# COL 351 : Analysis & Design of Algorithms

## LECTURE 22

## DYNAMIC PROGRAMMING I:

## WEIGHTED INDEPEDENT SET

SEPT 20, 2024 | ROHIT VAISH

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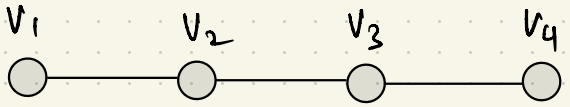

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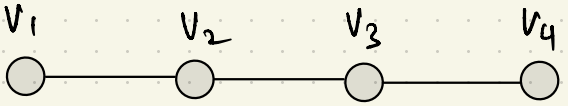

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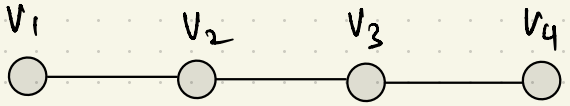

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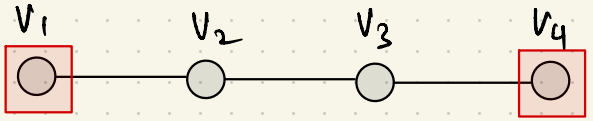

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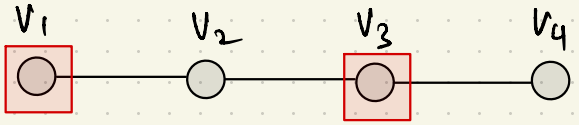

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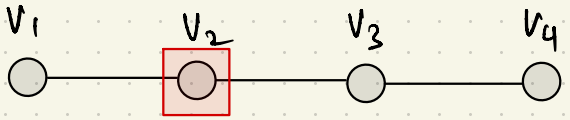

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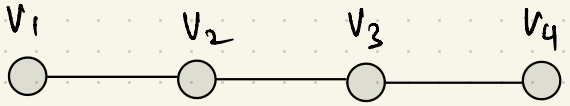

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How many independent sets? 8

$\emptyset$, four singletons, $\{v_1, v_3\}$, $\{v_2, v_4\}$, $\{v_1, v_4\}$

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in general: exponential in $n$ ☹

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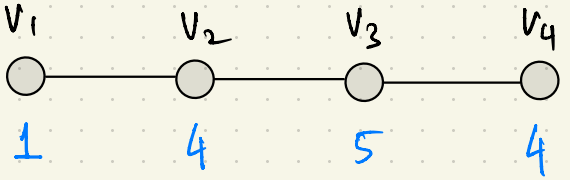

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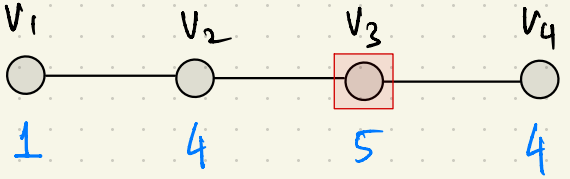

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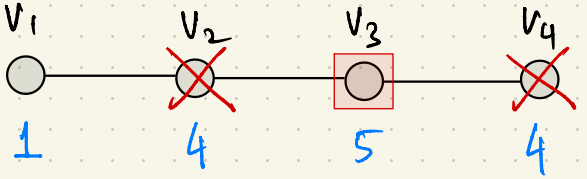

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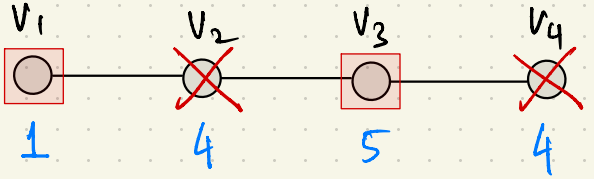


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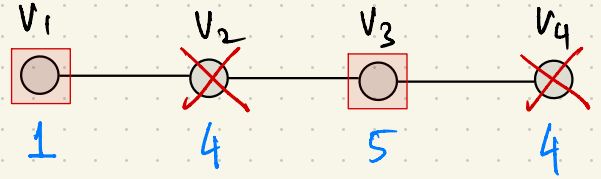


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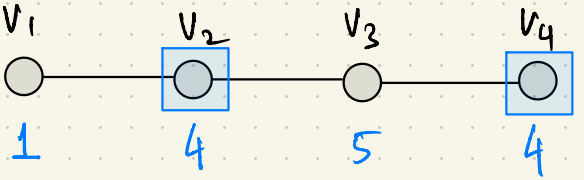


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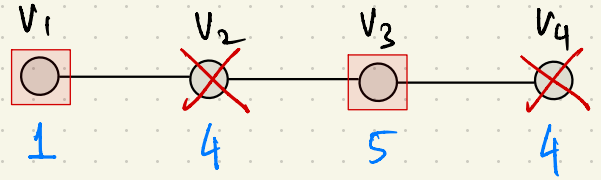


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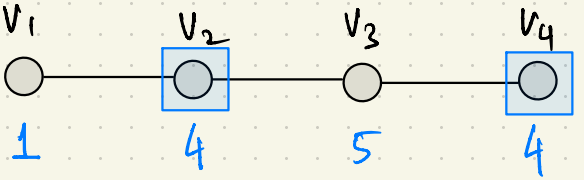


