COL 202: DISCRETE MATHEMATICAL STRUCTURES

LECTURE 2

PROPOSITIONAL LOGIC

JAN 03, 2024 | ROHIT VAISH

LAST TIME

PROOF

PROPOSITION

A statement that is either TRUE or FALSE

AXIOMS

Assumptions / Propositions that are "accepted" as TRUE

LOGICAL DEDUCTIONS

A collection of rules for proving new propositions using previously known ones.

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Every odd integer is a prime

The sum of angles of a triangle is 180°.

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p: Every odd integer is a prime

q: The sum of angles of a triangle is 180°.

TODAY

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Propositional Calculus

$\neg p$	Negation
$p \vee q$	Disjunction / Or
$p \wedge q$	Conjunction / And
$p \Rightarrow q$	Implication / If-Then

NEGATION

$\neg p$

FALSE if p is TRUE

TRUE if p is FALSE

NEGATION

$\neg p$ FALSE if p is TRUE
TRUE if p is FALSE

E.g., p : 3 is the smallest prime   (FALSE)

$\neg p$: 3 is not the smallest prime   (TRUE)

NEGATION

$\neg p$ FALSE if p is TRUE
TRUE if p is FALSE

E.g., p : 3 is the smallest prime (FALSE)

$\neg p$: 3 is not the smallest prime (TRUE)

Negation v/s "Opposite" 3 is the largest prime ✗

DISJUNCTION

$p \vee q$ TRUE if at least one of p or q is TRUE

FALSE if both p and q are FALSE

DISJUNCTION

$p \vee q$

TRUE if at least one of p or q is TRUE

FALSE if both p and q are FALSE

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

TRUTH TABLE

DISJUNCTION

$p \vee q$ TRUE if at least one of p or q is TRUE

FALSE if both p and q are FALSE

Eg.,

p : $1 + 2 = 3$

q : $3 + 2 = 5$

$p \vee q$: $1 + 2 = 3$ OR $3 + 2 = 5$

Logically TRUE, but strange in ordinary language
(either this or that)

DISJUNCTION

$p \vee q$ TRUE if at least one of p or q is TRUE

FALSE if both p and q are FALSE

Eg., $\exists n \in \mathbb{N}$, $\underbrace{n^2+n+41}_{P(n)}$ is composite

$$\underset{\text{FALSE}}{P(1)} \vee \underset{\text{FALSE}}{P(2)} \vee \underset{\text{FALSE}}{P(3)} \vee \cdots \vee \underset{\text{FALSE}}{P(39)} \vee \underset{\text{TRUE}}{P(40)} \vee \cdots$$

CONJUNCTION

$p \wedge q$ TRUE if both p and q are TRUE

FALSE o/w

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p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

CONJUNCTION

$p \wedge q$ TRUE if both p and q are TRUE

FALSE o/w

E.g., $\forall n \in \mathbb{N}$, $\underbrace{n^2+n+41}_{P(n)}$ is prime

$$\underset{\text{TRUE}}{\underset{\uparrow}{P(1)}} \wedge \underset{\text{TRUE}}{\underset{\uparrow}{P(2)}} \wedge \underset{\text{TRUE}}{\underset{\uparrow}{P(3)}} \wedge \cdots \wedge \underset{\text{TRUE}}{\underset{\uparrow}{P(39)}} \wedge \underset{\text{FALSE}}{\underset{\uparrow}{P(40)}} \wedge \cdots$$

IMPLICATION

$p \Rightarrow q$ TRUE if p is FALSE or q is TRUE

FALSE o/w

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

IMPLICATION

$p \Rightarrow q$ TRUE if p is FALSE or q is TRUE

FALSE o/w

p	q	$p \Rightarrow q$
T	T	T
T	F	F
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IMPLICATION

$p \Rightarrow q$ TRUE if p is FALSE or q is TRUE

FALSE o/w

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Eg., p : 6 is prime

q : $6^2 = 25$

$p \Rightarrow q$: 6 is prime $\Rightarrow 6^2 = 25$

IMPLICATION

$p \Rightarrow q$ TRUE if p is FALSE or q is TRUE

FALSE o/w

p	q	$p \Rightarrow q$
T	T	T
T	F	F
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Eg., p : 6 is prime (FALSE)

q : $6^2 = 25$ (FALSE)

$p \Rightarrow q$: 6 is prime $\Rightarrow 6^2 = 25$

(TRUE)

IMPLICATION

$p \Rightarrow q$ TRUE if p is FALSE or q is TRUE

FALSE o/w

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
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E.g., p : 6 is prime (FALSE)

q : $6^2 = 25$ (FALSE)

$p \Rightarrow q$: 6 is prime $\Rightarrow 6^2 = 25$

(TRUE)

"If pigs could fly, I will be king."

