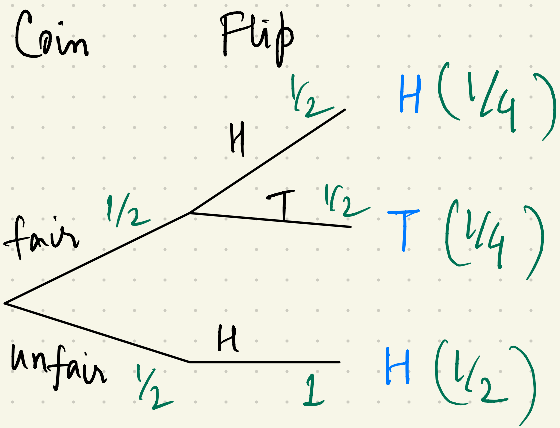

	A	B	$A \cap B$
H (1/4)	✓	✓	✓
T (1/4)	✓		
H (1/2)		✓	

2. Define events of interest

A = choose fair coin

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3. Analyze the probability

$$\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$

FAIR AND UNFAIR COINS

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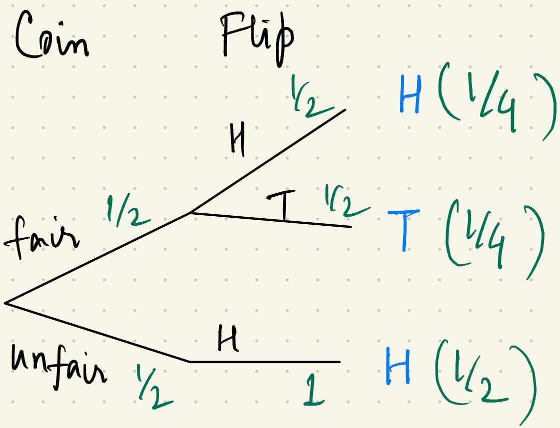


Outcome	A	B	A ∩ B
H (1/4)	✓	✓	✓
T (1/4)	✓		
H (1/2)		✓	

2. Define events of interest

A = choose fair coin

B = result is H

3. Analyze the probability

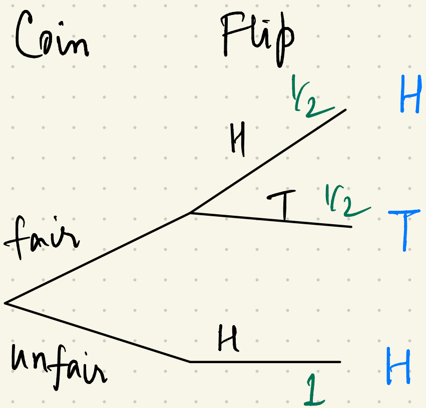
$$\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)} = \frac{1/4}{1/2 + 1/4} = \frac{1}{3}$$

FAIR AND UNFAIR COINS

What happens when the coins are chosen with unequal probability?

FAIR AND UNFAIR COINS

1. Draw the tree



A	B	$A \cap B$
✓	✓	✓
✓		
	✓	

2. Define events of interest

A = choose fair coin

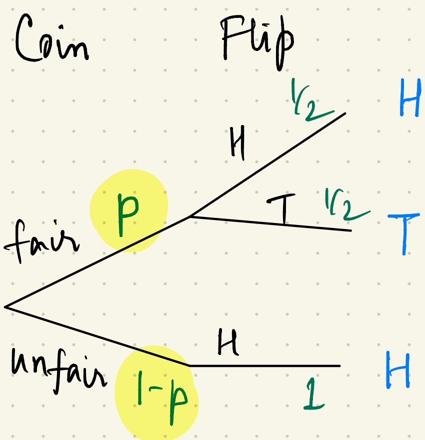
B = result is H

3. Analyze the probability

$$\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$

FAIR AND UNFAIR COINS

1. Draw the tree



A	B	$A \cap B$
✓	✓	✓
✓		
	✓	

2. Define events of interest

A = choose fair coin

B = result is H

3. Analyze the probability

$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$

Fair and Unfair Coins

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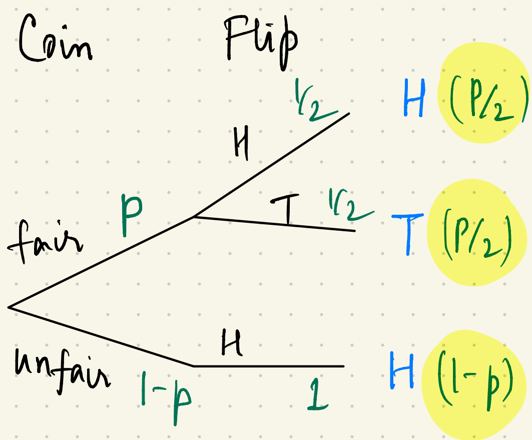


