COL 351 : ANALYSIS & DESIGN OF ALGORITHMS

LECTURE 23

DYNAMIC PROGRAMMING II :

WEIGHTED INDEPEDENT SET (CONTD.) & KNAPSACK

SEPT 24, 2024 | ROHIT VAISH

MAX WEIGHT INDEPENDENT SET ON PATHS

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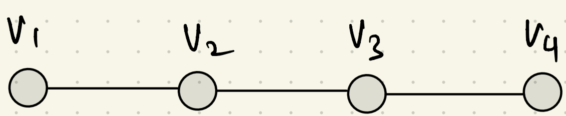

Brute force

Greedy

Divide and conquer

OPTIMAL SUBSTRUCTURE

Optimal solution of overall problem built up from optimal solution of subproblems in a prescribed way.

OPTIMAL SUBSTRUCTURE

$$G = v_1 — v_2 — \dots — v_{n-2} — v_{n-1} — v_n$$

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return S_1 or $S_2 \cup \{v_n\}$, whichever has higher weight

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$T(n) \geq \text{Fibonacci}(n) \sim \text{const.}^n$ No better than brute force ☹

Among the exponentially-many recursive calls, how many distinct subproblems are considered?

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$\Theta(n)$

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The prefixes of graph G $v_1 — v_2 — v_3 \cdots v_i — v_{i+1} \cdots v_n$

ELIMINATING REDUNDANCY

The first time we solve a subproblem, cache its solution in a global table for $O(1)$ time lookup later on.

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(i.e., memoization)

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Let $G_i :=$ first i vertices of graph G and corresponding edges

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Plan: populate array A left to right with $A[i]$ = value of MWIS of G_i.

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Running time: ?

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Correctness : same as recursive algorithm

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☹️

Computes the weight of Ind Set, but not the set itself

// main loop

for $i = 2, 3, \ldots, n$:

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Reconstruction Algorithm

RECONSTRUCTION ALGORITHM

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Vertex v_i belongs to MWIS of G_i $\iff$ w_i + MWIS of G_{i-2} $\geq$ MWIS of G_{i-1}

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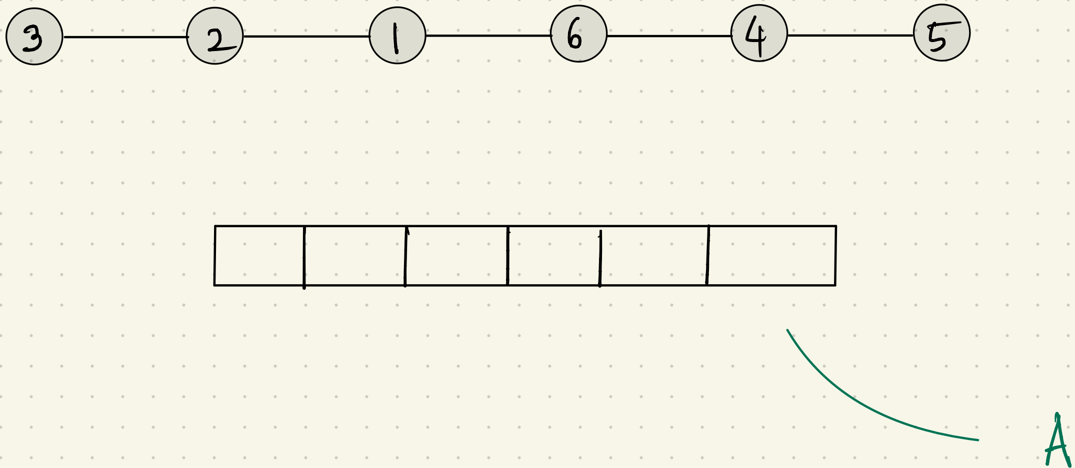
Key point:

vertex v_i belongs to MWIS of G_i $\Longleftrightarrow$ w_i + MWIS of G_{i-2} $\geq$ MWIS of G_{i-1}

```
S := ∅
while i ≥ 1                          // scan from right to left
    if A[i-1] ≥ A[i-2] + w_i         // v_i ∉ MWIS
        decrease i by 1
    else                             // v_i ∈ MWIS
        add v_i to S, decrease i by 2
return S
```

