COL 351 : Analysis & Design of Algorithms

LECTURE 38

NP-COMPLETENESS I : INTRODUCTION

NOV 05, 2024 | ROHIT VAISH

STORY SO FAR

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Problem	Algorithm	Running time
Sorting	Merge Sort	$O(n \log n)$
Strongly Connected Components	Kosaraju	$O(n+m)$
Shortest Paths	Dijkstra	$O((m+n) \log n)$
Minimum Spanning Tree	Kruskal	$O(m \log n)$
Sequence Alignment	Needleman-Wunsch	$O(mn)$
All Pairs Shortest Paths	Floyd-Warshall	$O(n^3)$

An Algorithmic Mystery

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with real-valued edge cost c_e

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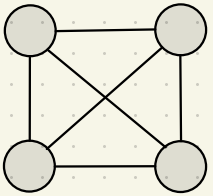

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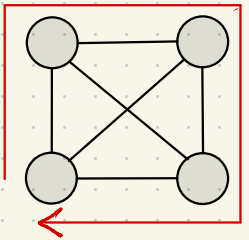

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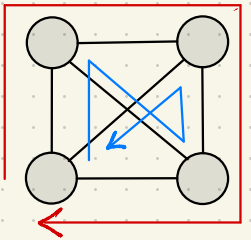

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Fact

No known fast (i.e., poly time) algorithm for TSP (despite decades of effort)

