COL 351 : ANALYSIS & DESIGN OF ALGORITHMS

LECTURE 11

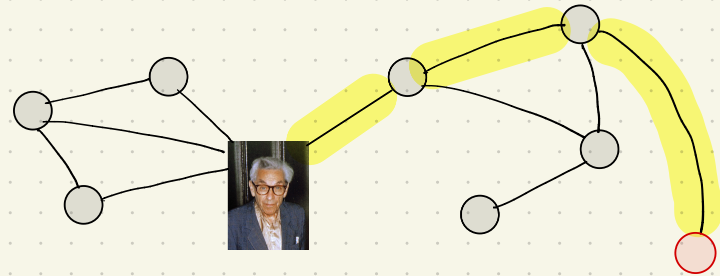
GRAPH ALGORITHMS IV :

TOPOLOGICAL ORDERING & STRONGLY CONNECTED COMPONENTS

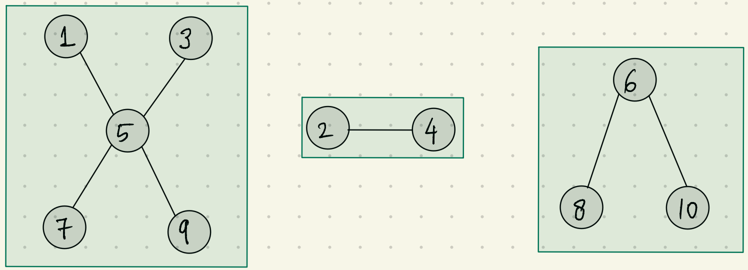
AUG 20, 2024 | ROHIT VAISH

APPLICATIONS OF BFS

Shortest paths



Connected Components



DEPTH-FIRST SEARCH

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Iterative version

DEPTH-FIRST SEARCH

Iterative version

mark all vertices as unexplored

$S :=$ a stack data structure (LIFO), initialized with s

while $S \neq \emptyset$

- remove the top node of S, say v ("pop")
- if v is unexplored
    - mark v as explored
    - for each (v, w) in adj. list of v
        - add w to the front of S ("push")

DEPTH-FIRST SEARCH

Recursive version

```
DFS (G, s)                    // all vertices unexplored before the call

    mark s as explored

    for each (s, v) in adj. list of s
    |       if v is unexplored
    |       |      DFS (G, v)
```

APPLICATIONS OF DFS

Topological ordering

Strongly Connected Components

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Topological ordering

Strongly Connected Components

TOPOLOGICAL ORDERING

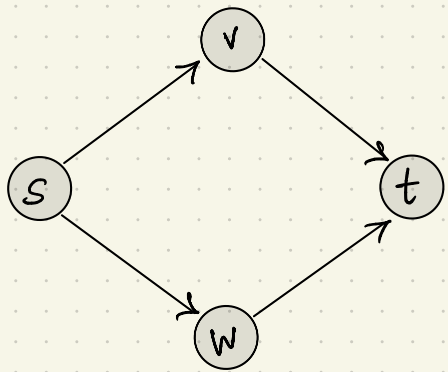
Directed graph $G = (V, E)$

A labeling f of G's vertices such that :

* unique $f(v) \in \{1, 2, \ldots, n\}$ for every $v \in V$

* for every $(v, w) \in E$ $\quad f(v) < f(w)$.

TOPOLOGICAL ORDERING



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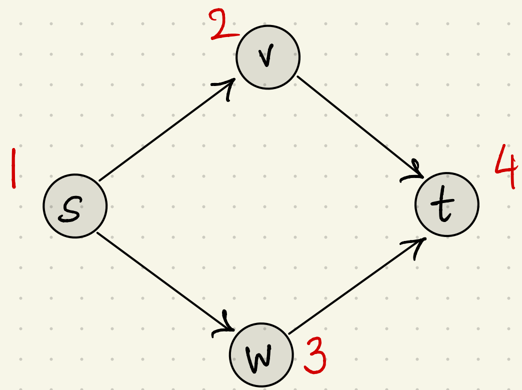

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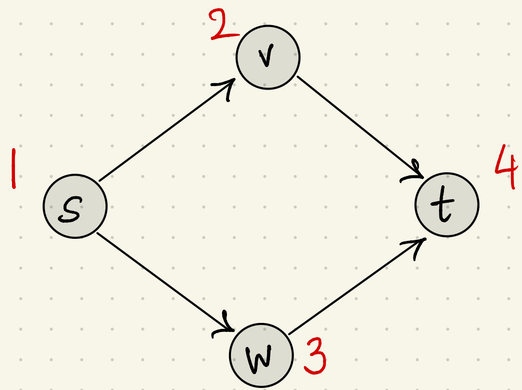

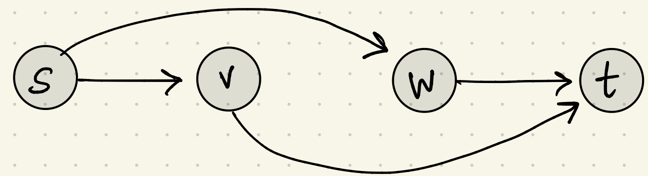

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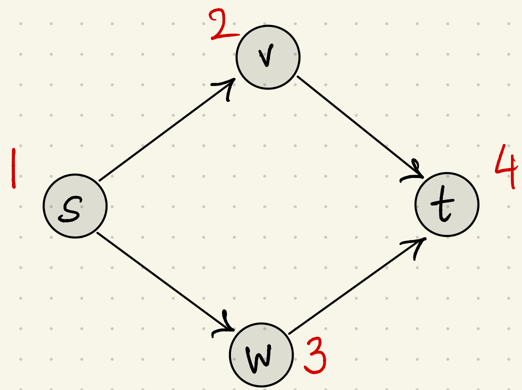

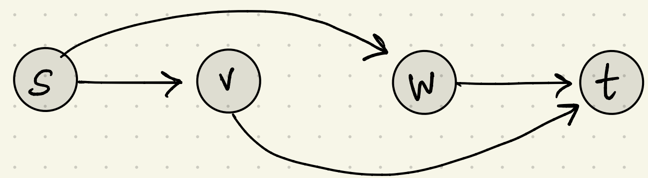

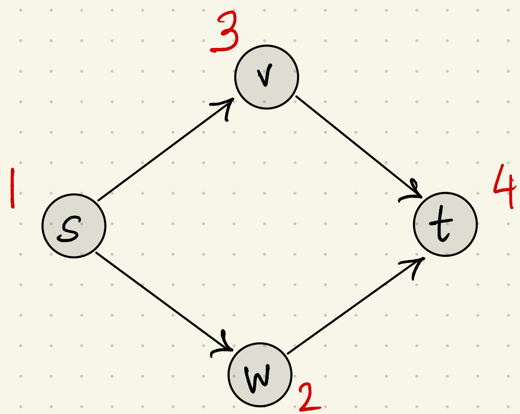

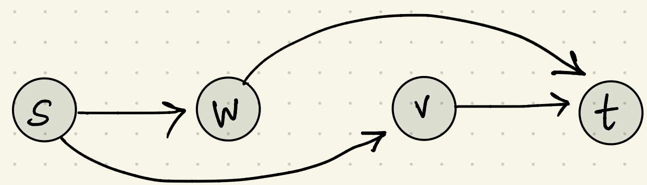

TOPOLOGICAL ORDERING

Theorem : Every directed acyclic graph has at least one topological ordering.

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Lemma: Every directed acyclic graph has at least one sink.

a vertex with no outgoing edges

TOPOLOGICAL ORDERING

Theorem: Every directed acyclic graph has at least one topological ordering.

Proof: Assign $f[v] = n$ to sink vertex v (exists!)

Recurse on $G \setminus \{v\} \longrightarrow$ must be directed acyclic. ∎

TOPOLOGICAL ORDERING

Theorem: Every directed acyclic graph has at least one topological ordering.

Algorithm

1. Find a sink vertex v.

2. Assign to it the largest available label and recurse on $G \setminus \{v\}$.

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Correctness

If $f[v] = i$, no edges from v to vertices with $f[v] < i$.

TOPOLOGICAL ORDERING

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Running time

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Running time

1. Find a sink vertex v. $\longrightarrow O(n)$

2. Assign to it the largest available label and recurse on $G \setminus \{v\}$.

$O(n^2)$

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1. Find a sink vertex v. $\longrightarrow$ $O(n)$
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Running time

$O(n^2)$

Can we do better for sparse graphs?

TOPOLOGICAL ORDERING via DFS

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DFS pseudocode

```
DFS (G, s)                    // all vertices unexplored before the call

    mark s as explored

    for each (s, v) in adj. list of s
        if v is unexplored
            DFS(G, v)
```

TOPOLOGICAL ORDERING via DFS

DFS (G, s) // all vertices initially unexplored

mark s as explored

for each (s, v) in adj. list of s

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DFS(G, v)

TOPOLOGICAL ORDERING via DFS

DFS - Loop (G)

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mark s as explored

for each (s, v) in adj. list of s

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        DFS (G, v)

TOPOLOGICAL ORDERING via DFS

DFS - Loop (G)

mark all vertices unexplored

current_label := |V| // labeling f

DFS (G, s) // all vertices initially unexplored

mark s as explored

for each (s, v) in adj. list of s

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TOPOLOGICAL ORDERING via DFS

DFS - Loop (G)

mark all vertices unexplored

current_label := $|V|$ // labeling f

for each $v \in V$

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TOPOLOGICAL ORDERING via DFS

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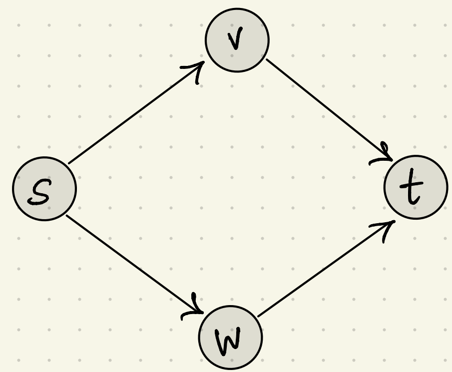
set $f[s]$ = current_label

decrease current_label by 1

TOPOLOGICAL ORDERING via DFS



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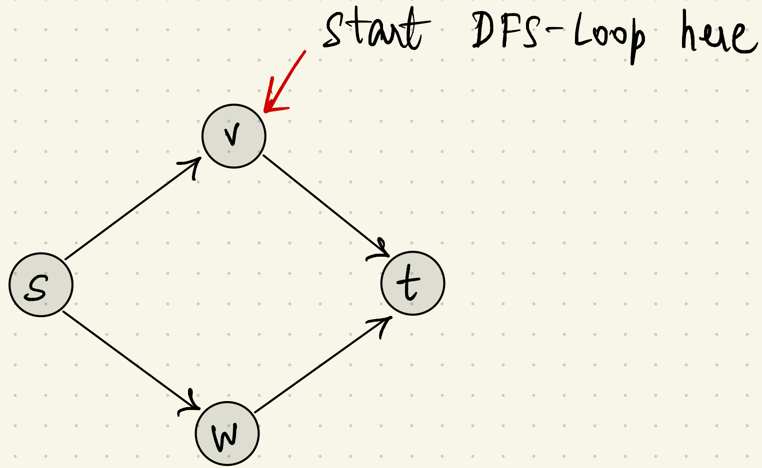

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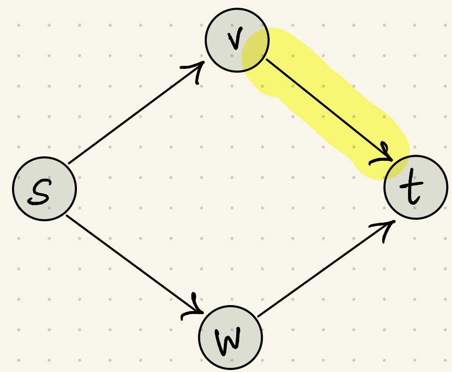