EXAMPLE



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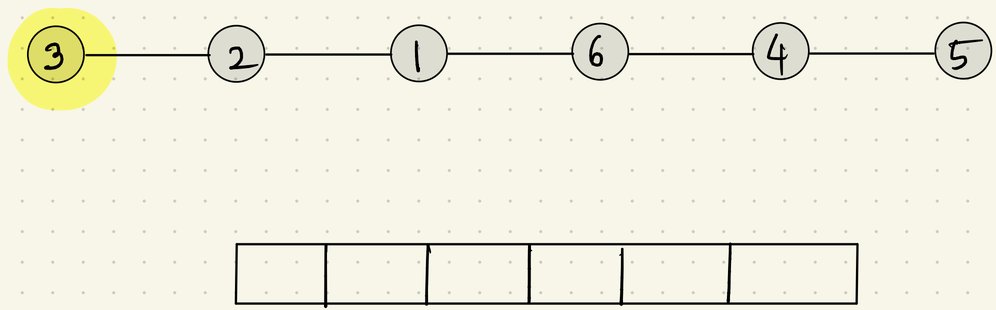

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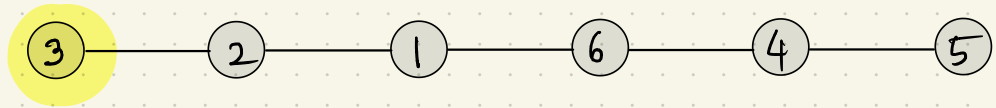

3					

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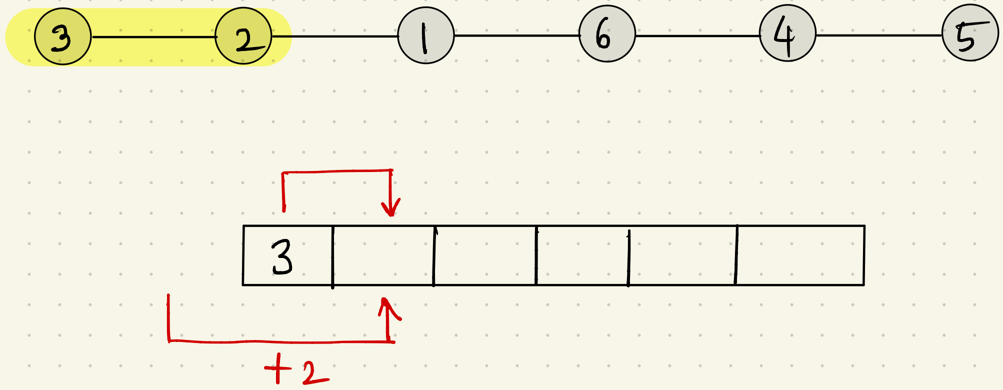

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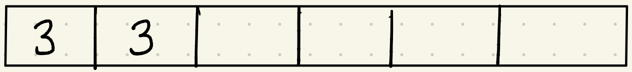

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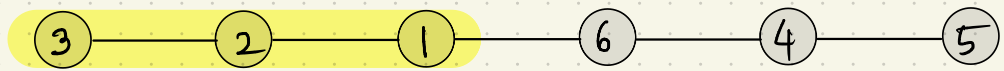
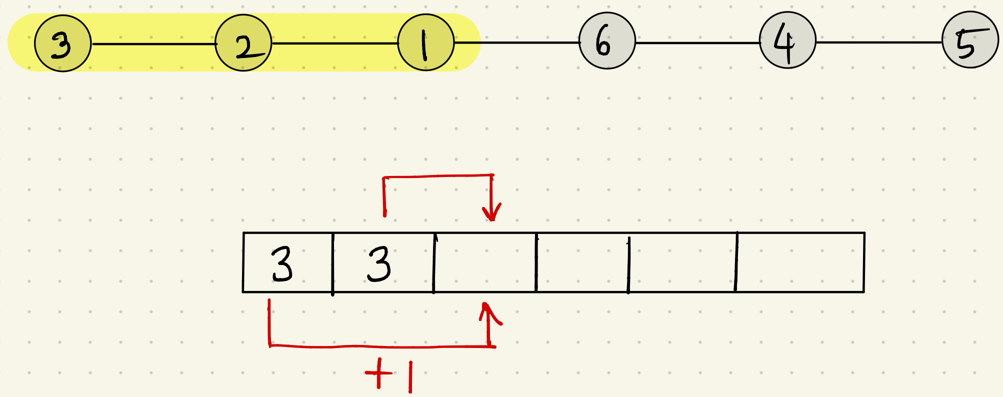