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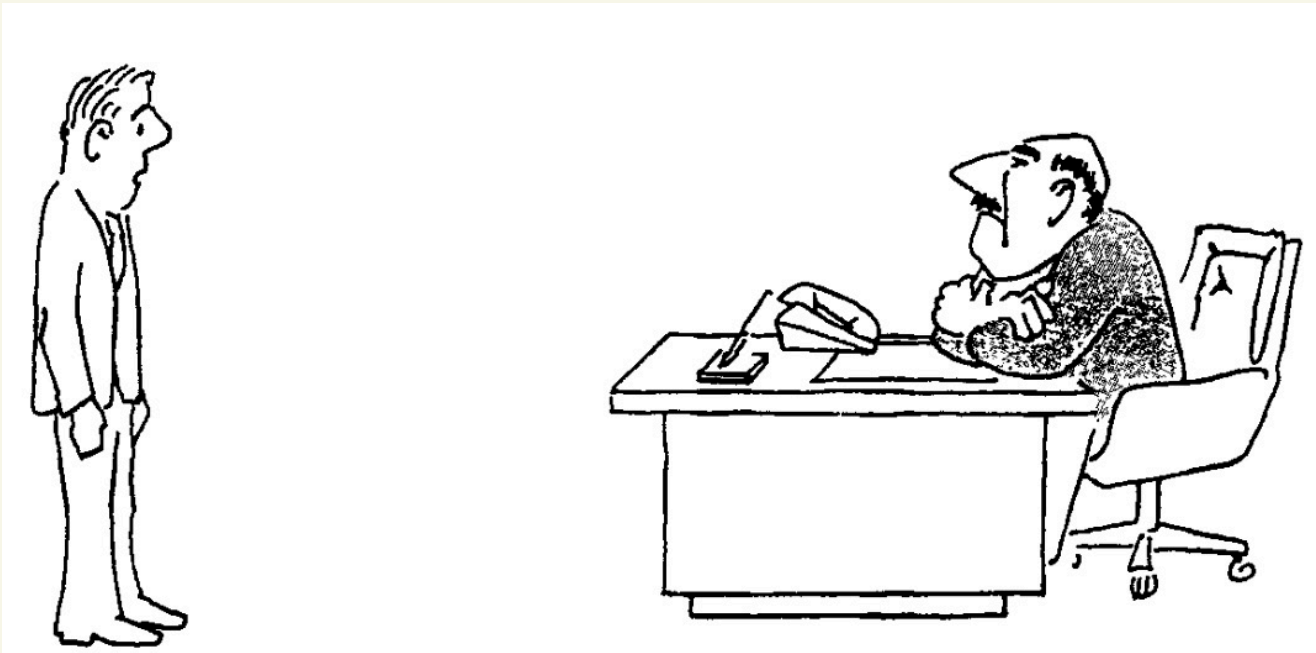
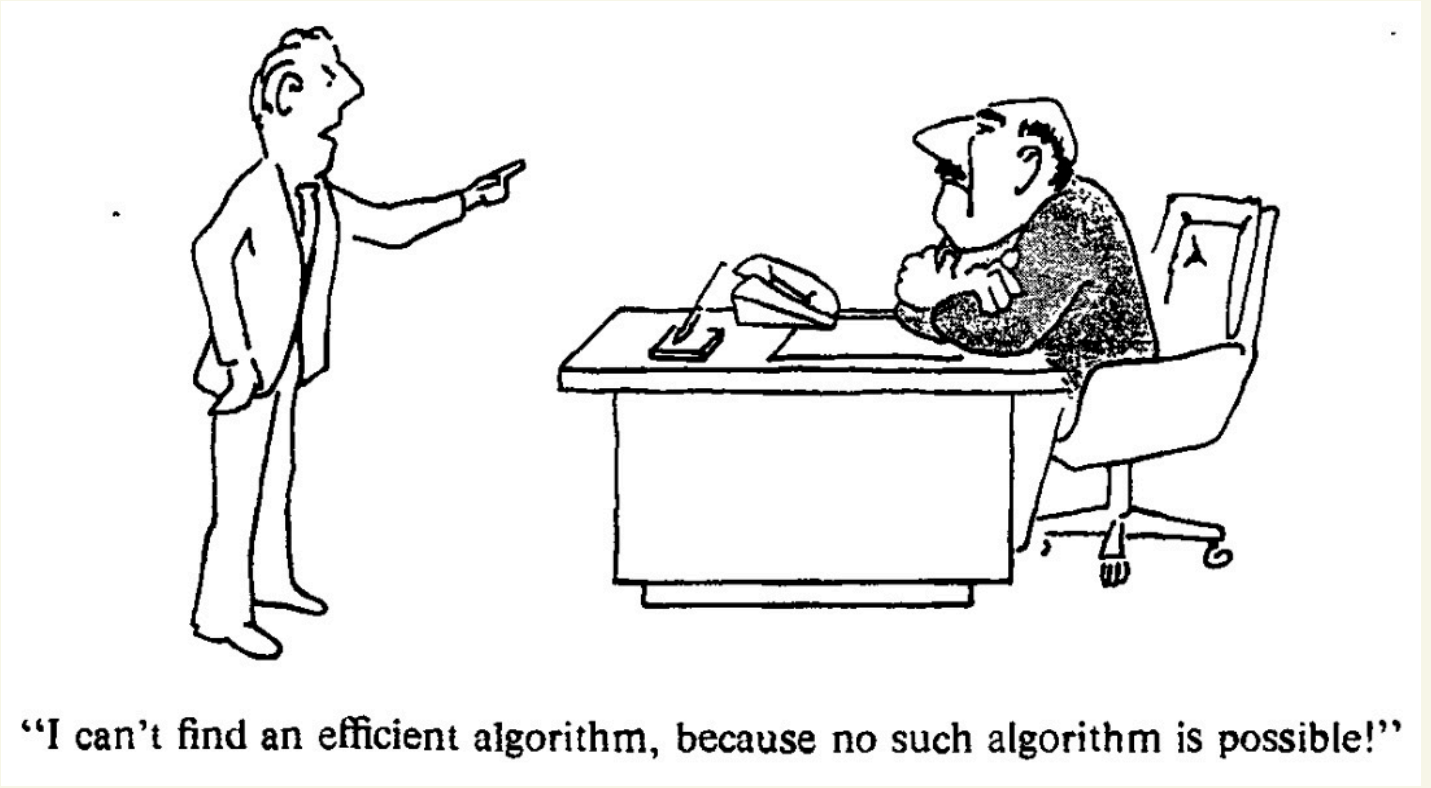


"I can't find an efficient algorithm, I guess I'm just too dumb."

Source : Garey and Johnson (1979)



"I can't find an efficient algorithm, because no such algorithm is possible!"

