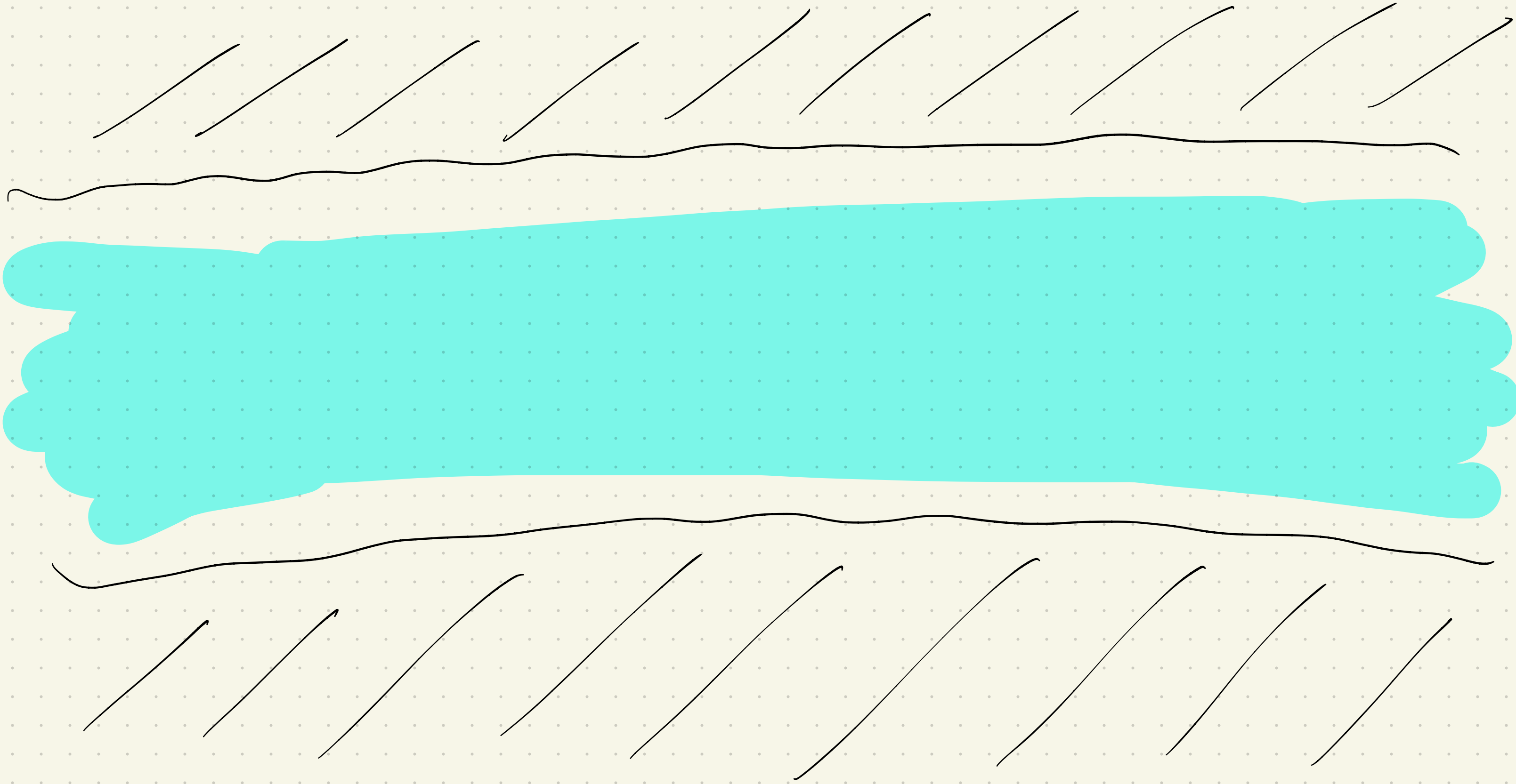
COL 202: DISCRETE MATHEMATICAL STRUCTURES

LECTURE 24

EULER TOURS

MAR 06, 2024 | ROHIT VAISH

Seven Bridges of Königsberg



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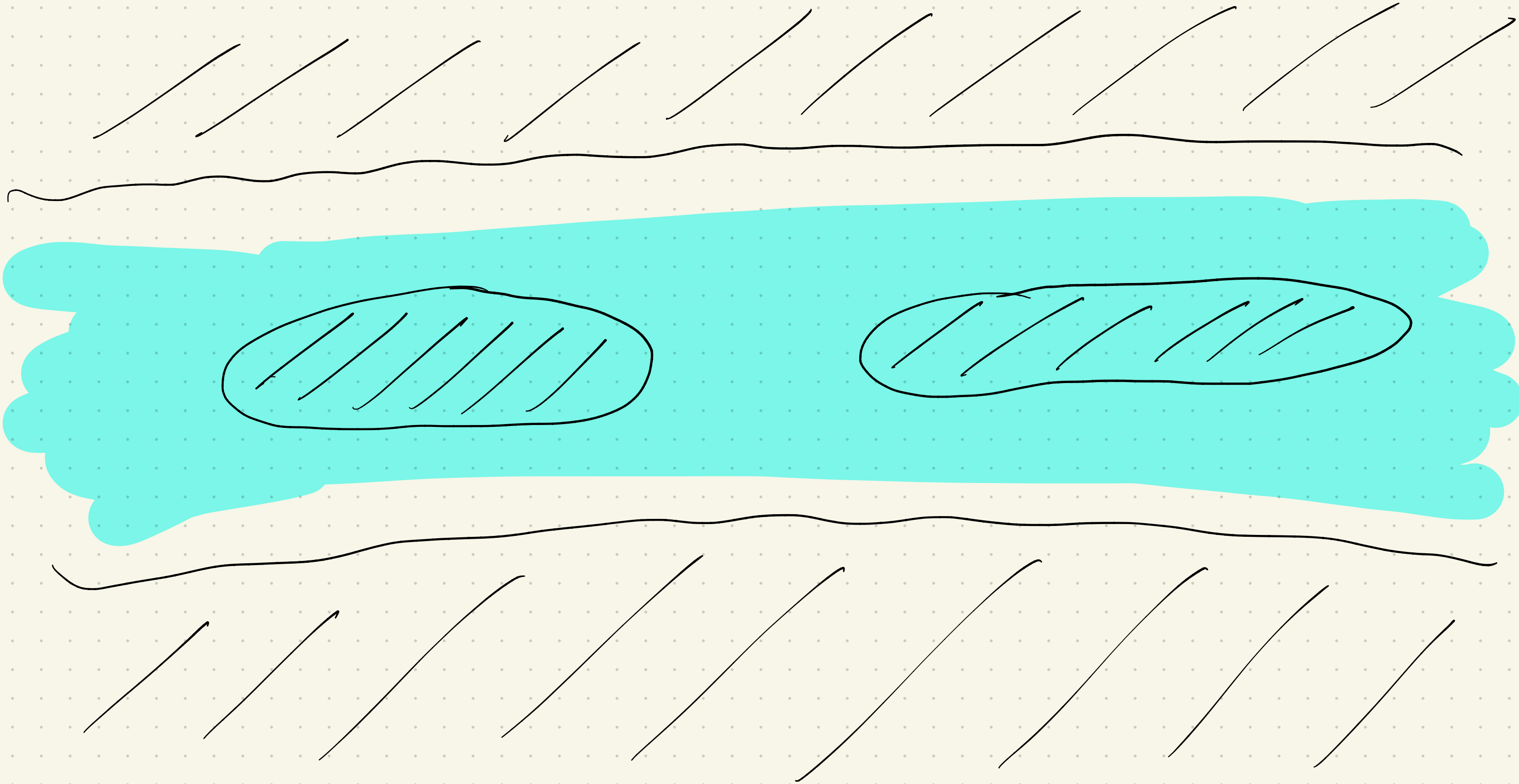

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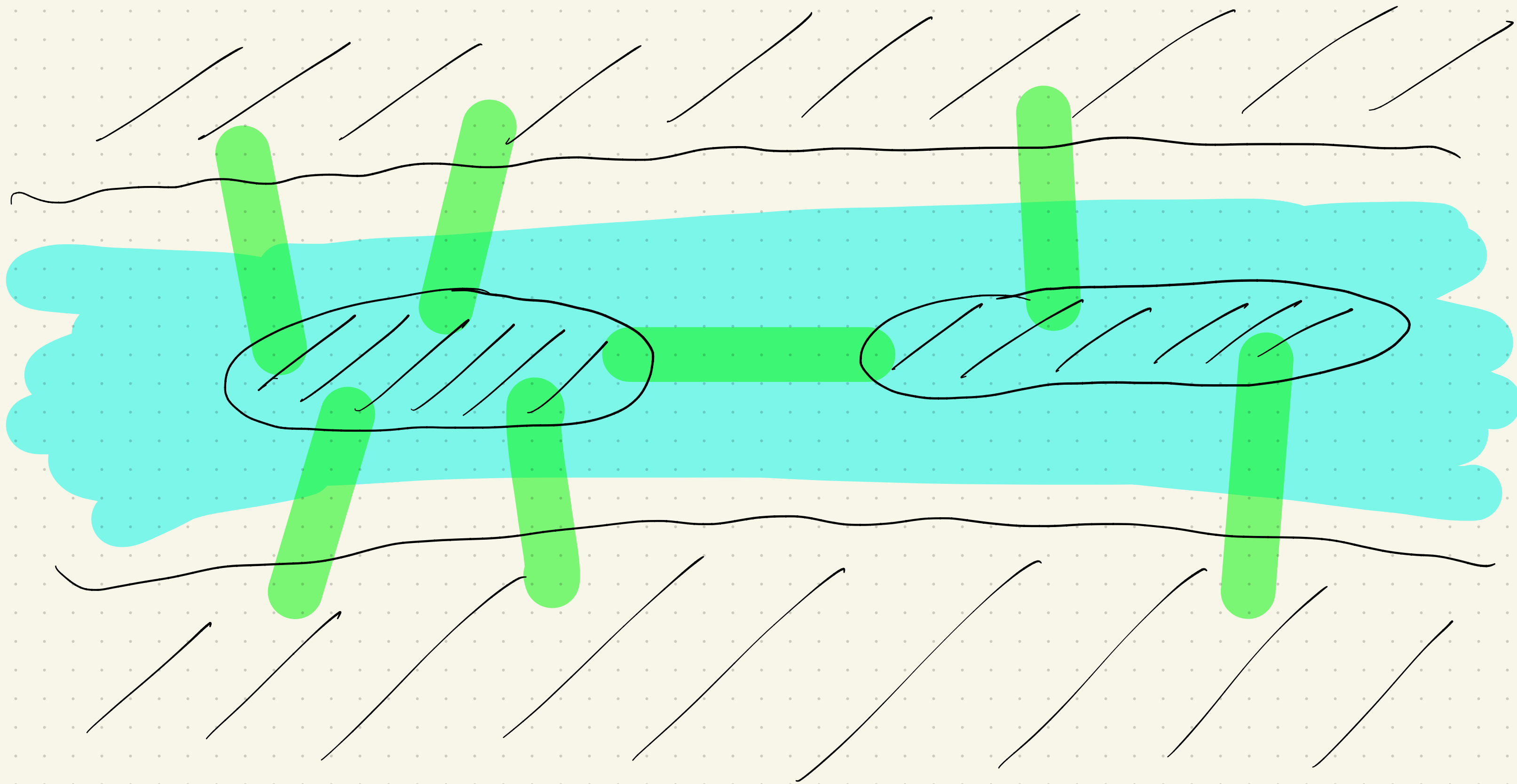