3	3				

Example



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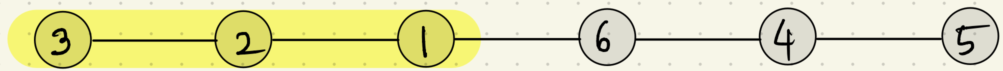

3	3	4			

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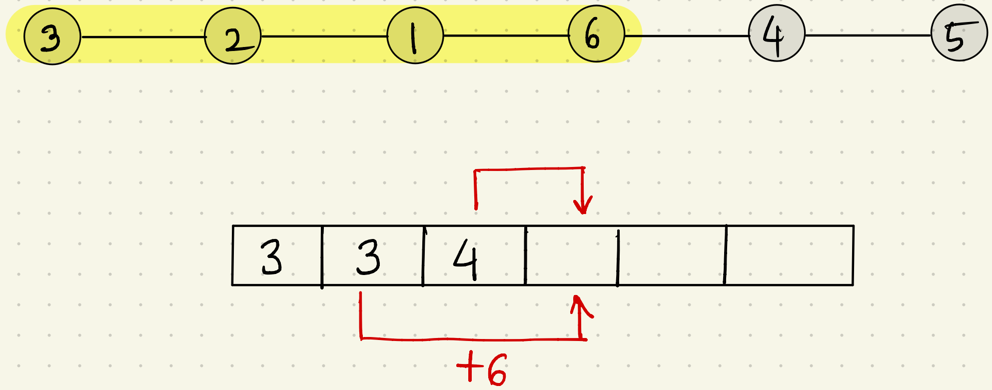

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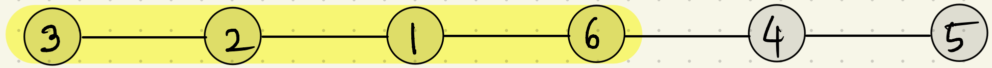

3	3	4	9		

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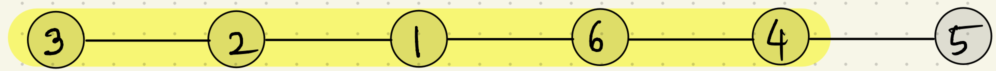

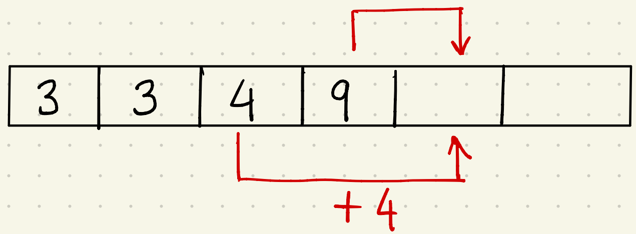

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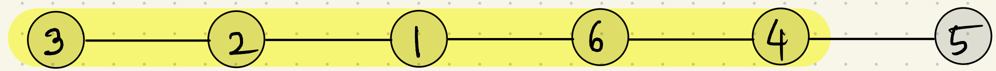

3	3	4	9	9	

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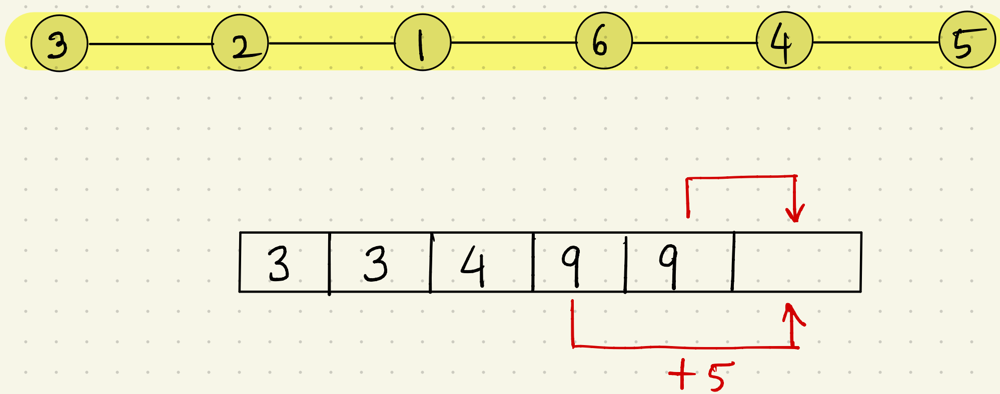