Outcome	A	B	A ∩ B
H $(p/2)$	✓	✓	✓
T $(p/2)$	✓		
H $(1-p)$		✓	

2. Define events of interest

A = choose fair coin

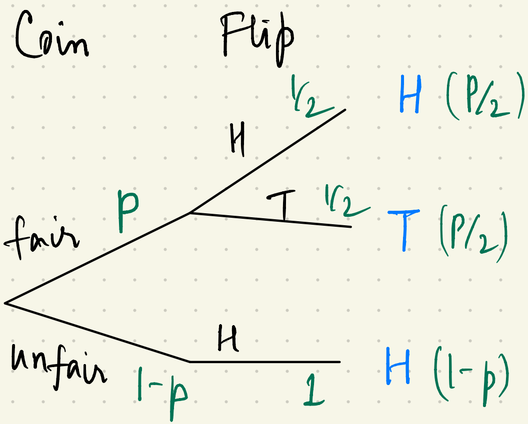
B = result is H

3. Analyze the probability

$$\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$

FAIR AND UNFAIR COINS

1. Draw the tree



	A	B	A ∩ B
H (p/2)	✓	✓	✓
T (p/2)	✓		
H (1-p)		✓	

2. Define events of interest

A = choose fair coin

B = result is H

3. Analyze the probability

$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)} = \frac{p/2}{p/2 + 1-p} = \frac{p}{2-p}$$

FAIR AND UNFAIR COINS

If unfair coin is picked for sure

FAIR AND UNFAIR COINS

If unfair coin is picked for sure

$$\Rightarrow \quad p = 0$$

$$\Rightarrow \quad \Pr(\text{fair coin} \mid \text{result is } H) = \frac{0}{2-0} = 0 .$$

FAIR AND UNFAIR COINS

If unfair coin is picked for sure

$\Rightarrow p = 0$

$$\Rightarrow Pr(\text{fair coin} \mid \text{result is } H) = \frac{0}{2-0} = 0.$$

If fair coin is picked for sure

$\Rightarrow p = 1$

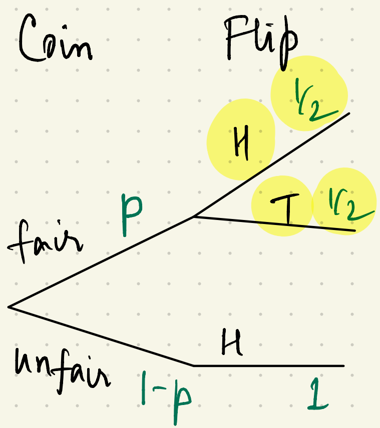
$$\Rightarrow Pr(\text{fair coin} \mid \text{result is } H) = \frac{1}{2-1} = 1.$$

FAIR AND UNFAIR COINS

Now imagine the coin is flipped 100 times, and it lands H every time.

Fair and Unfair Coins

1. Draw the tree



A	B	$A \cap B$
✓	✓	✓
✓		
	✓	

2. Define events of interest

A = choose fair coin

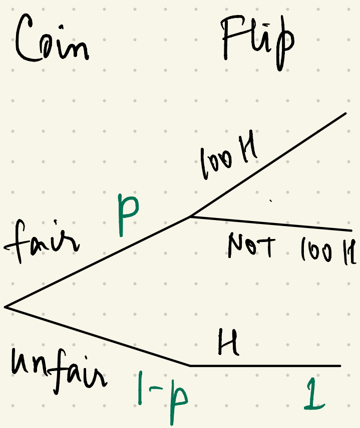
B = result is H

3. Analyze the probability

$$\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)} =$$

FAIR AND UNFAIR COINS

1. Draw the tree



A	B	$A \cap B$
✓	✓	✓
✓		
	✓	

2. Define events of interest

A = choose fair coin

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3. Analyze the probability

$$\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)} =$$

FAIR AND UNFAIR COINS

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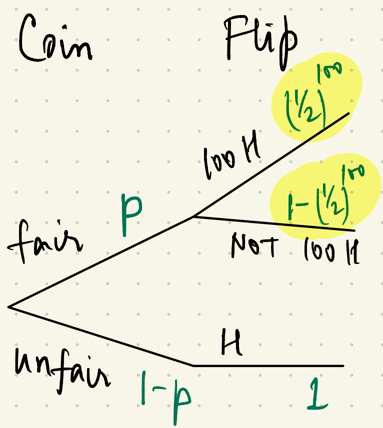

A	B	$A \cap B$
✓	✓	✓
✓		
	✓	

2. Define events of interest

A = choose fair coin

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3. Analyze the probability

$$\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)} =$$

FAIR AND UNFAIR COINS

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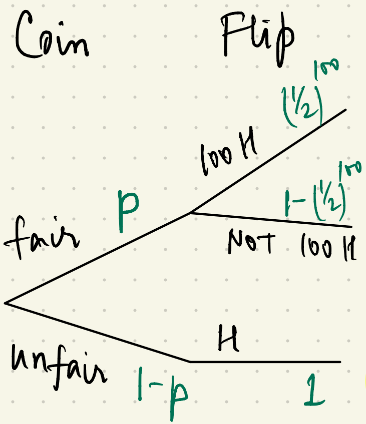

Outcome	A	B	$A \cap B$
$p \cdot \left(\frac{1}{2}\right)^{100}$	✓	✓	✓
$p\left(1-\left(\frac{1}{2}\right)^{100}\right)$	✓		
$1-p$		✓	

2. Define events of interest

A = choose fair coin

B = result is H

3. Analyze the probability

$$\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)} =$$

FAIR AND UNFAIR COINS

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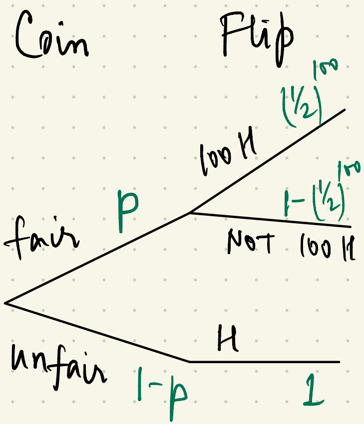

