COL 351 : Analysis & Design of Algorithms

LECTURE 39

NP-COMPLETENESS II : THE CLASS NP

NOV 06, 2024 | ROHIT VAISH

An Algorithmic Mystery

Minimum Spanning Tree

Traveling Salisman Problem

An Algorithmic Mystery

Minimum Spanning Tree

input: an undirected graph $G = (V, E)$
with real-valued edge cost c_e
for each edge e

Traveling Salesman Problem

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algorithms: Prim's, Kruskal's, $O(m \log n)$

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algorithms: 

Easy : Problems with poly-time algorithms

Hard : Problems without any polynomial-time algorithm

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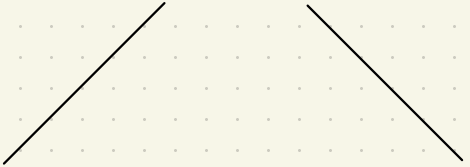

because one doesn't exist?

because we have failed to find one?

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Conjecture [Edmonds '67]

TSP doesn't have a poly-time algo.

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Relative hardness and reductions!

"I can't find an efficient algorithm, but neither can all these famous people."

Source : Garey and Johnson (1979)

Building a case with reductions

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reduction

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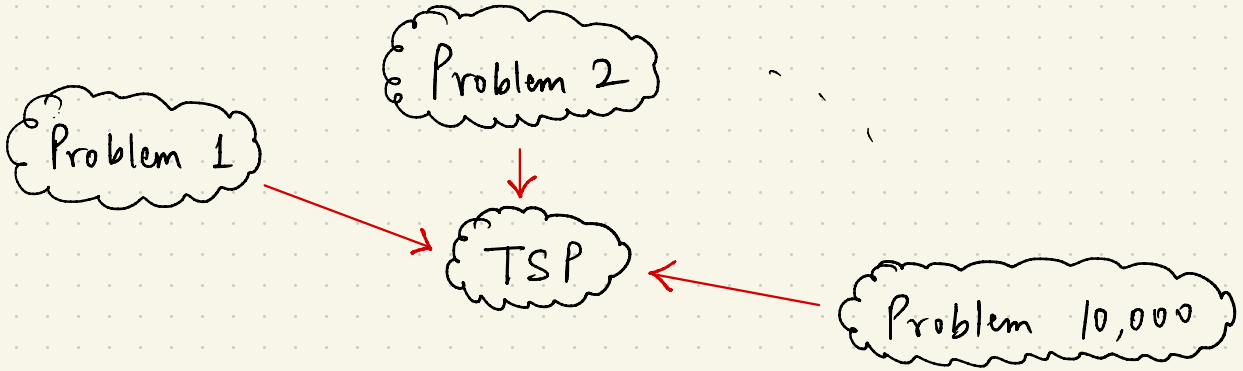

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reduction

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If you are willing to believe that problems in C are "hard", then you should also believe that TSP is hard.

Bigger the set C $\Rightarrow$ stronger evidence for TSP's intractability

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(Turing, 1936) HALTING problem is undecidable.

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Too ambitious ! Includes the HALTING problem.

"Given a program, determine whether it goes into an infinite loop or eventually halts."

(Turing, 1936) HALTING problem is undecidable.

Cannot be solved by a computer in any finite amount of time
(not even exponential, or factorial, ...)

ON COMPUTABLE NUMBERS, WITH AN APPLICATION TO THE ENTSCHEIDUNGSPROBLEM

By A. M. TURING.

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* Birth of computer science
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By A. M. TURING.

* Birth of computer science

* Formulated Turing machines — formal model of general-purpose computers

* Showed undecidability of HALTING problem.

Thus, computer scientists have been aware of computers' limitations literally from Day 1.

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Some problems (e.g. HALTING) can't reduce to TSP.

because TSP is solvable in finite time via exhaustive search

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The class NP

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e.g., is there a TSP tour of cost ≤ 100?

no need to actually produce a tour if one exists

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e.g., return any TSP tour of cost ≤ 100, if one exists.

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e.g., return the min cost TSP tour

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NOTE: every optimization has a natural search version.

e.g., TSP tour with cost ≤ 100,

Knapsack solution with value ≥ 100, etc.

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Search reduces to optimization : easy! (Exercise)

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Search reduces to optimization: easy! (Exercise)

Optimization reduces to search: via binary search (Exercise)

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NP: Commonly defined for decision problems

We will define it for search problems

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↳ technically, functional NP (FNP)

THE CLASS NP

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Recall our goal : To define the class of problems solvable via exhaustive search.

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What are the minimal ingredients necessary to solve a problem via naive exhaustive search?

THE CLASS NP

Efficient recognition of alleged solutions.

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Efficient recognition of alleged solutions.

e.g.,

5	3			7				
6			1	9	5			
	9	8					6	
8				6				3
4			8		3			1
7				2				6
	6					2	8	
			4	1	9			5
				8			7	9

5	3	4	6	7	8	9	1	2
6	7	2	1	9	5	3	4	8
1	9	8	3	4	2	5	6	7
8	5	9	7	6	1	4	2	3
4	2	6	8	5	3	7	9	1
7	1	3	9	2	4	8	5	6
9	6	1	5	3	7	2	8	4
2	8	7	4	1	9	6	3	5
3	4	5	2	8	6	1	7	9

THE CLASS NP

Efficient recognition of alleged solutions.

e.g., checking whether $v_5 \to v_{17} \to v_{105} \to v_2 \to \cdots \to v_5$

is a valid TSP tour of cost ≤ 100.

THE CLASS NP

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(2) For every instance and every candidate solution, the feasibility of the solution can be confirmed or denied in time polynomial in the input size.

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(1) For every instance of the problem, every candidate solution has description length (in bits) bounded above by a polynomial function of the input size.

e.g., any sequence of vertices is of polynomial size

(2) For every instance and every candidate solution, the feasibility of the solution can be confirmed or denied in time polynomial in the input size.

e.g., checking validity of a sequence of vertex possible in poly time

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A search problem belongs to the complexity class NP if it can be solved by a Nondeterministic Turing machine in Polynomial time.

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NP ≠ Not Polynomial

Examples of Problems in NP

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1. Search version of TSP

EXAMPLES OF PROBLEMS IN NP

(1) Search version of TSP

input: An undirected graph $G = (V, E)$ with edge costs $\{c_e\}_{e \in E}$.

output: Return a tour with cost $\leq t$ if one exists
report "no solution" otherwise

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(2) Search version of MST

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(2) Search version of MST

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EXAMPLES OF PROBLEMS IN NP

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* checking whether a given subgraph is a valid spanning tree and has cost $\leq t$ can be done in polynomial time

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(3) Search version of sequence alignment

EXAMPLES OF PROBLEMS IN NP

(3) Search version of sequence alignment

input: Two strings X and Y, gap penalty α_{gap}, mismatch penalty α_{xy}

output: Return an alignment with cost $\leq t$ if one exists
report "no solution" otherwise

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(3) Search version of sequence alignment

input : Two strings X and Y, gap penalty α_{gap}, mismatch penalty α_{xy}

output: Return an alignment with cost $\leq t$ if one exists
report "no solution" otherwise

Why in NP?

* Any alleged alignment can be described using $O(n+m)$ bits.
* checking whether a given pair of strings constitute a valid alignment and has cost $\leq t$ can be done in polynomial time