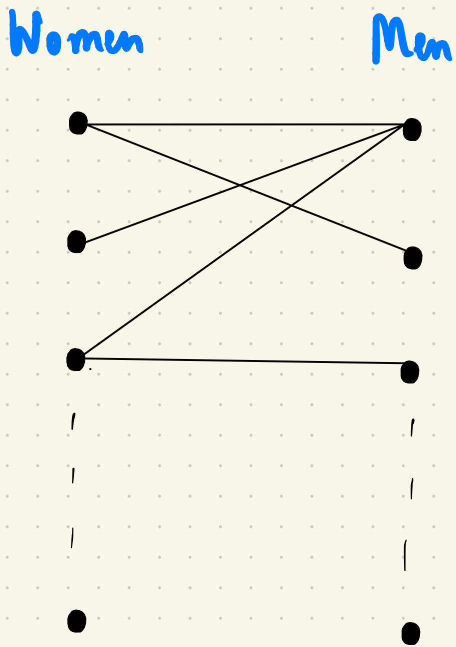
LECTURE 18

PERFECT MATCHINGS

FEB 09, 2024 | ROHIT VAISH

GRAPH THEORY

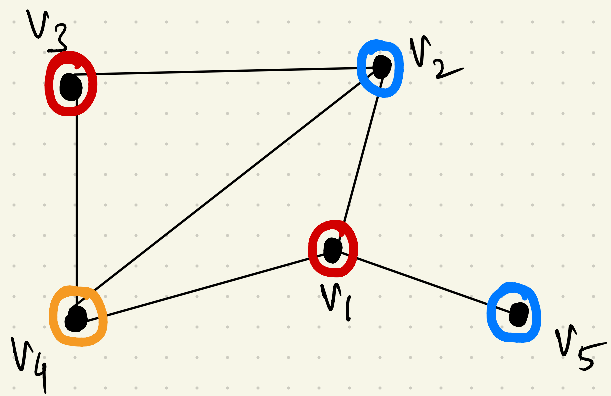
On average, who has more opposite-gender partners: men or women?



$$\frac{\text{Avg. deg of men}}{\text{Avg. deg. of women}} = \frac{\text{No. of women}}{\text{No. of men}}$$

On average, men have more opposite-gender partners than women if # women > # men.

GRAPH COLORING



Properly color a given graph with as few colors as possible.

GRAPH COLORING

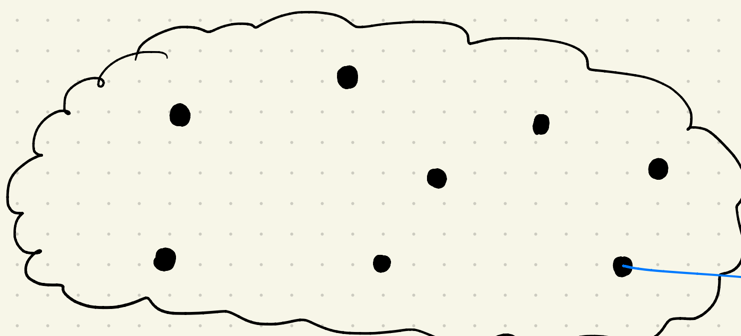
proper ≤ k colors

Given a coloring, checking if it is a valid solution : Easy!

Given graph G and k, determine if G admits a proper k-coloring: Hard!

NP- complete

A class of "equally hard" problems



Graph coloring

BASIC GRAPH COLORING ALGORITHM

1. Order the vertices $v_1, v_2, \ldots, v_n$

2. Order the colors $c_1, c_2, \ldots$

3. For $i = 1, 2, \ldots, n$

   Assign the least-index legal color to v_i

BASIC GRAPH COLORING ALGORITHM

Theorem: If every vertex in G has degree at most d, then Basic Algo. uses at most $d+1$ colors.

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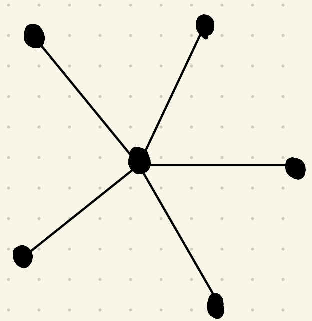

"Star" graph

degree $= n-1$

$\chi(G) = 2$

Basic algo will also use two colors under any ordering of vertices.