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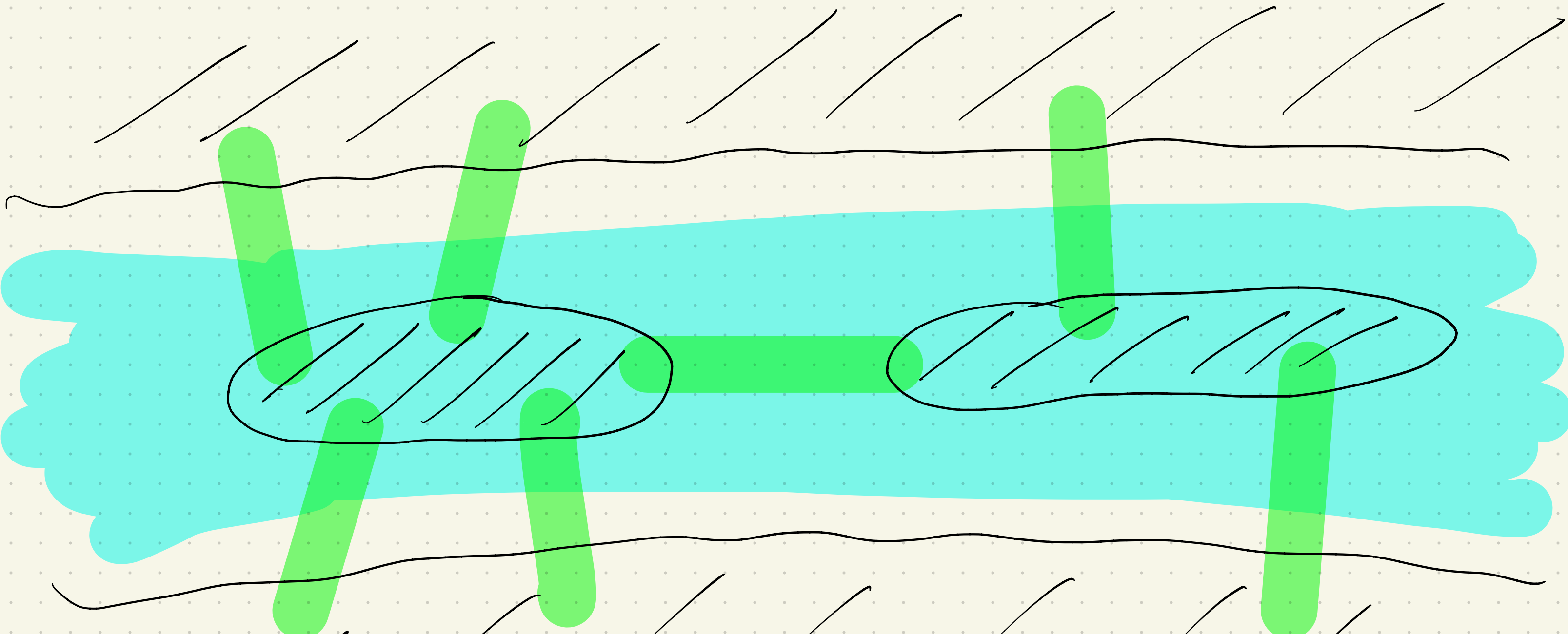
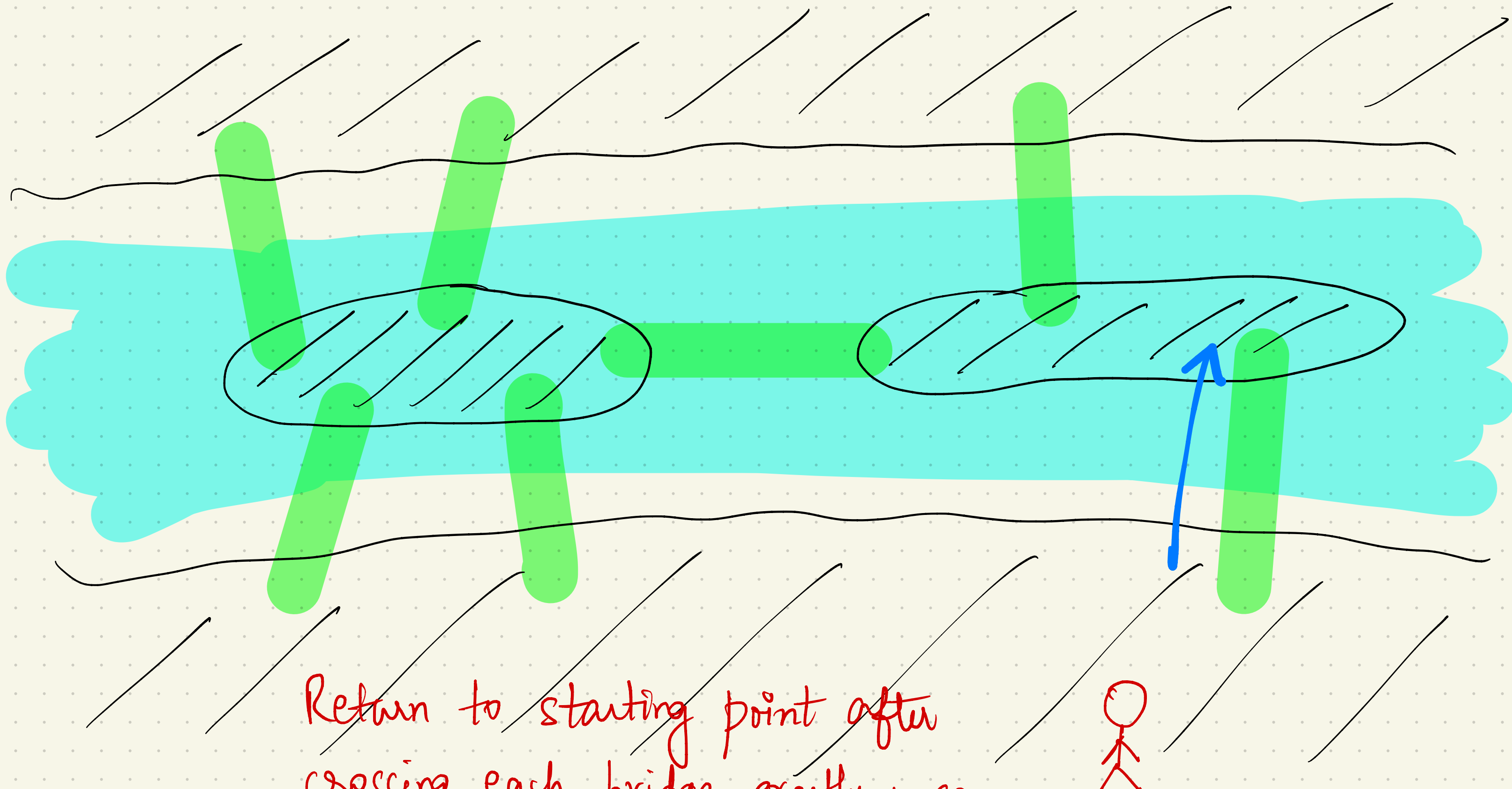


Return to starting point after crossing each bridge exactly once.

SEVEN BRIDGES OF KÖNIGSBERG



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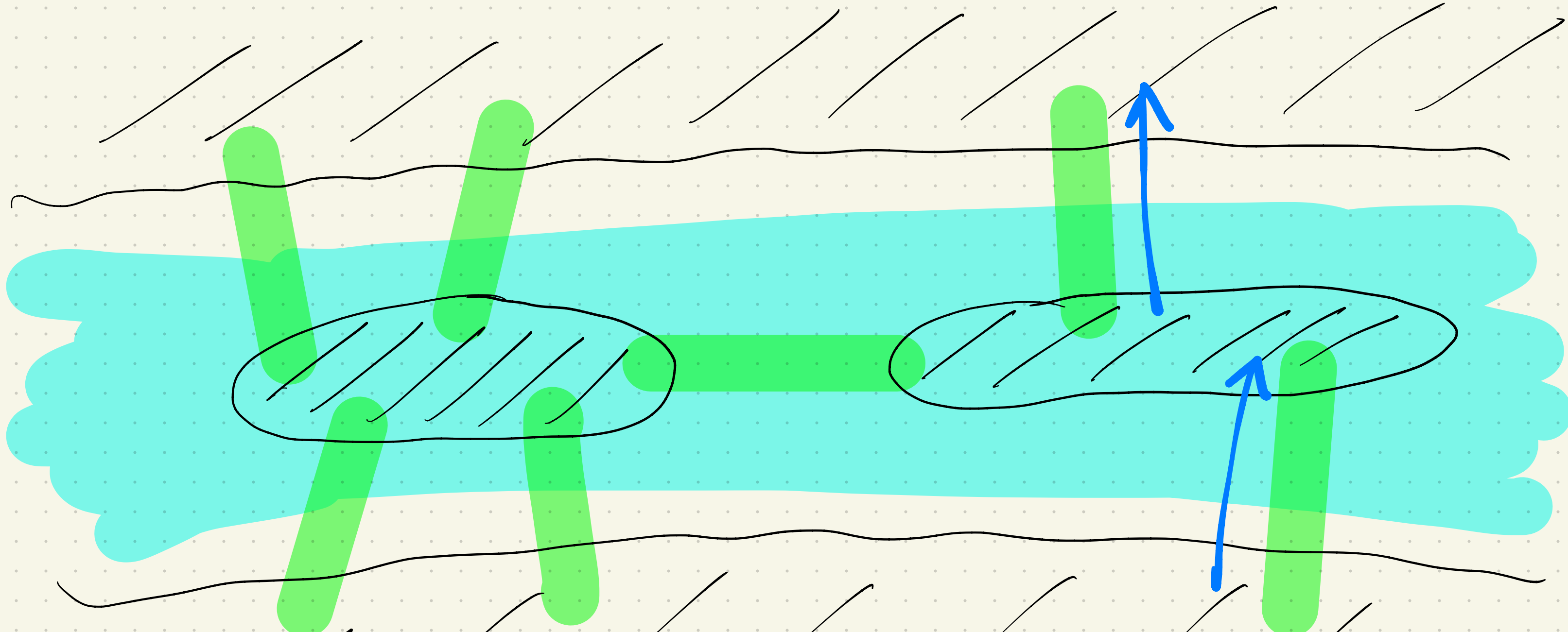

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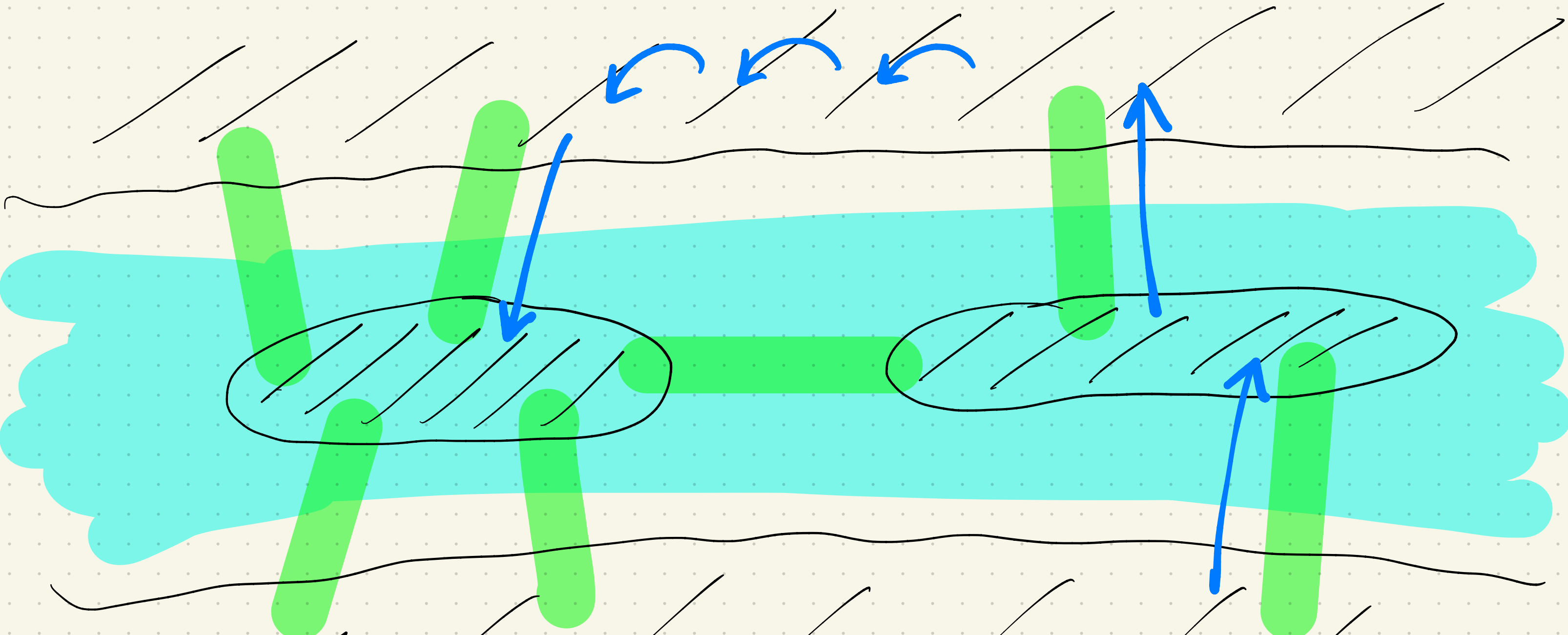