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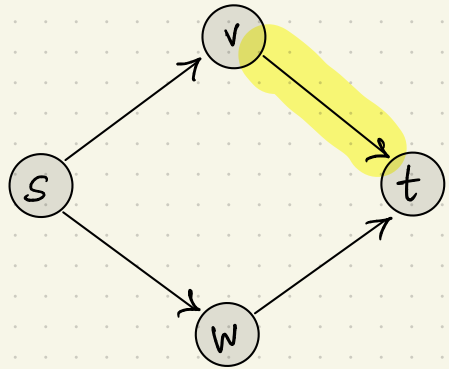
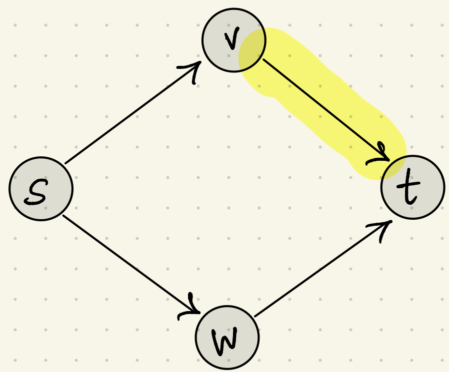


nowhere to go from t

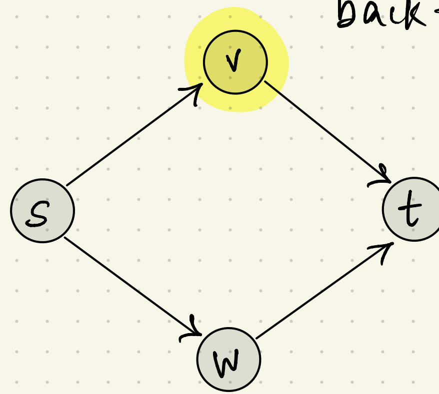
TOPOLOGICAL ORDERING via DFS



$f[t] = 4$

TOPOLOGICAL ORDERING via DFS

backtrack to v



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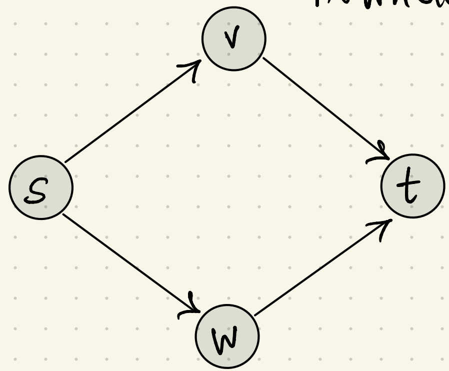
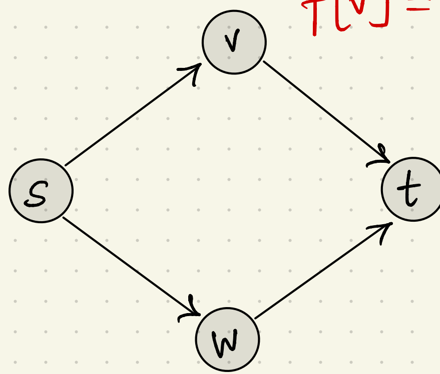
TOPOLOGICAL ORDERING via DFS



nowhere else to go from v

$f[t] = 4$

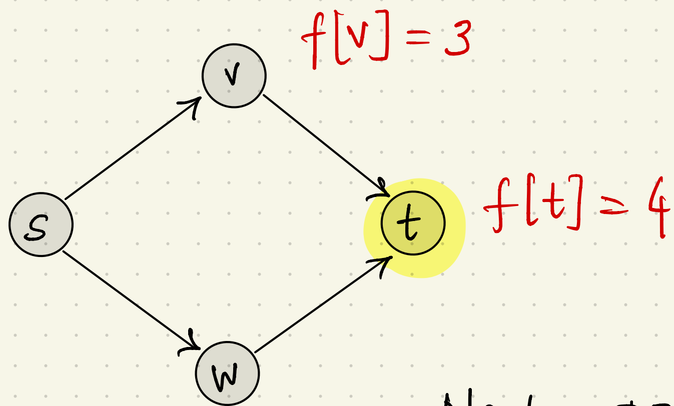
TOPOLOGICAL ORDERING via DFS



$f[v] = 3$

$f[t] = 4$

TOPOLOGICAL ORDERING via DFS



Next, DFS-Loop considers t

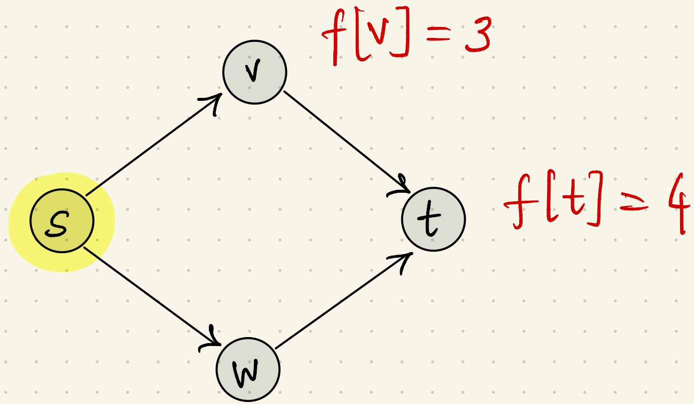
but it is already explored

so skip it.

TOPOLOGICAL ORDERING via DFS

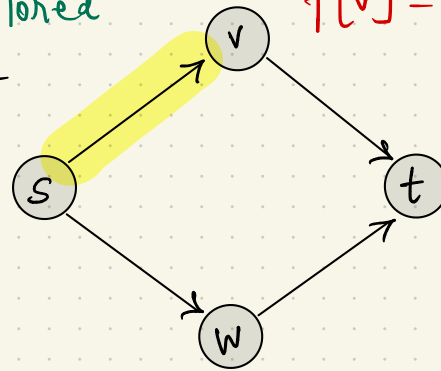
Next, DFS-Loop

Considers s.



TOPOLOGICAL ORDERING via DFS

v is explored
so skip it



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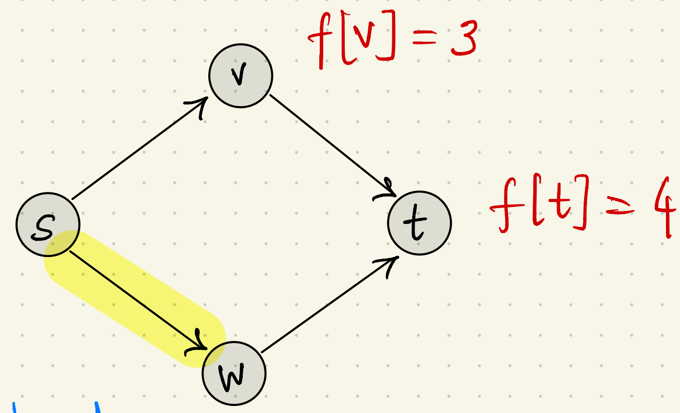
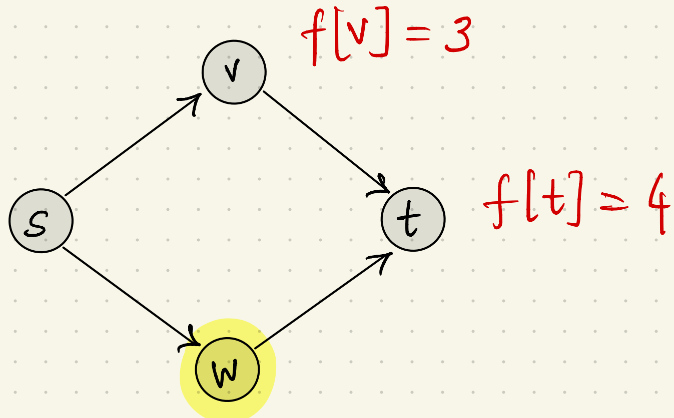


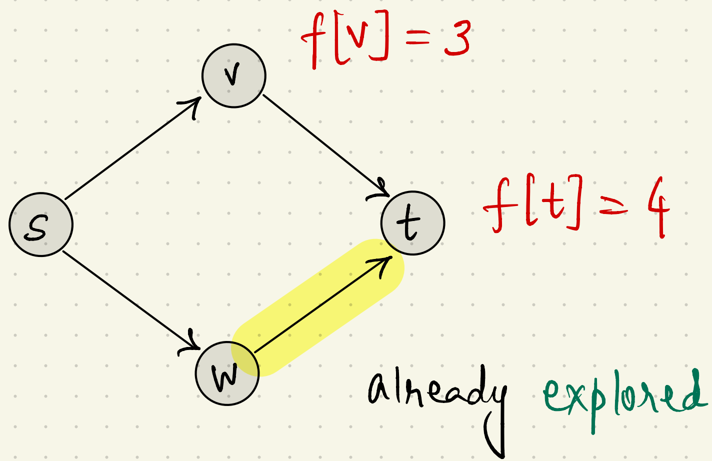
w is unexplored

TOPOLOGICAL ORDERING via DFS



Call DFS on w

TOPOLOGICAL ORDERING via DFS



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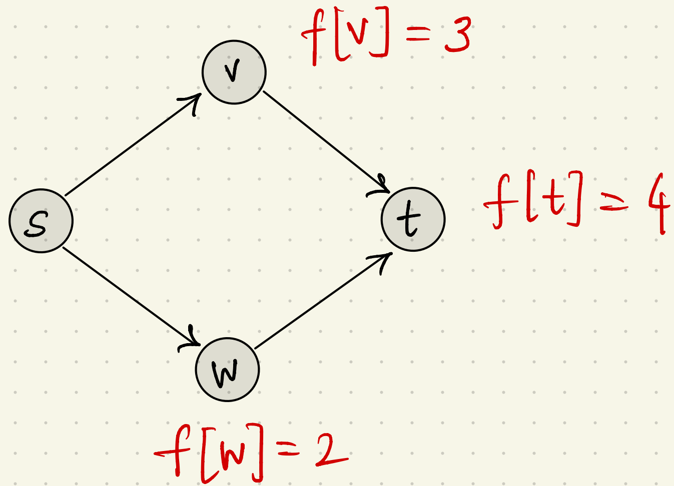
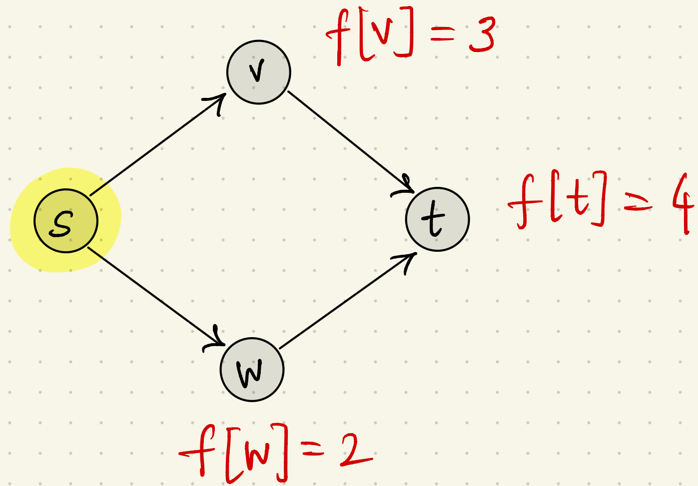


TOPOLOGICAL ORDERING via DFS

backtrack to s



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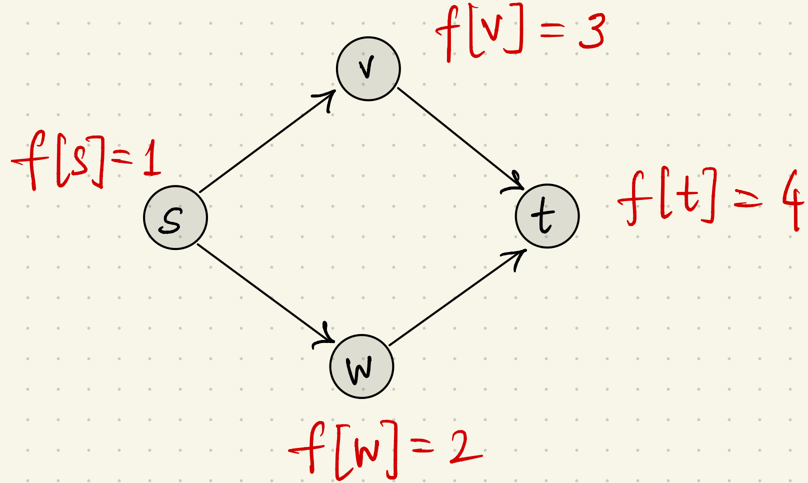

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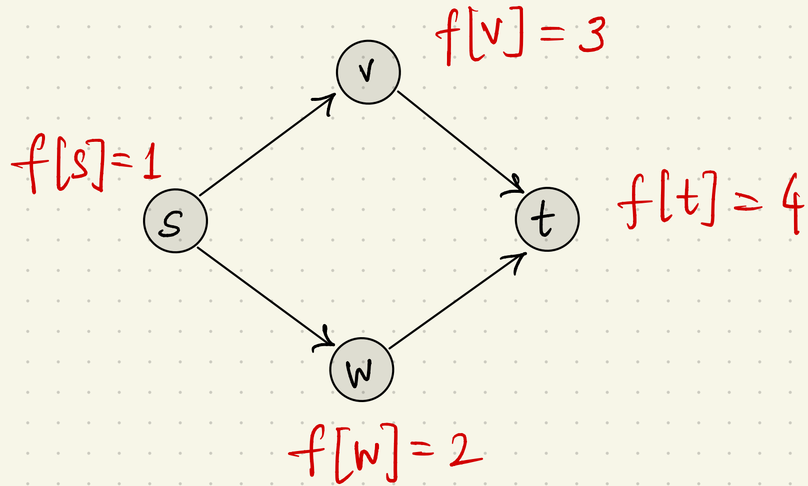

DFS-Loop runs out of vertices ⇒ done!

