COL 351 : Analysis & Design of Algorithms

Lecture 15

Greedy Algorithms II :

Job Scheduling (contd.) and Huffman Coding

Aug 28, 2024 | Rohit Vaish

Greedy Algorithms

Do what looks best right now and hope everything works out at the end

JOB SCHEDULING

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input : a set of n jobs with positive lengths $l_1, l_2, \ldots, l_n$ and positive weights $w_1, w_2, \ldots, w_n$

output : a job schedule (or sequence) that minimizes the sum of weighted completion times $\sum_{j=1}^{n} w_j \cdot C_j$.

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Brute force : $O(n!)$ ☹

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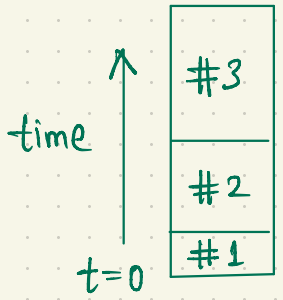
E.g., $l_1 = 1$, $l_2 = 2$, $l_3 = 3$



$C_1 = 1$ $C_2 = 3$ $C_3 = 6$

OBJECTIVE FUNCTION

goal : minimize the weighted sum of completion times

$$\min \sum_{j=1}^{n} w_j \cdot C_j$$

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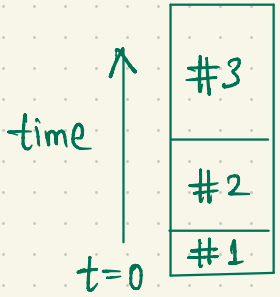
weights $w_1 = 3$, $w_2 = 2$, $w_3 = 1$



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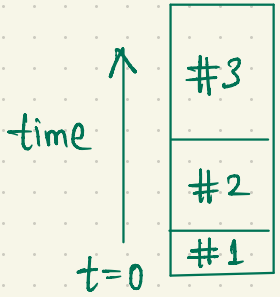

$$\sum_{j=1}^{n} w_j \cdot C_j = \underset{w_1}{3} \times \overset{C_1}{1} + \underset{w_2}{2} \times \overset{C_2}{3} + \underset{w_3}{1} \times \overset{C_3}{6}$$

$$= 15$$

SPECIAL CASES

goal : min $\sum_{j=1}^{n} w_j \cdot C_j$

(I) All lengths equal : schedule heavier job earlier

(II) All weights equal : schedule shorter job earlier

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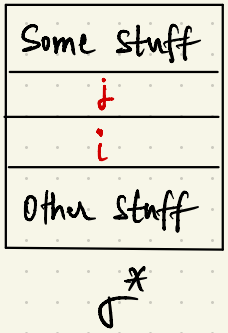

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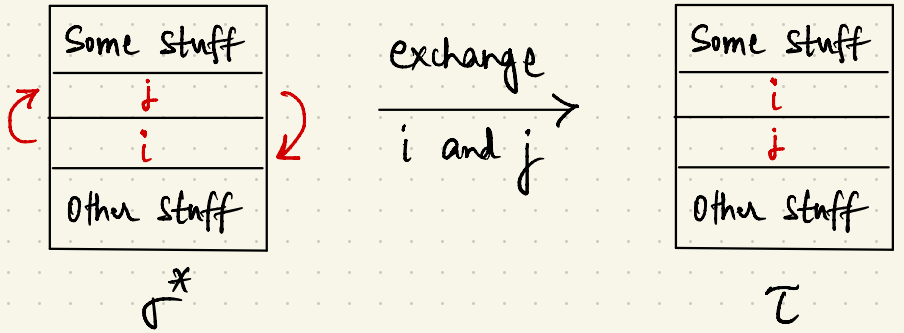