	A	B	$A \cap B$
$p \cdot \left(\frac{1}{2}\right)^{100}$	✓	✓	✓
$p\left(1-\left(\frac{1}{2}\right)^{100}\right)$	✓		
$1-p$		✓	

2. Define events of interest

A = choose fair coin

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3. Analyze the probability

$$\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)} =$$

FAIR AND UNFAIR COINS

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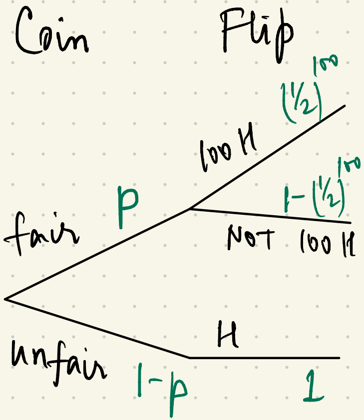


	A	B	$A \cap B$
$p \cdot (1/2)^{100}$	✓	✓	✓
$p\left(1-\left(\frac{1}{2}\right)^{100}\right)$	✓		
$1-p$		✓	

2. Define events of interest

A = choose fair coin

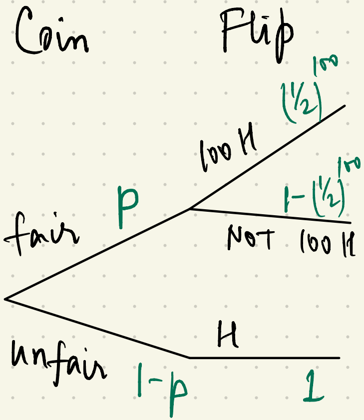
B = result is $\underbrace{H, H, \ldots, H}_{100}$

3. Analyze the probability

$$\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)} =$$

FAIR AND UNFAIR COINS

1. Draw the tree



	A	B	$A \cap B$
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$1-p$		✓	

2. Define events of interest

A = choose fair coin

B = result is $\underbrace{H, H, \ldots, H}_{100}$

3. Analyze the probability

$$\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)} = \frac{p}{p + 2^{100}(1-p)}$$

FAIR AND UNFAIR COINS

Now imagine the coin is flipped 100 times, and it lands H every time.

What is the value of p?

FAIR AND UNFAIR COINS

Now imagine the coin is flipped 100 times, and it lands H every time.

$$\Pr\left(\text{fair coin} \,\middle|\, \text{result is } \underbrace{H, H, \cdots, H}_{100 \text{ times}}\right) = \frac{p}{p + 2^{100}(1-p)}$$

FAIR AND UNFAIR COINS

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$$\Pr\Big(\text{fair coin} \;\Big|\; \text{result is } \underbrace{H, H, \cdots, H}_{100 \text{ times}}\Big) = \frac{p}{p + 2^{100}(1-p)}$$

$$\approx 0 \quad \text{when } p = 0.9999$$

$$= 1 \quad \text{when } p = 1$$

FAIR AND UNFAIR COINS

Now imagine the coin is flipped 100 times, and it lands H every time.

$$\Pr\Big(\text{fair coin} \;\Big|\; \text{result is } \underbrace{H, H, \cdots, H}_{100 \text{ times}}\Big) = \frac{p}{p + 2^{100}(1-p)}$$

either very confident that the coin was unfair ≈ 0 when $p = 0.9999$

OR

really unlucky with our experiment $= 1$ when $p = 1$

FAIR AND UNFAIR COINS

Now imagine the coin is flipped 100 times, and it lands H every time.

$$\Pr\left(\text{fair coin} \;\middle|\; \text{result is } \underbrace{H, H, \cdots, H}_{100 \text{ times}}\right) = \frac{p}{p + 2^{100}(1-p)}$$

Same as election polls!

either very confident that the coin was unfair

OR

really unlucky with our experiment

≈ 0 when $p = 0.9999$

$= 1$ when $p = 1$

Medical Testing

MEDICAL TESTING

10% of the population has the disease

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If a person has the disease, 10% chance test is negative

False negative

MEDICAL TESTING

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False positive

—— doesn't have ———, 30 % ——————— positive

MEDICAL TESTING

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False negative

If a person has the disease, 10% chance test is negative

False positive

—— doesn't have ——, 30% —— positive

Suppose a random person tests positive.

MEDICAL TESTING

10% of the population has the disease

If a person has the disease, 10% chance test is negative → False negative

——— doesn't have ———, 30% ——— positive → False positive

Suppose a random person tests positive.

What is the probability they have the disease?

