COL 351 : Analysis & Design of Algorithms

Lecture 35

Quiz 4

Oct 29, 2024 | Rohit Vaish

Problem 1

Consider the following greedy algorithm for the maximum flow problem.

ALGORITHM 1: Greedy Flow

Input: Directed graph $G = (V, E)$, edge capacities $\{u_e\}_{e \in E}$, source $s \in V$, destination $t \in V$.

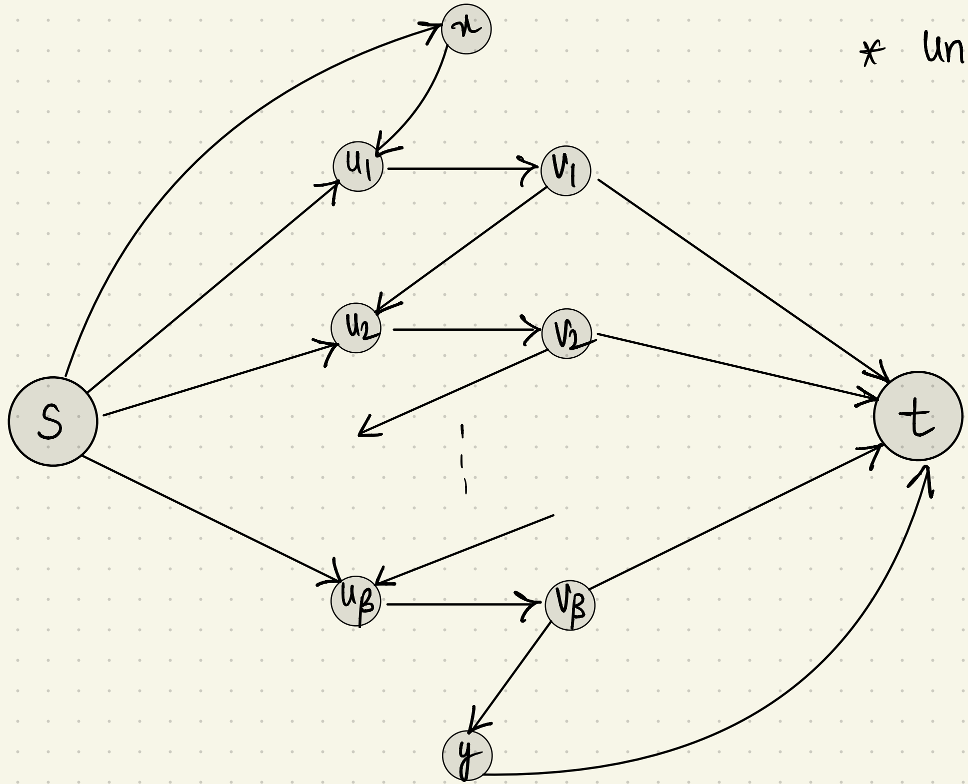
Output: A flow f.

For every $e \in E$, initialize $f_e = 0$.
while *there is a path from s to t in G* **do**
    $P \leftarrow$ an arbitrary path from s to t in G
    $\Delta \leftarrow \min_{e \in P} u_e$ `// minimum capacity of any edge in` P
    for *every edge e in P* **do**
        $f_e \leftarrow f_e + \Delta$
        if $u_e = \Delta$ **then**
            remove e from G
        else
            $u_e \leftarrow u_e - \Delta$
return f

Show that Greedy Flow fails to compute even a good approximation to the maximum flow even on unit-capacity networks. That is, for any constant $\alpha > 1$, show that there is a flow network G where $u_e = 1$ for every edge $e \in E$ such that the value of the maximum flow is more than α times the value of the flow computed by Greedy Flow.

* $\beta = \lceil \alpha \rceil + 1$

* unit capacities



* GREEDY FLOW can pick $s \to x \to u_1 \to v_1 \to u_2 \to v_2 \cdots u_\beta \to v_\beta \to y \to t$ in the first iteration to send a flow of 1 unit.

* After this step, there is no (s,t) path left, so the algo. terminates.

* The max flow has value β: Send flow along $s \to u_i \to v_i \to t$ for $i \in \{1, 2, \ldots, \beta\}$.

* Thus, max flow $> \alpha \cdot$ flow returned by GREEDY FLOW.

Problem 2

Let (A, B) and (A', B') be minimum (s, t)-cuts in some flow network G. Prove that $(A \cap A', B \cup B')$ and $(A \cup A', B \cap B')$ are also minimum (s, t)-cuts in G.

