COL 351 : ANALYSIS & DESIGN OF ALGORITHMS

LECTURE 37

MAX FLOW APPLICATIONS

NOV 01, 2024 | ROHIT VAISH

1. MINIMUM CUT PROBLEM

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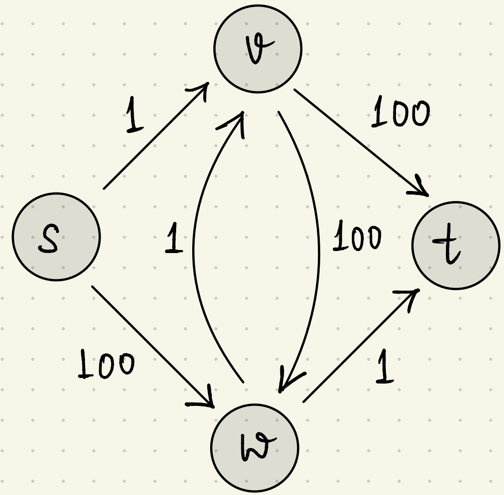

Goal : compute (s,t) - cut (A,B) minimizing $\sum_{e \in \delta^+(A)} u_e$.

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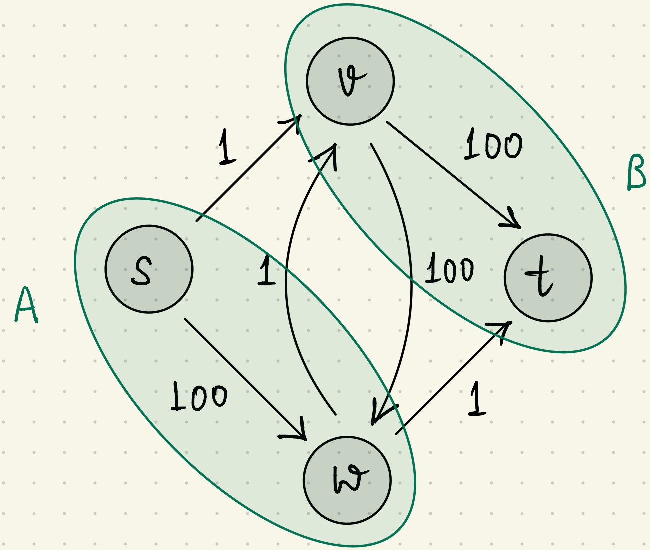

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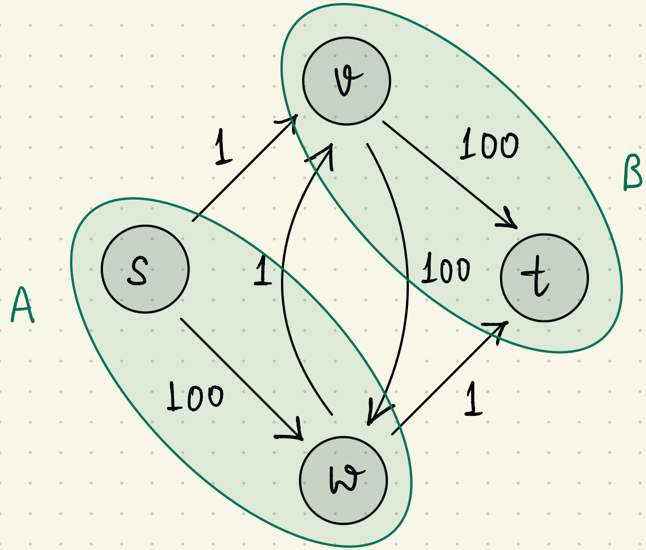
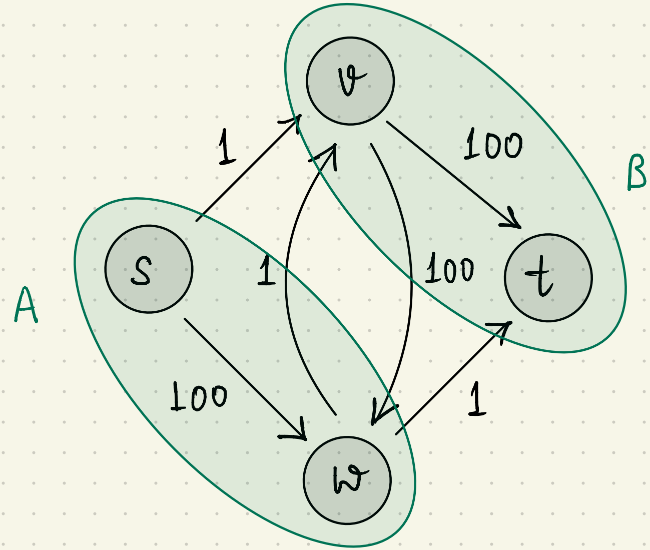


Goal: compute (s,t)-cut (A,B) minimizing $\sum_{e \in \delta^+(A)} u_e$.

Recall: (s,t)-minimum cut problem reduces to max flow in linear time.

MINIMUM CUT PROBLEM



Application in image segmentation (Additional reading)

2. BIPARTITE MATCHING

BIPARTITE MATCHING

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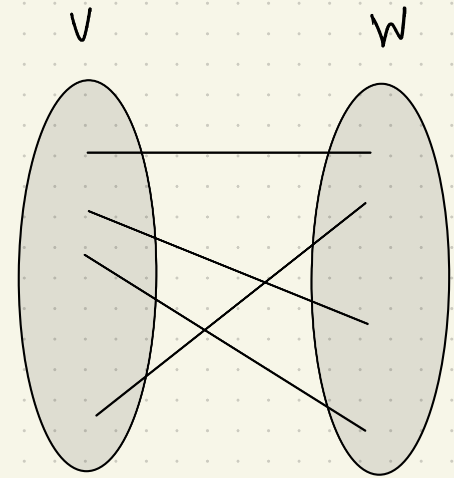
input : an undirected bipartite graph
$G = (V \cup W, E)$.

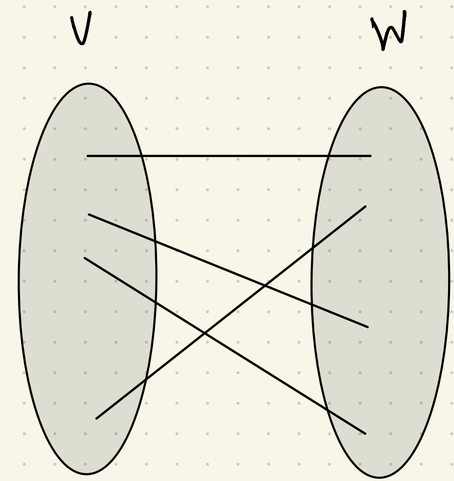


BIPARTITE MATCHING

input : an undirected bipartite graph
$G = (V \cup W, E)$.

goal : a maximum cardinality matching

subset of edges
with no shared endpoints



BIPARTITE MATCHING

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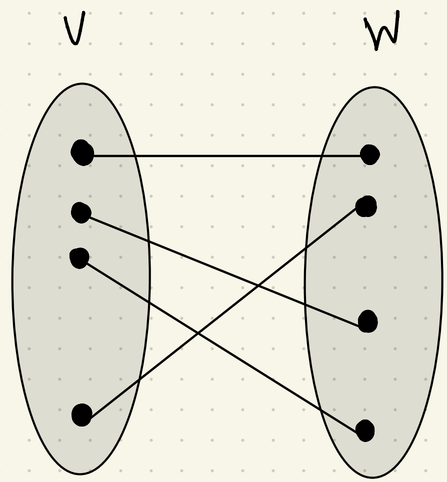

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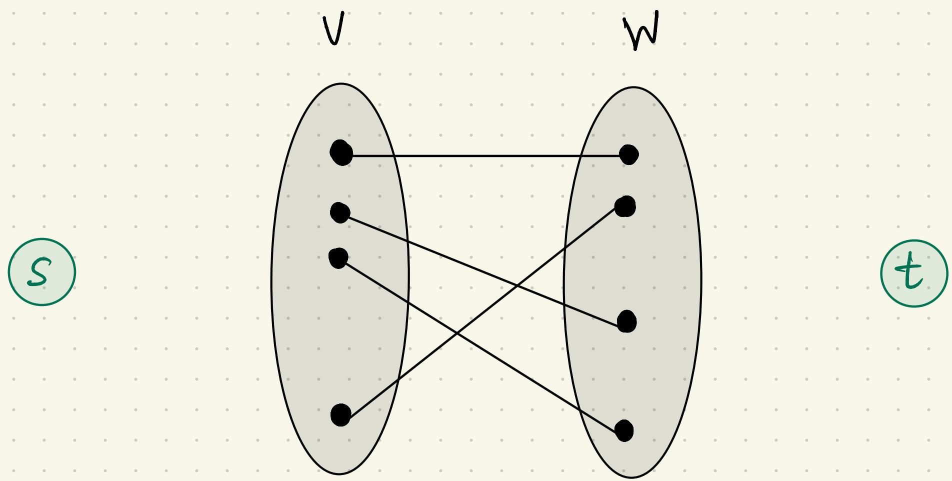