MEDICAL TESTING

A : Event that the person has disease

B : Event that the person tests positive

MEDICAL TESTING

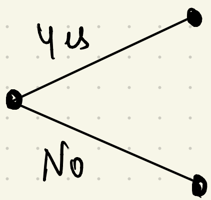
A : Event that the person has disease

B : Event that the person tests positive

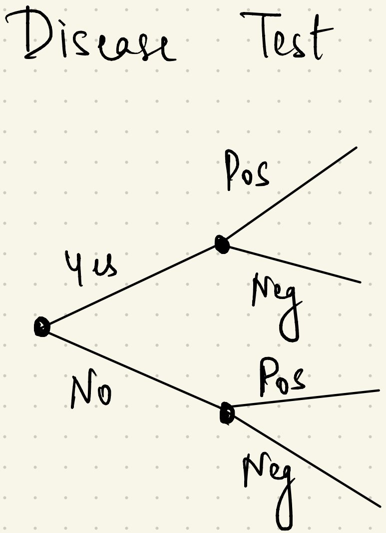
$$\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)} = \ ?$$

MEDICAL TESTING

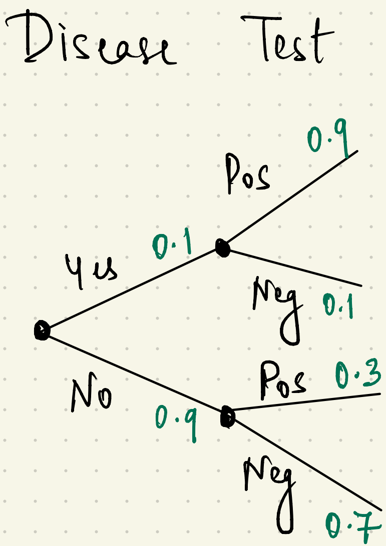
Disease



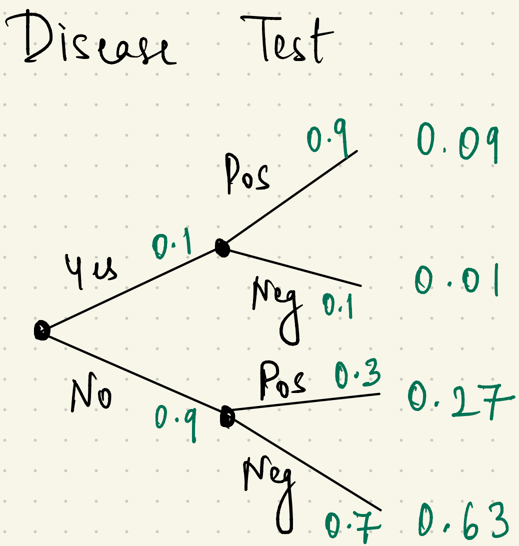
MEDICAL TESTING



MEDICAL TESTING

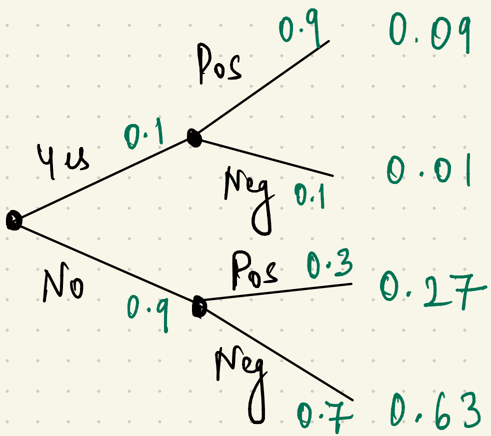


MEDICAL TESTING



Medical Testing

Disease Test



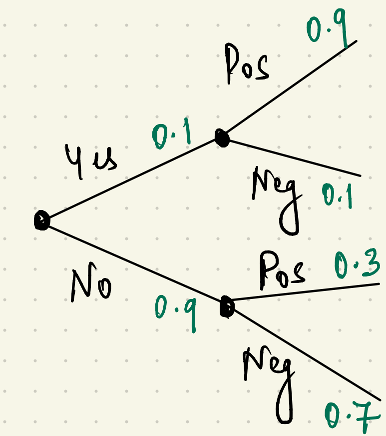
A has disease

B tests Pos

$A \cap B$

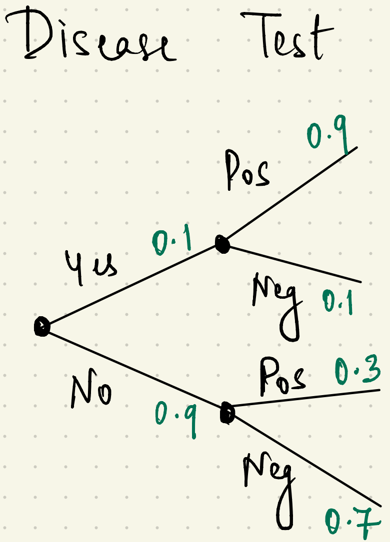
MEDICAL TESTING

Disease Test



	A — has disease	B — tests Pos	$A \cap B$
0.09	✓	✓	✓
0.01	✓		
0.27		✓	
0.63			

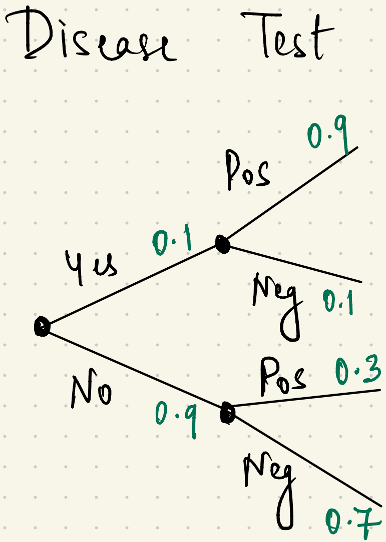
Medical Testing



	A — has disease	B — tests Pos	$A \cap B$
0.09	✓	✓	✓
0.01	✓		
0.27		✓	
0.63			

$$\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$

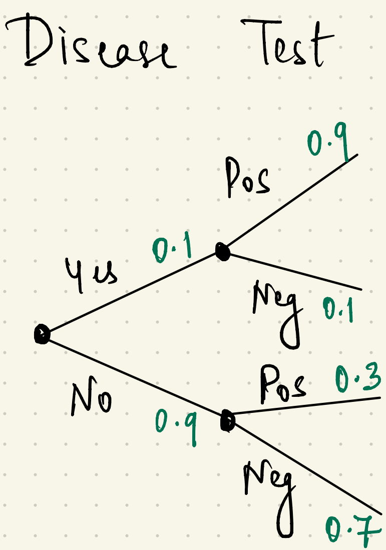
Medical Testing



	A — has disease	B — tests Pos	$A \cap B$
0.09	✓	✓	✓