We will use the max flow - min cut theorem.

Consider any max flow f in the network G (well-defined because a feasible flow always exists).

Since (A, B) is a minimum (s,t) cut, we have that:

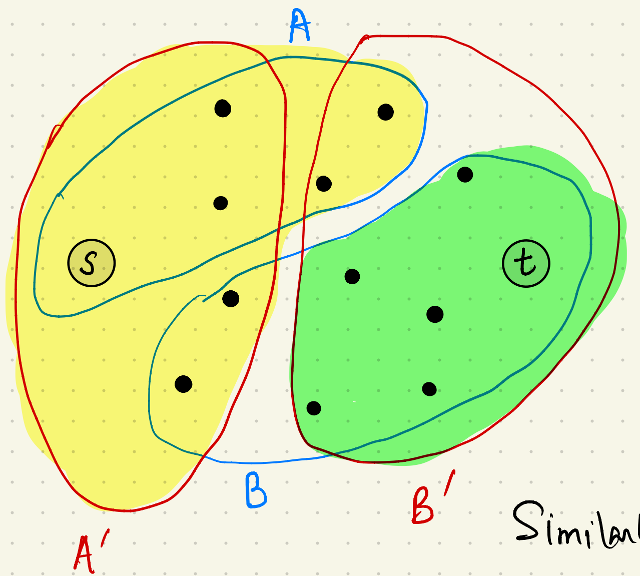
$$\text{value of } f = \text{capacity of } (A, B).$$

$\Rightarrow$ (A, B) is a tight cut.

$\Rightarrow$ every edge $e \in \delta^{+}(A)$ is saturated (i.e., $u_e = f_e$), and

" " " $\in \delta^{-}(A)$ " zeroed out (i.e., $f_e = 0$). — ①

Same observation holds for the (s,t) cut (A', B'). — ②



Consider the (s,t)-cut $(A \cup A', B \cap B')$.

Consider any edge $e = (u,v) \in \delta^+(A \cup A')$.

Then, $u \in A \cup A'$ and $v \in B \cap B'$

① + ② $\Rightarrow$ any such edge is saturated.

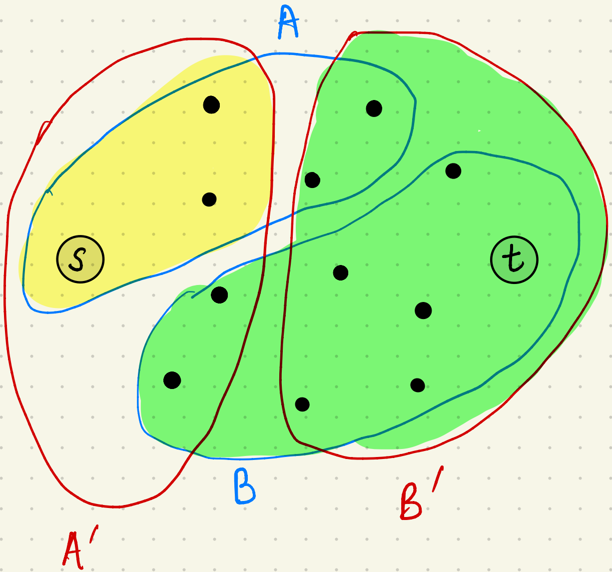
Similarly, consider any edge $e = (u,v) \in \delta^-(A \cup A')$.

Then, $u \in B \cap B'$ and $v \in A \cup A'$.

Again, by ① + ② $\Rightarrow$ any such edge is zeroed out.

We know that: value of $f = \sum_{e \in \delta^+(A \cup A')} f_e - \sum_{e \in \delta^-(A \cup A')} f_e =$ capacity of the (s,t) cut $(A \cup A', B \cap B')$.

$\Rightarrow (A \cup A', B \cap B')$ must be a minimum (s,t)-cut.



Consider the (s,t)-cut $(A \cap A', B \cup B')$.

From ①: We get that any edge $e \in \delta^+(A \cap A')$ going into B must be saturated.

From ②: We get that any edge $e \in \delta^+(A \cap A')$ going into B' must be saturated.

Thus, any edge $e \in \delta^+(A \cap A')$ going into $B \cup B'$ must be saturated.

Similarly, any edge $e \in \delta^-(A \cap A')$ coming from $B \cup B'$ must be zeroed out.

It follows that $(A \cap A', B \cup B')$ is a minimum (s,t)-cut.

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