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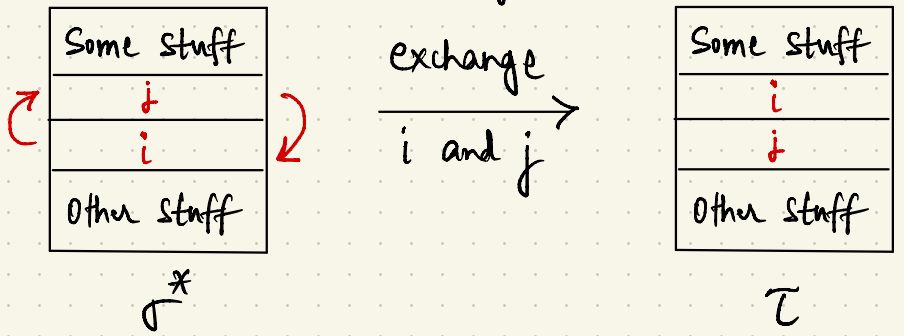

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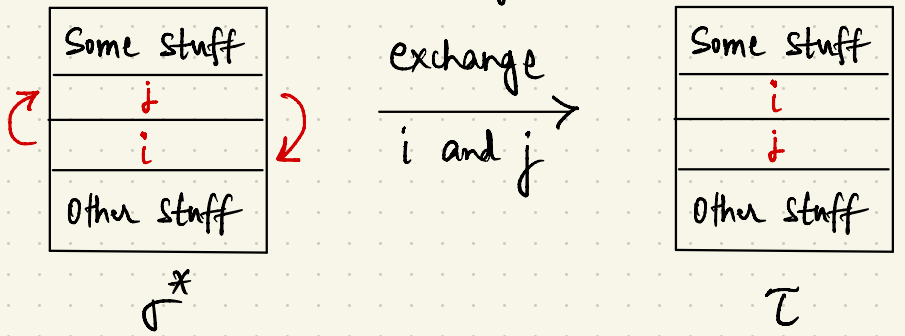

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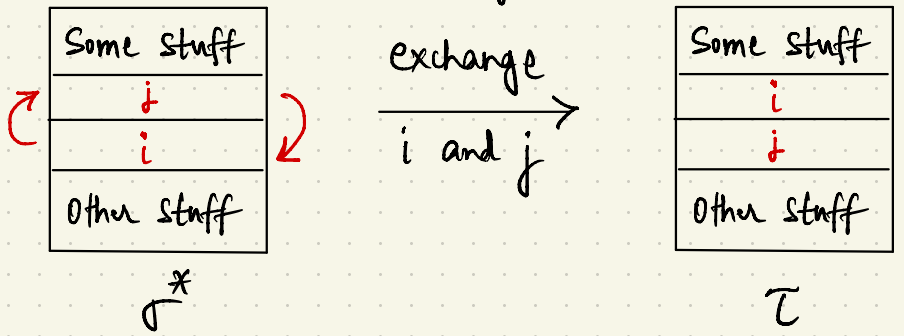

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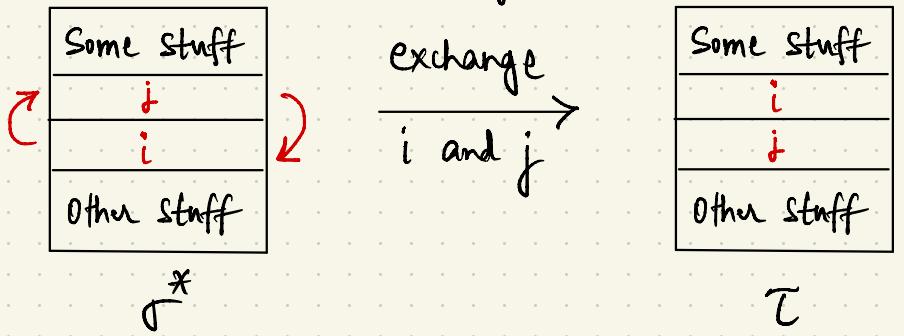

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Will show that σ is at least as good as τ

$\Rightarrow$ greedy schedule is optimal.

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↳ consecutive jobs i, j in τ such that $i > j$.

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Observe,

$$\sum_{k=1}^{n} w_k \cdot C_k(\tau^{new}) = \sum_{K=1}^{n} w_k \cdot C_k(\tau) + w_i l_j - w_j l_i$$

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$\Rightarrow$ σ must be optimal

HUFFMAN CODING

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e.g.,

	fixed length		variable length	
A	00	A	0	Ambiguous ☹
B	01	B	01	How to interpret 001?
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variable length

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How to compute an optimal prefix-free code ?