COL866: Special Topics in Algorithms

Lecture 9

Fairness and Efficiency

Sep 01, 2023 | Rohit Vaish

The Model

	A	B	C	D	E
	4	1	2	2	2
	1	0	5	1	1
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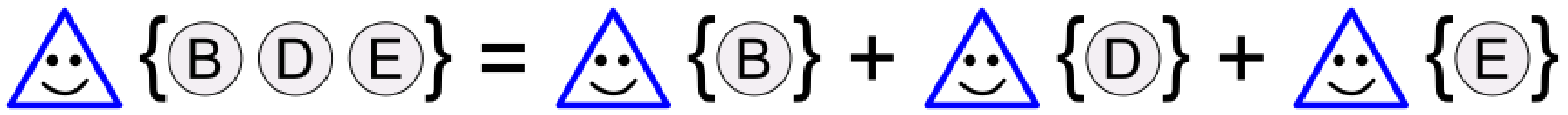
Additive valuations



= 0+1+1 = 2

Envy-Freeness Up To One Good [Budish, 2011]

Envy can be eliminated by removing some good in the envied bundle.

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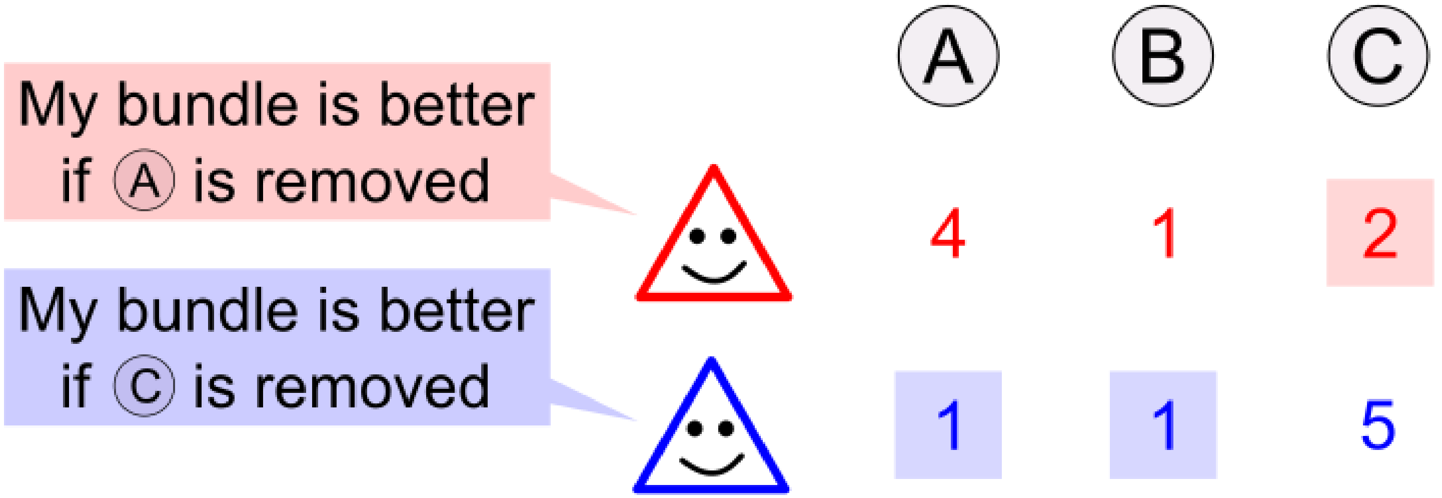

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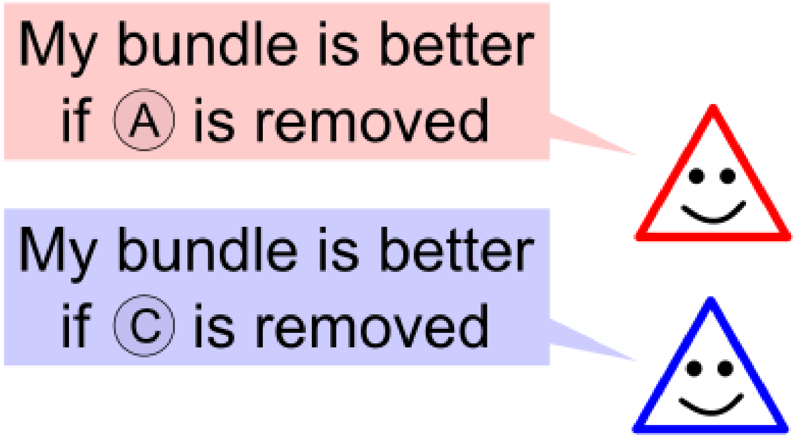