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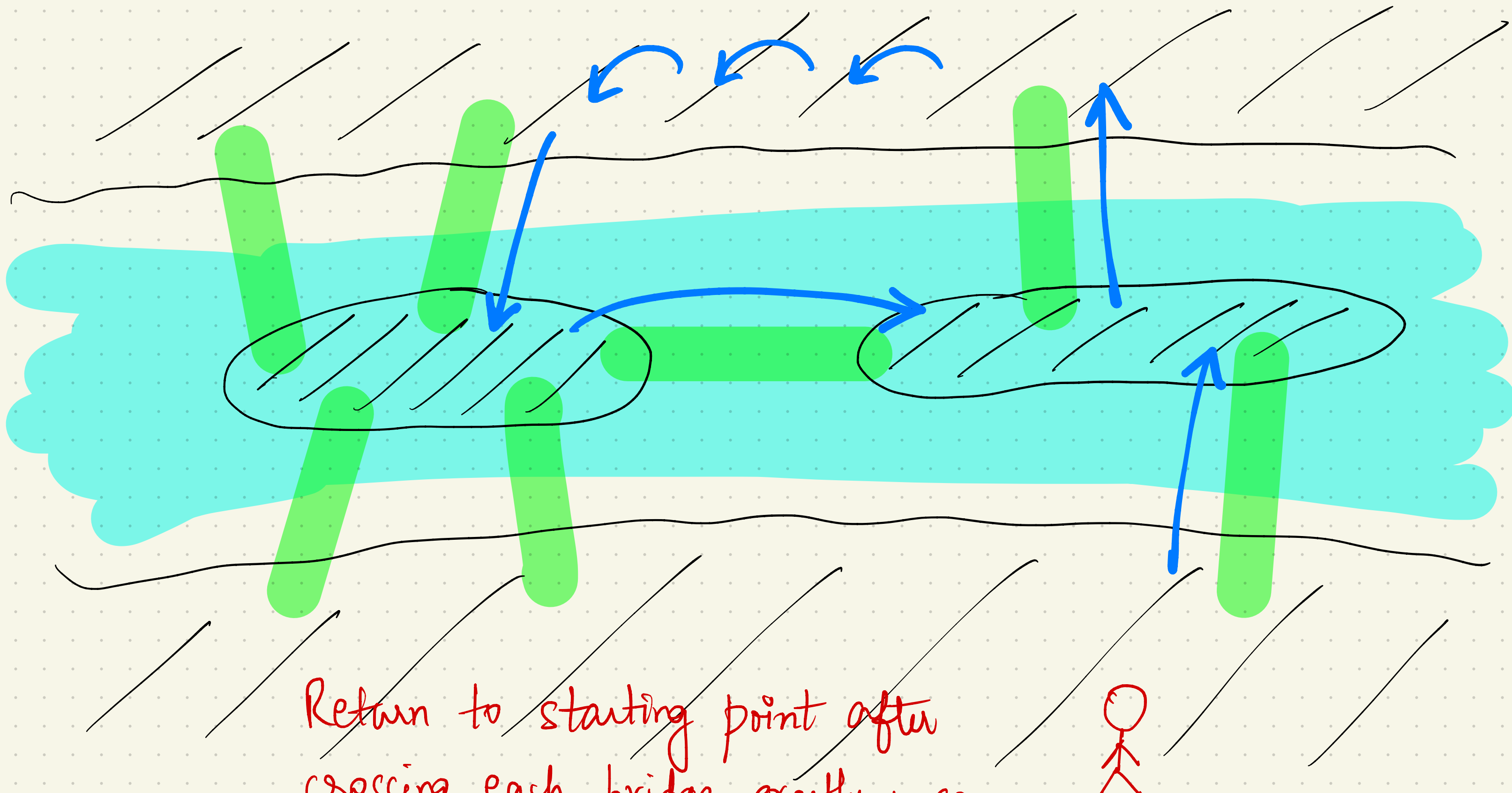

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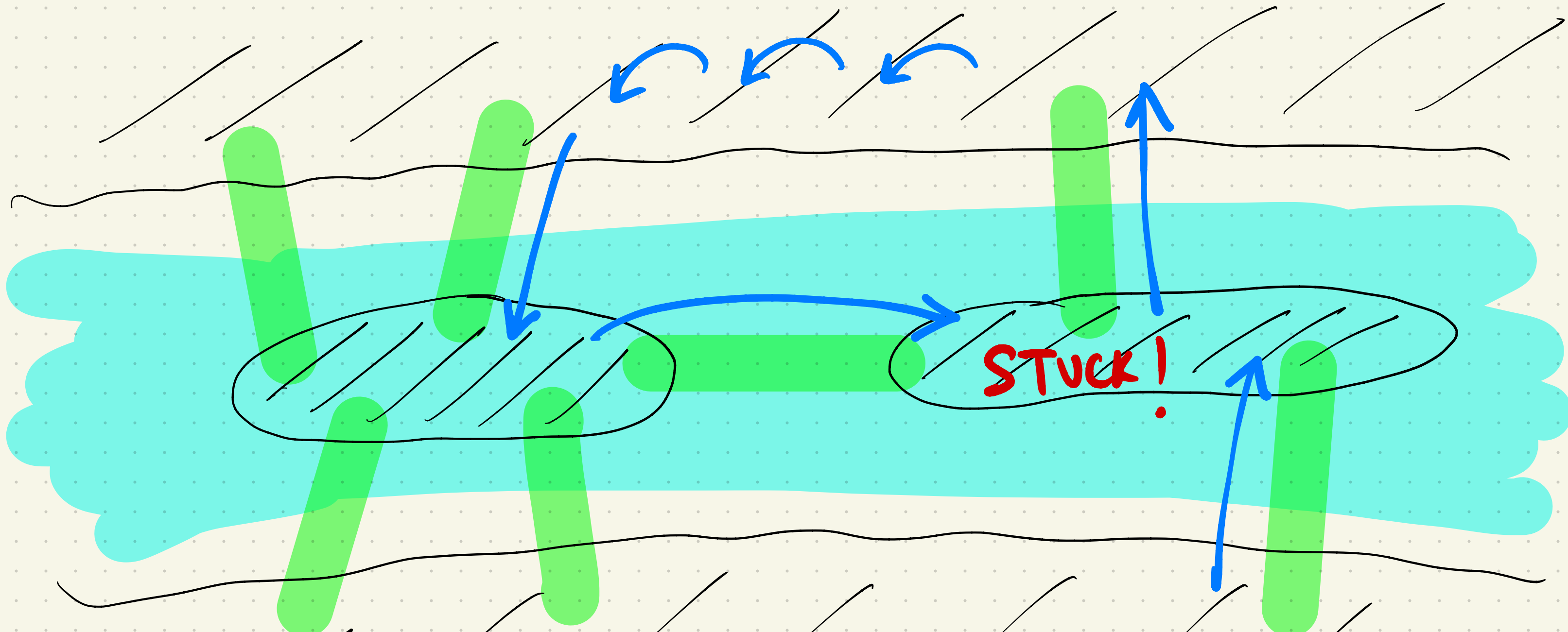

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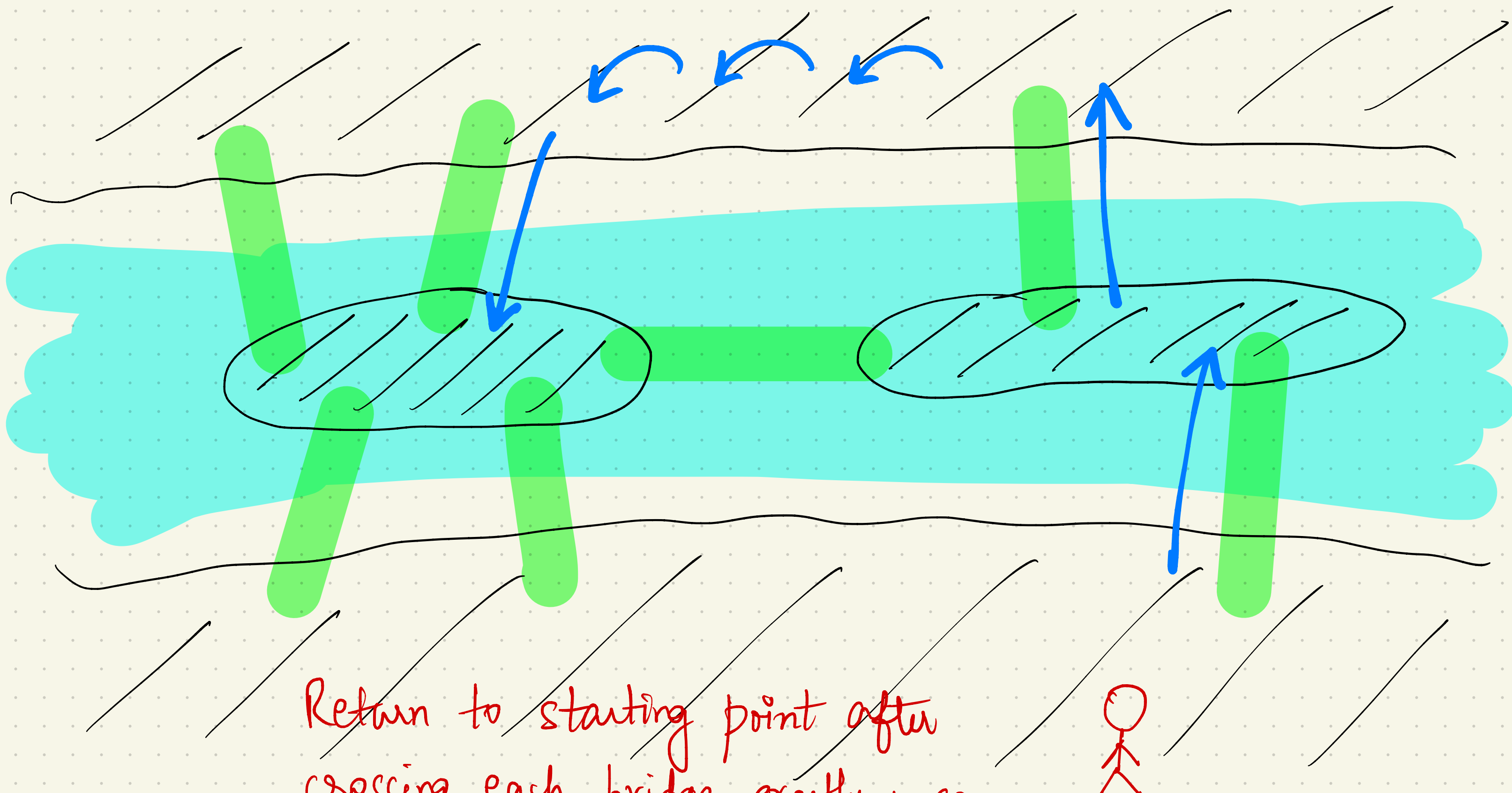