TOPOLOGICAL ORDERING via DFS

Claim 1: DFS-Loop algorithm runs in $O(m+n)$ time.

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Claim 2: Under DFS-Loop,

if $(u, v) \in E$ then $f[u] < f[v]$.

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TOPOLOGICAL ORDERING via DFS

NOTE: DFS-Loop algorithm terminates on any input graph and returns some labeling f.

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not necessarily a DAG

not topological ordering

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STRONGLY CONNECTED COMPONENTS

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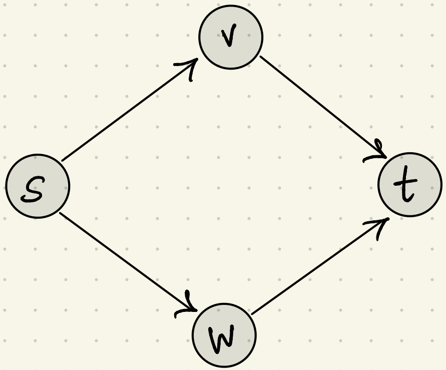
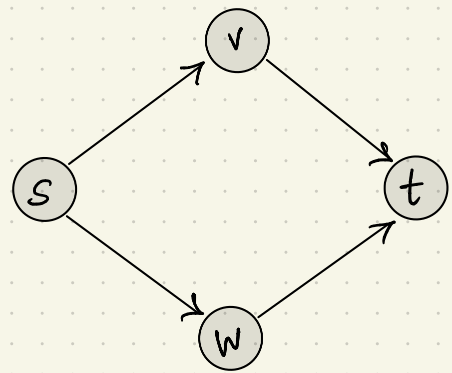


Connected ?

STRONGLY CONNECTED COMPONENTS

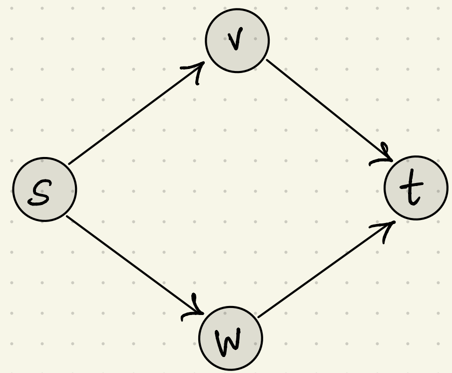


Connected?

Yes if one looks at the underlying undirected graph

No if one cares about reachability via directed edges

STRONGLY CONNECTED COMPONENTS



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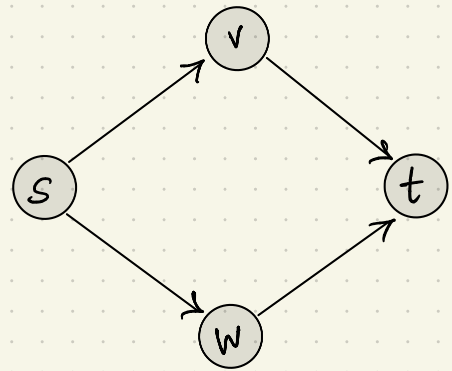
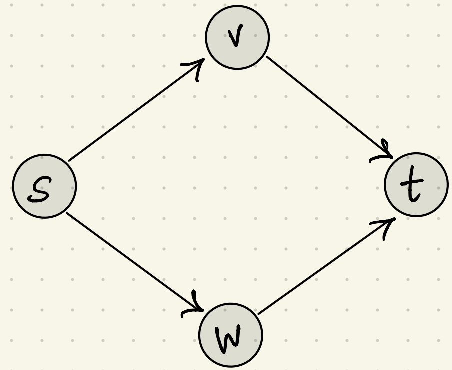


A directed graph is strongly connected if every vertex can be reached from every other vertex by a directed path.

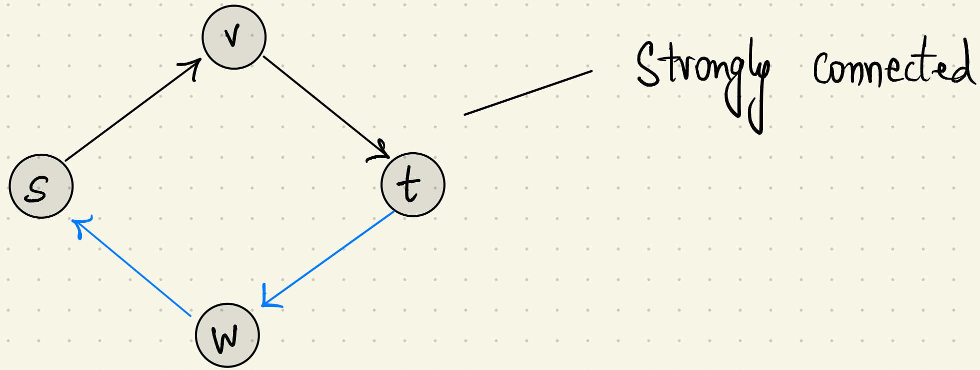
STRONGLY CONNECTED COMPONENTS



Not strongly connected

A directed graph is strongly connected if every vertex can be reached from every other vertex by a directed path.

STRONGLY CONNECTED COMPONENTS



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STRONGLY CONNECTED COMPONENTS

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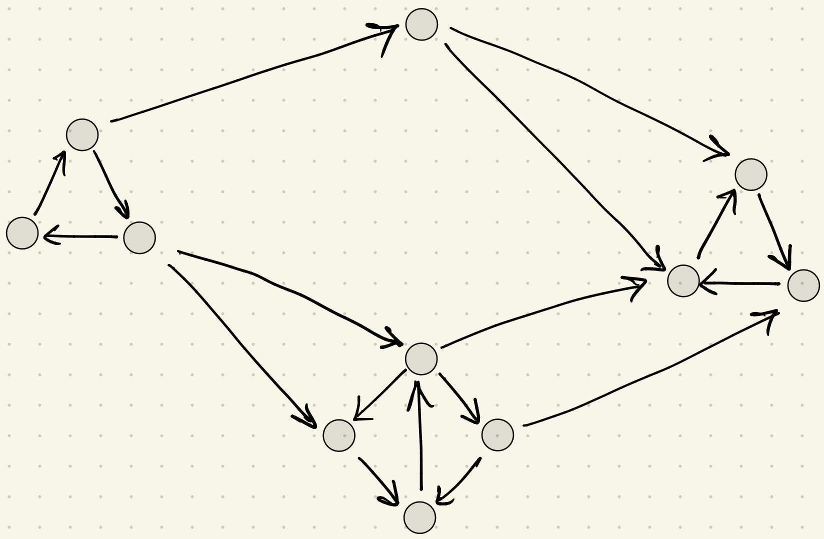

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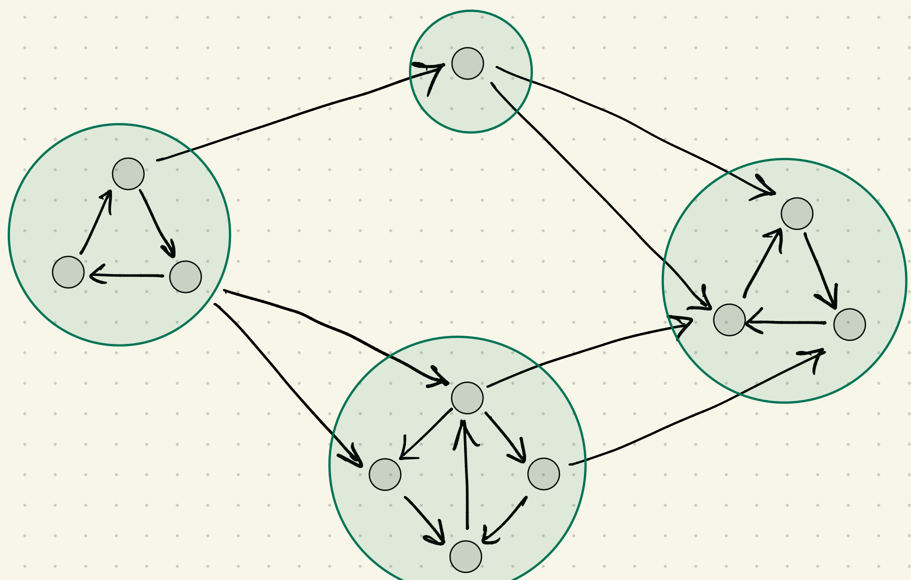

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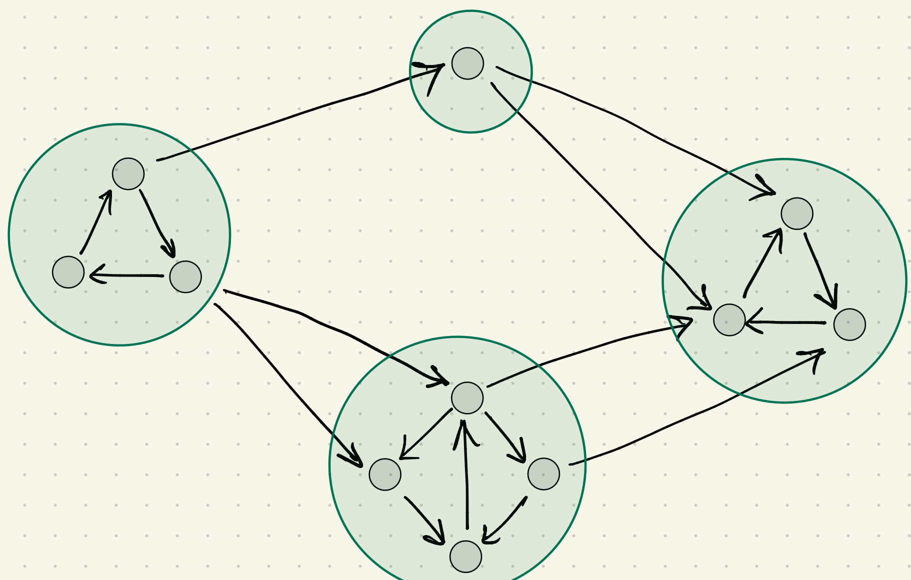

A DAG!