IMPLICATION

$p \Rightarrow q$ TRUE if p is FALSE or q is TRUE

FALSE o/w

p	q	$p \Rightarrow q$
T	T	T
T	F	F
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Eg., $\forall n \in \mathbb{N},\ n \geq 2 \Rightarrow n^2 \geq 4.$

$P(1) \wedge P(2) \wedge P(3) \wedge \cdots$

$P(1): \quad 1 \geq 2 \Rightarrow 1 \geq 4$ (TRUE)

$P(2): \quad 2 \geq 2 \Rightarrow 4 \geq 4$ (TRUE)

$P(3): \quad 3 \geq 2 \Rightarrow 9 \geq 4$ (TRUE)

$\vdots$

IF AND ONLY IF

$p \Leftrightarrow q$ TRUE if both p and q are TRUE or both are FALSE

FALSE o/w

p	q	$p \Rightarrow q$	$q \Rightarrow p$	$p \Leftrightarrow q$
T	T	T	T	T
T	F	F	T	F
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IF AND ONLY IF

$p \Leftrightarrow q$ TRUE if both p and q are TRUE or both are FALSE

FALSE o/w

E.g., $\forall n \in \mathbb{Z}$, $n \geq 2 \Longleftrightarrow n^2 \geq 4$.

Integers $\{0, 1, -1, 2, -2, \ldots\}$

IF AND ONLY IF

$p \Leftrightarrow q$ TRUE if both p and q are TRUE or both are FALSE

FALSE o/w

E.g., $\forall n \in \mathbb{Z}$, $n \geq 2 \Leftrightarrow n^2 \geq 4$.

$\Rightarrow$ is TRUE $\Leftarrow$ is FALSE

Hence, $\Leftrightarrow$ is FALSE

IF AND ONLY IF

$p \Leftrightarrow q$ TRUE if both p and q are TRUE or both are FALSE

FALSE o/w

$p \Leftrightarrow q$ is **logically equivalent** to $(p \Rightarrow q) \wedge (q \Rightarrow p)$.

p	q	$p \Rightarrow q$	$q \Rightarrow p$	$p \Leftrightarrow q$
T	T	T	T	T
T	F	F	T	F
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F	F	T	T	T

LOGICAL EQUIVALENCE

Two compound propositions are logically equivalent if they are TRUE in exactly the same cases.

$p \vee q$	$r \wedge (p \vee q)$	p	q	r	$r \wedge p$	$r \wedge q$	$(r \wedge p) \vee (r \wedge q)$
T	T	T	T	T	T	T	T
T	F	T	T	F	F	F	F
T	T	T	F	T	T	F	T
T	F	T	F	F	F	F	F
T	T	F	T	T	F	T	T
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The propositions $r \wedge (p \vee q)$ and $(r \wedge p) \vee (r \wedge q)$ are logically equivalent. $\longrightarrow$ **DISTRIBUTIVE LAW**

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Other examples of logical equivalence:

$$\left.\begin{array}{lll} \neg(p \vee q) & \text{equiv. to} & \neg p \wedge \neg q \\ \neg(p \wedge q) & \text{equiv. to} & \neg p \vee \neg q. \end{array}\right\} \textbf{ DE MORGAN'S LAWS}$$

LOGICAL EQUIVALENCE

Two compound propositions are logically equivalent if they are TRUE in exactly the same cases.

$p \Rightarrow q$ equiv. to $\neg q \Rightarrow \neg p$

If I am hungry, then I am grumpy.

If I am not grumpy, then I am not hungry.

ORDER OF QUANTIFIERS

Recall : $\forall$ denotes "for every"

$\exists$ denotes "there exists"

P : $\forall\, n \in \mathbb{N}$, $\exists\, m \in \mathbb{N}$ such that $m = n^2$.

Q : $\exists\, m \in \mathbb{N}$ such that $\forall\, n \in \mathbb{N}$, $m = n^2$.

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depends on n

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Same m for all n

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Same m for all n

Socks before shoes!

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Enough about propositions.

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Let's talk about axioms.

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A proposition that is assumed to be TRUE.

E.g., There is a straight line segment between any two points.

If $a = b$ and $b = c$, then $a = c$.

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Proof of $2+2=4$ requires more than 20,000 steps!

Foundational, but also primitive

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* This course : Use facts from high-school mathematics as axioms.

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* Modern mathematics is based on Zermelo - Frankel axioms (ZFC)
* This course : Use facts from high-school mathematics as axioms.
* Tutorial / Exam problems cannot be assumed as axioms 

GUIDING PRINCIPLES FOR AXIOMS

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No axiomatic system can be simultaneously consistent and complete.

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Maybe Goldbach's Conjecture is TRUE but cannot be proved?

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Next time!