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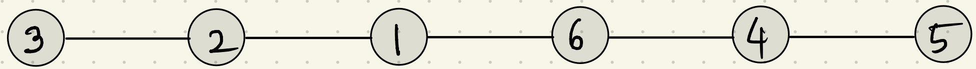

3	3	4	9	9	14

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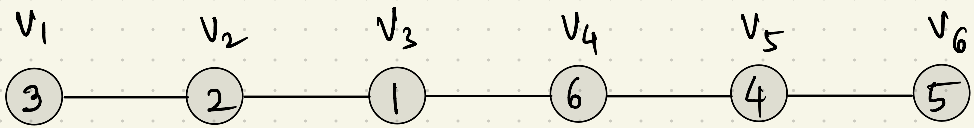

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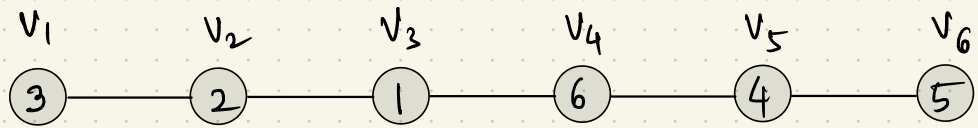
Reconstruction $S = \{ \quad \}$

EXAMPLE



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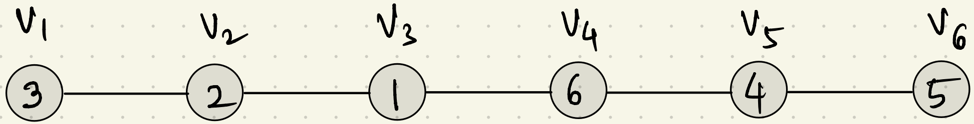

exclude v_6

3	3	4	9	9	14

+5

include v_6

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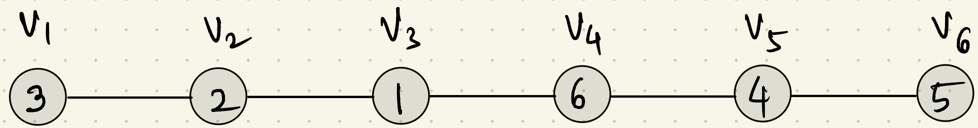

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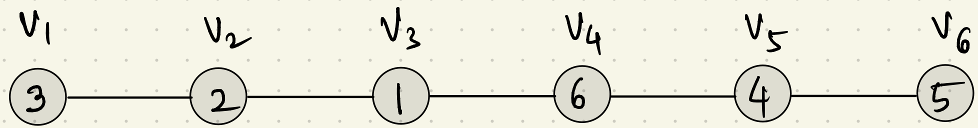

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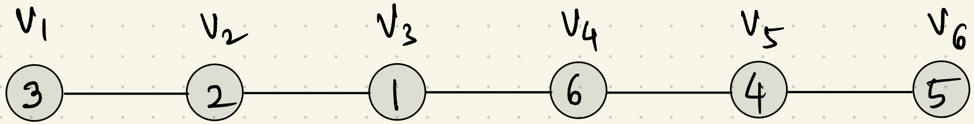

exclude v_4

3	3	4	9	9	14

+6

include v_4

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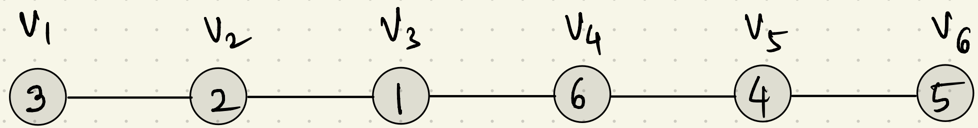

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+6

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Reconstruction $S = \{ \quad v_4 \quad v_6 \}$

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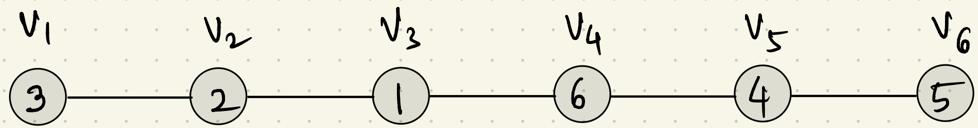