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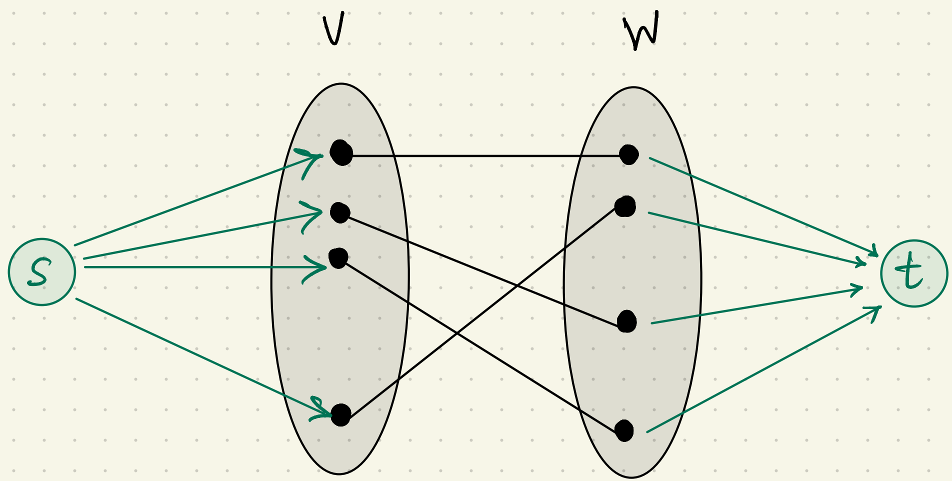

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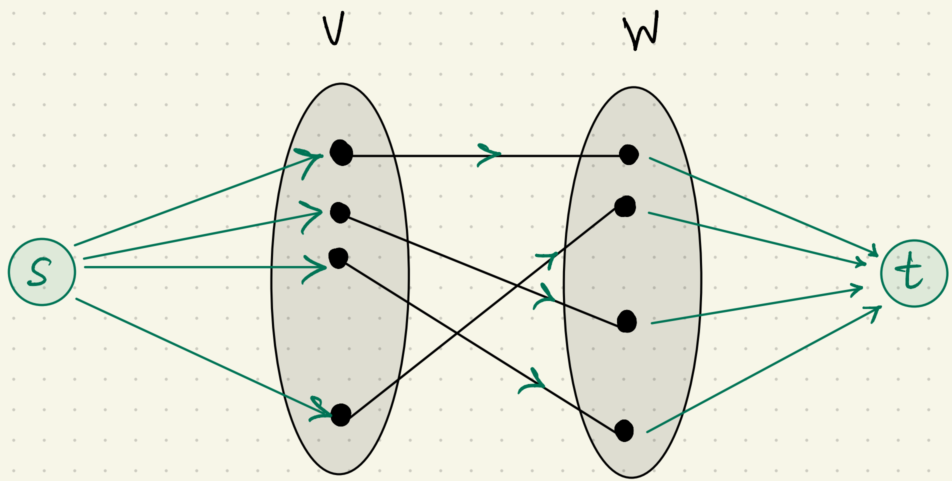

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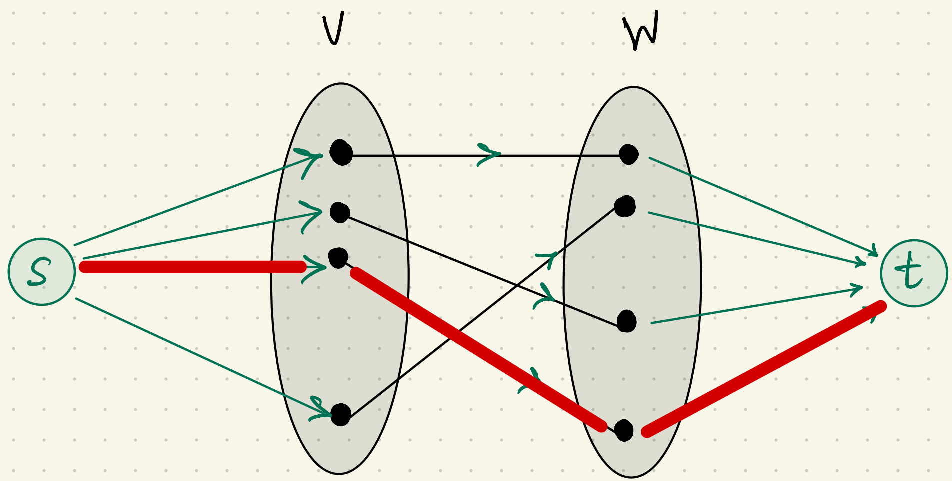

NOTE: every (s,t) flow path includes exactly one edge of given bipartite graph

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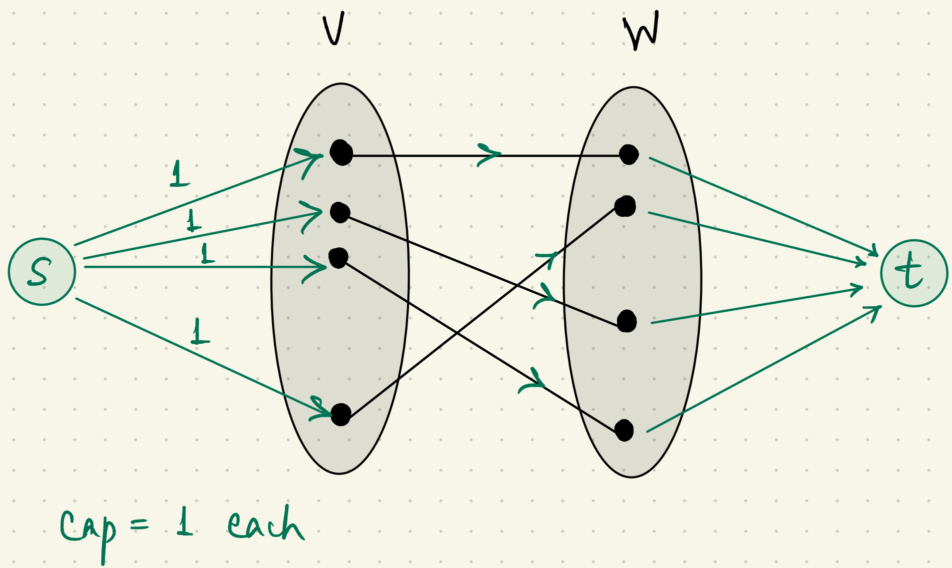

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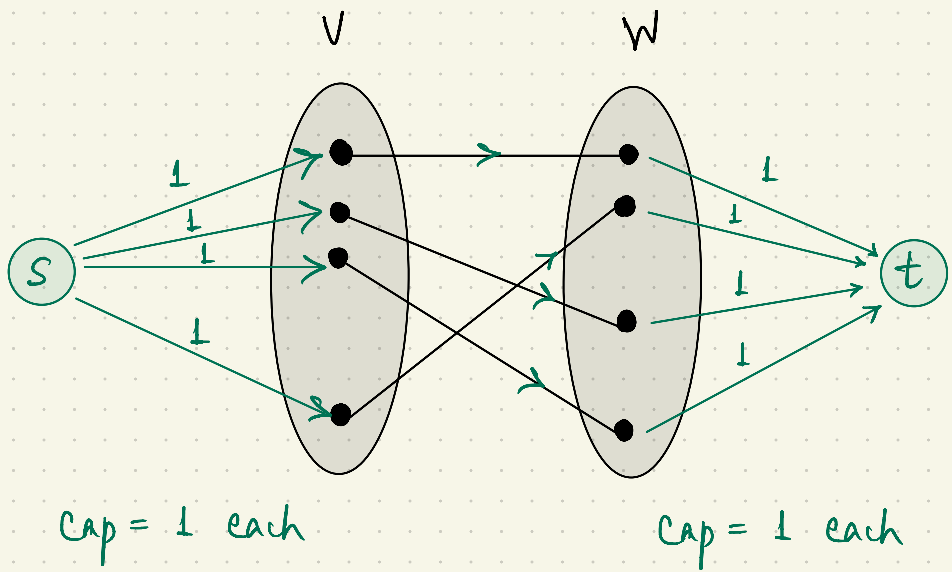

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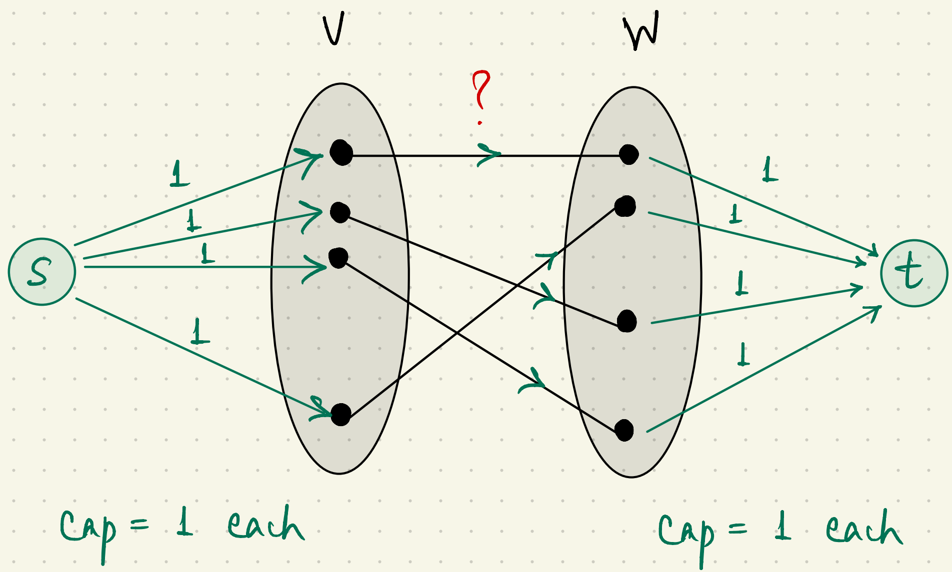