0.01	✓		
0.27		✓	
0.63			

$$\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)} = \frac{0.09}{0.09 + 0.27} = \frac{1}{4}$$

MEDICAL TESTING



	A — has disease	B — tests Pos	$A \cap B$
0.09	✓	✓	✓
0.01	✓		
0.27		✓	
0.63			

Whaaaat??

$$\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)} = \frac{0.09}{0.09 + 0.27} = \frac{1}{4}$$

MEDICAL TESTING

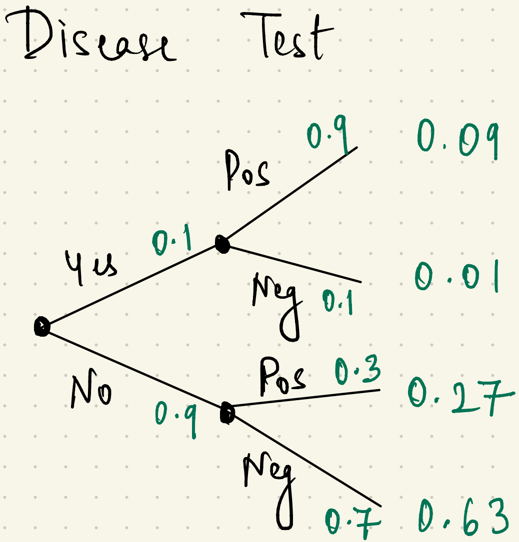
Disease	Test		A — has disease	B — tests Pos	$A \cap B$
Yes 0.1	Pos 0.9	0.09	✓	✓	✓
Yes 0.1	Neg 0.1	0.01	✓		
No 0.9	Pos 0.3	0.27		✓	
No 0.9	Neg 0.7	0.63			

Whaaaat??

$$\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)} = \frac{0.09}{0.09 + 0.27} = \frac{1}{4}$$

Most people (90%) don't have the disease. Positives are mostly disease-free.

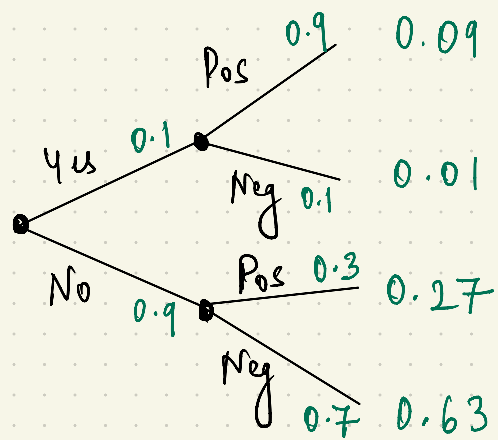
MEDICAL TESTING



Pr (Test is correct)

MEDICAL TESTING

Disease Test

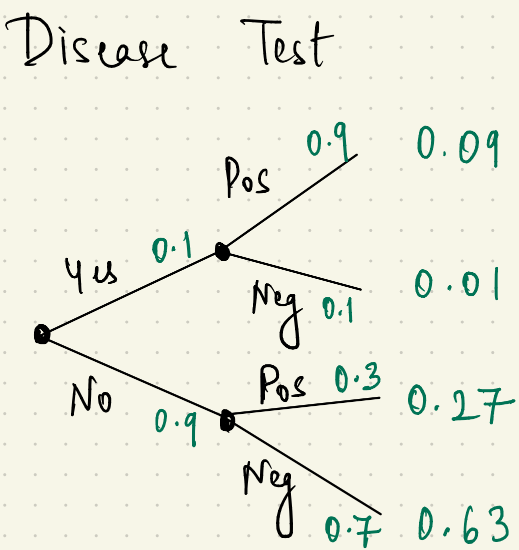


$$\Pr(\text{Test is correct})$$

$$= 0.09 + 0.63$$

$$= 0.72$$

MEDICAL TESTING



$$\Pr(\text{Test is correct})$$

$$= 0.09 + 0.63$$

$$= 0.72$$

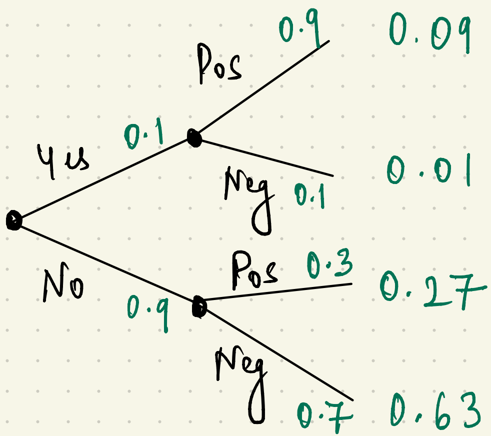
The test is likely to be right.