STRONGLY CONNECTED COMPONENTS

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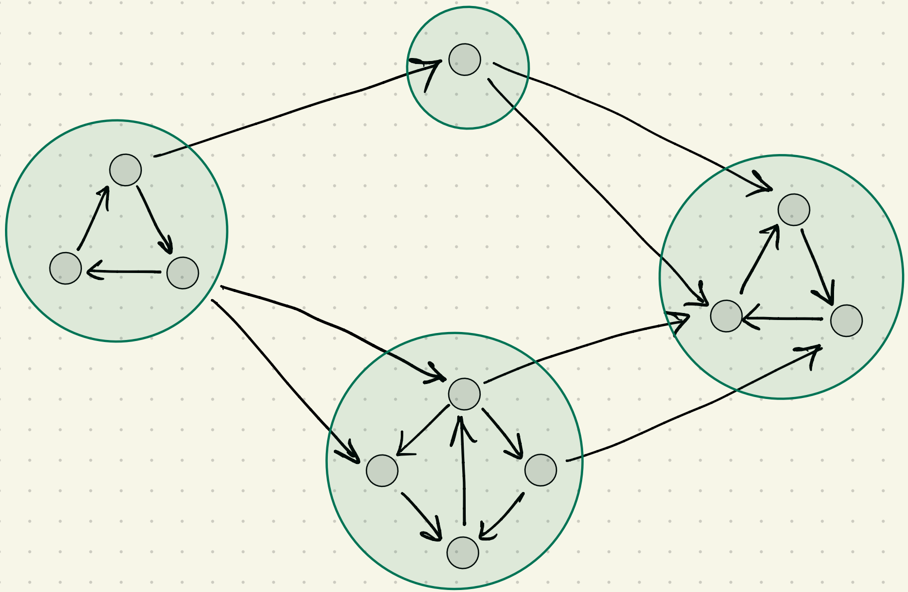
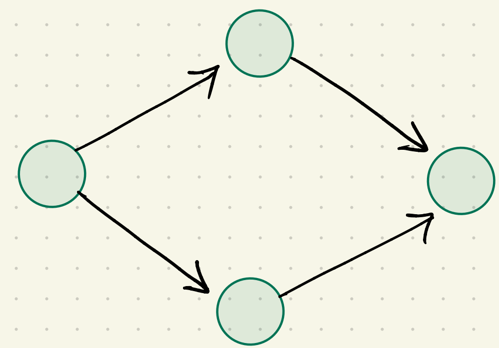
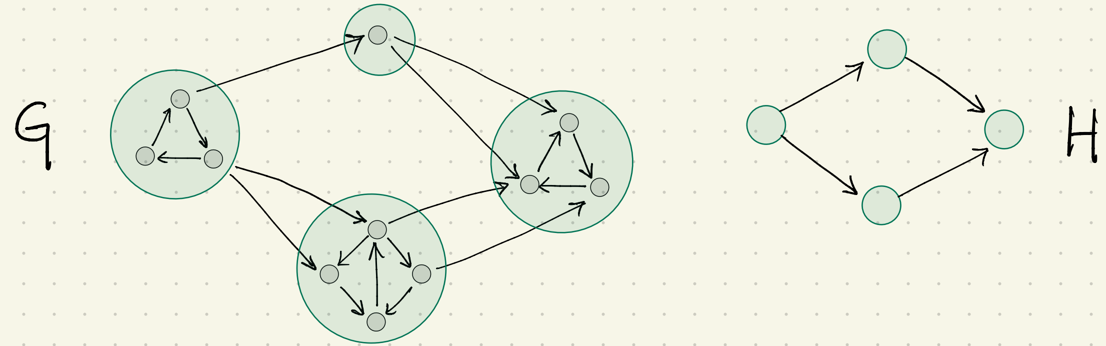




STRONGLY CONNECTED COMPONENTS

Theorem: Given a directed graph G, define a "meta" graph $H = (X, F)$ that has a vertex for every SCC of G, and an edge $(x, y) \in F$ if there is an edge from some vertex in SCC corresponding to x to some vertex in SCC corresponding to y.

Then, H is a DAG.



STRONGLY CONNECTED COMPONENTS via DFS

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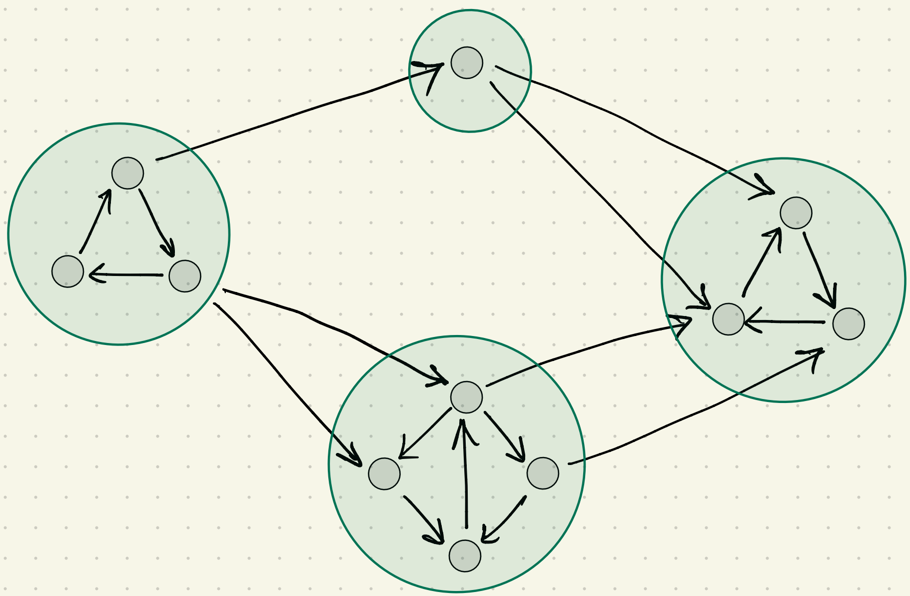

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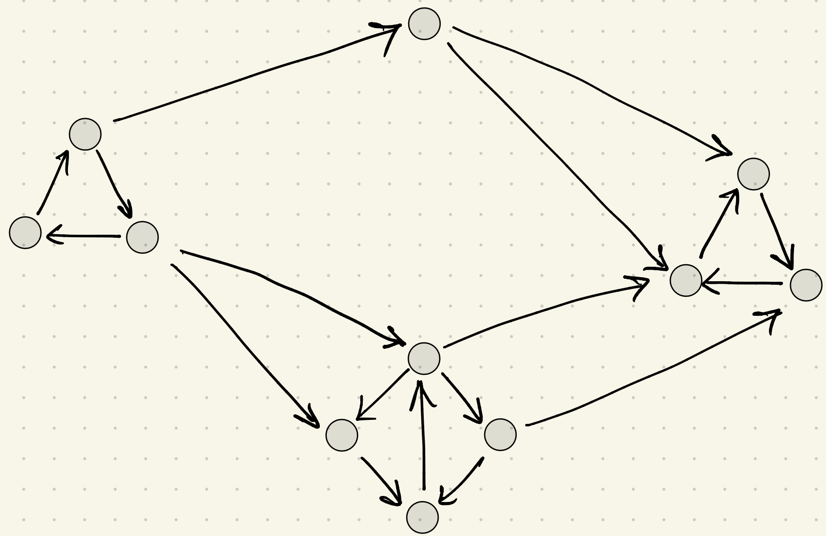

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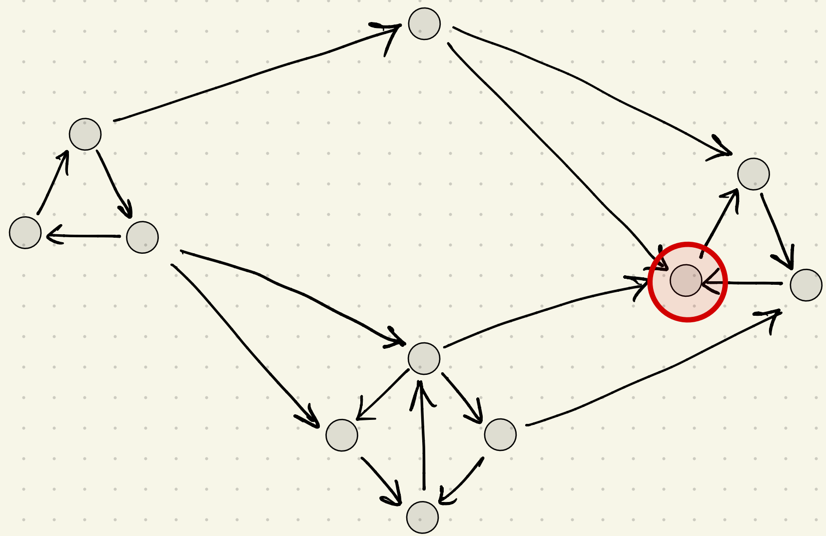

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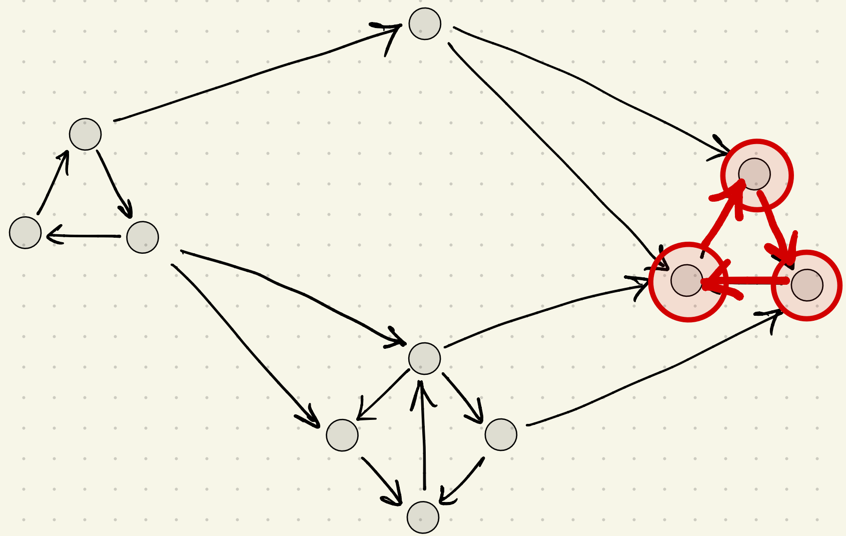
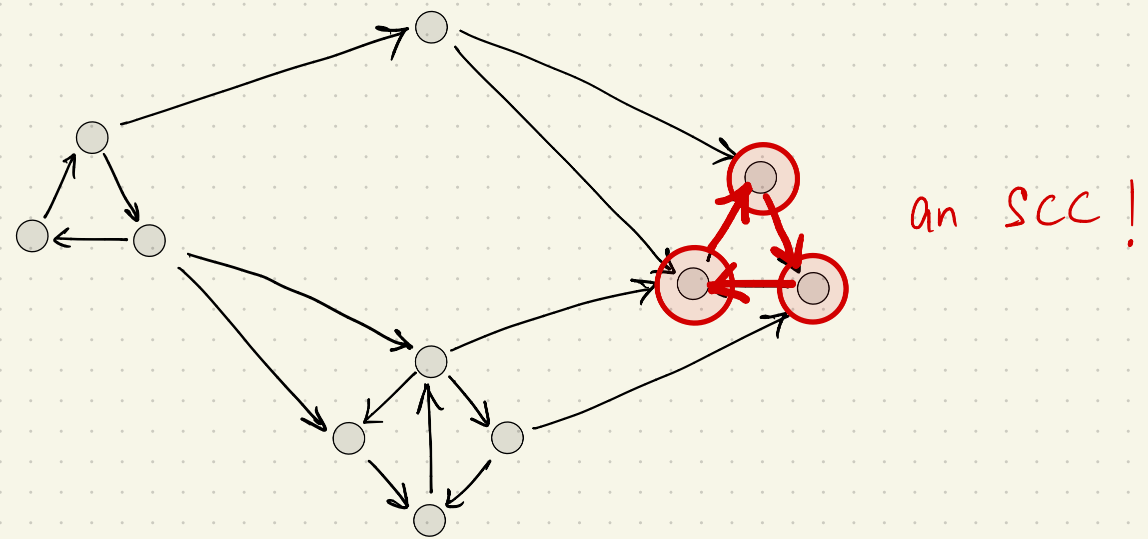


STRONGLY CONNECTED COMPONENTS via DFS



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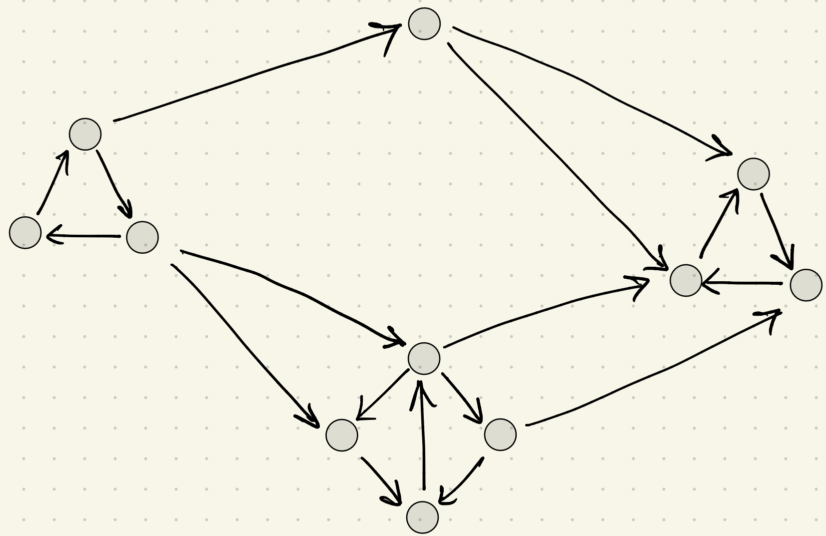

🤔 Maybe do a DFS from every node to identify all SCCs ?