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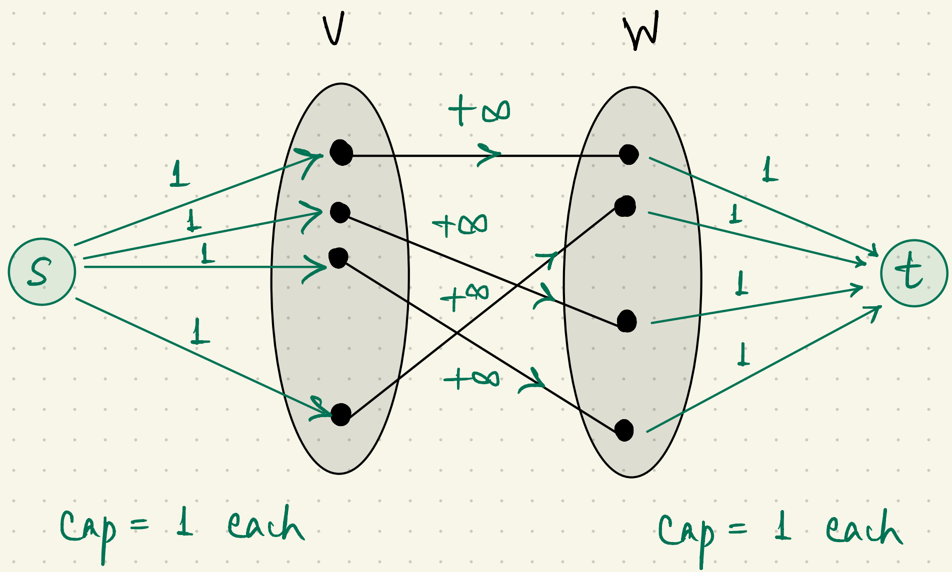

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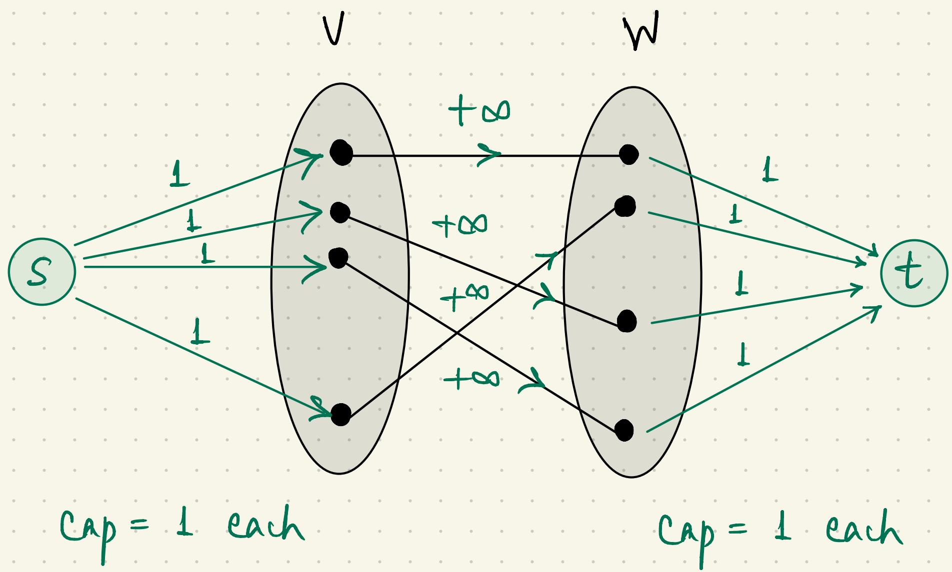

Bipartite matching $\longleftrightarrow$ Integral Flow (Check!)

Size of matching = Value of flow

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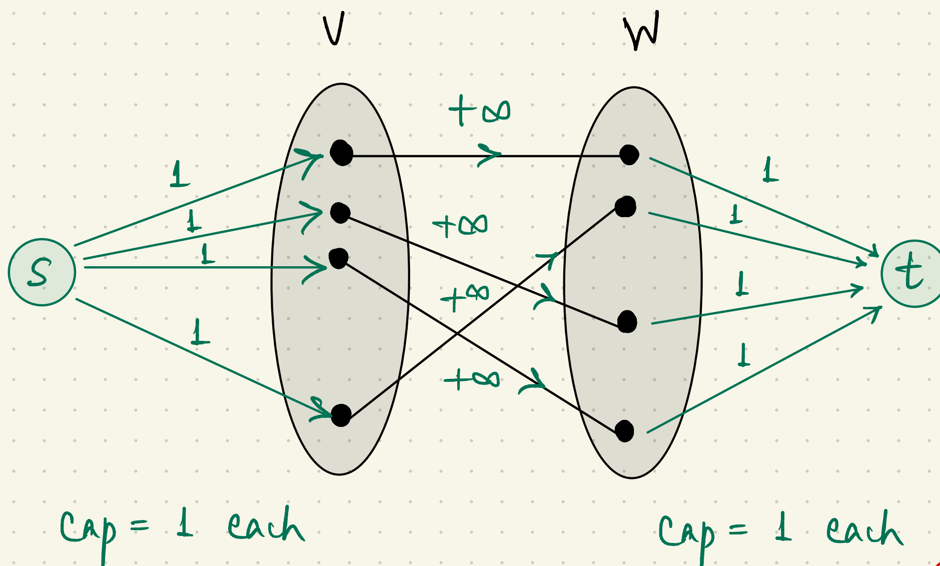

achieved by Ford-Fulkerson and Edmonds-Karp

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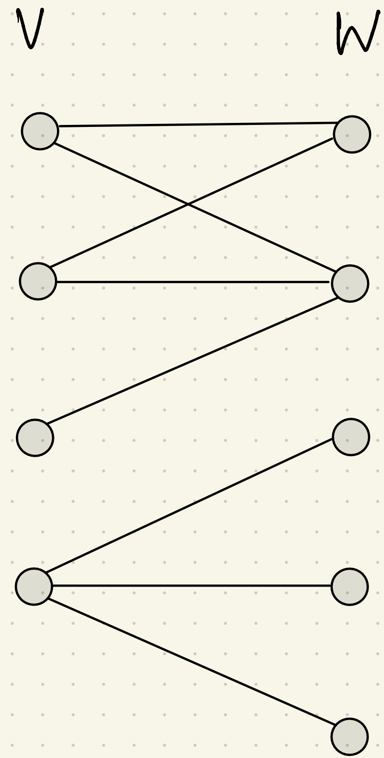
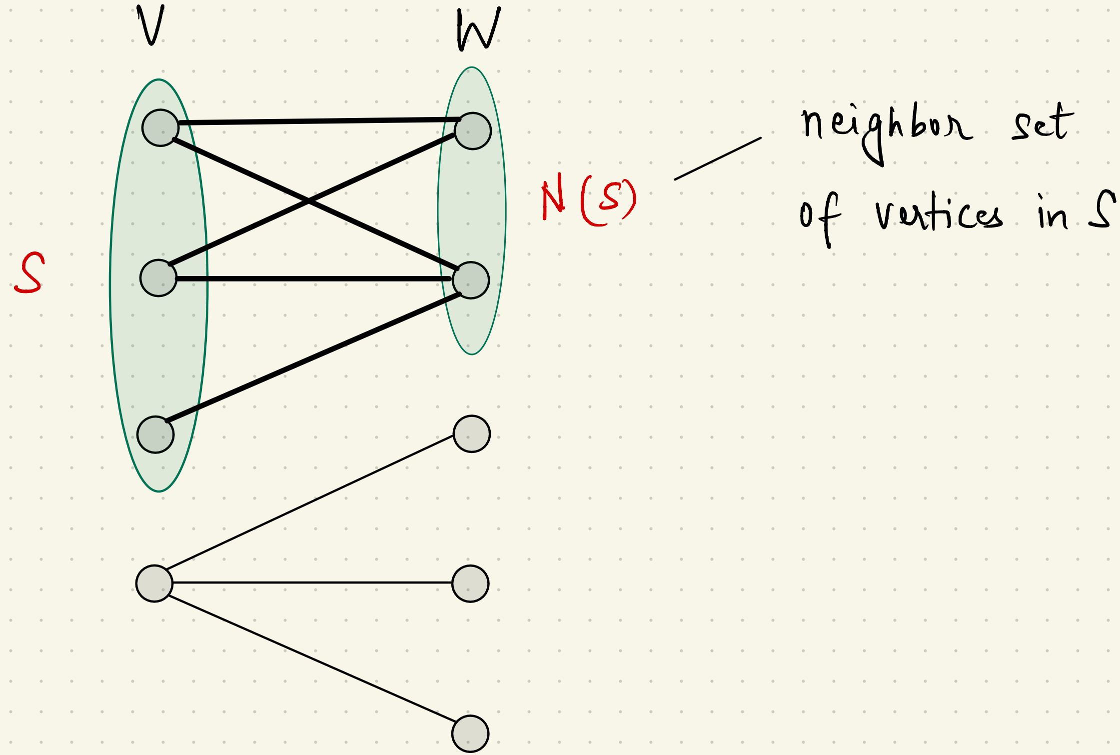
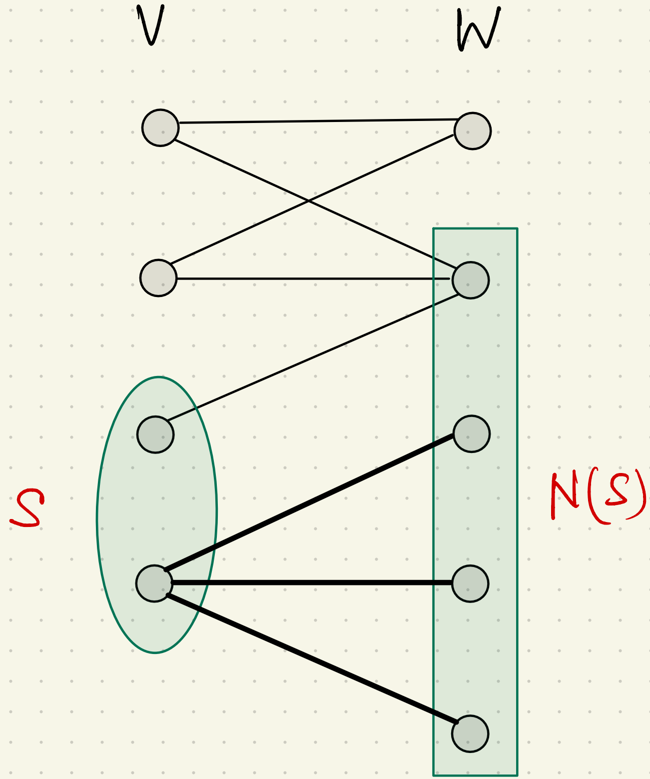
Size of matching = Value of flow