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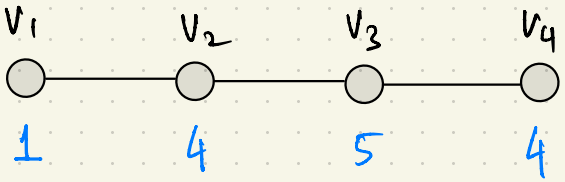

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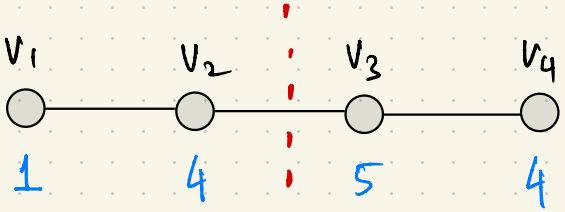

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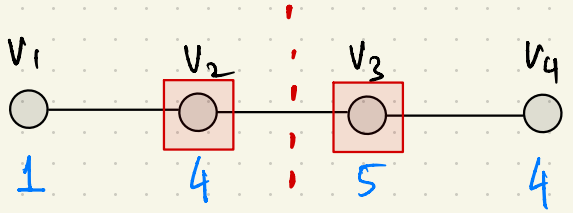

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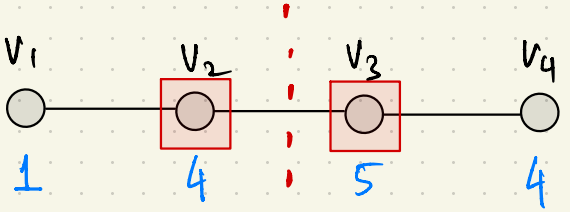


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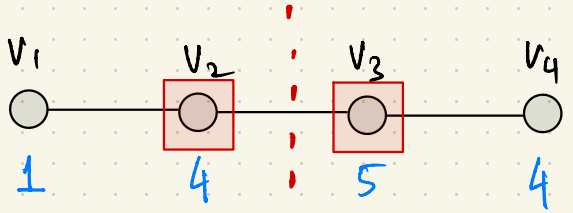


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🤔 $O(n^2)$ algorithm with four recursive calls.

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**Hope:** Need to consider only a few subproblems — manageable!

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👎 We don't know which case we are in.

👍 Only need to think about two subproblems!

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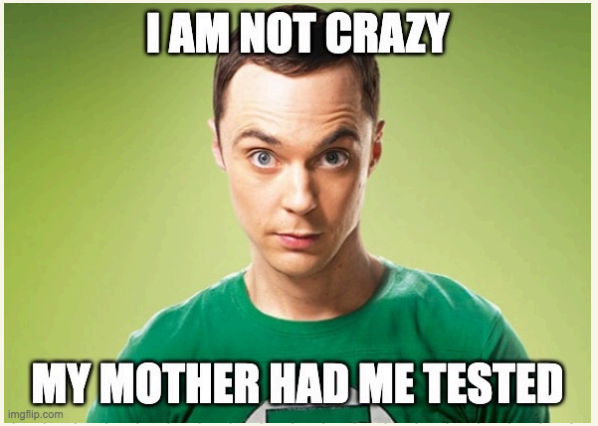

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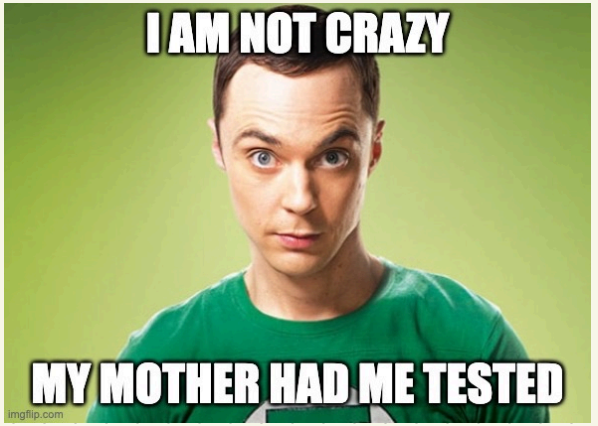


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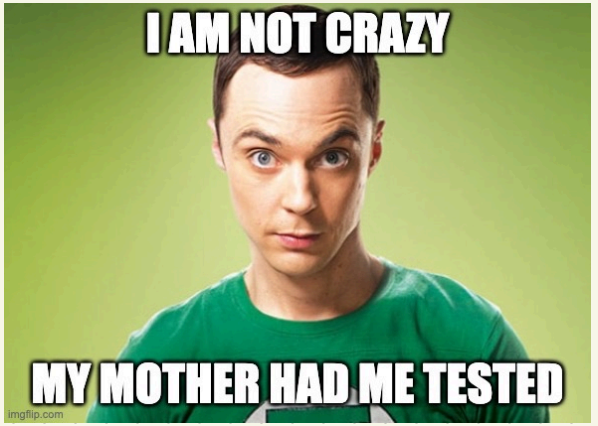


This is recursively organized brute force.

By eliminating redundant computations, we can solve the problem in linear time.