But if it says "You have the disease"

it is likely to be wrong!

MEDICAL TESTING

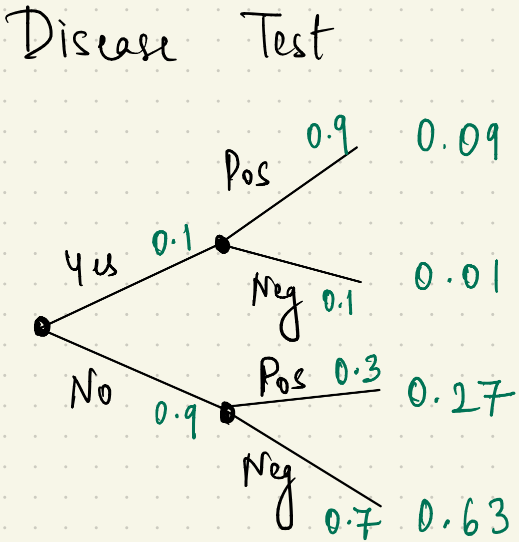
Disease Test



A naive test that always says "negative" will be right 90% of the time.

Is the naive test "better"?

MEDICAL TESTING



A naive test that always says "negative" will be right 90% of the time.

Is the naive test "better"?

Accuracy ≠ Utility, in general

False negatives can sometimes be much costlier than false positives.

PARADOX #1 : CASINO DICE

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* You pick a number $N \in \{1, 2, \dots, 6\}$

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* You win if N shows up on at least one die.

PARADOX #1 : CASINO DICE

* You pick a number $N \in \{1, 2, \dots, 6\}$

* Roll 3 dice (each die is 6-sided and fair)

* You win if N shows up on at least one die.

What is Prob of win ?

PARADOX #1 : CASINO DICE

A_i : Event that i^{th} die shows N.

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$$= Pr(A_1) + Pr(A_2) + Pr(A_3)$$

PARADOX #1 : CASINO DICE

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$$\Pr(\text{win}) = \Pr(A_1 \cup A_2 \cup A_3)$$

$$= \Pr(A_1) + \Pr(A_2) + \Pr(A_3)$$

$$= \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{1}{2}$$

PARADOX #1 : CASINO DICE

A_i : Event that i^{th} die shows N.

$$\Pr(\text{win}) = \Pr(A_1 \cup A_2 \cup A_3)$$

Problem!

$$= \Pr(A_1) + \Pr(A_2) + \Pr(A_3)$$

$$= \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{1}{2}$$

PARADOX #1 : CASINO DICE

A_i : Event that i^{th} die shows N.

$$Pr(\text{win}) = Pr\left(A_1 \cup A_2 \cup A_3\right)$$

$$= Pr(A_1) + Pr(A_2) + Pr(A_3)$$

$$= \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{1}{2}$$

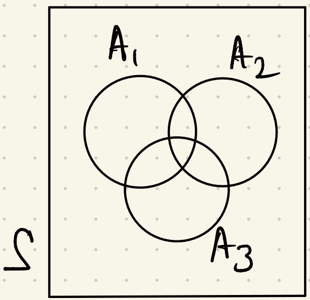
Problem!

Must take intersection into account

PARADOX #1 : CASINO DICE

A_i : Event that i^{th} die shows N.

$$\Pr(\text{win}) = \Pr\left(A_1 \cup A_2 \cup A_3\right)$$



$$= \Pr(A_1) + \Pr(A_2) + \Pr(A_3)$$

$$- \Pr(A_1 \cap A_2) - \Pr(A_2 \cap A_3) - \Pr(A_1 \cap A_3)$$

$$+ \Pr(A_1 \cap A_2 \cap A_3)$$

PARADOX # 1 : CASINO DICE

A_i : Event that i^{th} die shows N.

$$\Pr(\text{win}) = \Pr(A_1 \cup A_2 \cup A_3)$$

$$= \Pr(A_1) + \Pr(A_2) + \Pr(A_3)$$

$$- \Pr(A_1 \cap A_2) - \Pr(A_2 \cap A_3) - \Pr(A_1 \cap A_3)$$

$$+ \Pr(A_1 \cap A_2 \cap A_3)$$

$$= \frac{1}{6} + \frac{1}{6} + \frac{1}{6} - \frac{1}{36} - \frac{1}{36} - \frac{1}{36} + \frac{1}{216}$$

PARADOX #1 : CASINO DICE

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$$\Pr(\text{win}) = \Pr(A_1 \cup A_2 \cup A_3)$$

$$= \Pr(A_1) + \Pr(A_2) + \Pr(A_3)$$

$$- \Pr(A_1 \cap A_2) - \Pr(A_2 \cap A_3) - \Pr(A_1 \cap A_3)$$

$$+ \Pr(A_1 \cap A_2 \cap A_3)$$

$$= \frac{1}{6} + \frac{1}{6} + \frac{1}{6} - \frac{1}{36} - \frac{1}{36} - \frac{1}{36} + \frac{1}{216}$$

$$= 0.421$$

PARADOX # 1 : CASINO DICE

A_i : Event that i^{th} die shows N.

$$\begin{aligned}
Pr(\text{win}) &= Pr(A_1 \cup A_2 \cup A_3) \\
&= Pr(A_1) + Pr(A_2) + Pr(A_3) \\
&\quad - Pr(A_1 \cap A_2) - Pr(A_2 \cap A_3) - Pr(A_1 \cap A_3) \\
&\quad + Pr(A_1 \cap A_2 \cap A_3) \\
&= \frac{1}{6} + \frac{1}{6} + \frac{1}{6} - \frac{1}{36} - \frac{1}{36} - \frac{1}{36} + \frac{1}{216} \\
&= 0.421
\end{aligned}$$

Unfair game !