3. HALL'S THEOREM



assume WOLOG $|V| \le |W|$ (i.e., shorter side is on the left).


V
W
S
N(S)
neighbor set
of vertices in S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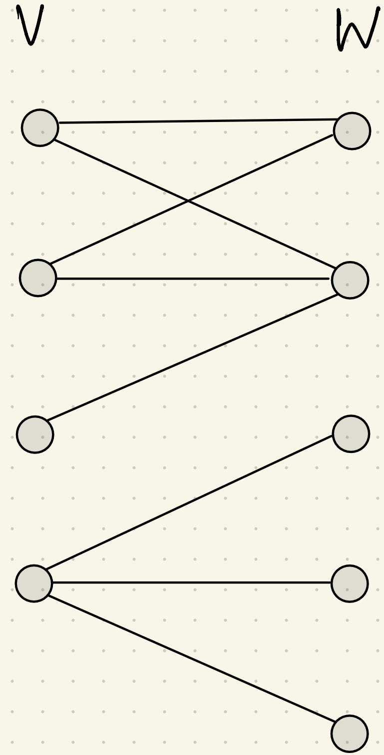
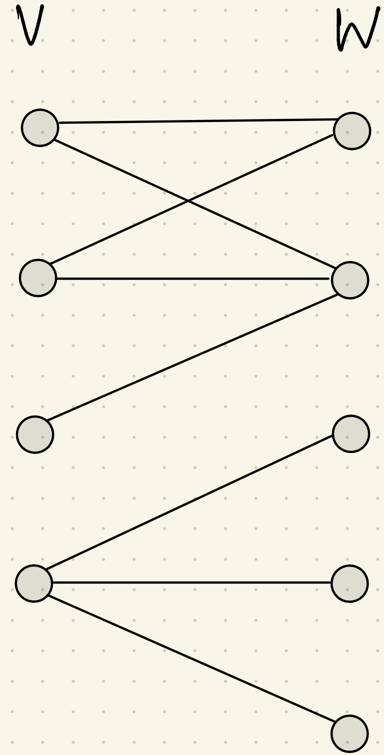
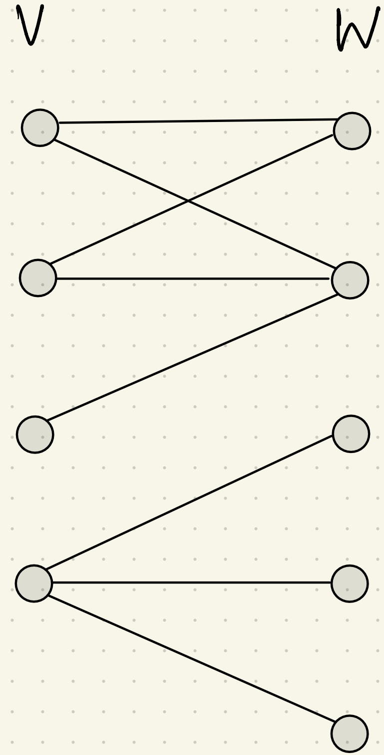
W
S
N(S)



Perfect matching : all vertices in V are matched.

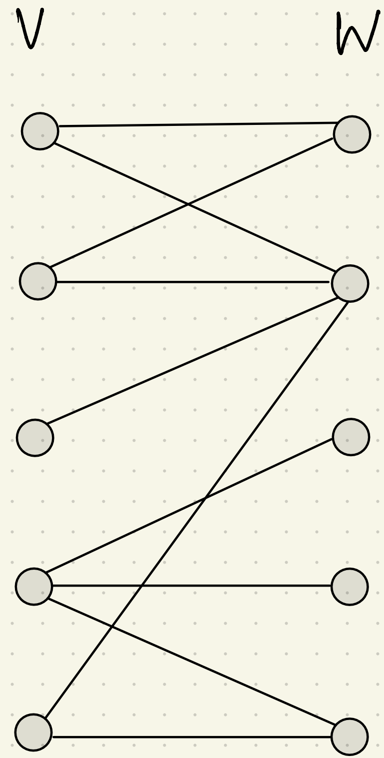


Does this graph have a perfect matching?



Imbalanced vertex sets

Does this graph have a perfect matching? NO!



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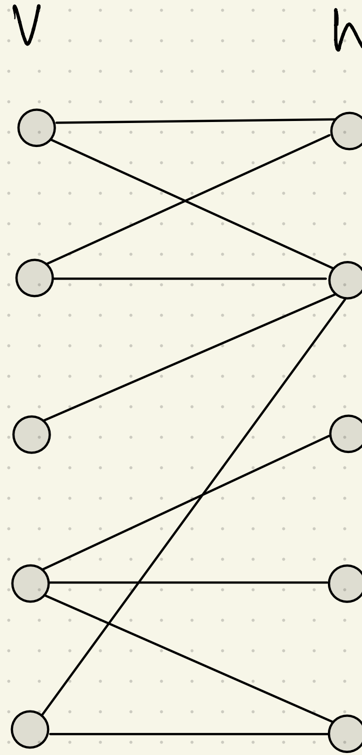
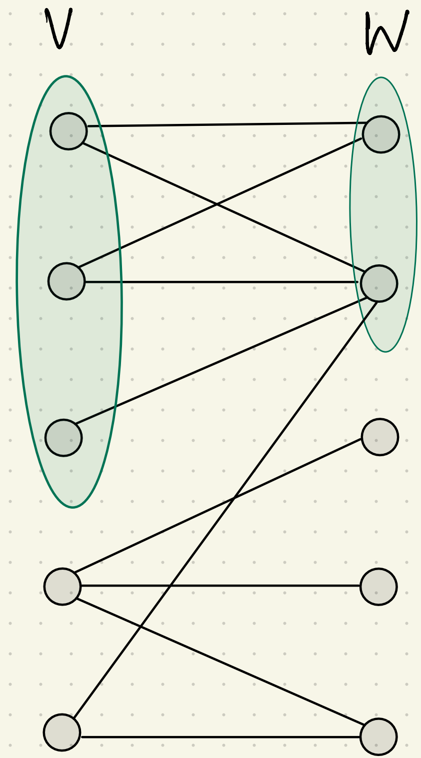


Does this graph have a perfect matching? No.