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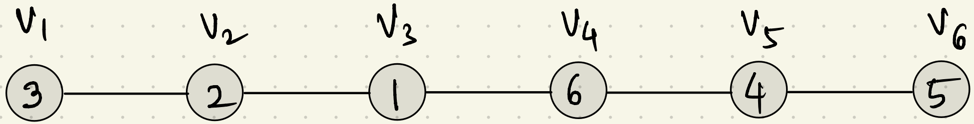


exclude v_2

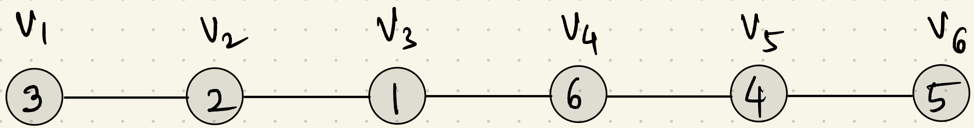
3	3	4	9	9	14

include v_2

+2

Reconstruction $S = \{ \quad v_4 \quad v_6 \}$

EXAMPLE



exclude v_2

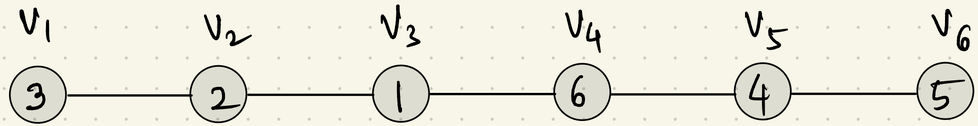
3	3	4	9	9	14

$+2$

include v_2

Reconstruction $S = \{ \cancel{v_2} \; v_4 \; v_6 \}$

EXAMPLE



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Reconstruction $S = \{ v_1 \ v_4 \ v_6 \}$

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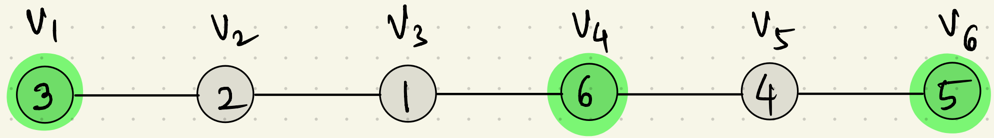

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A DAG VIEW OF DYNAMIC PROGRAMMING

1.

2.

3.

A DAG VIEW OF DYNAMIC PROGRAMMING

Subproblem 1   Subproblem 2   Subproblem 3   Subproblem 4   Subproblem 5

1. Identify a small number of subproblems

2.

3.

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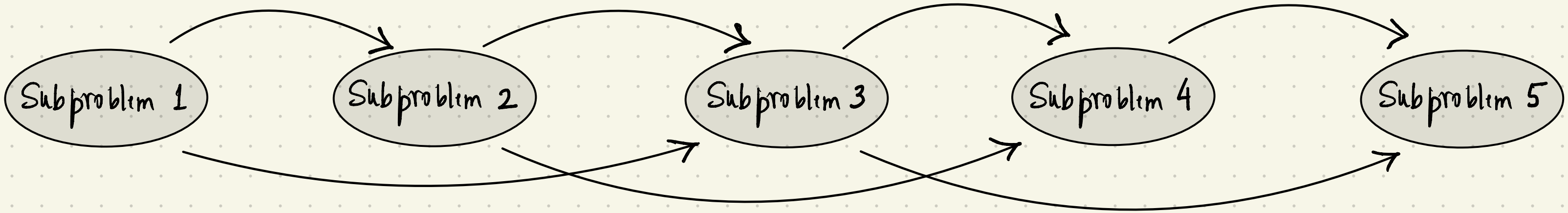

1. Identify a small number of subproblems

2. Show how to quickly and correctly solve "large" subproblems given solution of "smaller" ones.

3.

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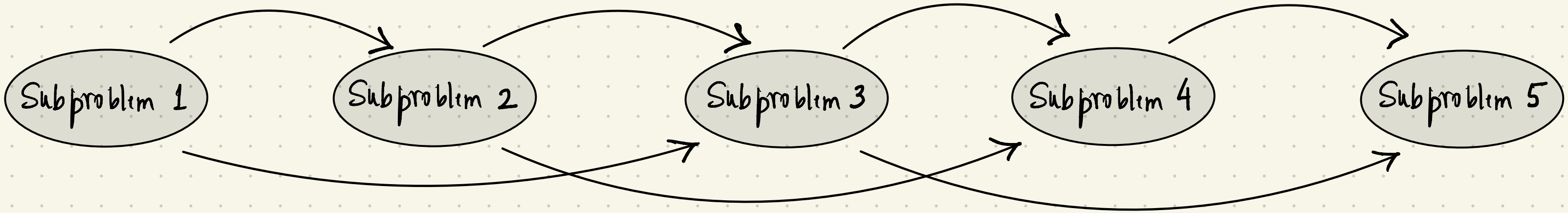

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3. " " " " " final problem given solutions of all other subproblems.