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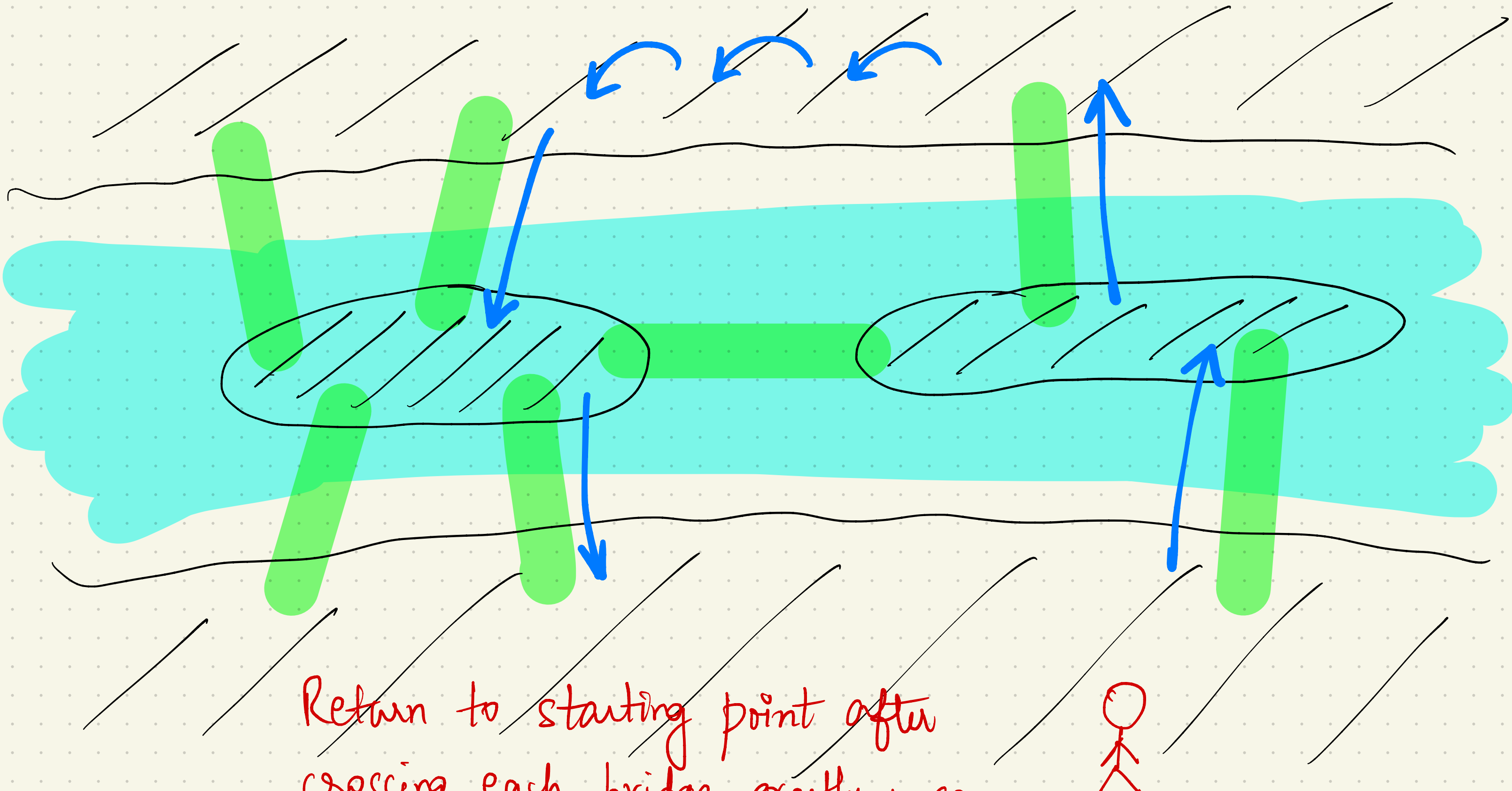

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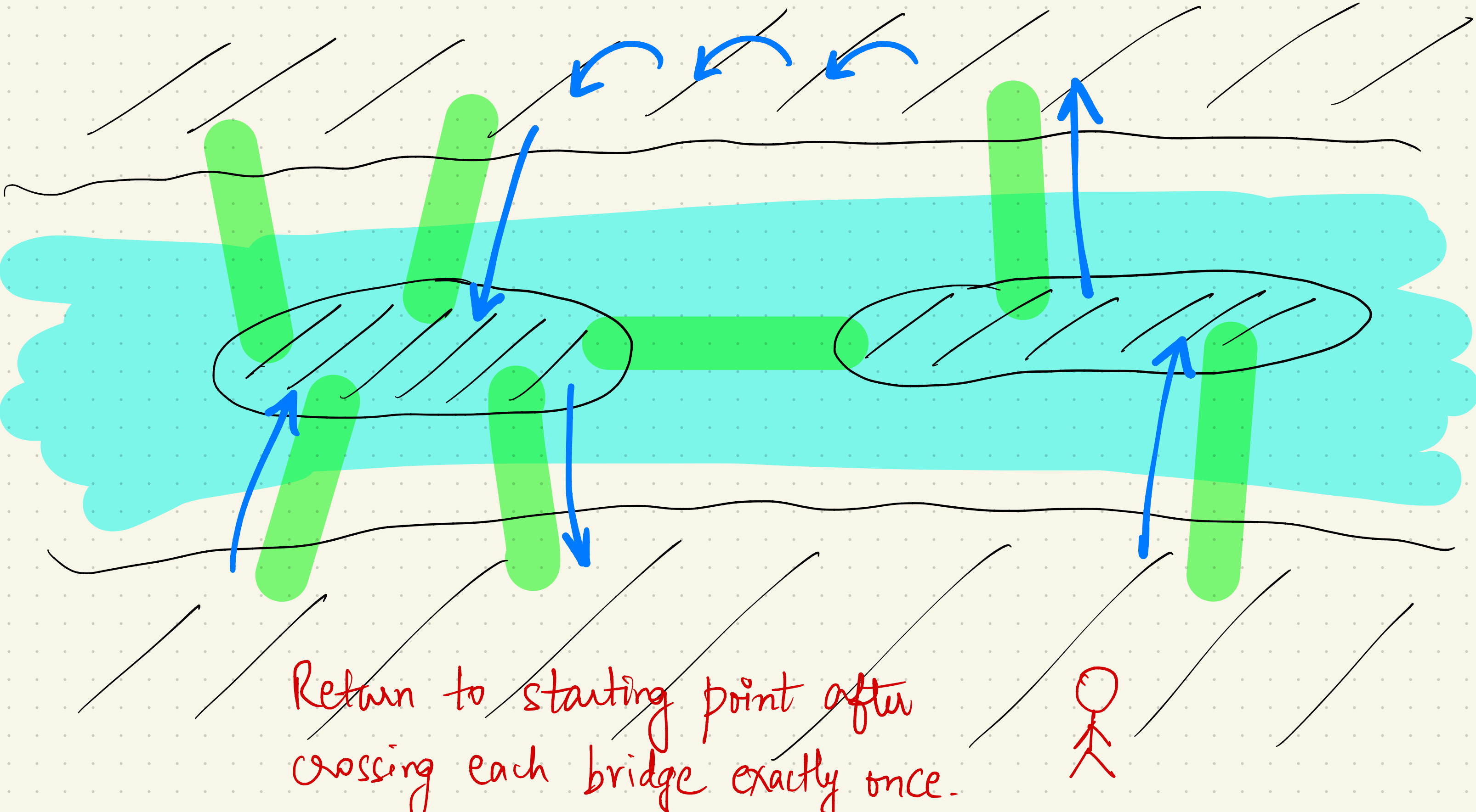

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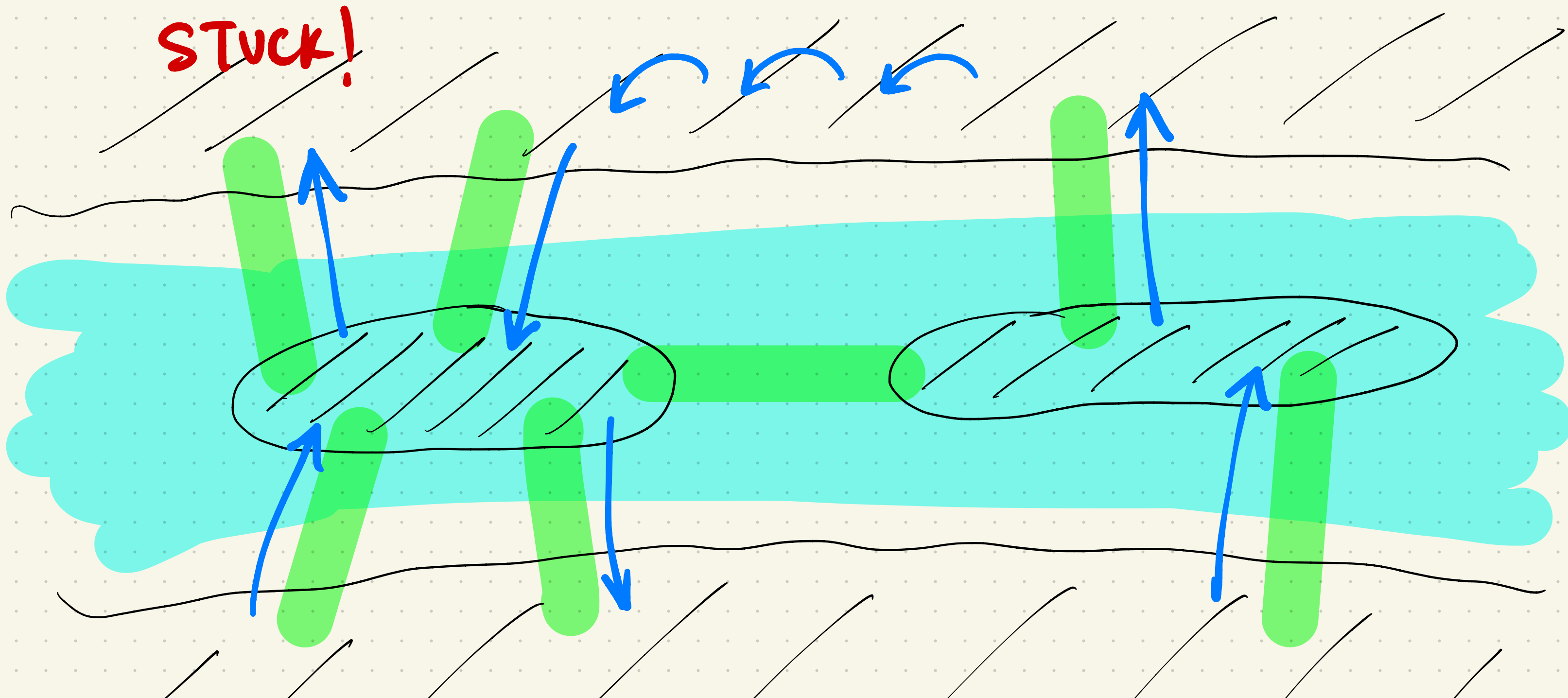

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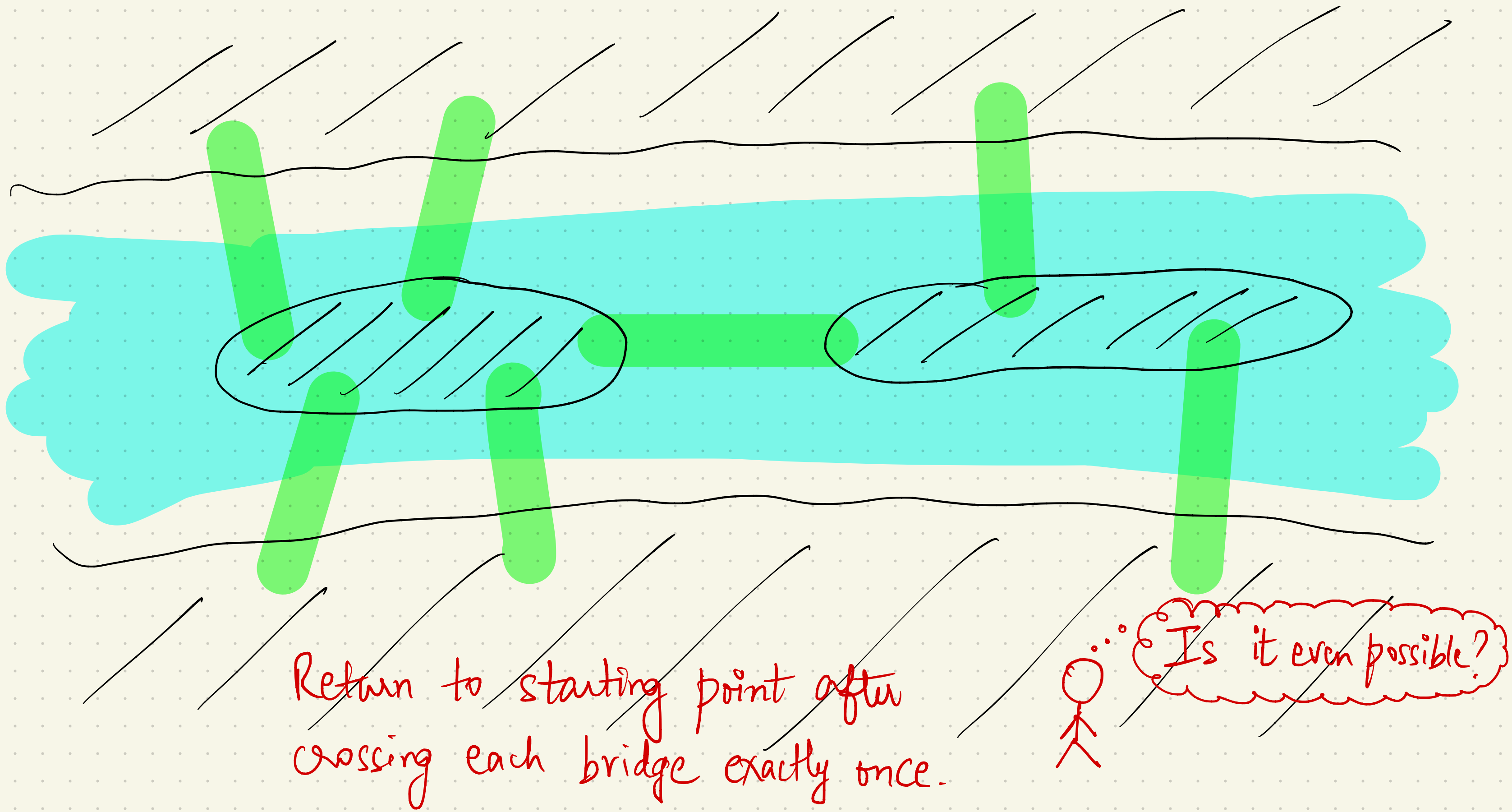