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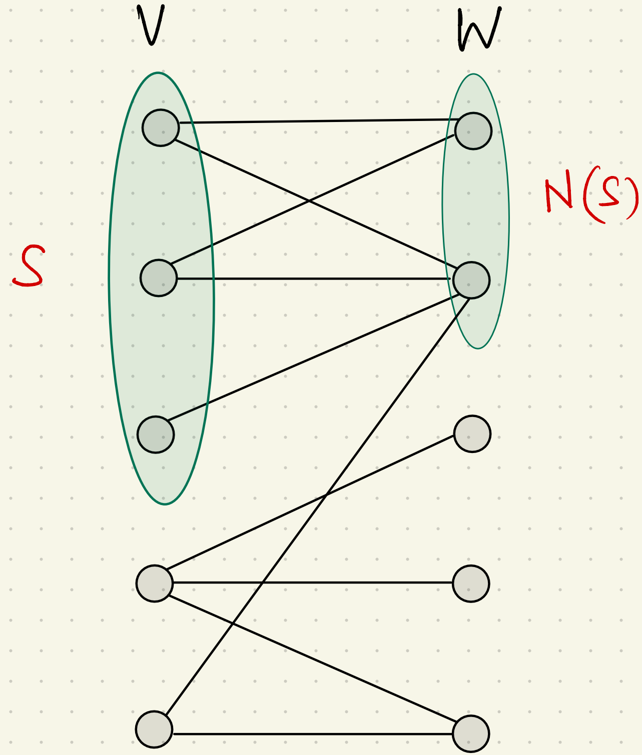

$|N(S)| < |S|$

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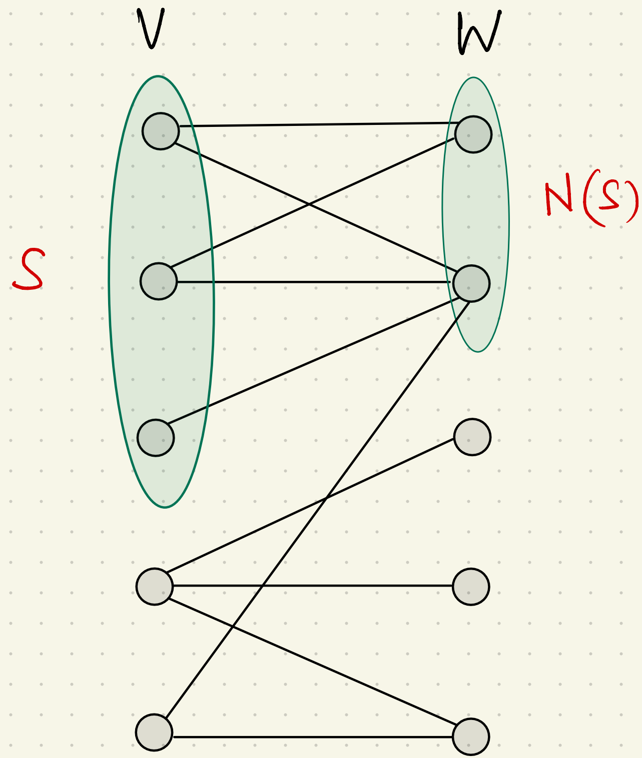

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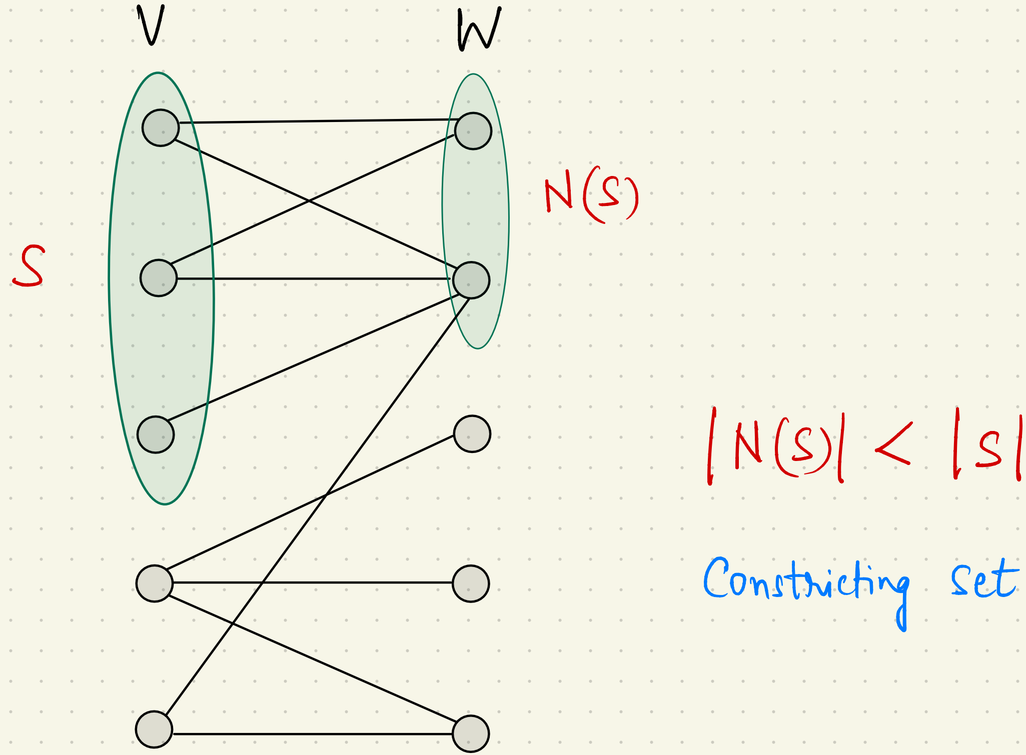
Constricting set

Does this graph have a perfect matching? No.



Hall's theorem: Constricting sets are the only obstacle to perfect matchings.

HALL'S THEOREM

A bipartite graph $G = (V \cup W, E)$ with $|V| \leq |W|$ has a left-perfect matching if and only if $\forall\, S \subseteq V$, $|N(S)| \geq |S|$.

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possibly exponential time?

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Proof : $(\Rightarrow)$

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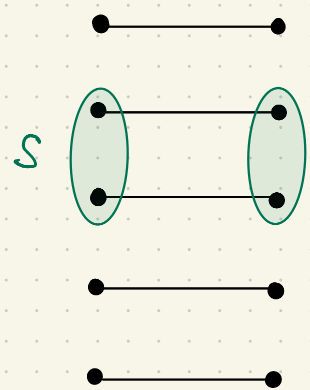
Proof : $(\Rightarrow)$ easy !



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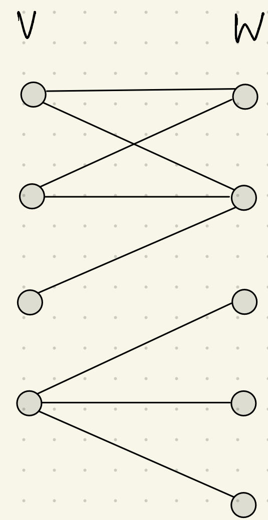

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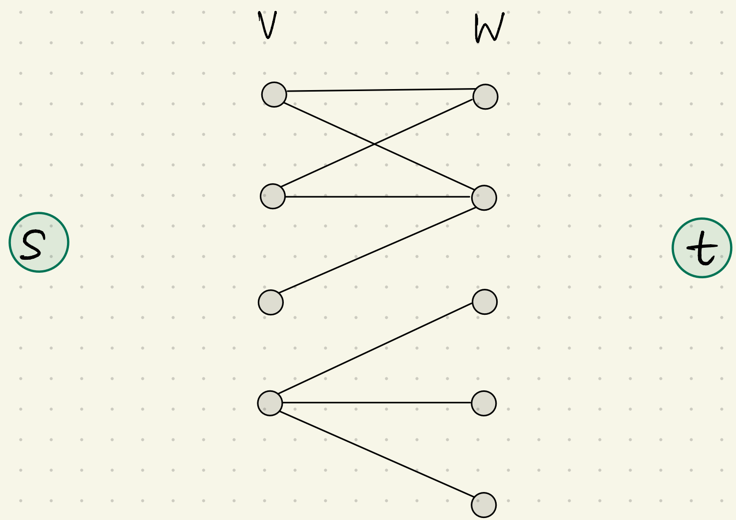

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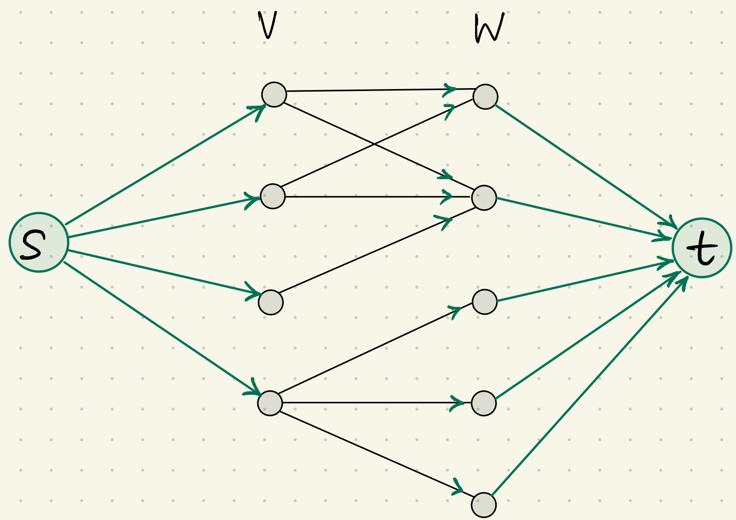