BASIC GRAPH COLORING ALGORITHM

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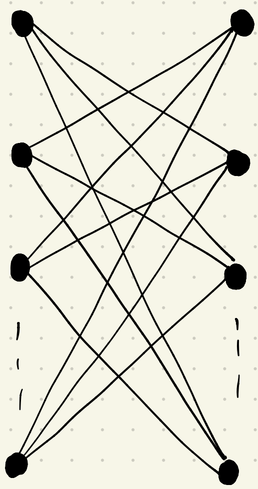

all edges except the horizontal ones

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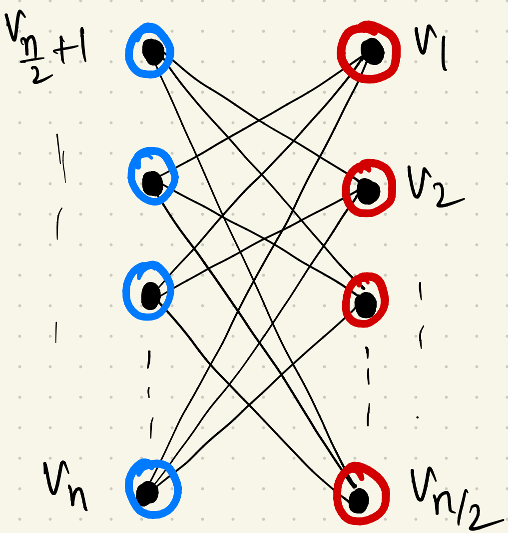
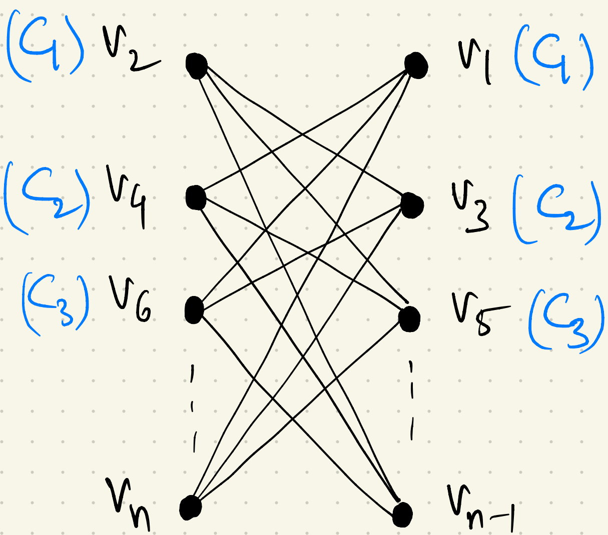


A good ordering $\rightarrow$ 2 colors

all edges except the horizontal ones

BASIC GRAPH COLORING ALGORITHM

Theorem: If every vertex in G has degree at most d, then Basic Algo. uses at most d+1 colors.

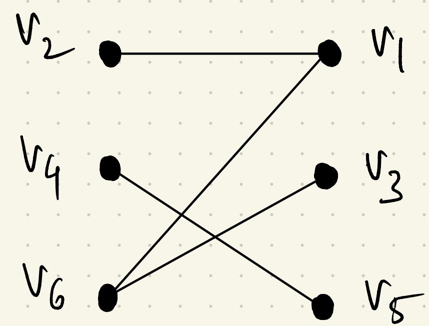


A bad ordering $\longrightarrow$ $n/2$ colors

Ordering matters!

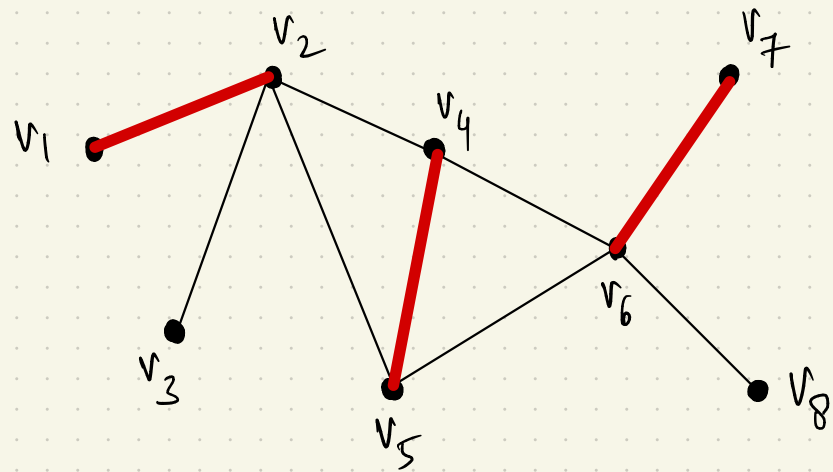
all edges except the horizontal ones

A graph $G = (V, E)$ is bipartite if V can be partitioned into two sets V_L and V_R such that all edges in E connect a vertex in V_L with a vertex in V_R.



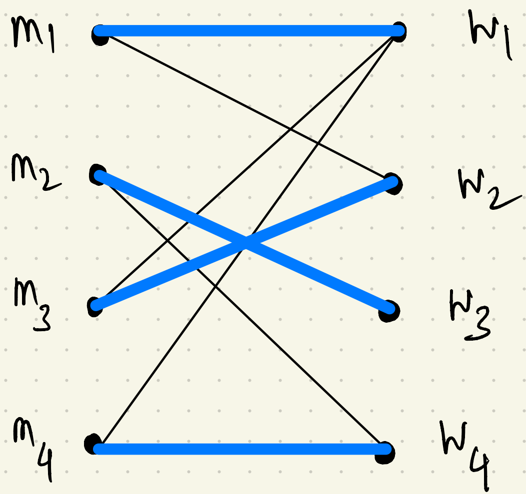
MATCHING

Given a graph $G = (V, E)$ that is not necessarily bipartite a matching is a subgraph of G where every vertex has degree one.