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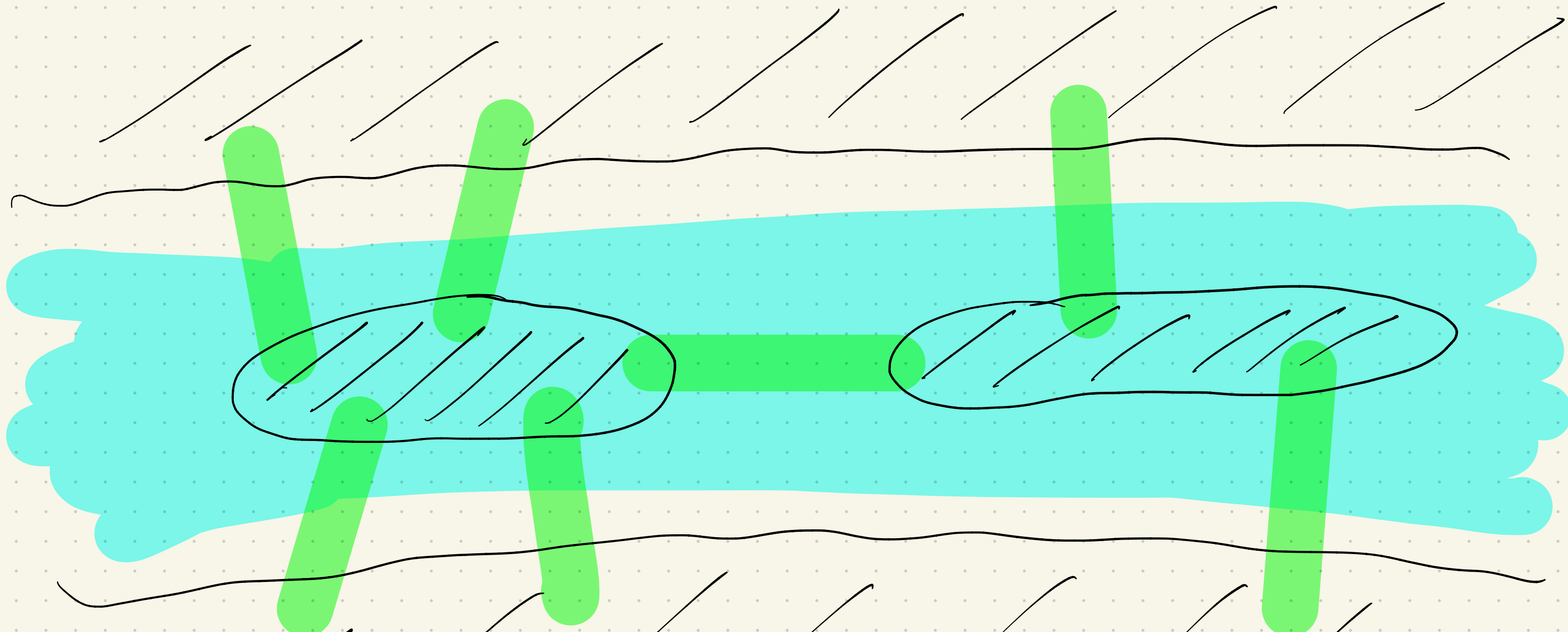
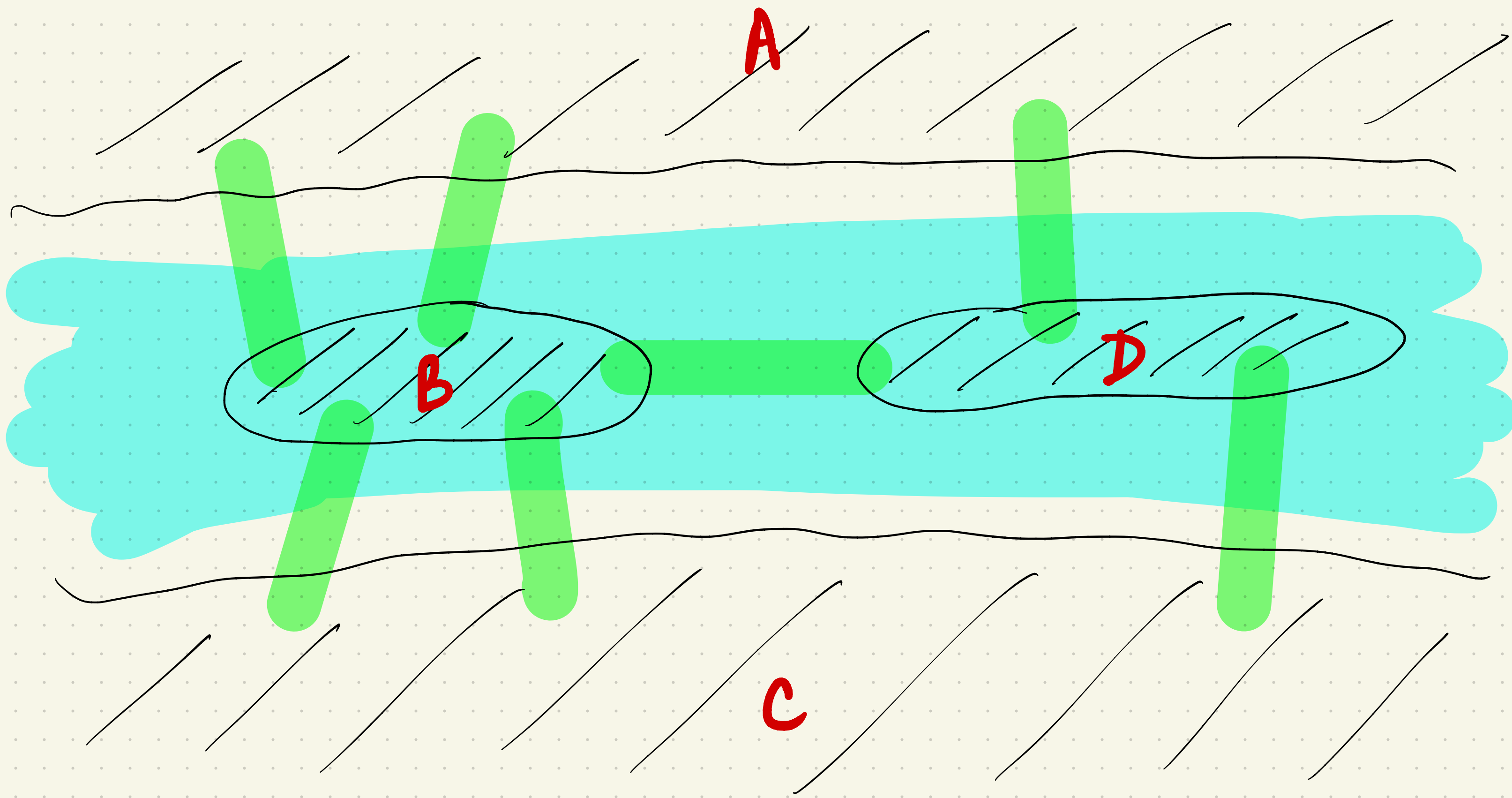


Return to starting point after crossing each bridge exactly once.

"Is it even possible?"

"No!" — Euler

Seven Bridges of Königsberg



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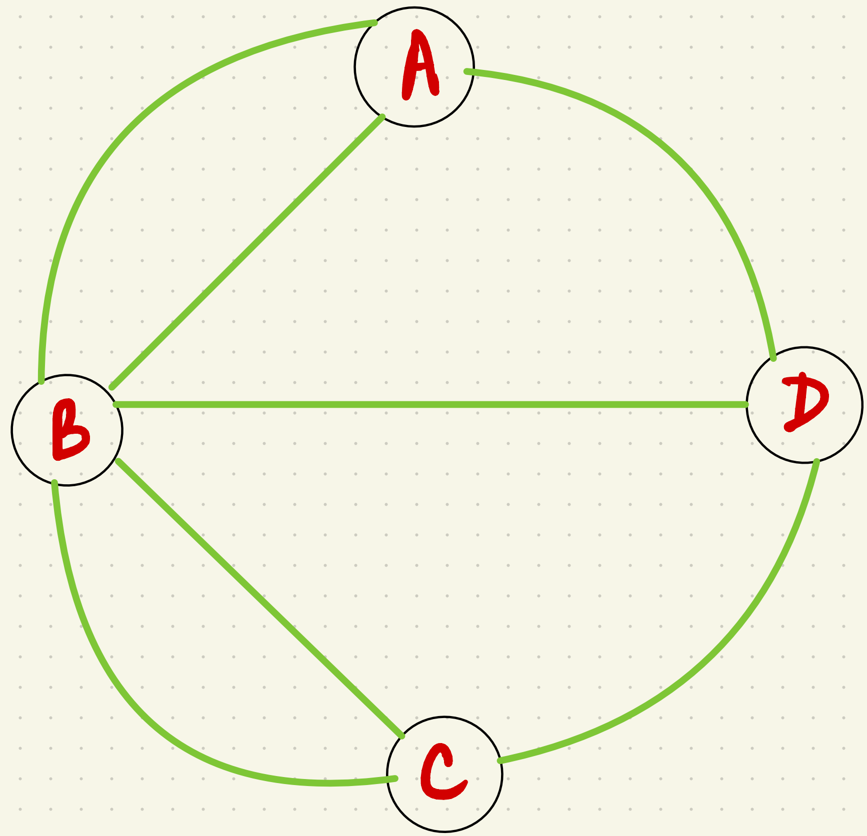