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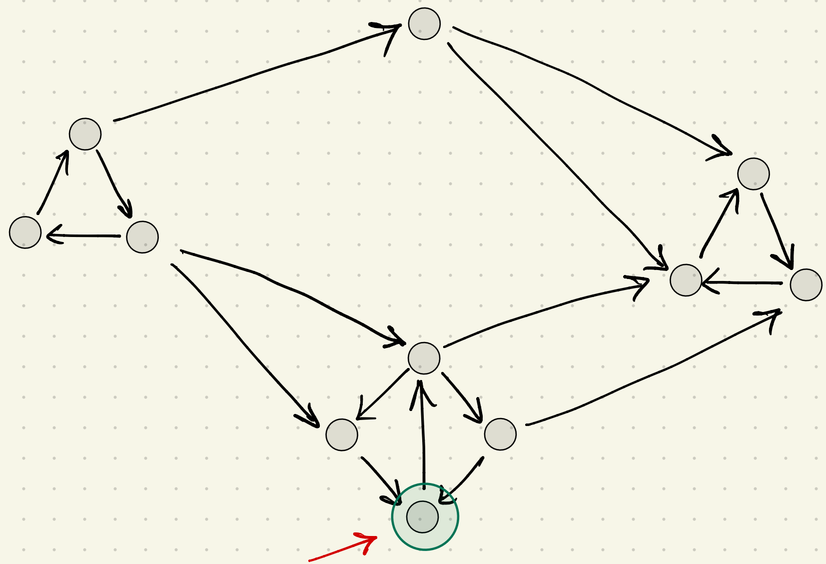

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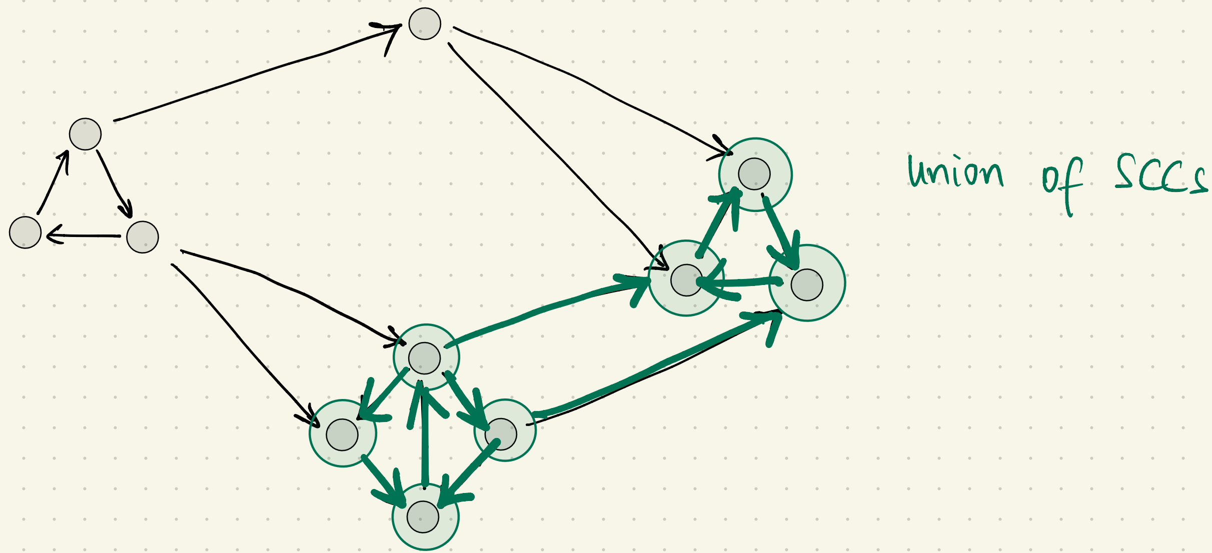

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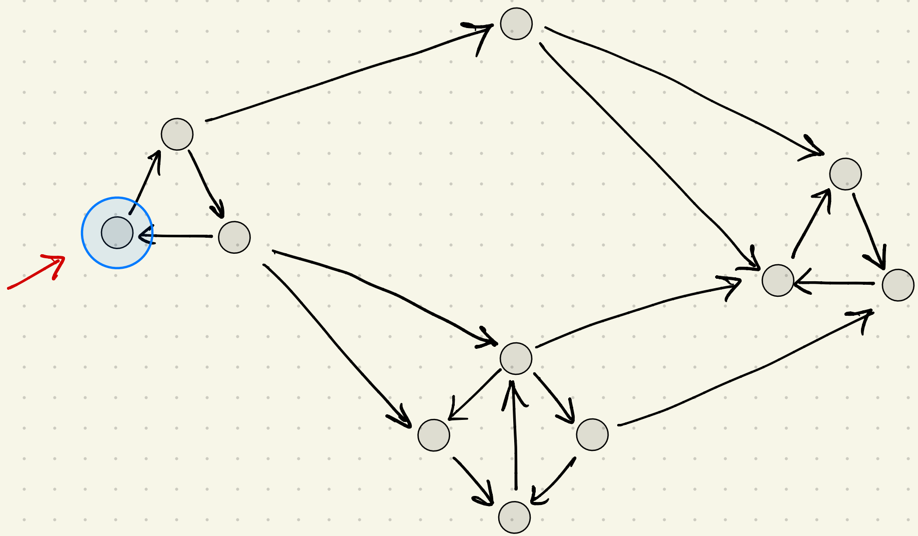

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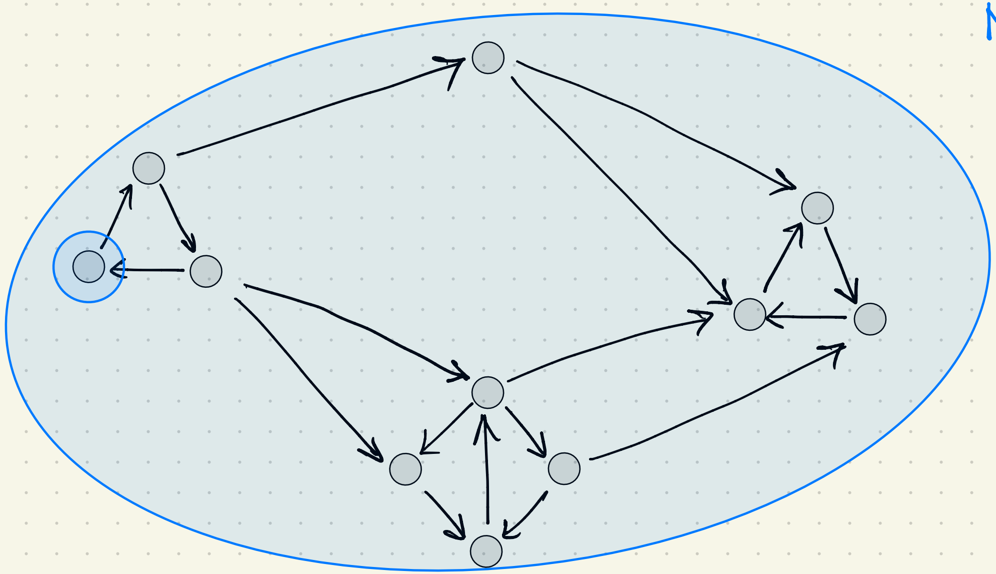
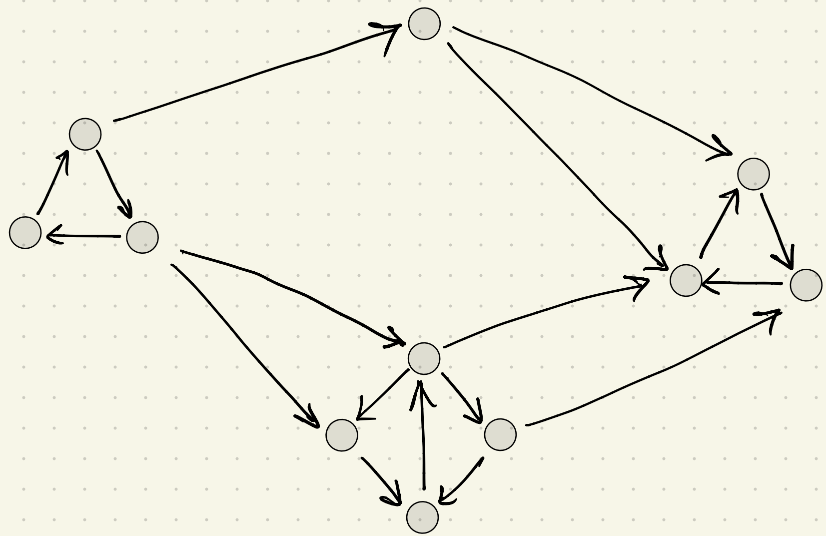


No information at all!

🤔 Maybe do a DFS from every node to identify all SCCs?

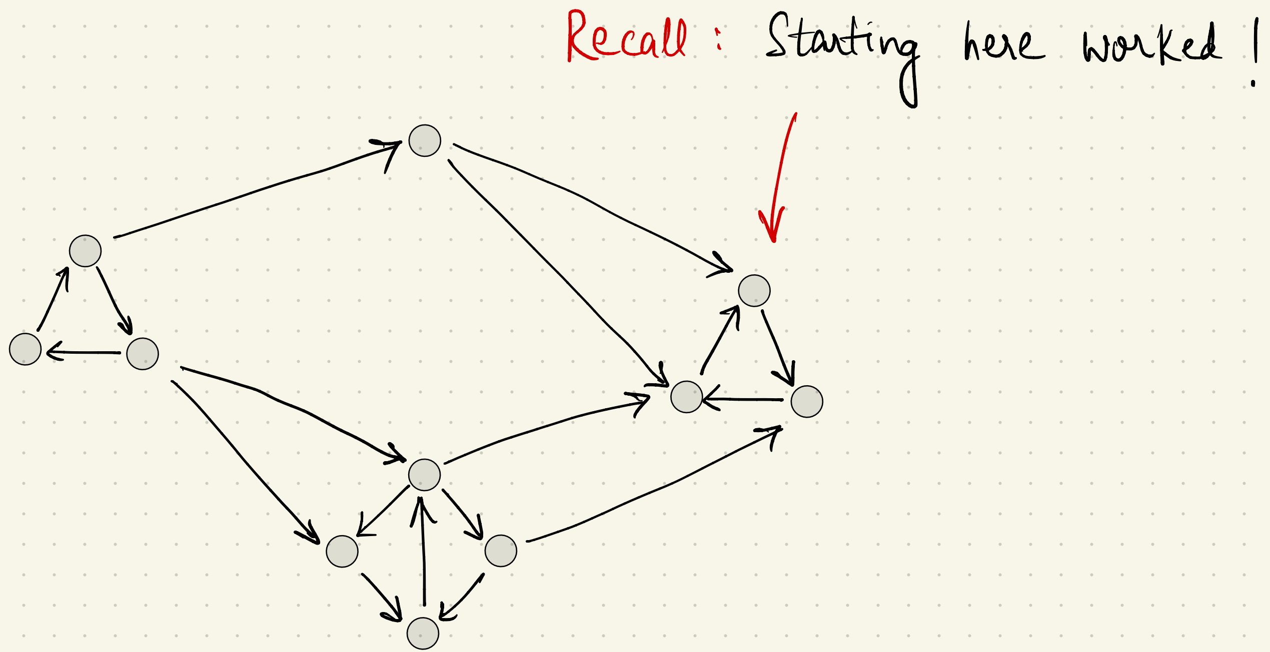
STRONGLY CONNECTED COMPONENTS via DFS



Starting point matters!