MATCHING

A matching is perfect if it has size $|V|/2$.



MATCHING

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When does a bipartite graph have a perfect matching?

BIPARTITE PERFECT MATCHING

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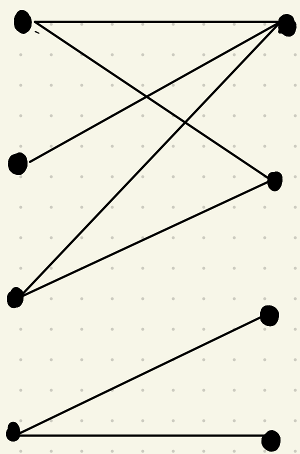
Bipartite Perfect Matching

When does a bipartite graph obviously NOT have a perfect matching?

BIPARTITE PERFECT MATCHING

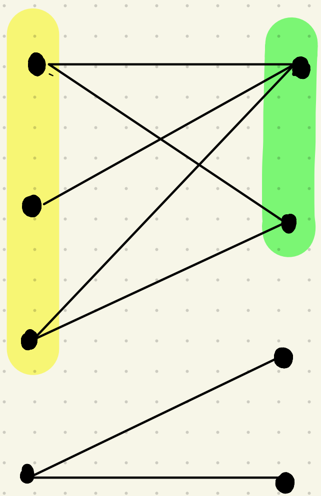
obviously NOT

When does a bipartite graph have a perfect matching?



BIPARTITE PERFECT MATCHING

When does a bipartite graph obviously NOT have a perfect matching?



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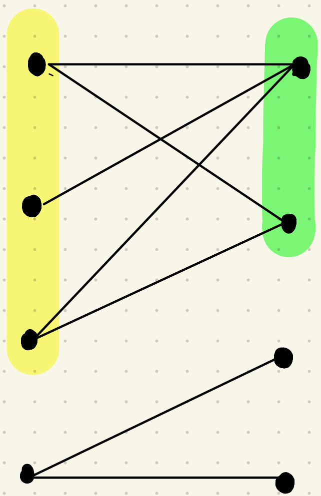

No perfect matching because the yellow vertices do not have "sufficiently many" neighbors

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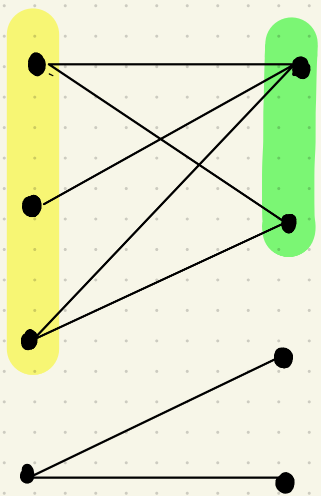

A bare minimum condition

Every subset of left vertices $S \subseteq V_L$ has at least $|S|$ neighbors on the right.

BIPARTITE PERFECT MATCHING

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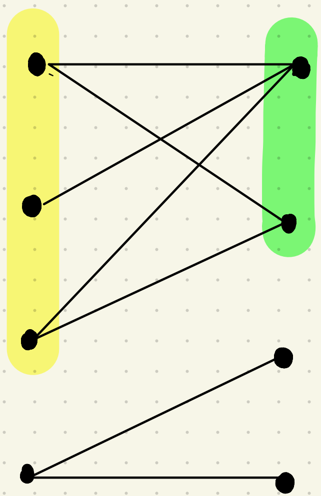

A ~~bare minimum~~ necessary condition

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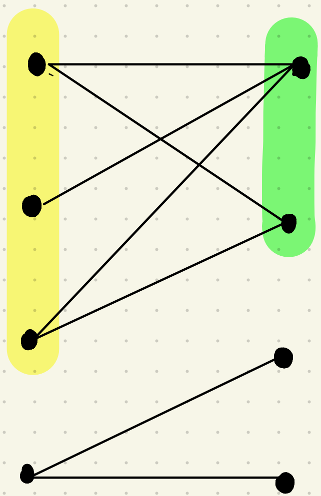

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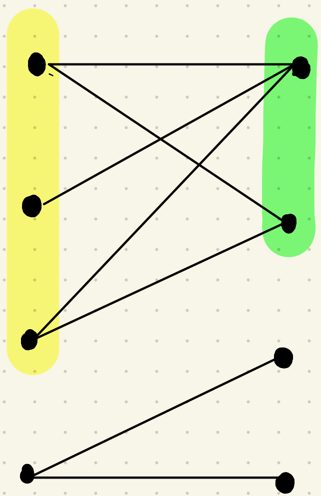

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Is it sufficient ?