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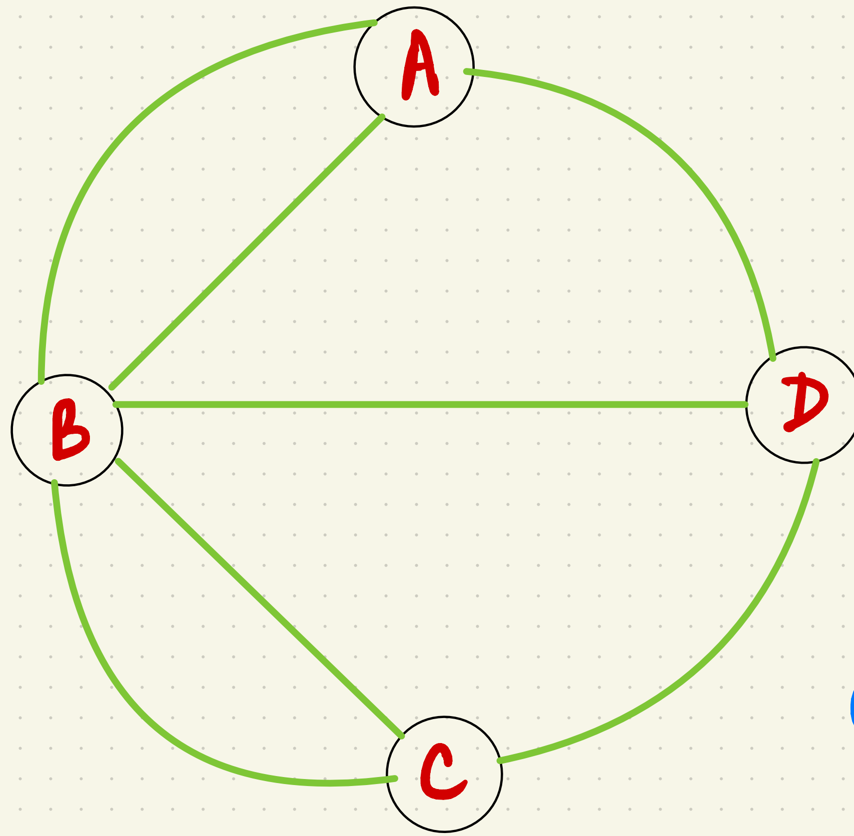
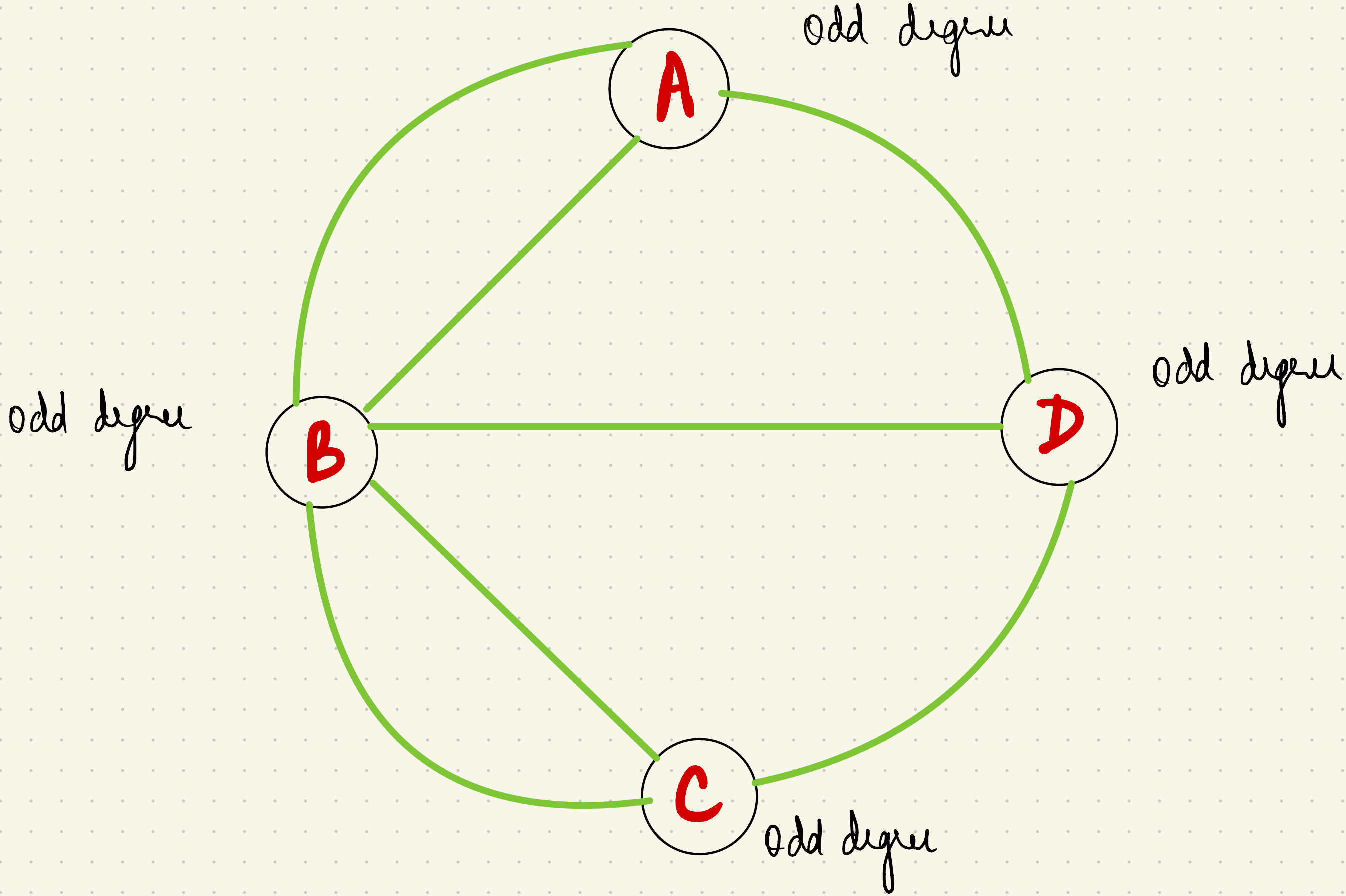


Birth of
Graph theory!
(1736)

Seven Bridges of Königsberg



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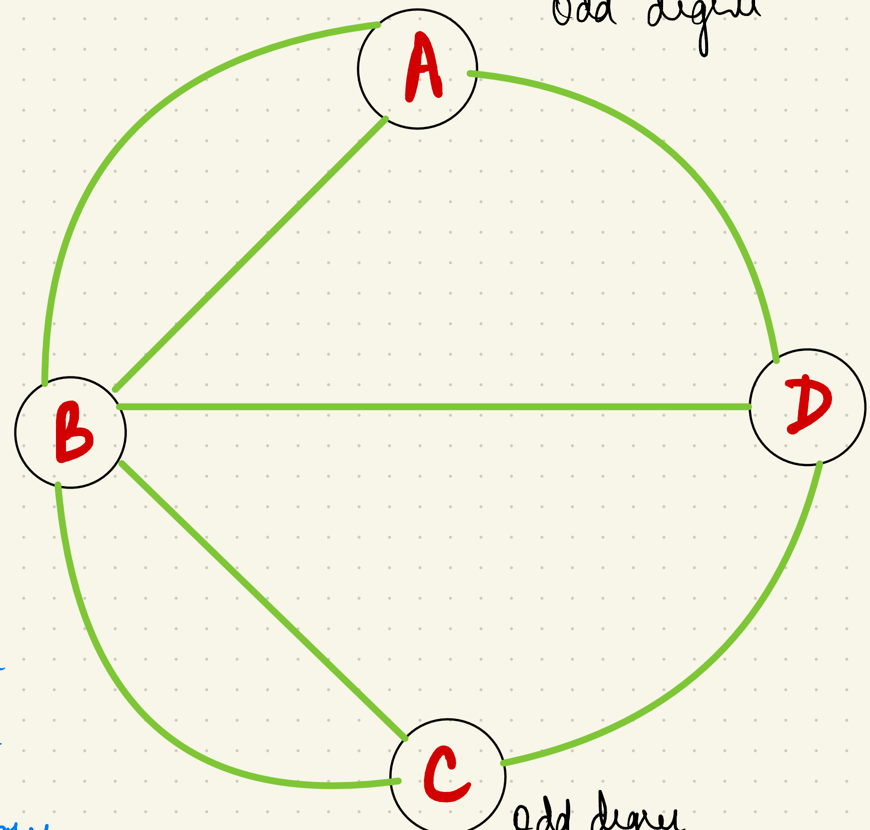


The entry and exit edges should be paired

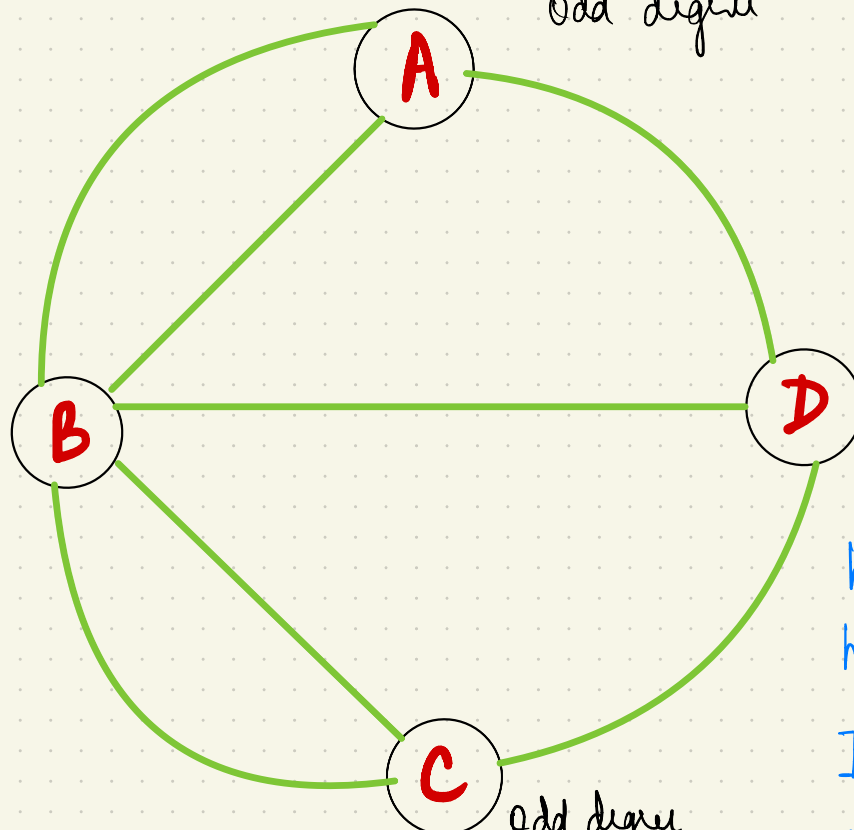
⇒ Desired walk not possible with odd degree vertices

Seven Bridges of Königsberg

odd degree

odd degree

odd degree

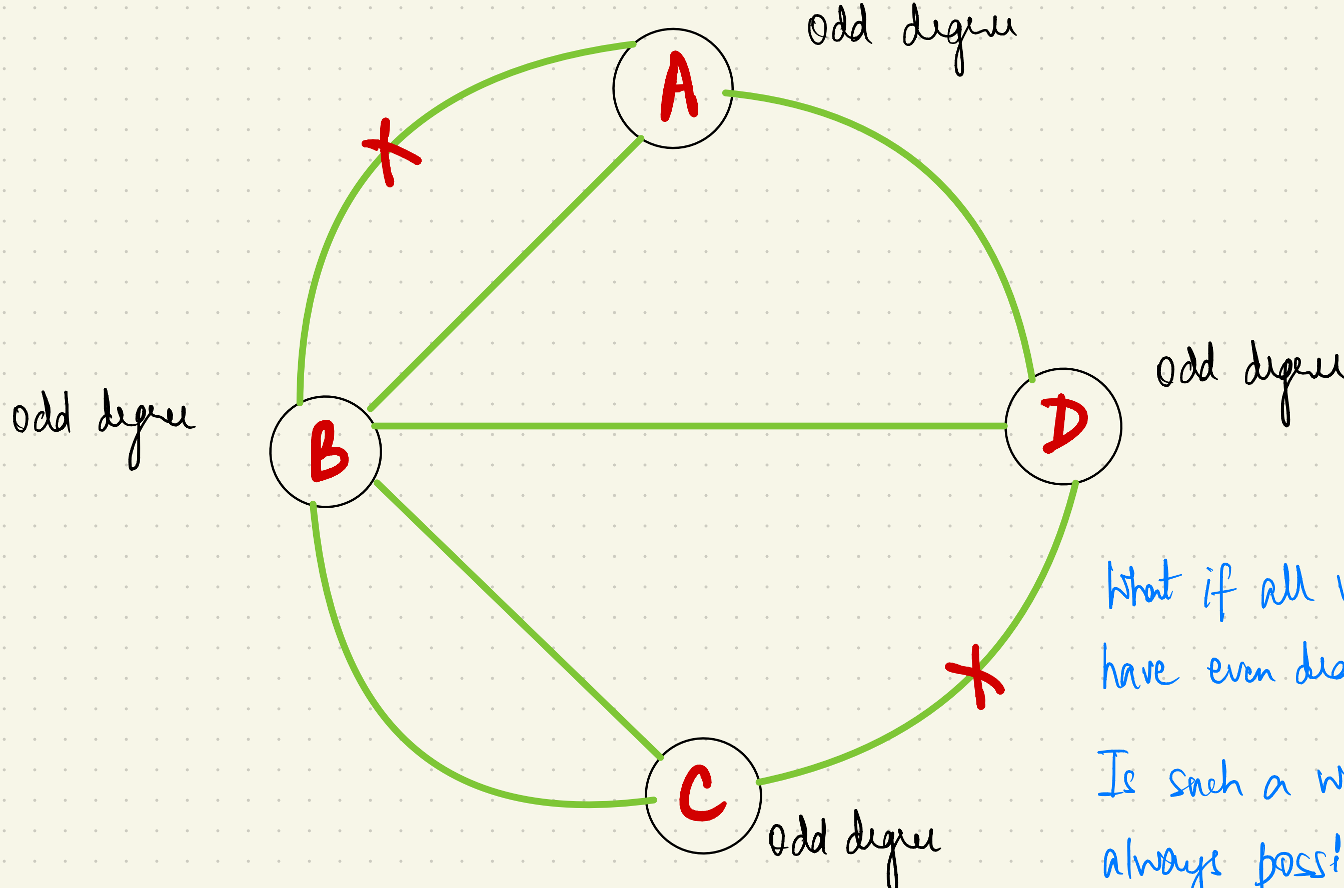


odd degree

What if all vertices have even degree?

Is such a walk always possible?