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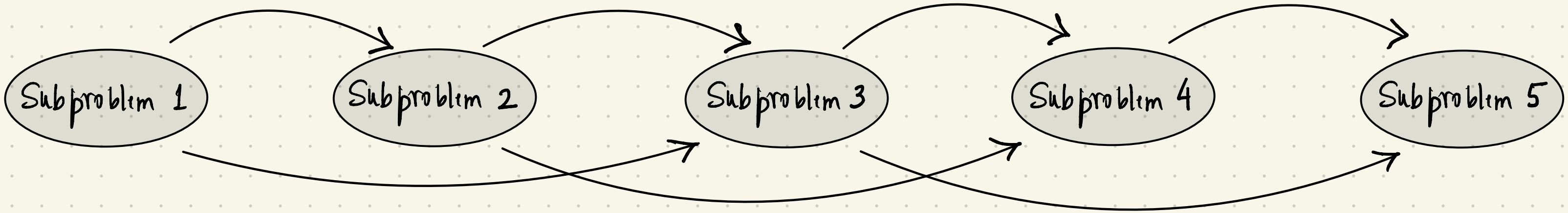

1. Identify a small number of subproblems

   e.g., max wt independent set of G_i for $i \in \{0, 1, 2, \ldots, n\}$

2. Show how to quickly and correctly solve "large" subproblems given solution of "smaller" ones.

   e.g., recurrence $A[i] = \max\{A[i-1], A[i-2] + w_i\}$.

3. " " " " " final problem given solutions of all other subproblems.

   e.g., return $A[n]$

KNAPSACK

KNAPSACK

input : 1

2

KNAPSACK

input : ① n items, each having a

- value v_i (non negative)
- size s_i (non negative and integral)

②

KNAPSACK

input : ① n items, each having a
- value v_i (non negative)
- size s_i (non negative and integral)

② capacity C (non negative and integral)

KNAPSACK

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output : a subset $S \subseteq \{1, 2, \dots, n\}$

that maximizes $\sum_{i \in S} v_i$

KNAPSACK

input : (1) n items , each having a

- value v_i (non negative)
- size s_i (non negative and integral)

(2) capacity C (non negative and integral)

output : a subset $S \subseteq \{1, 2, \dots, n\}$

that maximizes $\sum_{i \in S} v_i$

subject to $\sum_{i \in S} s_i \leq C$

EXAMPLE

item	value	size
1	3	4
2	2	3
3	4	2
4	4	3

capacity = 6

EXAMPLE

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APPLICATIONS

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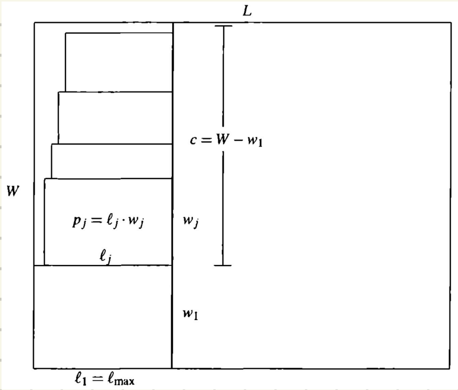
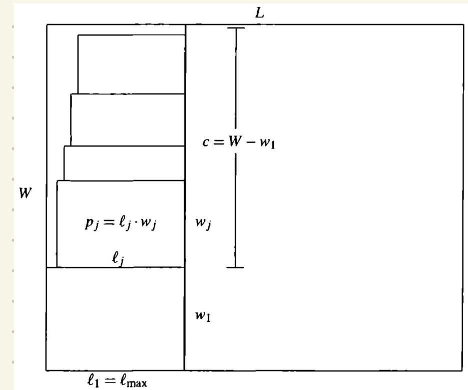
Scheduling advertisements



Profit - maximizing cutting

APPLICATIONS

Scheduling advertisements



Profit - maximizing cutting

Other applications : Credit assignment, Cryptosystems, ...

OPTIMAL SUBSTRUCTURE

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$\Rightarrow$ S must be optimal for the first $(n-1)$ items

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buffer for item n