First Attempt

FIRST ATTEMPT



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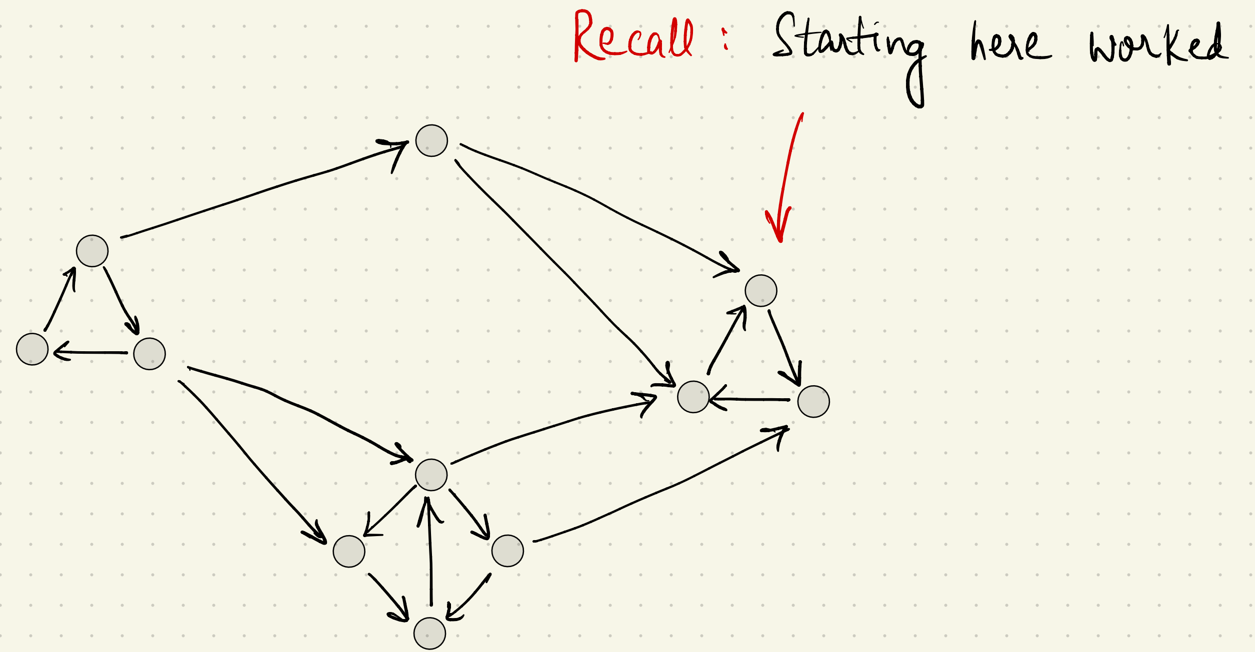
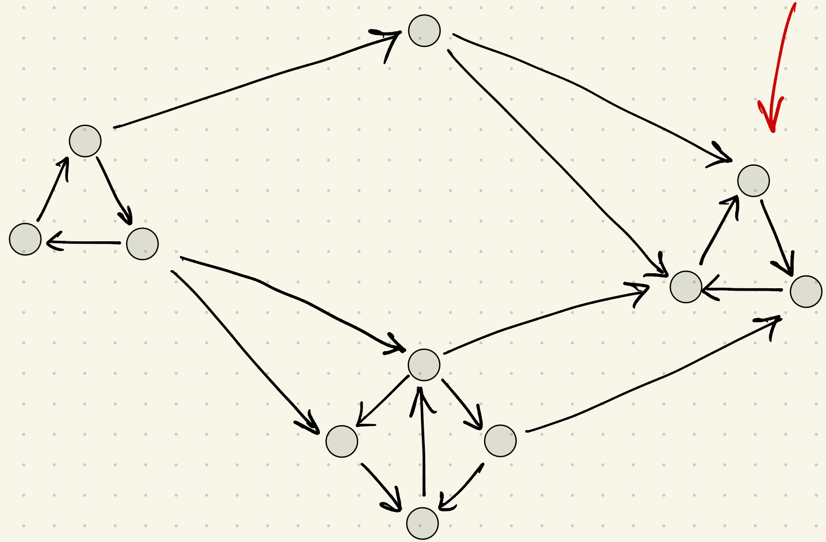


Start DFS at the "last" vertex
(as per topological ordering algo)

FIRST ATTEMPT

Recall : Starting here worked !



maybe it lies in the "sink" SCC ?

Start DFS at the "last" vertex
(as per topological ordering algo)

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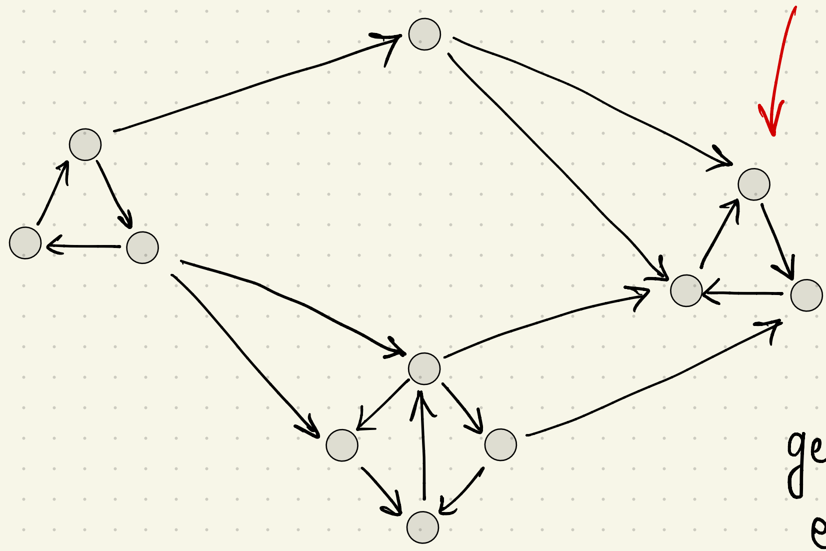
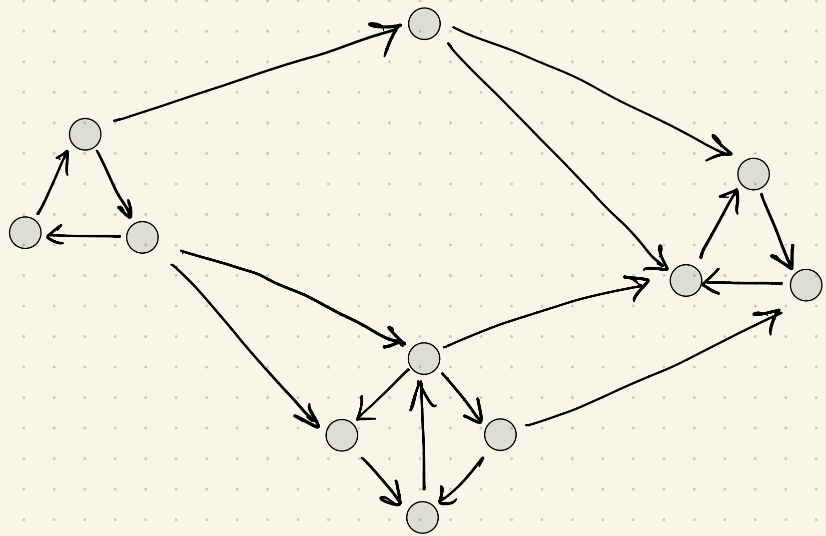
Recall : Starting here worked !



generates some ordering
even for cyclic graphs

Start DFS at the "last" vertex
(as per topological ordering algo)

FIRST ATTEMPT



Let's run topological ordering (via DFS) and see what happens.

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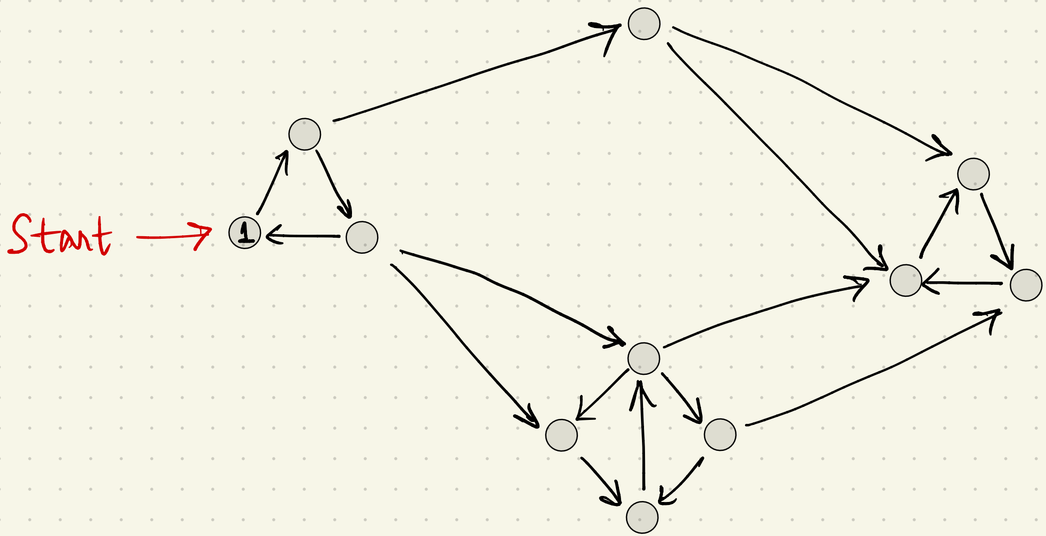

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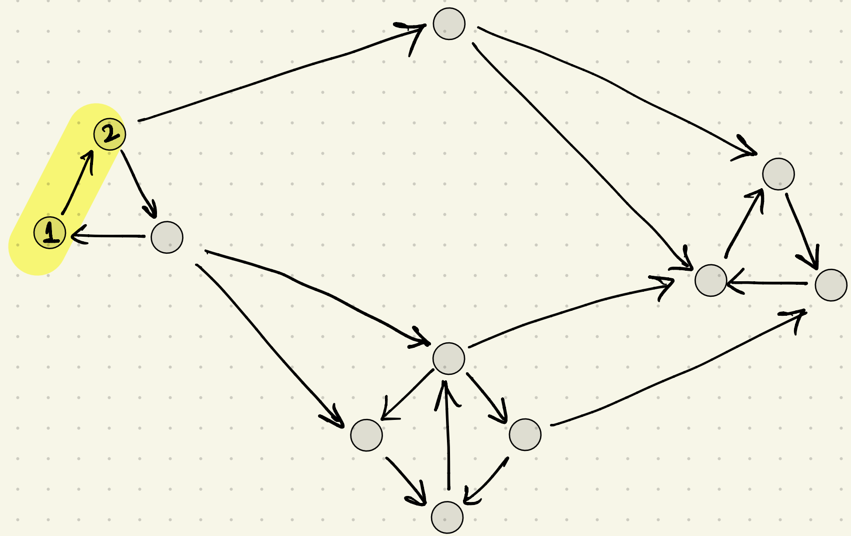

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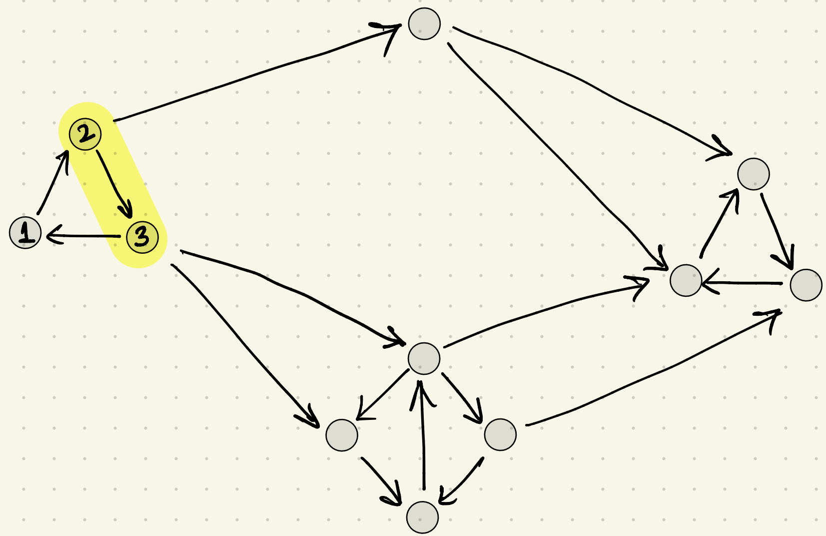

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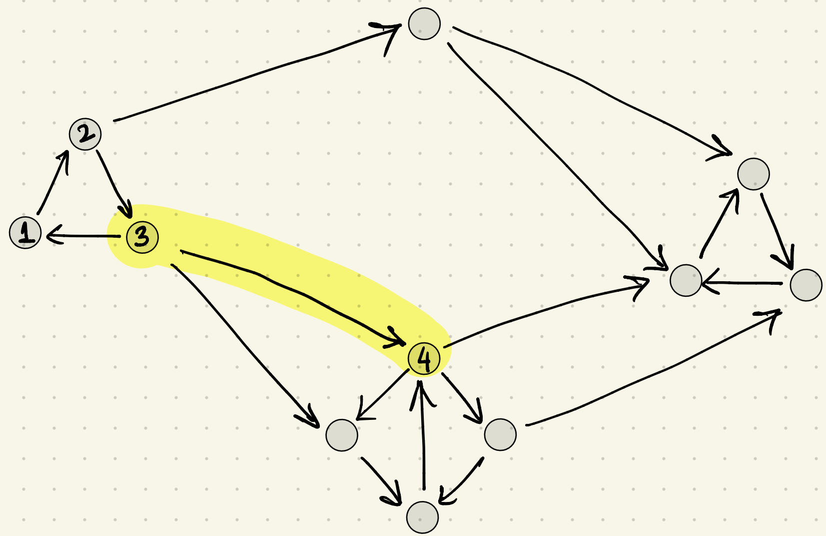