Seven Bridges of Königsberg



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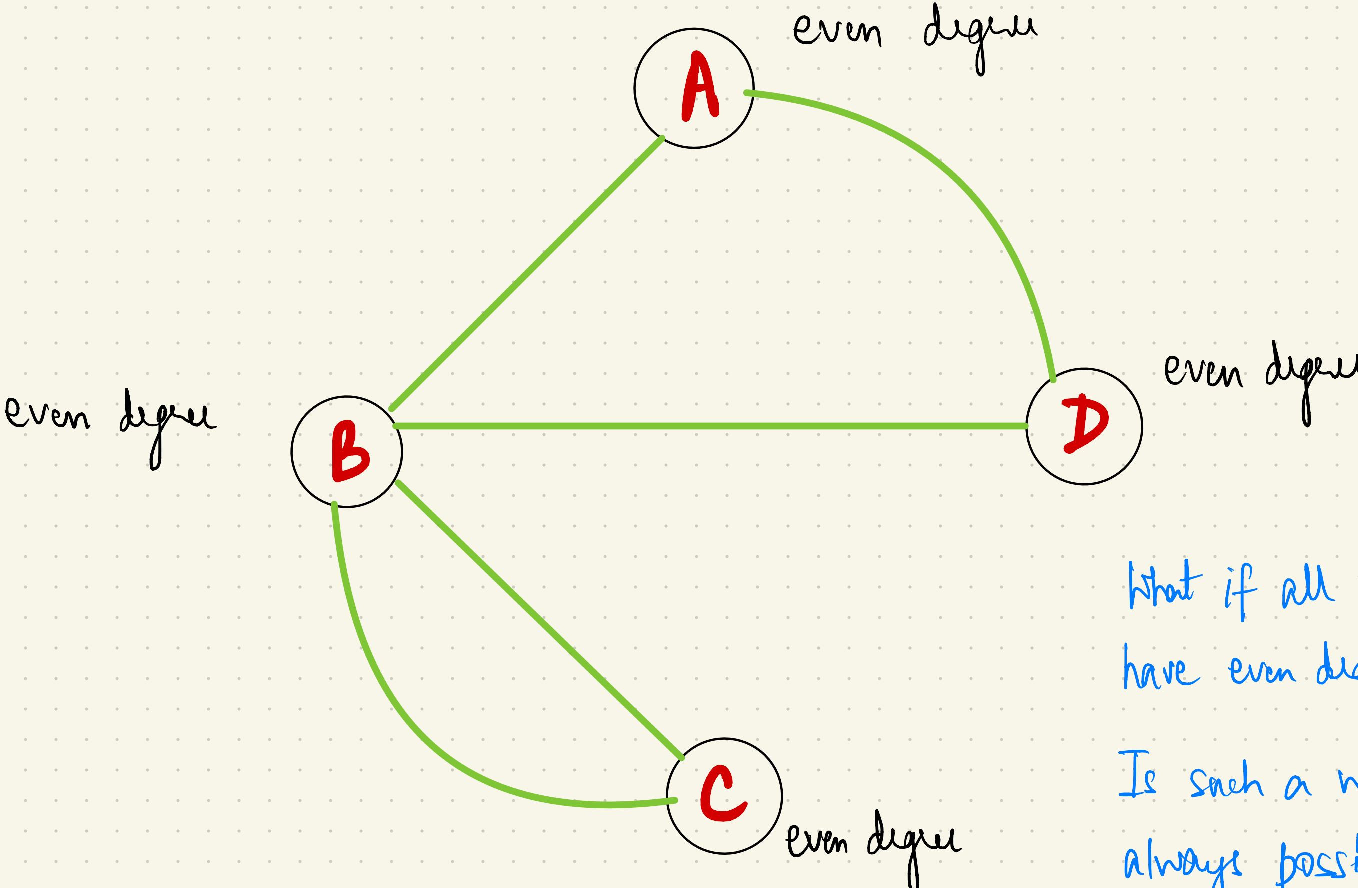

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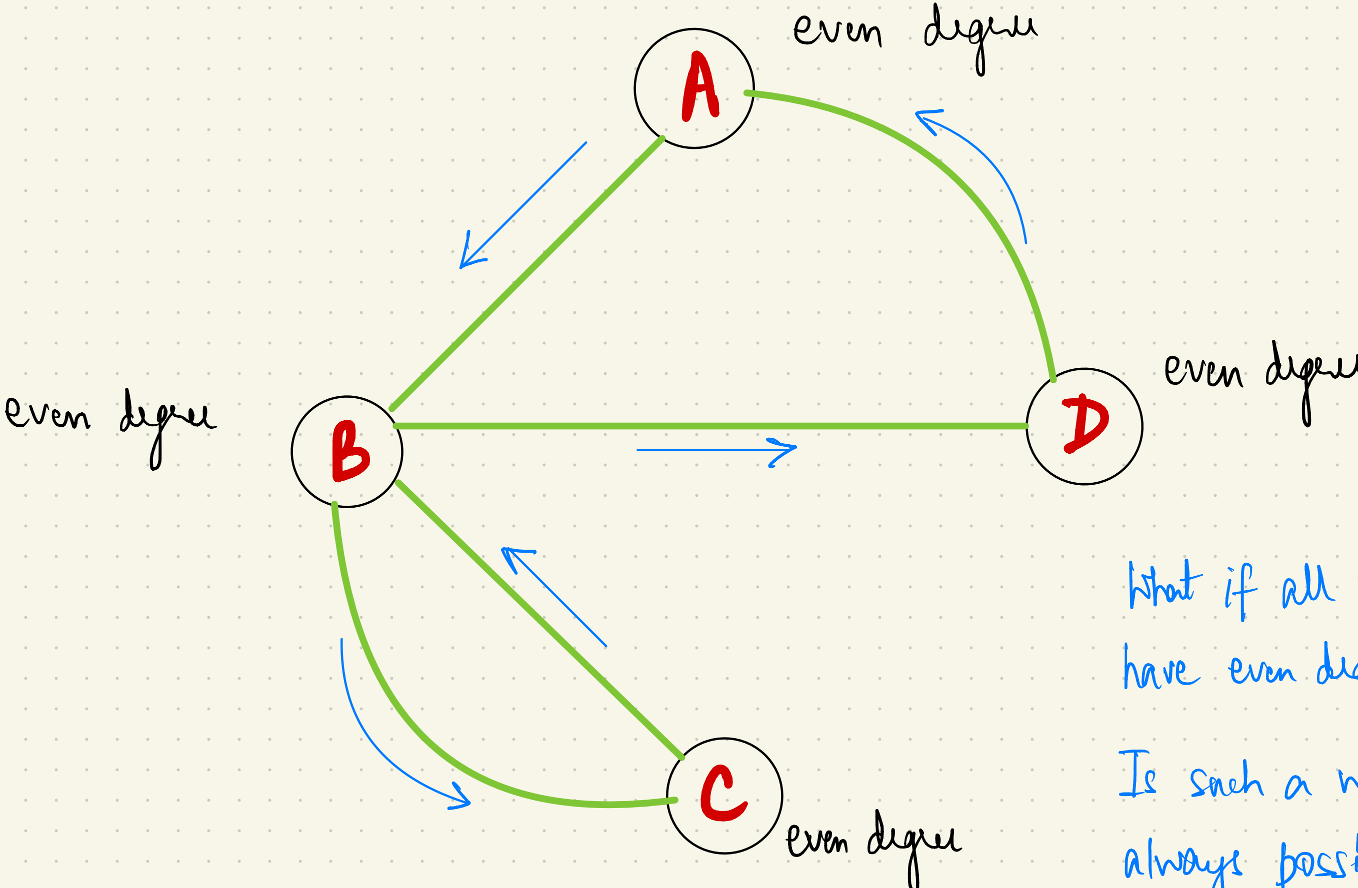

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EULER TOUR

A tour is a closed walk that visits every vertex at least once.

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$\updownarrow$

a closed walk that visits every vertex at least once and every edge exactly once.

When does a graph admit an Euler tour?

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Theorem: A connected graph has an Euler tour if and only if every vertex has an even degree.

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$\downarrow$ individual degrees

$\downarrow$ ability to traverse the graph

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* "Local" information gives us "global" information

  Local information → individual degrees

  Global information → ability to traverse the graph

* Do not need to know how the nodes are connected

* Easy to check if a graph has an Euler tour.

  Easy → polynomial time

Theorem: A connected graph has an Euler tour if and only if every vertex has an even degree.

Proof: ($\Longrightarrow$) Suppose $G=(V, E)$ has an Euler tour.

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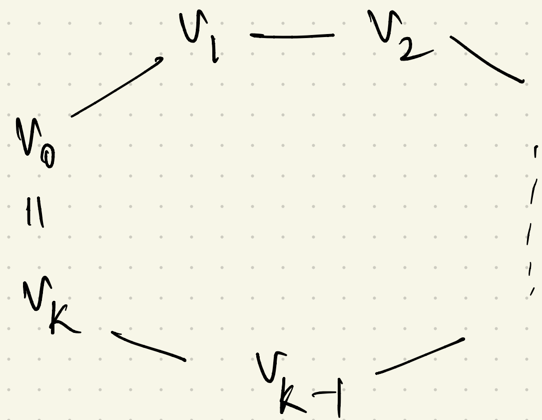

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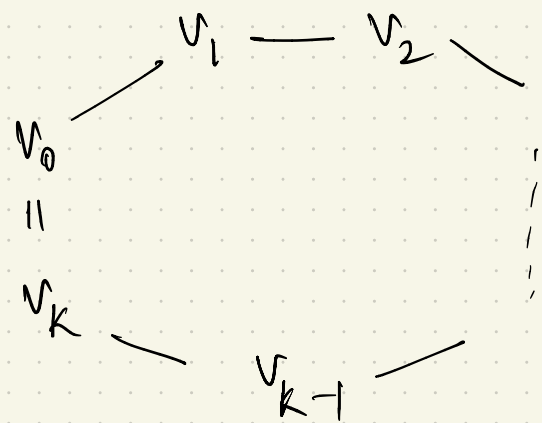

Every edge in E is traversed exactly once $\Longrightarrow$

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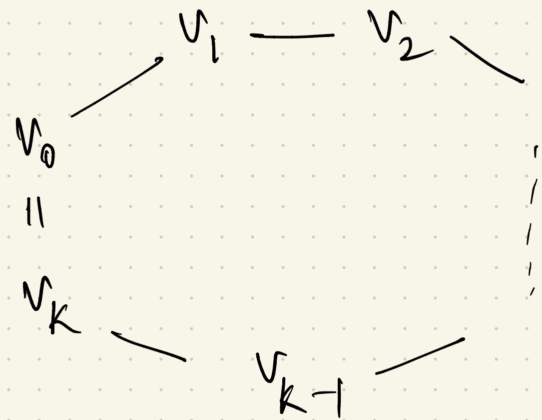

Every edge in E is traversed exactly once $\Rightarrow$

$\deg(v) = 2 \times \#$ times v appears in the tour $v_0, \dots, v_{k-1}, v_k$

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Let $W = v_0 - v_1 - \cdots - v_k$ be the longest walk that traverses any edge at most once. (well-defined)

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Obs 1: Any edge in E that is incident to v_k is covered by W.

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(Otherwise W may be extended — not possible)

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Obs 1: Any edge in E that is incident to v_k is covered by W.
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Obs 2: $v_0 = v_k$
(Otherwise v_k has odd no. of edges in $W \Rightarrow$ odd degree)

Theorem : A connected graph has an Euler tour if and only if every vertex has an even degree.

Proof : ($\Leftarrow$) For $G=(V, E)$, assume $deg(v)$ is even for all $v \in V$.

Let $W = v_0 - v_1 - \cdots - v_k$ be the longest walk that traverses any edge at most once.

We now want to show that W is an Euler tour.

Theorem: A connected graph has an Euler tour if and only if every vertex has an even degree.

Proof: ($\Leftarrow$) For $G=(V,E)$, assume $\deg(v)$ is even for all $v \in V$.

Let $W = v_0 - v_1 - \cdots - v_k$ be the longest walk that traverses any edge at most once.

Suppose W is not an Euler tour.

Theorem: A connected graph has an Euler tour if and only if every vertex has an even degree.

Proof: $(\Leftarrow)$ For $G=(V,E)$, assume $\deg(v)$ is even for all $v \in V$.

Let $W = v_0 - v_1 - \cdots - v_k$ be the longest walk that traverses any edge at most once.

Suppose W is not an Euler tour.

$\Rightarrow$ There is an edge, say e, in E such that e is not in W

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Why?