Source : Garey and Johnson (1979)

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Vol. 71B, No. 4, October–December 1967

Optimum Branchings*

Jack Edmonds

traveling saleman problem [cf. 4]. I conjecture that there is no good algorithm for the traveling saleman problem. My reasons are the same as for any mathematical conjecture: (1) It is a legitimate mathematical possibility, and (2) I do not know.

good = polynomial-time

So far, we haven't managed to prove Edmonds' conjecture.

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We have identified thousands of "TSP-like" problems:

* Graph coloring
* Integer Programming
* Ising Models
* Longest Paths
* Protein Folding
* Rubik's cube
* Number Partitioning
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RELATIVE INTRACTABILITY

"I can't find an efficient algorithm, but neither can all these famous people."

Source : Garey and Johnson (1979)

EASY PROBLEM

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A problem is polynomial-time solvable if there is a polynomial-time algorithm that correctly solves it on all inputs.

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Equivalently:

A problem is polynomial-time solvable if there is an algorithm for which the solvable input size (for a fixed time budget) scales multiplicatively with increasing computational power.

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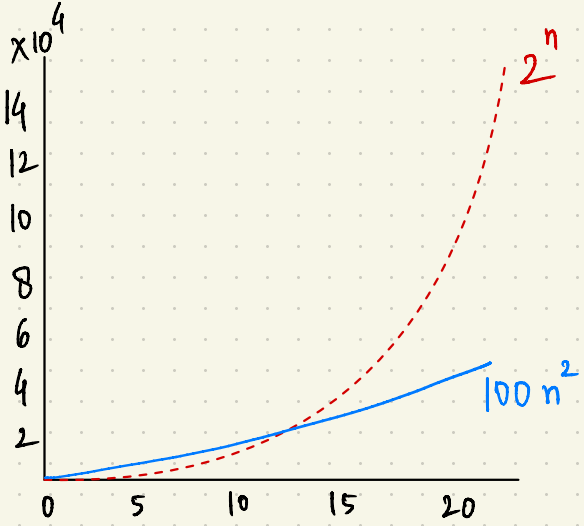

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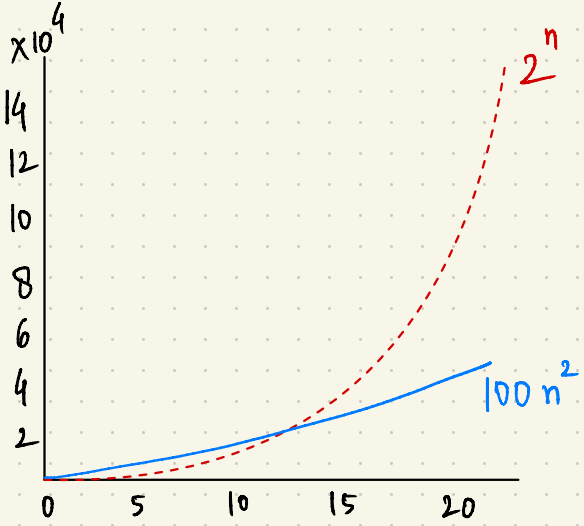

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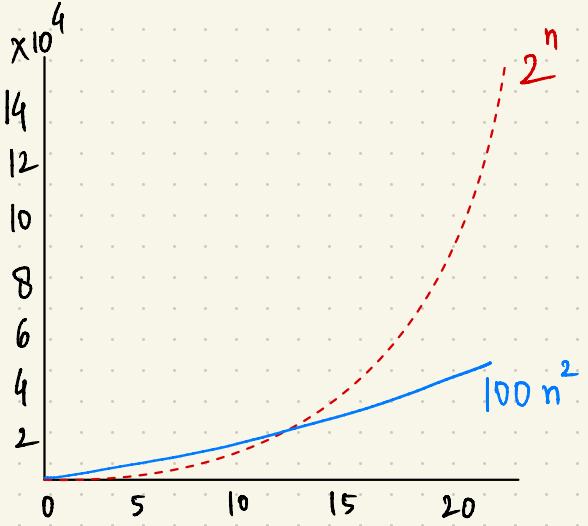