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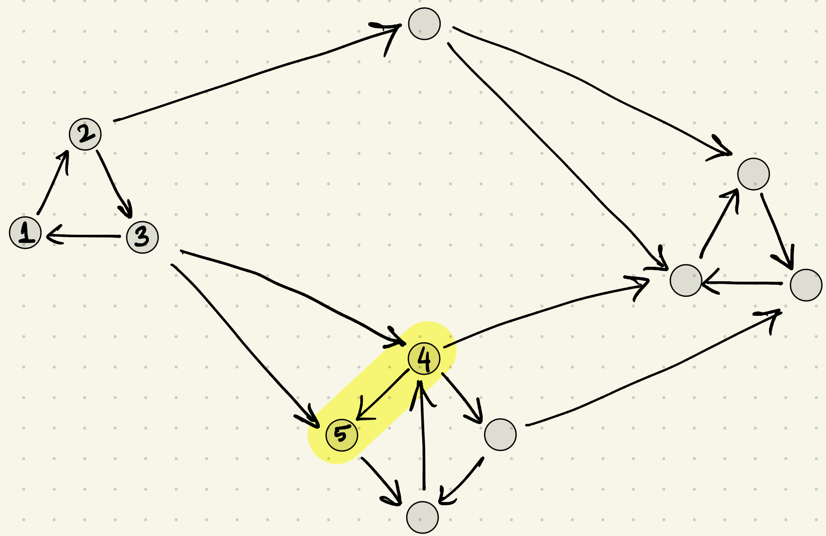

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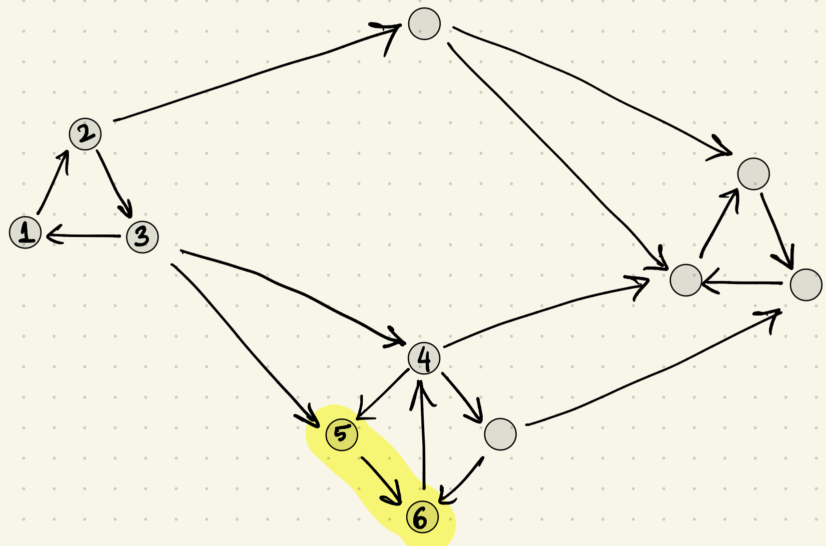

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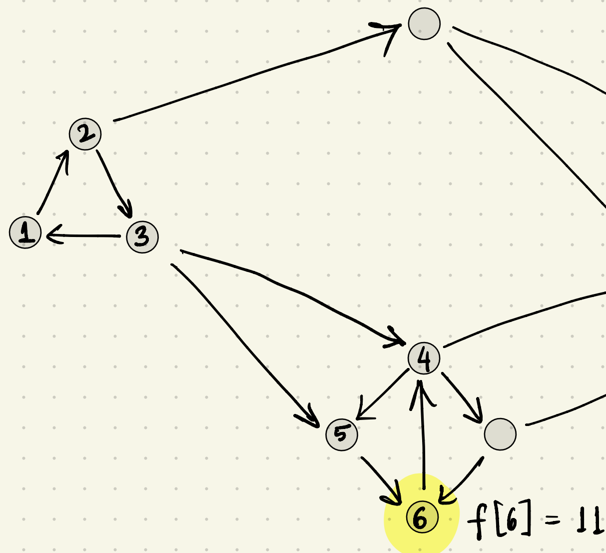

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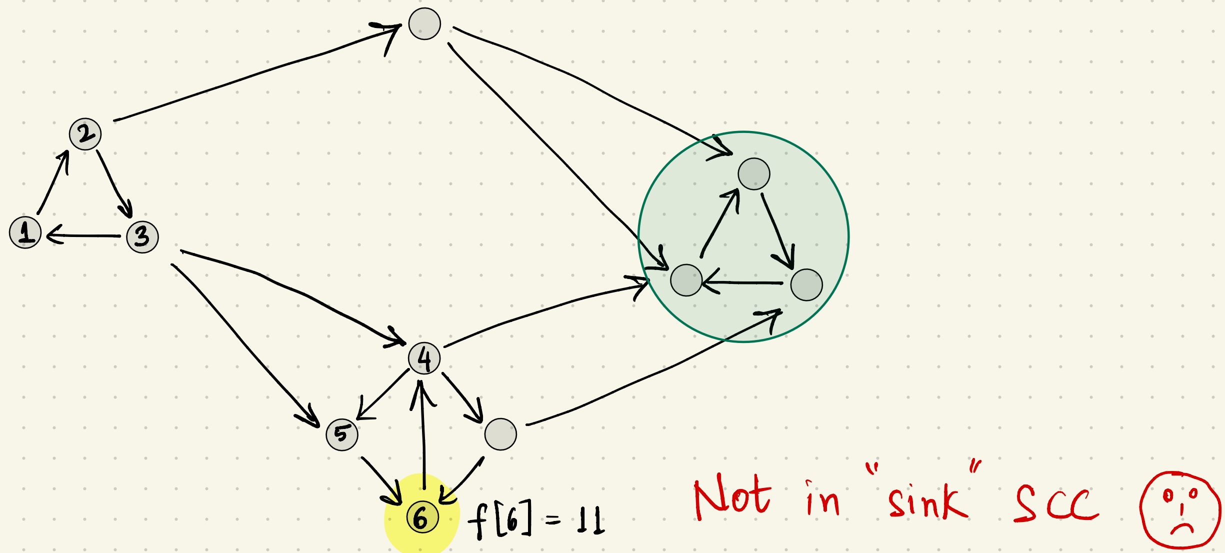

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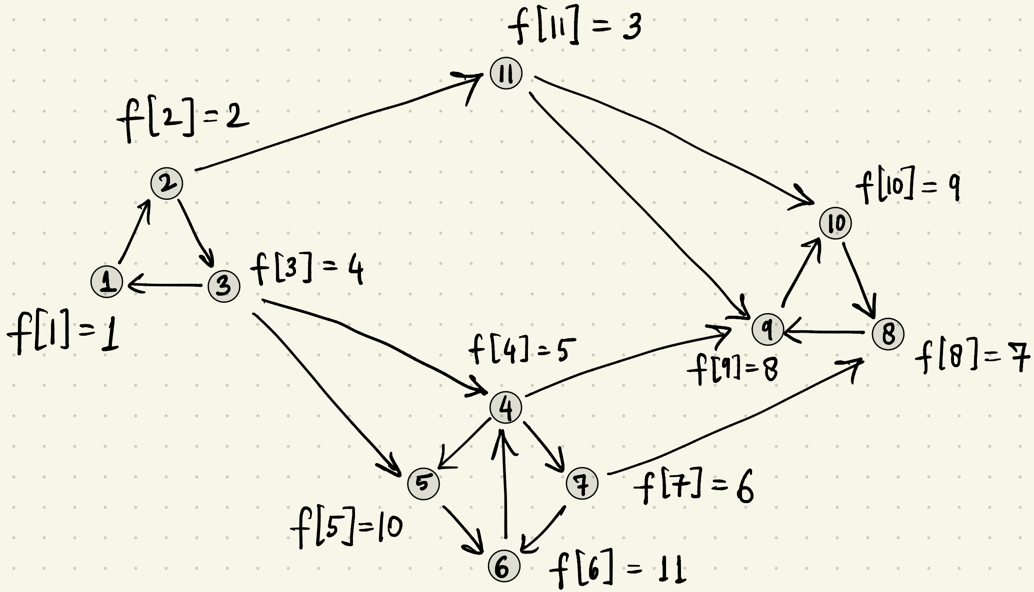

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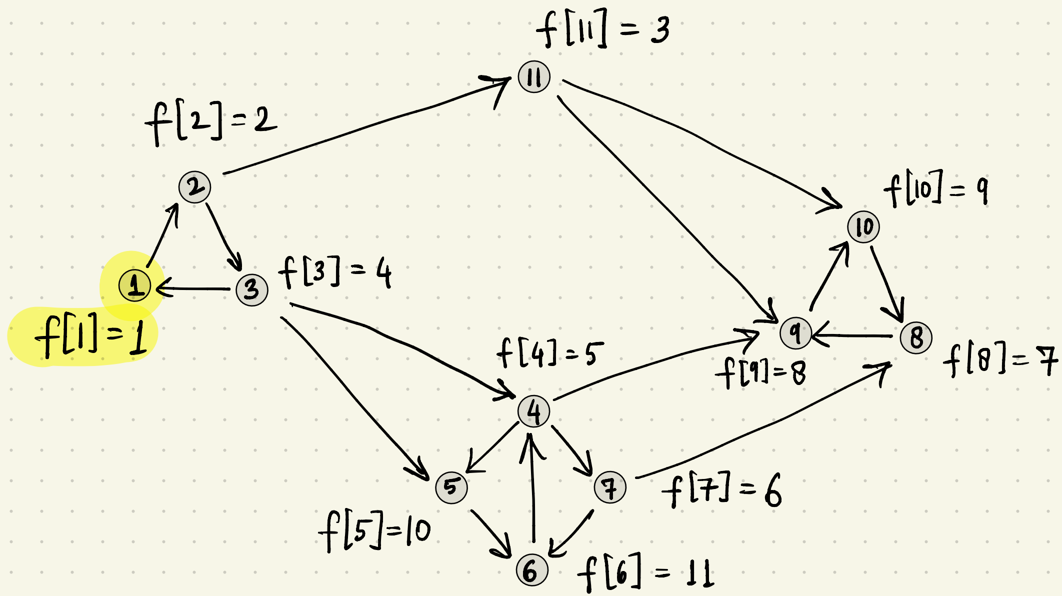