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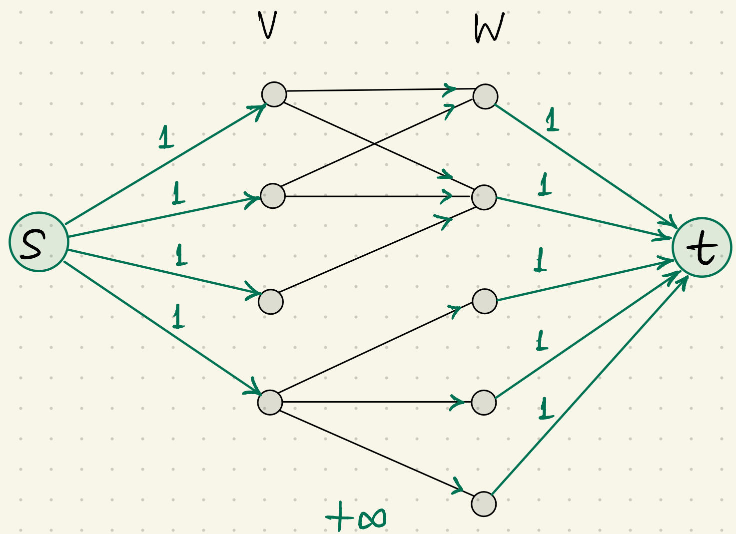

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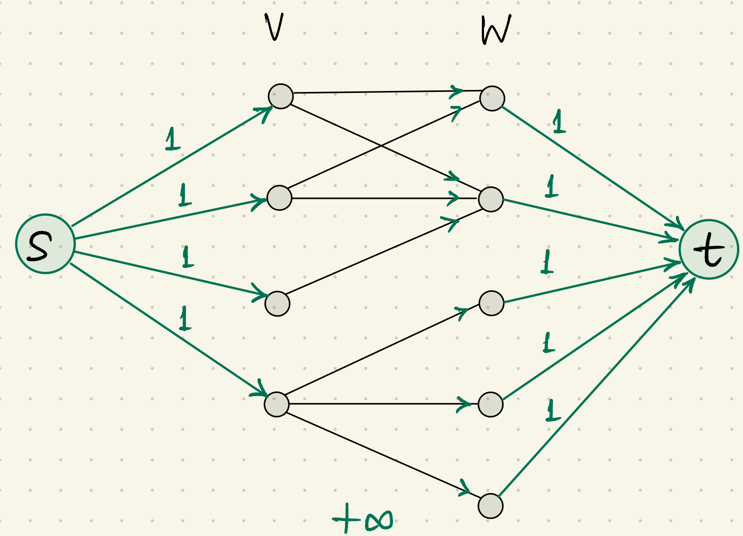

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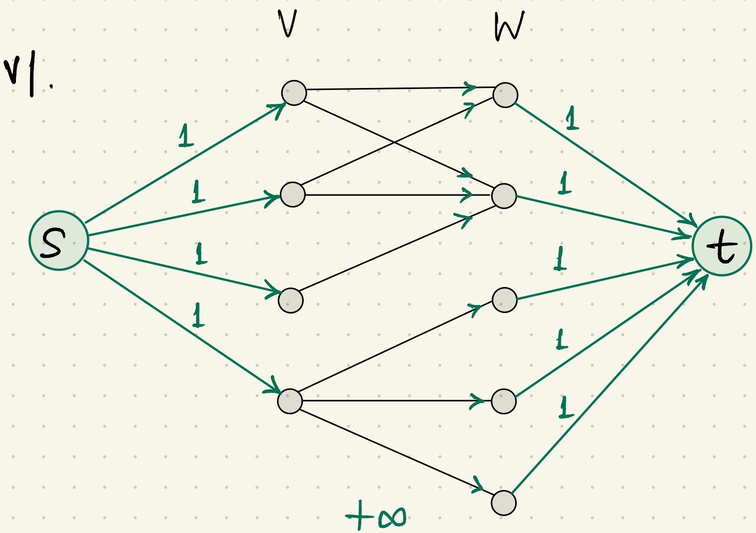
Proof : $(\Leftarrow)$ Reduction to max flow.

Claim : Every (s,t) cut has capacity $\geqslant |V|$.



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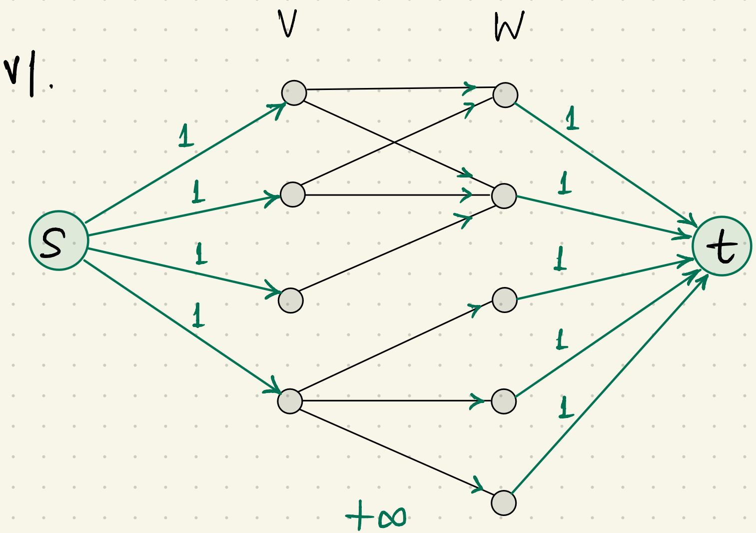
Theorem follows from claim since:



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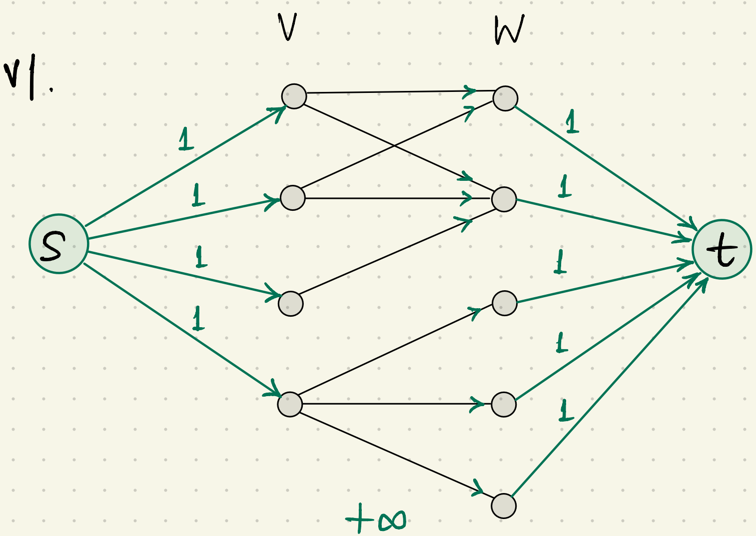

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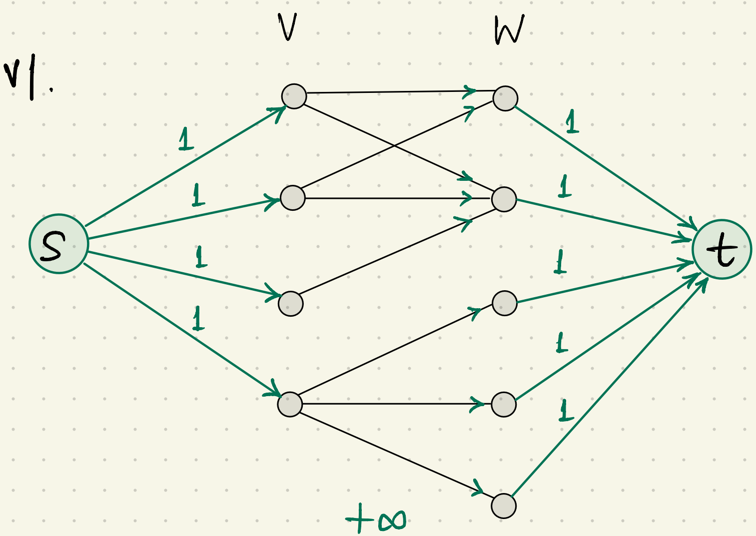

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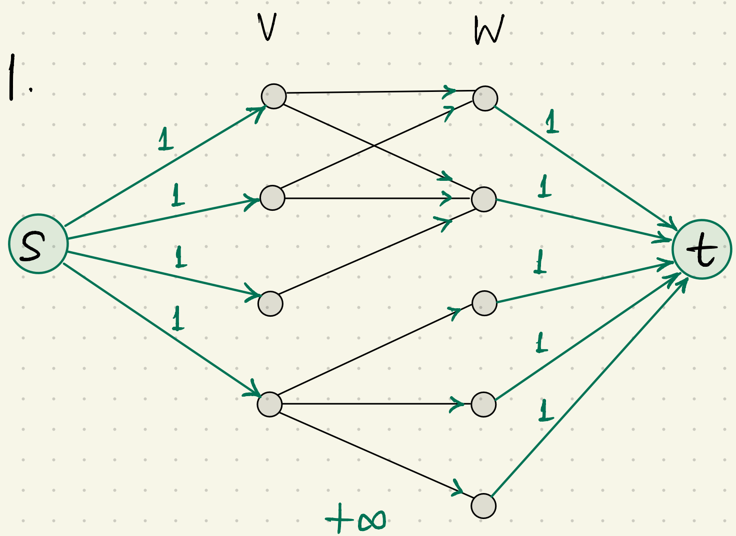