Probability rules :

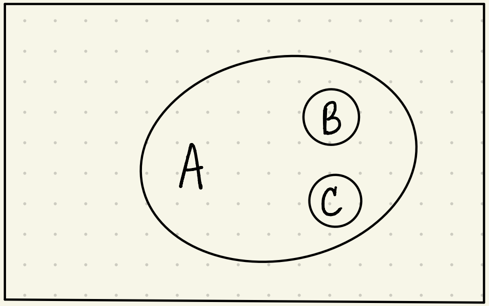
$$* \ Pr(A_1 \cup A_2 \cup A_3) = Pr(A_1) + Pr(A_2) + Pr(A_3) - Pr(A_1 \cap A_2) - Pr(A_2 \cap A_3) - Pr(A_1 \cap A_3) + Pr(A_1 \cap A_2 \cap A_3)$$

$$* \ Pr(A \cup B \mid C) = Pr(A \mid C) + Pr(B \mid C) - Pr(A \cap B \mid C)$$

NOT TRUE :

If $B \cap C = \phi$, then $\Pr(A \mid B \cup C) = \Pr(A \mid B) + \Pr(A \mid C)$

Why is this wrong ?



$\Pr(A \mid B) = 1$

$\Pr(A \mid C) = 1$

$\Pr(A \mid B \cup C) = 1$

Paradox #2: Simpson's Paradox

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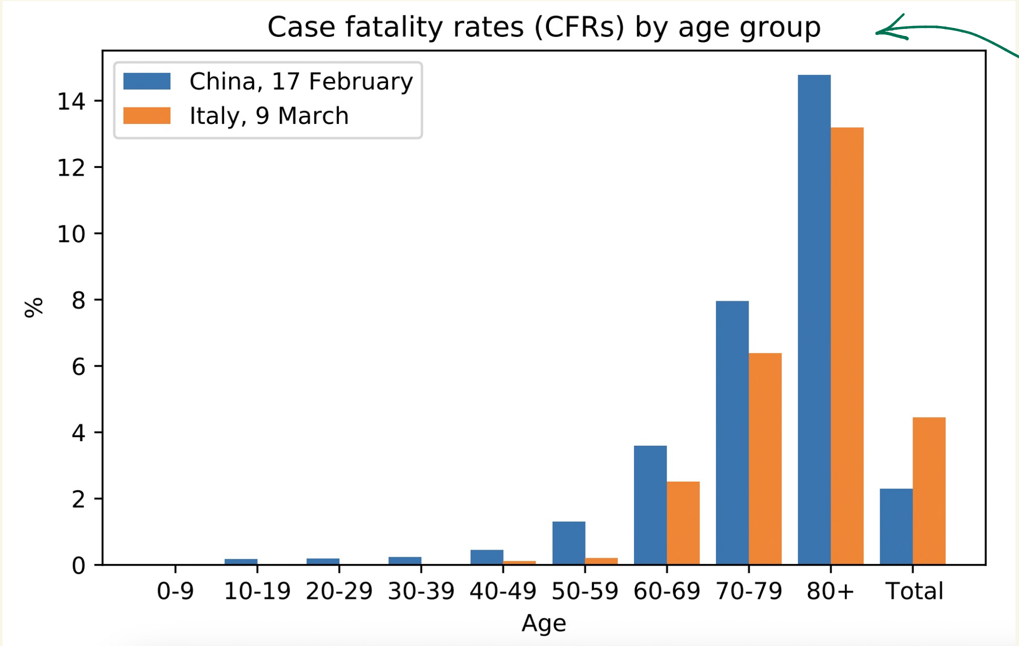
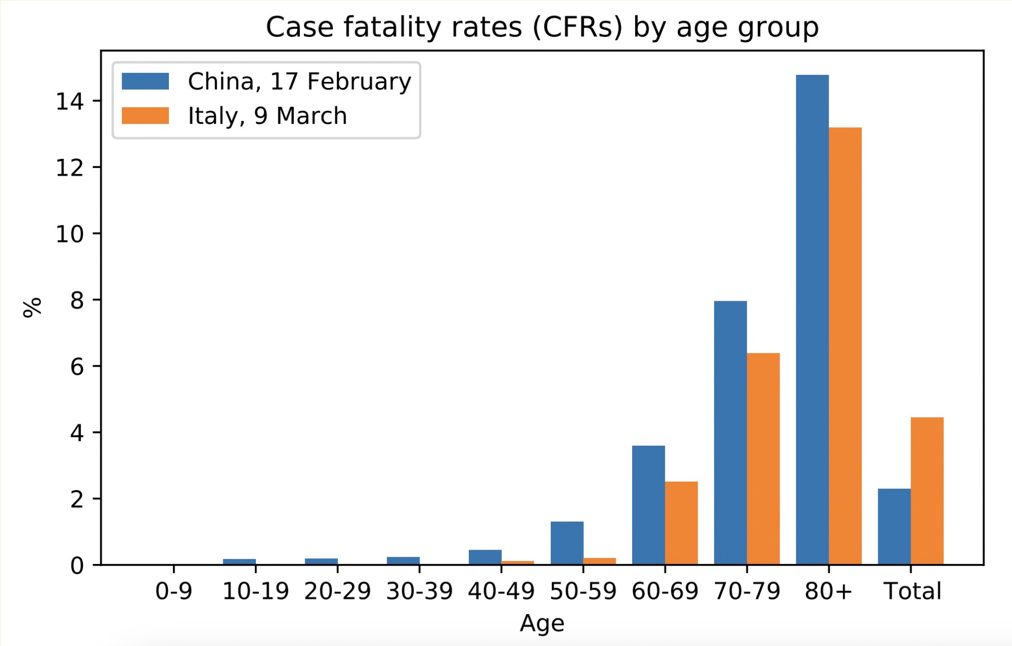