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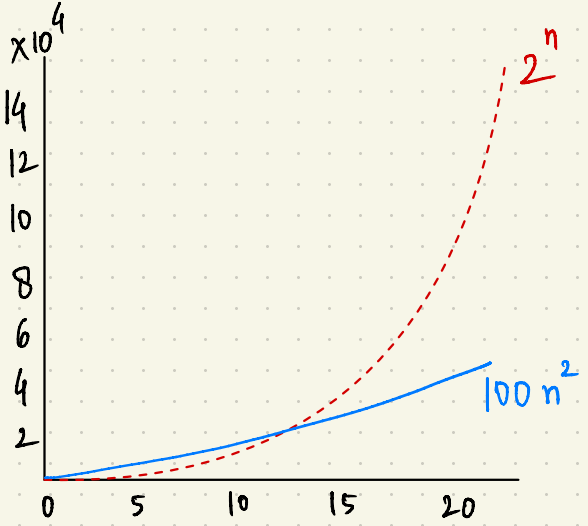

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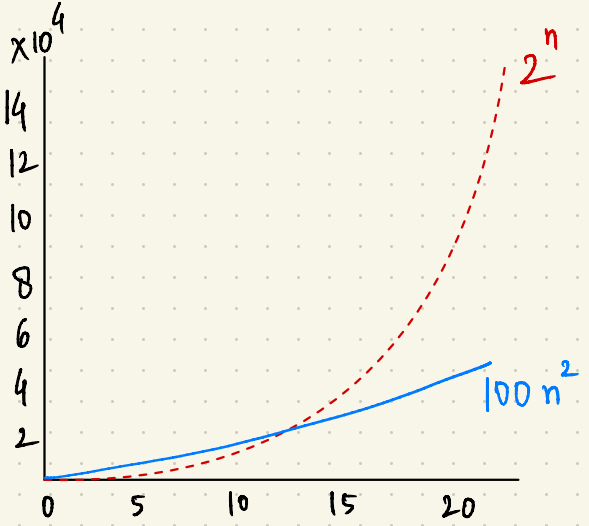

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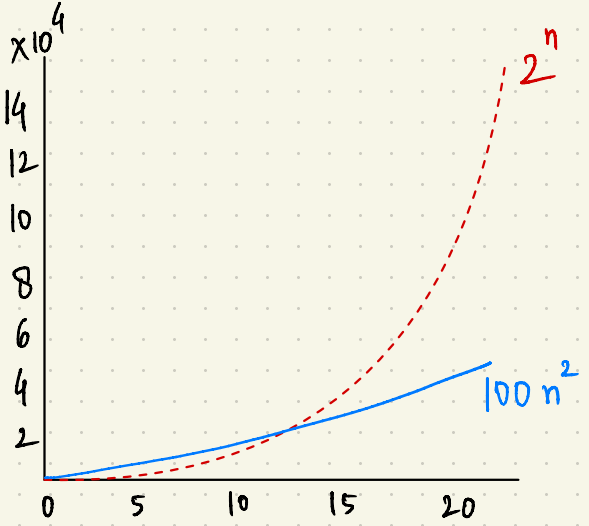

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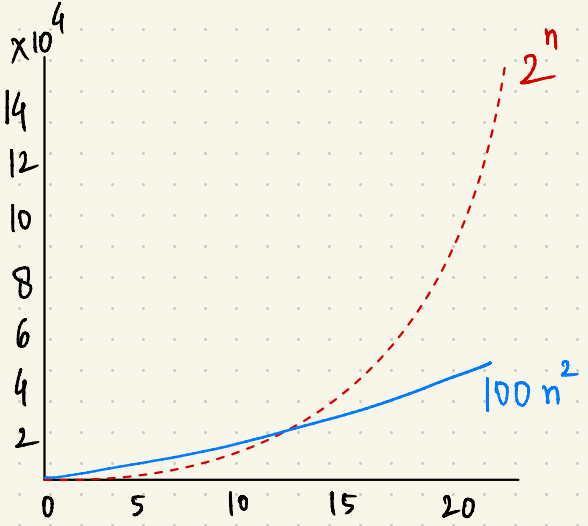

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because there isn't one ?

or because we haven't found one yet ?

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  - we'll never admit to this !

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Seems intimidating to prove this

or because we haven't found one yet ?