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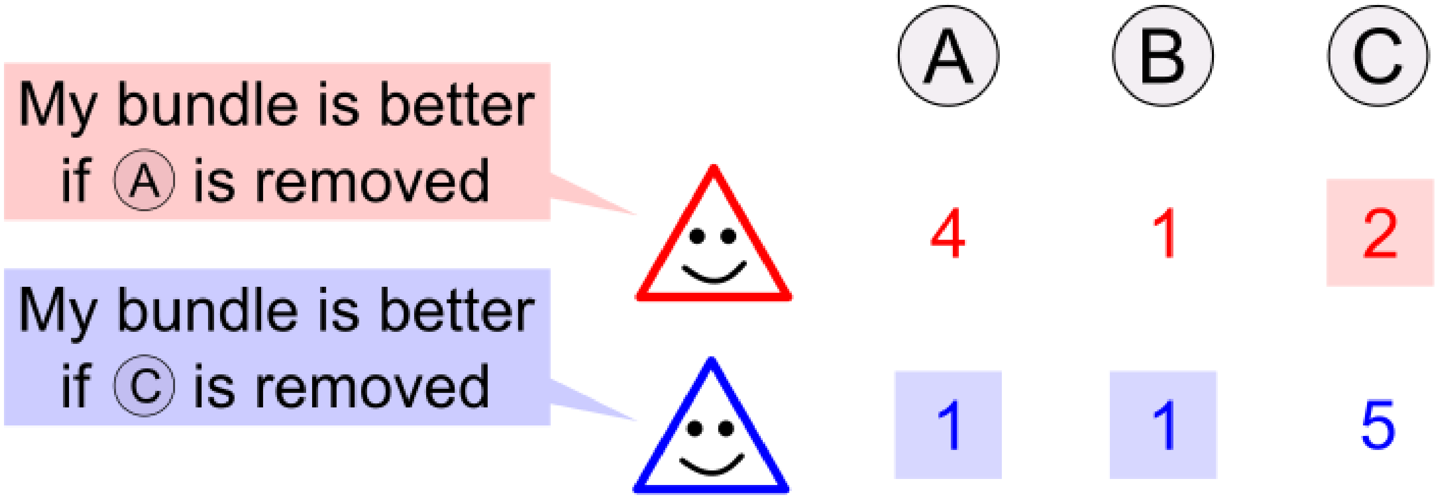

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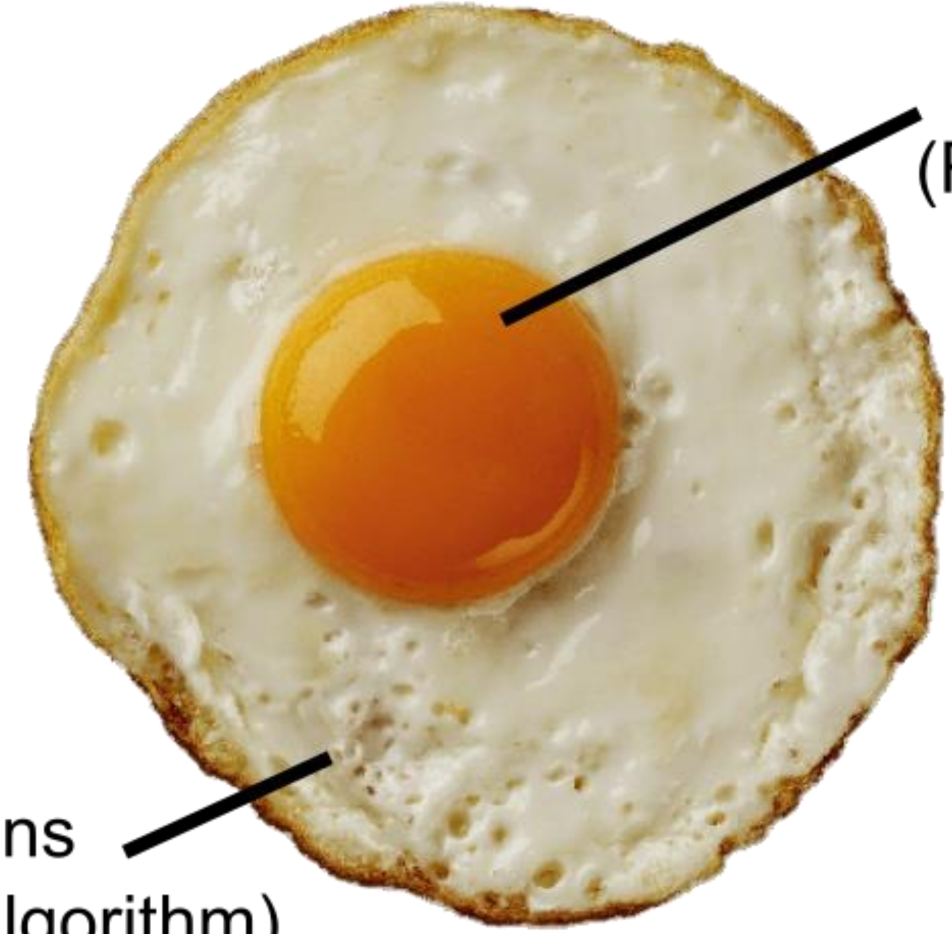
Guaranteed to exist and efficiently computable

Last Time

Algorithms for finding an EF1 allocation



A trivial way of achieving fairness: Don't allocate anything!