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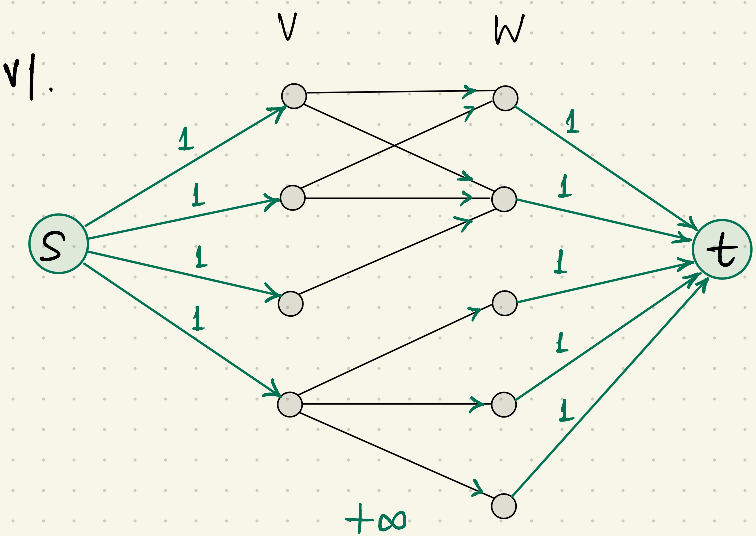

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min cut capacity $\geq |V|$

$\Rightarrow$ max flow value $\geq |V|$

$\Rightarrow$ flow out of s is saturated.

$\Rightarrow$ left-perfect matching.

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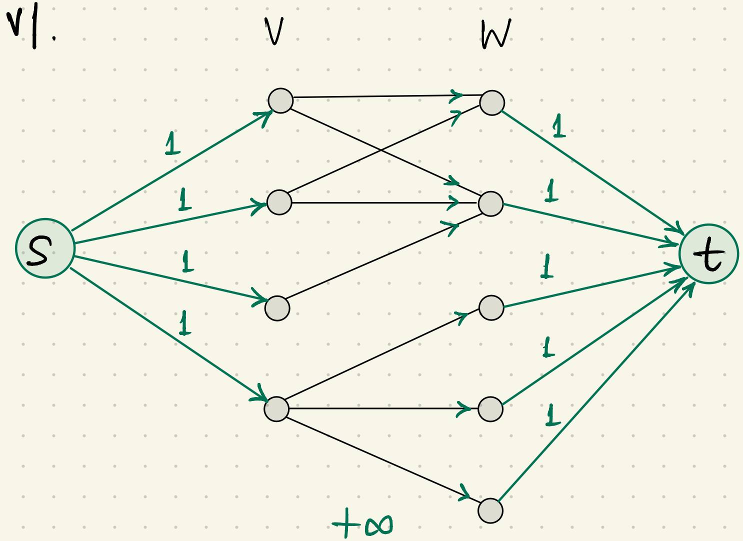

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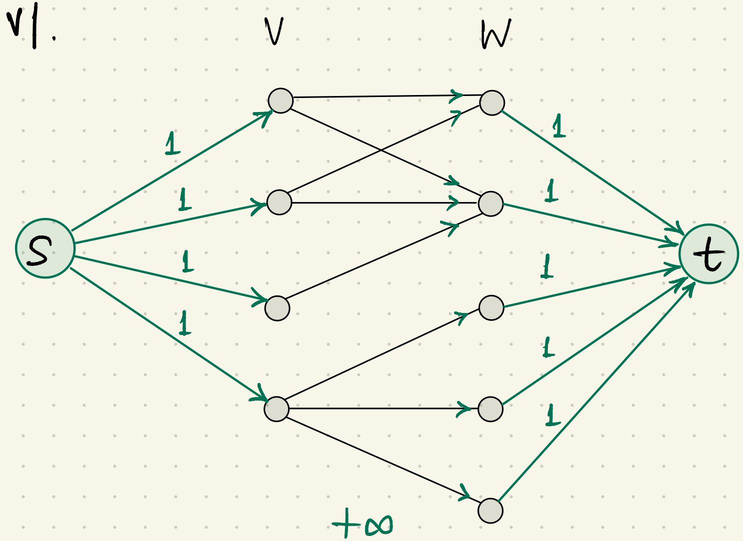

Proof: Fix any (s,t)-cut (A,B).

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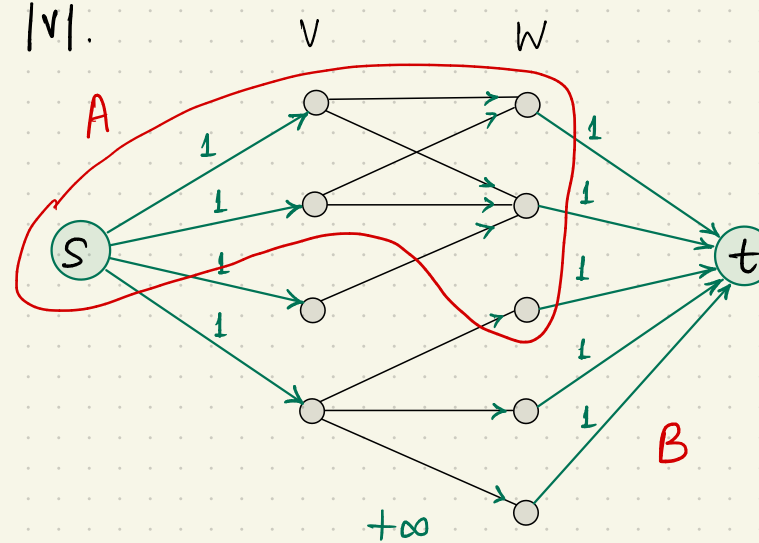

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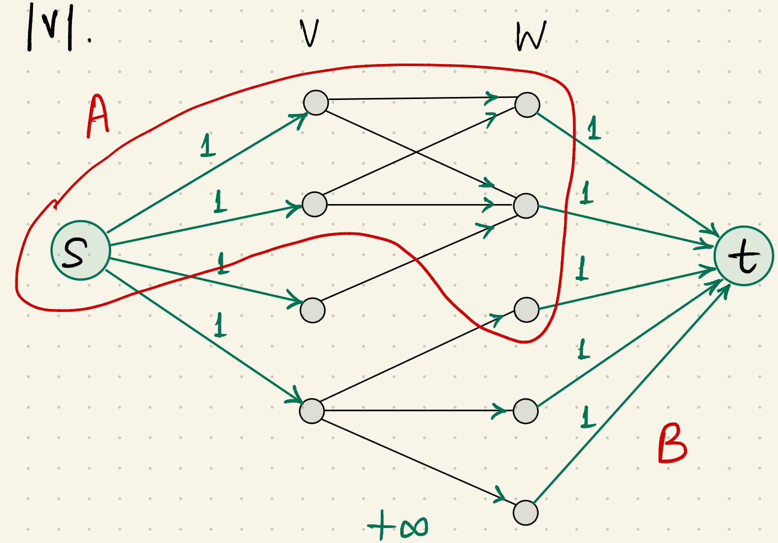
Let $V' := V \cap A$.



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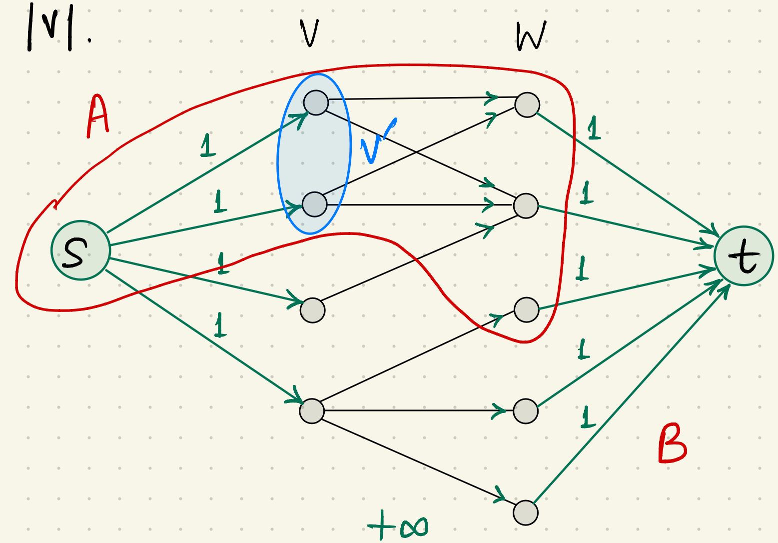

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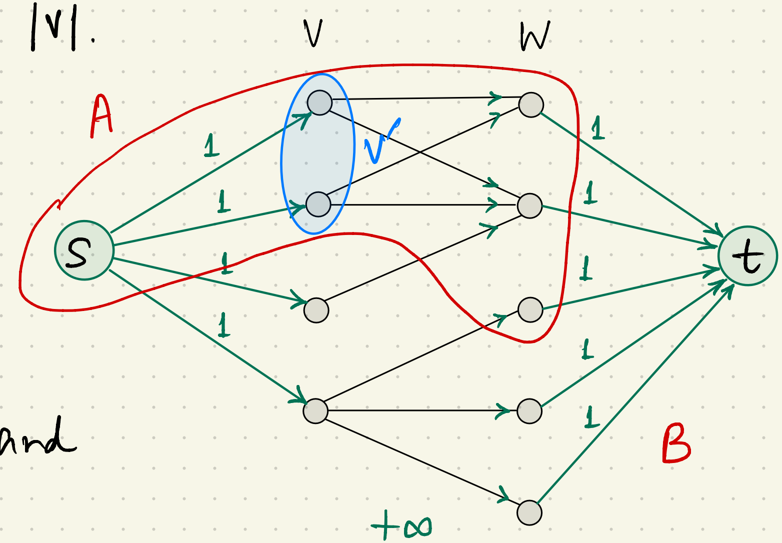
Proof: Fix any (s,t)-cut (A,B).