probability of dying if you get infected

Source: von Kügelgen, Gresele, and Schölkopf, IEEE Trans. on AI Feb 2021

PARADOX # 2 : SIMPSON'S PARADOX



Italy better than China in every age-group
but China is better overall.

WHY?

Source: von Kügelgen, Gresele, and Schölkopf, IEEE Trans. on AI Feb 2021

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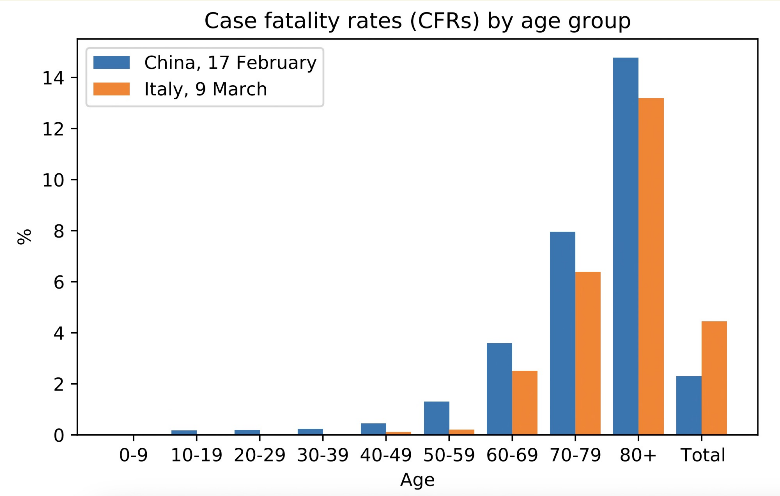

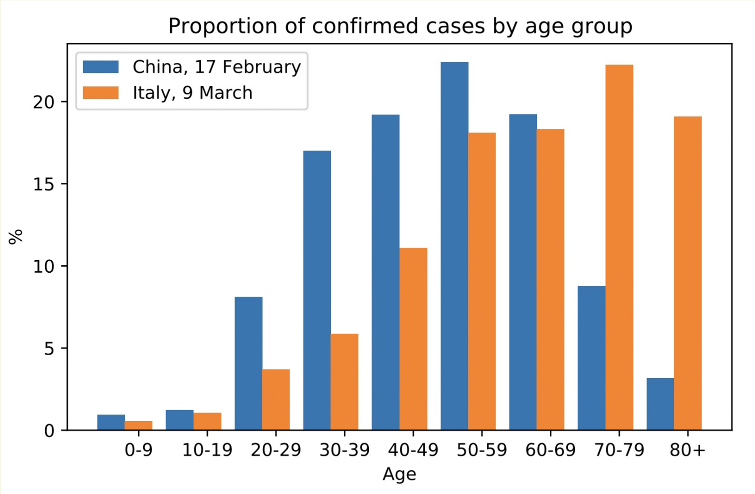

Italy's cases are mostly older people — less likely to survive

China's ———————— younger people — more ————————

Source: von Kügelgen, Gresele, and Schölkopf, IEEE Trans. on AI Feb 2021

PARADOX # 2 : SIMPSON'S PARADOX

(Fictitious data)

	Young	Old
Italy	1	4
China	4	1

PARADOX # 2 : SIMPSON'S PARADOX

(Fictitious data)

	Young	Old
Italy	1/1	1/4
China	3/4	0/1

people survived

PARADOX # 2 : SIMPSON'S PARADOX

(Fictitions data)

	Young	Old
Italy	1/1 = 100%	1/4 = 25%
China	3/4 = 75%	0/1 = 0%

PARADOX # 2 : SIMPSON'S PARADOX

(Fictitions data)

	Young	Old
Italy	$1/1$ $= 100\%$	$1/4$ $= 25\%$
China	$3/4$ $= 75\%$	$0/1$ $= 0\%$

better

better

PARADOX # 2 : SIMPSON'S PARADOX

(Fictitious data)

	Young	Old	Overall
Italy	$1/1 = 100\%$	$1/4 = 25\%$	$2/5 = 40\%$
China	$3/4 = 75\%$	$0/1 = 0\%$	$3/5 = 60\%$

better

better

worse