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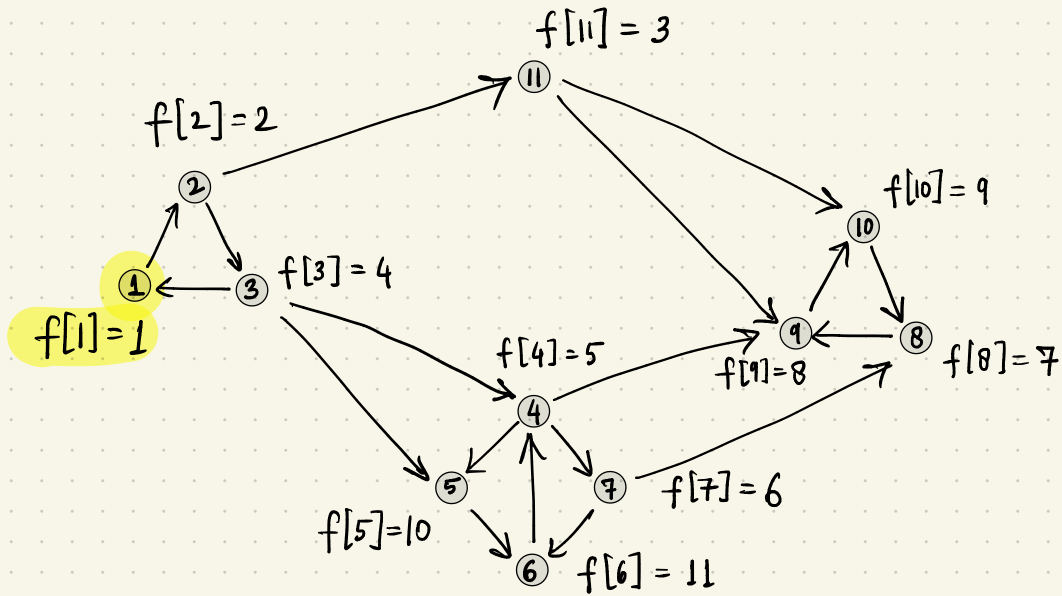
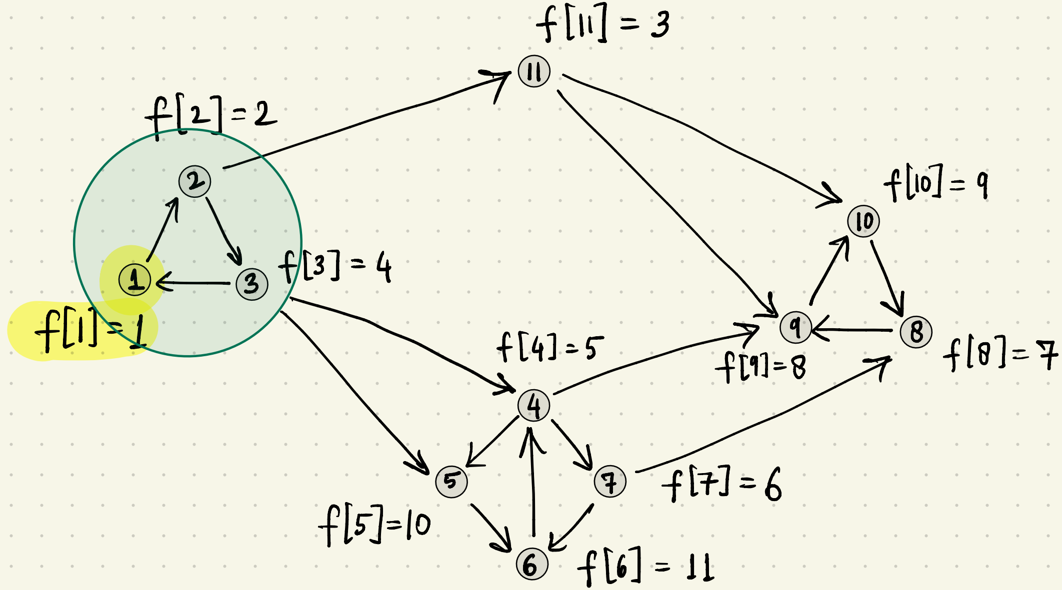


Perhaps the "smallest-label" vertex is always in "source" SCC?
(as per topological ordering)

FIRST ATTEMPT



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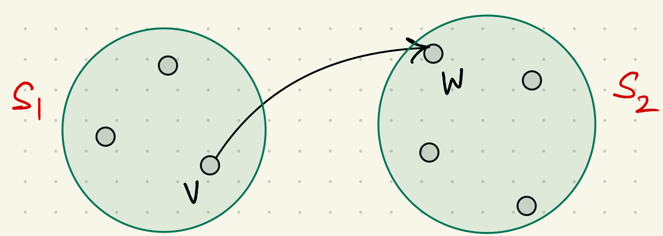
KEY OBSERVATION

Theorem: Let f be the labeling of directed graph G generated by the topological ordering algorithm on G (arbitrary ordering of vertices).

Let S_1, S_2 be two "adjacent" SCCs of G, i.e., there is an edge (v, w) with $v \in S_1$ and $w \in S_2$. Then,

$$\min_{x \in S_1} f(x) < \min_{y \in S_2} f(y).$$



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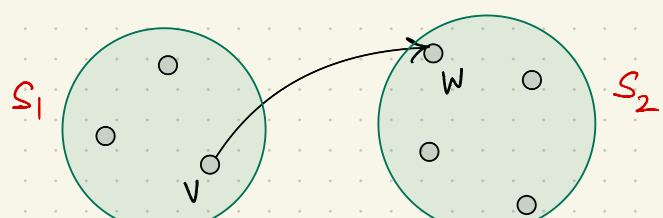

recursive call of some vertex in S_1 finishes after that of all vertices in S_2.

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Generalization of "every DAG has a topological ordering".

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$$\min_{x \in S_1} f(x) < \min_{y \in S_2} f(y).$$

Corollary: The vertex v with $f[v] = 1$ must lie in source SCC.

Proof:

Proof: Case I : Topological algo. discovers some vertex in S_1 before S_2.

Case II : Topological algo. discovers some vertex in S_2 before S_1.

Proof: Case I : Topological algo. discovers some vertex in S_1 before S_2.

All vertices in S_2 reachable from some vertex in S_1.

Case II : Topological algo. discovers some vertex in S_2 before S_1.

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What we have: A way of identifying a vertex in source SCC.

What we want: A way of identifying a vertex in sink SCC.

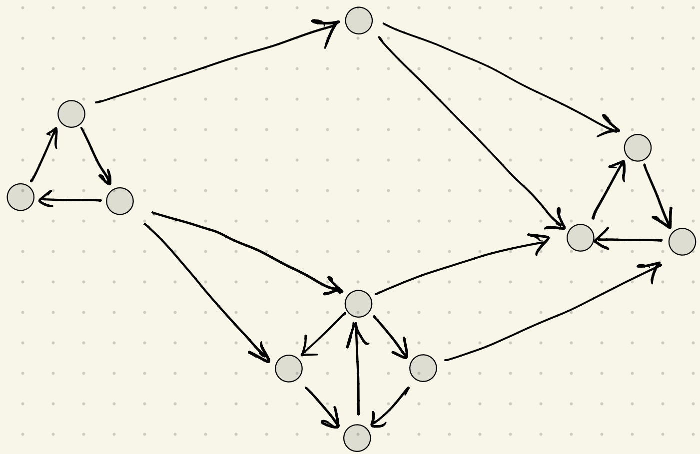
What's the fix:

What we have : A way of identifying a vertex in source SCC.

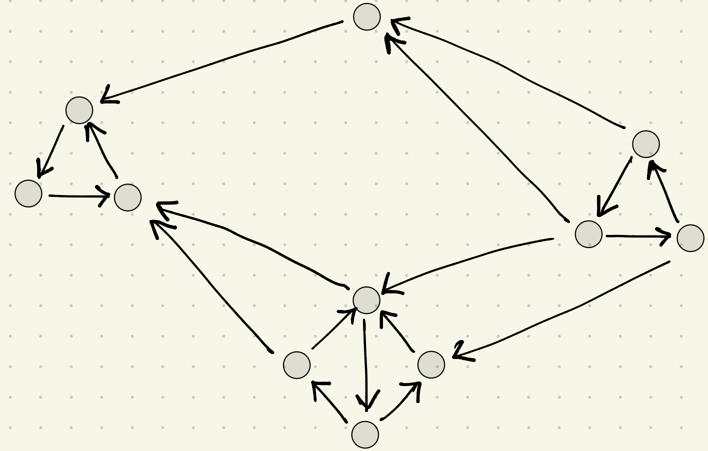
What we want : A way of identifying a vertex in sink SCC.

What's the fix : Reverse the graph!

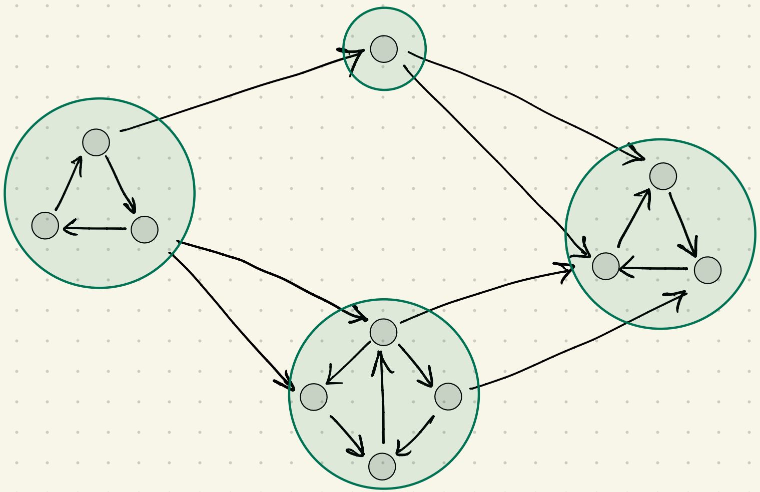
G



G^{rev}



G



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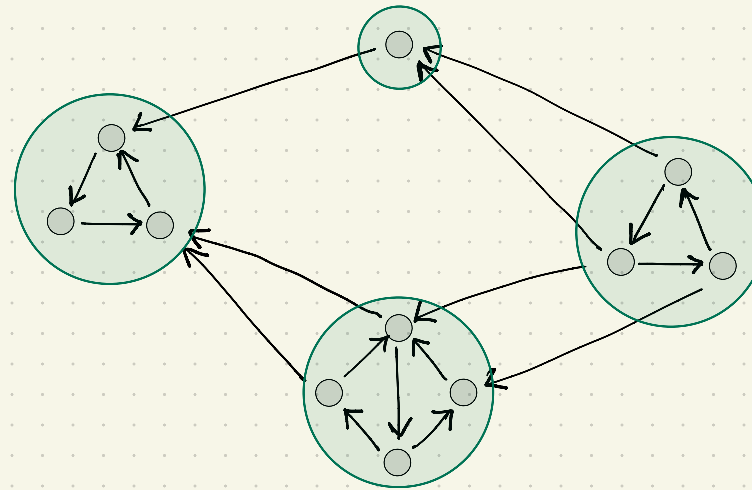
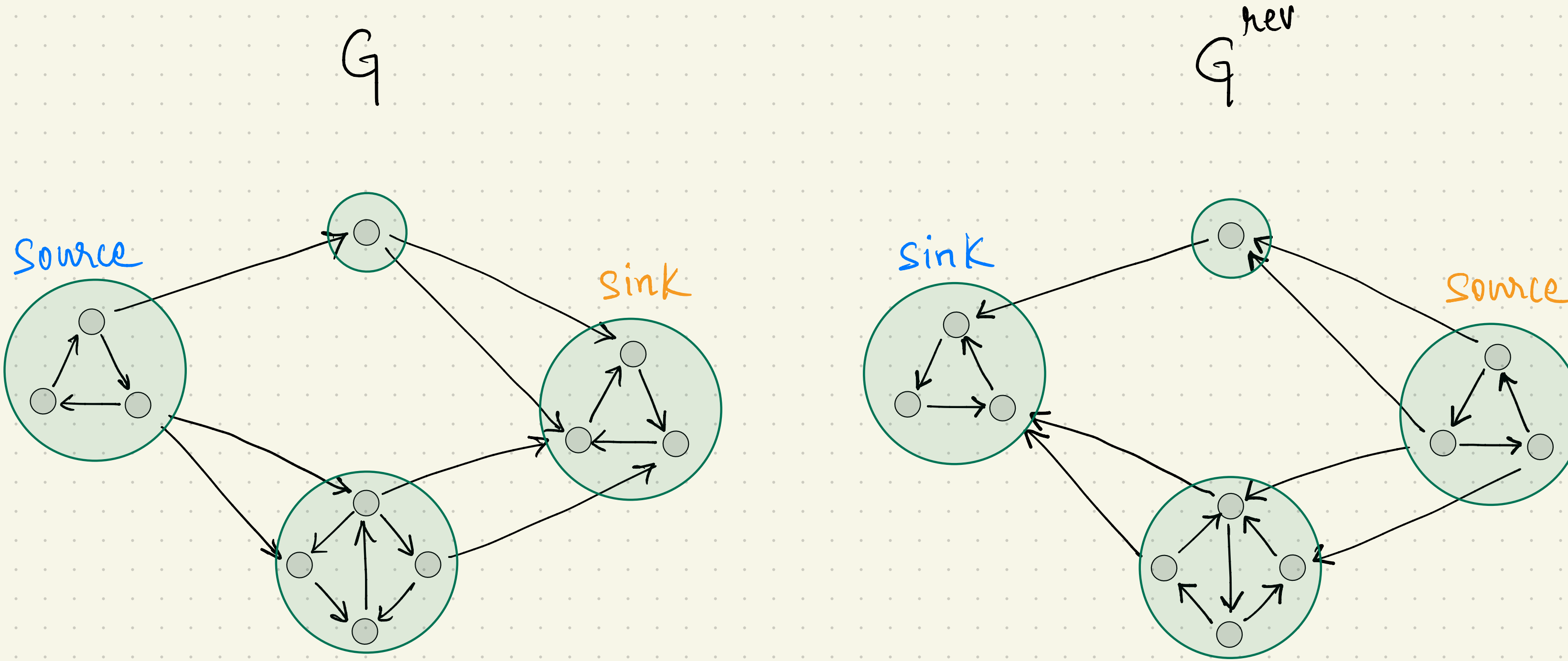


SCCs stay the same!



Source / sink in meta graph of G

Sink / Source " " " G^{rev}