Let $V' := V \cap A$.

Then, $N(V') \subseteq A$

(Otherwise, capacity of A is $+\infty$ and the claim follows).



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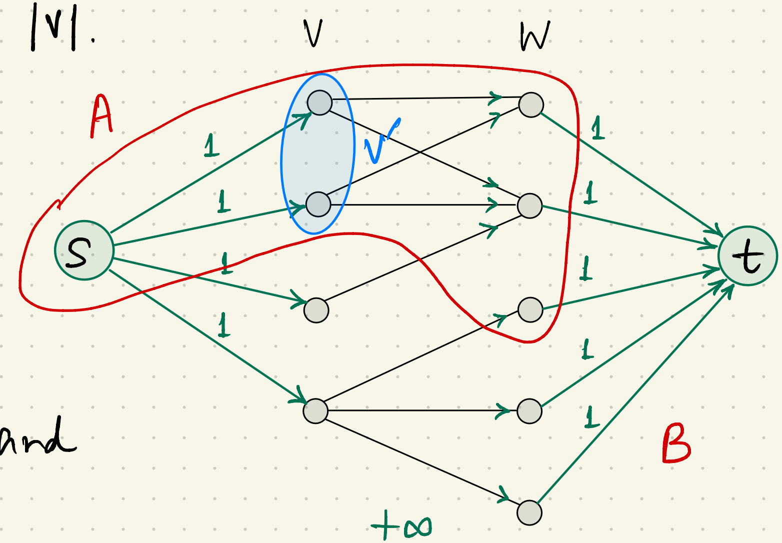
$\Rightarrow$ capacity of (A,B) $\geq$



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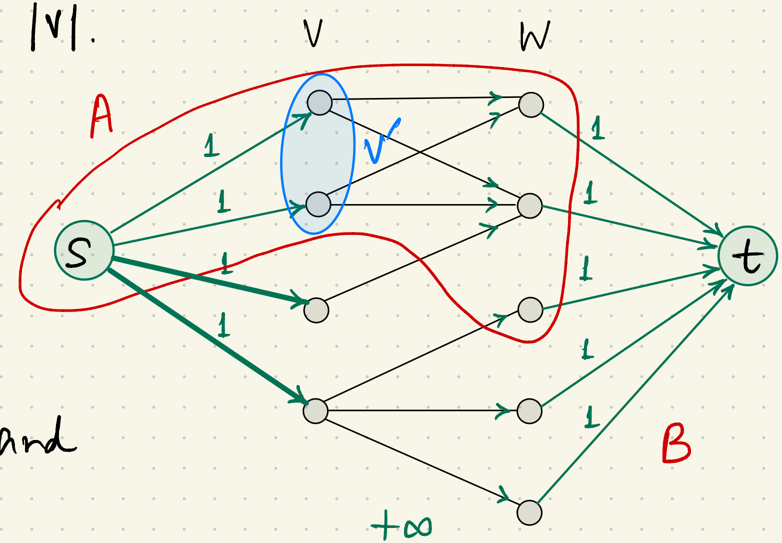

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$$\Rightarrow \text{capacity of } (A,B) \geq \underbrace{|V| - |V'|}_{\substack{\text{edges from } s \\ \text{to } V \setminus V'}}$$

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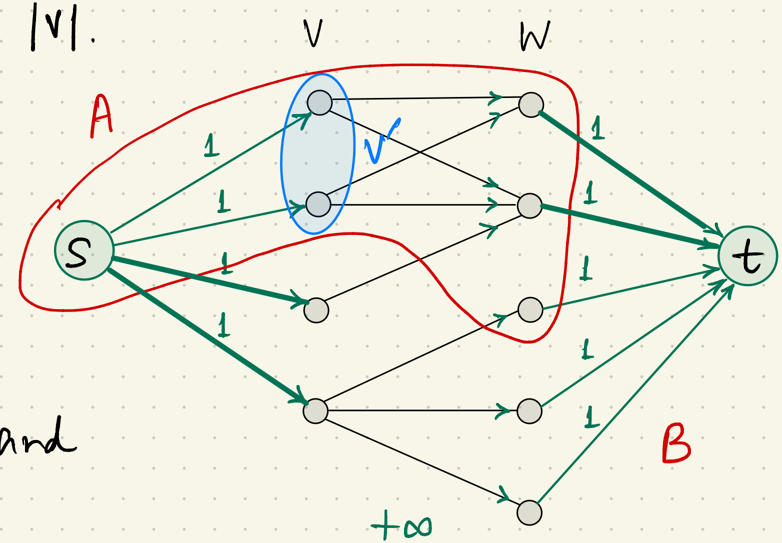

$$\Rightarrow \text{capacity of } (A,B) \geq \underbrace{|V| - |V'|}_{\substack{\text{edges from } s \\ \text{to } V \setminus V'}} + \underbrace{|N(V')|}_{\substack{\text{edges from } N(V') \\ \text{to } t}}$$

HALL's THEOREM

Claim: Every (s,t) cut has capacity $\geq |V|$.

Proof: Fix any (s,t)-cut (A,B).

Let $V' := V \cap A$.

Then, $N(V') \subseteq A$

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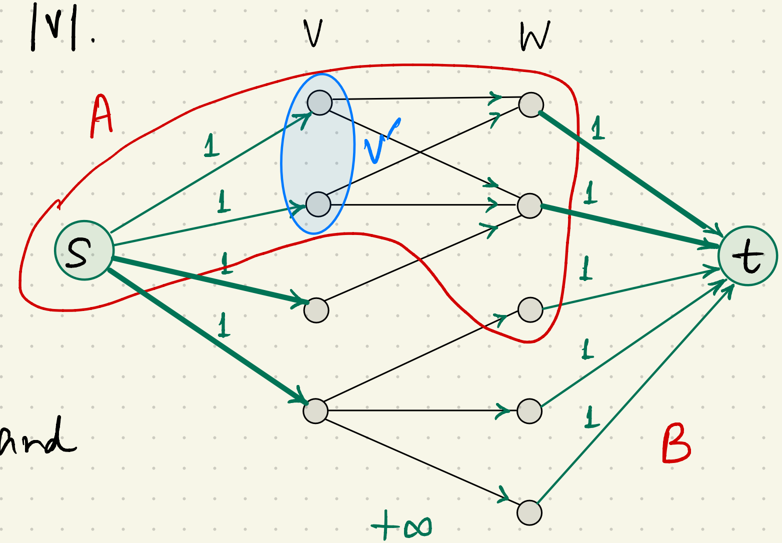

$\Longrightarrow$ capacity of $(A,B) \geq |V| - |V'| + |N(V')| \geq$?

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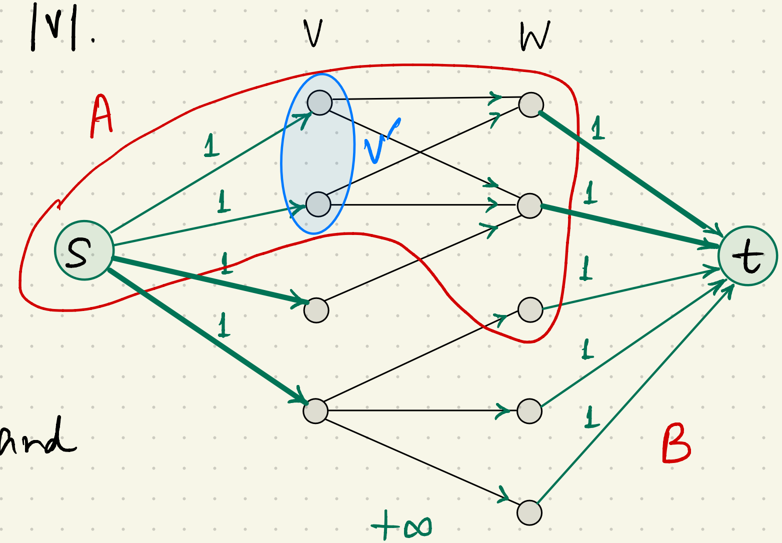

$$\Rightarrow \text{capacity of } (A,B) \geq |V| - |V'| + |N(V')| \geq |V|$$

by Hall's condition.

$\blacksquare$