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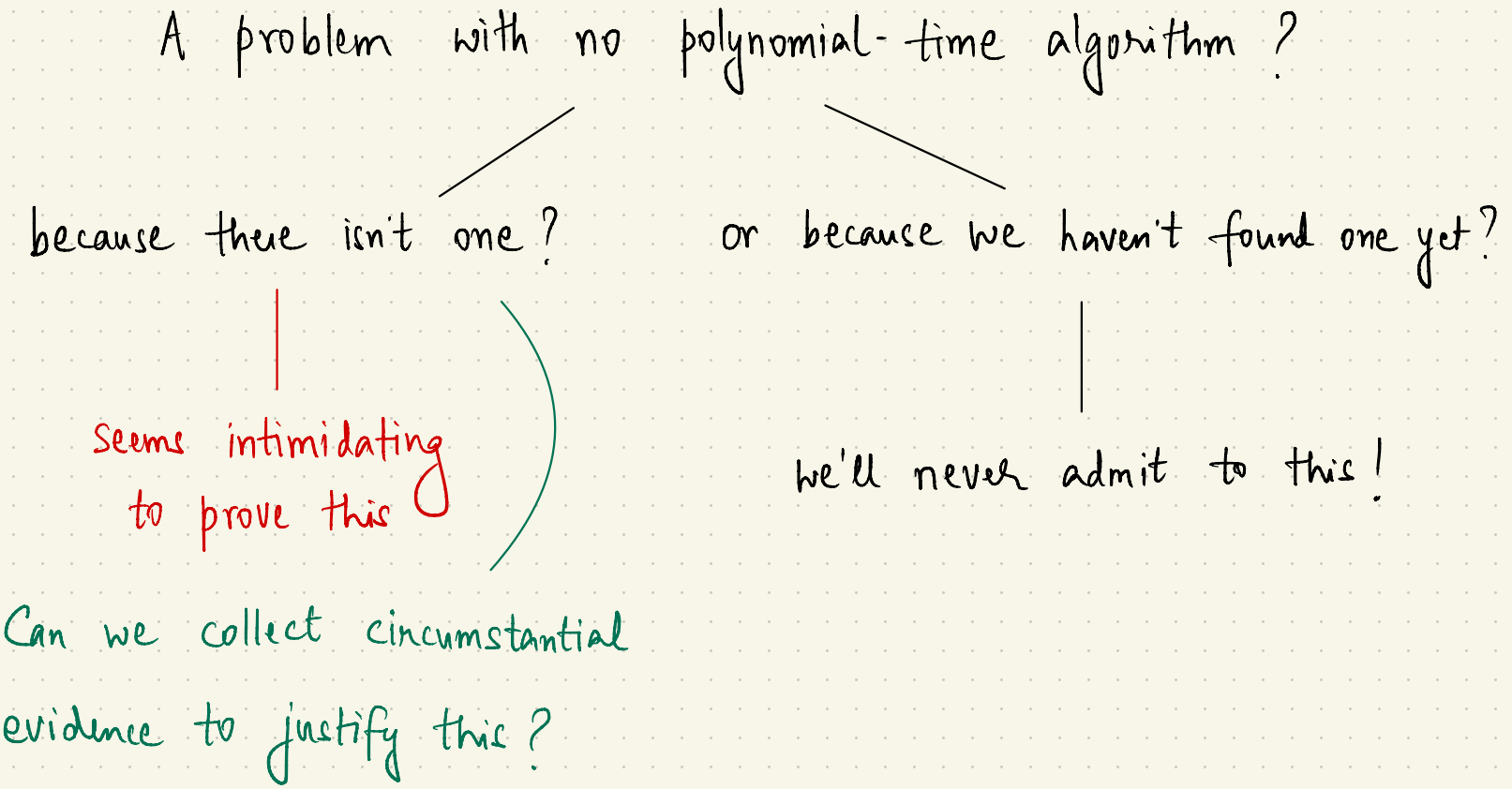

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A poly-time algorithm for Traveling Salesman Problem would solve thousands of problems that have resisted the efforts of tens of thousands of brilliant minds over many decades.

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NP-hard

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Bigger the set C $\Longrightarrow$ stronger evidence for TSP's intractability