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Why? Connectedness of graph G

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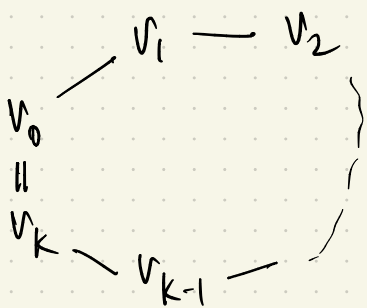

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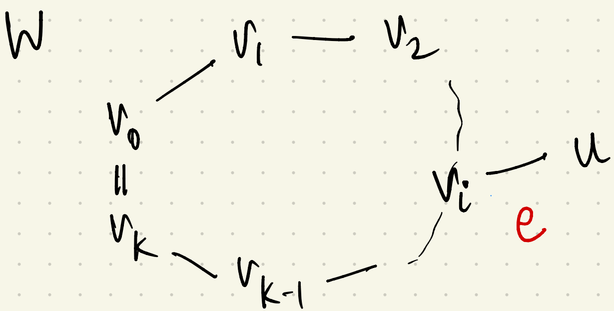

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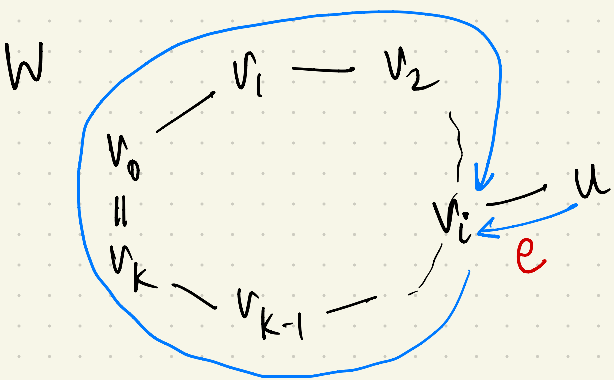

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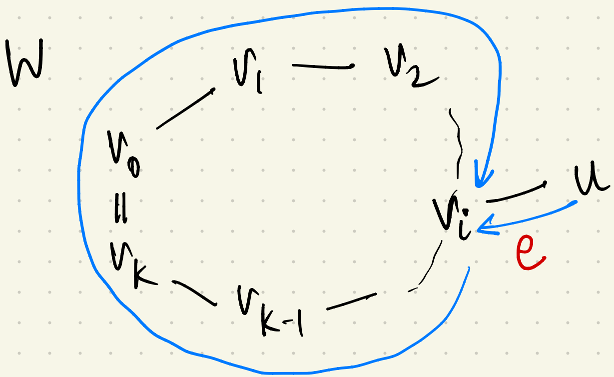

Can make W longer!

$$u - v_i - v_{i+1} - \cdots v_k = v_0 - - - - v_i$$

Theorem: A connected graph has an Euler tour if and only if every vertex has an even degree.

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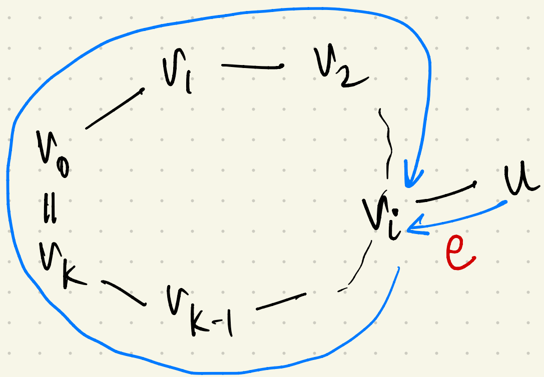

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Contradiction! $\Rightarrow$ W must be an Euler tour.

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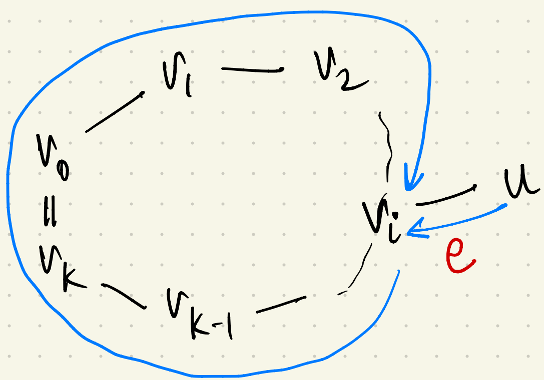

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AN APPLICATION OF EULER TOURS

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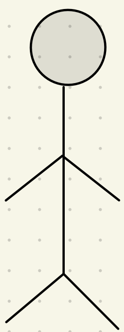
Genome Sequencing

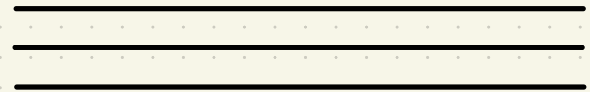


= ... TACGAGACAGTACA...

~ 3 billion letters

AN APPLICATION OF EULER TOURS

Multiple identical copies
of a genome



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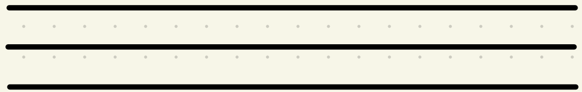

Shatter the genome into reads

AN APPLICATION OF EULER TOURS

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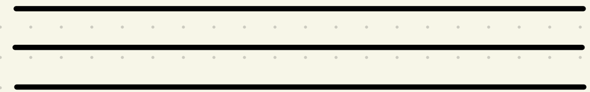


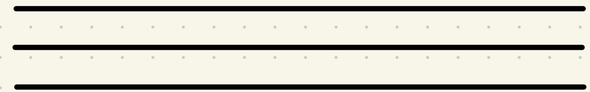
Shatter the genome into reads

Sequence the reads

AGATATCA CGATCCAT

An Application of Euler Tours

Multiple identical copies of a genome



Shatter the genome into reads

Sequence the reads

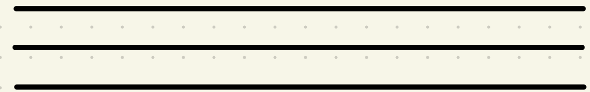
AGATATCA CGATCCAT

Assemble the genome using overlapping reads

```
  AGATCCG
   GATCCGA
 TAGATCC
 ---------
 TAGATCCGA
```

AN APPLICATION OF EULER TOURS

Multiple identical copies of a genome



Shatter the genome into reads

Sequence the reads

AGATATCA   CGATCCAT

Assemble the genome using overlapping reads

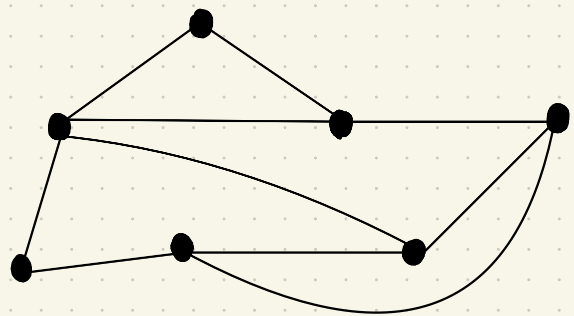
A G A T C C G
  G A T C C G A
T A G A T C C

T A G A T C C G A

Euler tours!

HAMILTONIAN CYCLE

A cycle that visits every vertex exactly once



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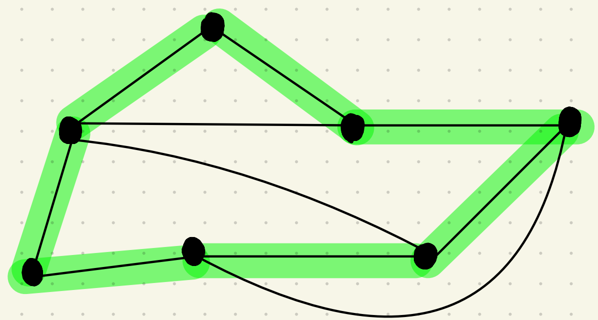

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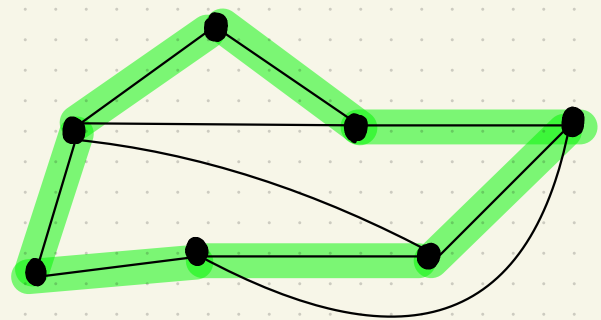
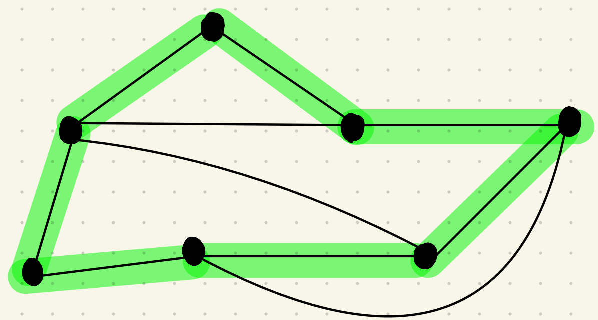


How to check if a graph has a Hamiltonian cycle?

HAMILTONIAN CYCLE

A cycle that visits every vertex exactly once

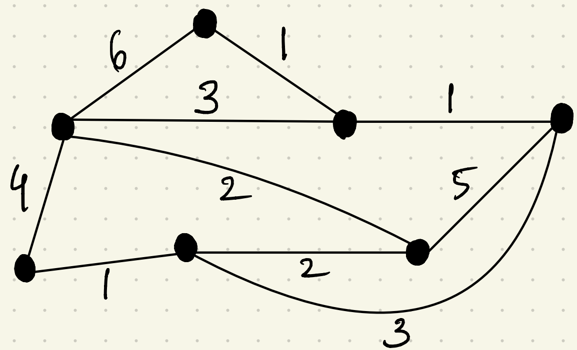


How to check if a graph has a Hamiltonian cycle?

NP-complete

HAMILTONIAN CYCLE

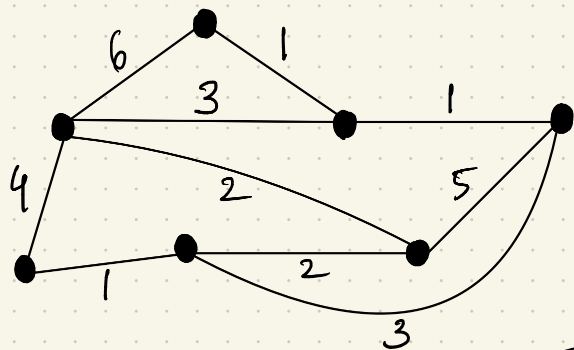
A cycle that visits every vertex exactly once



How to find a minimum-weight Hamiltonian cycle?

HAMILTONIAN CYCLE

A cycle that visits every vertex exactly once

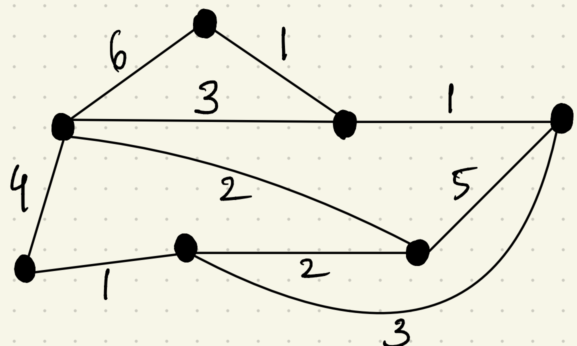


e.g., weight ≤ 15

How to find a minimum-weight Hamiltonian cycle?

HAMILTONIAN CYCLE

A cycle that visits every vertex exactly once



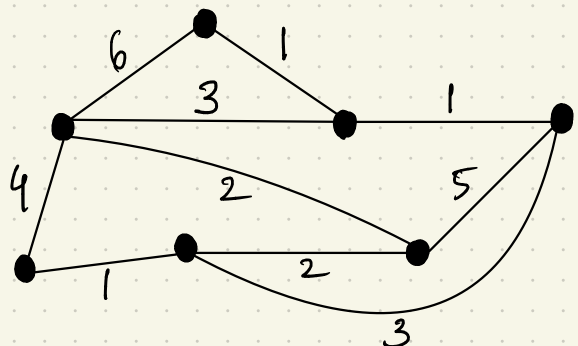
→ e.g., weight ≤ 15

How to find a minimum-weight Hamiltonian cycle?

NP-complete ☹

HAMILTONIAN CYCLE

A cycle that visits every vertex exactly once



→ e.g., weight ≤ 15

How to find a minimum-weight Hamiltonian cycle?

→ Traveling Salesperson Problem

NP-complete ☹