A bare minimum efficiency requirement: Completeness

WHEN A COMPLETE ALLOCATION

SIMPLY ISN'T ENOUGH

"Obvious" Improvement

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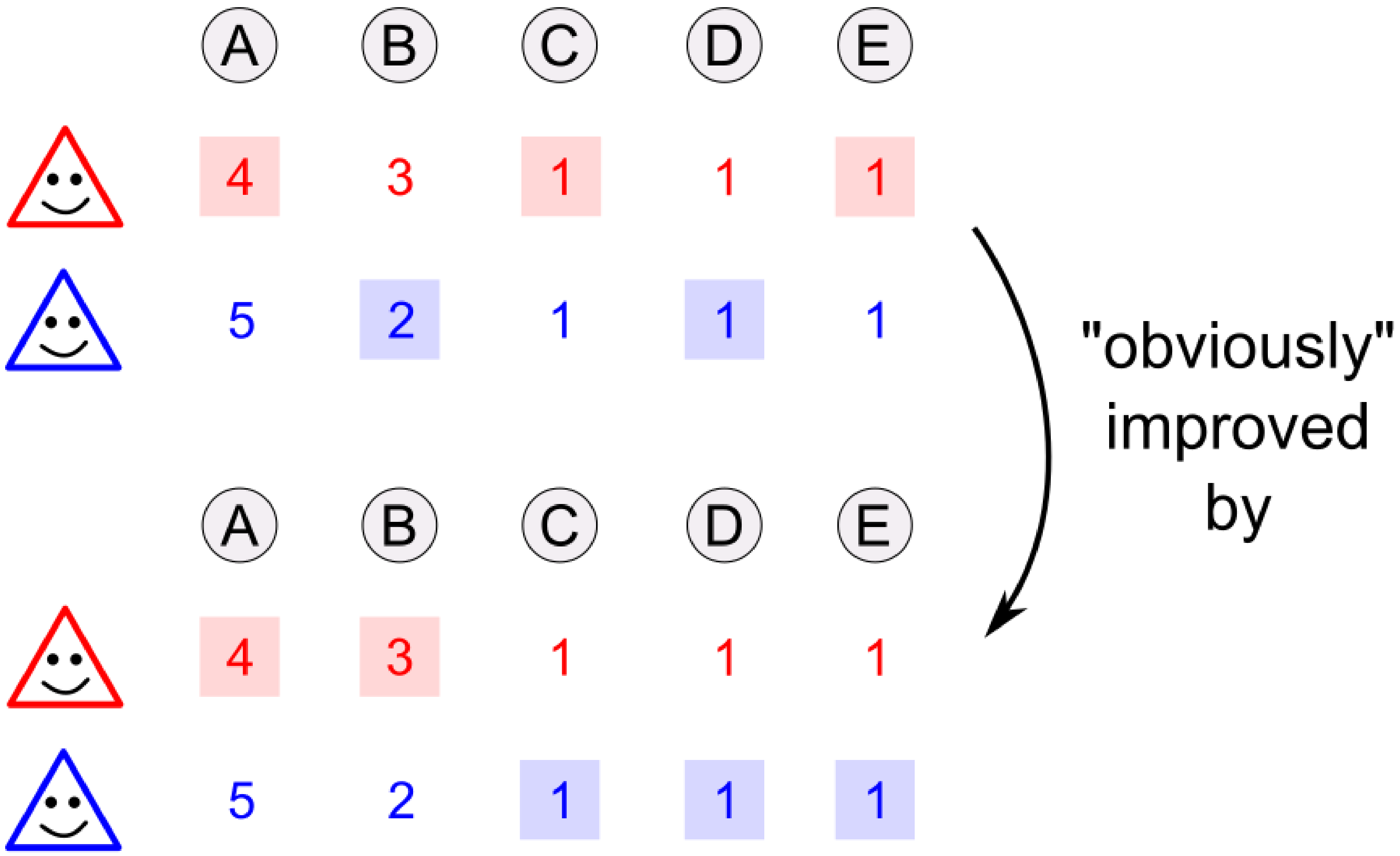

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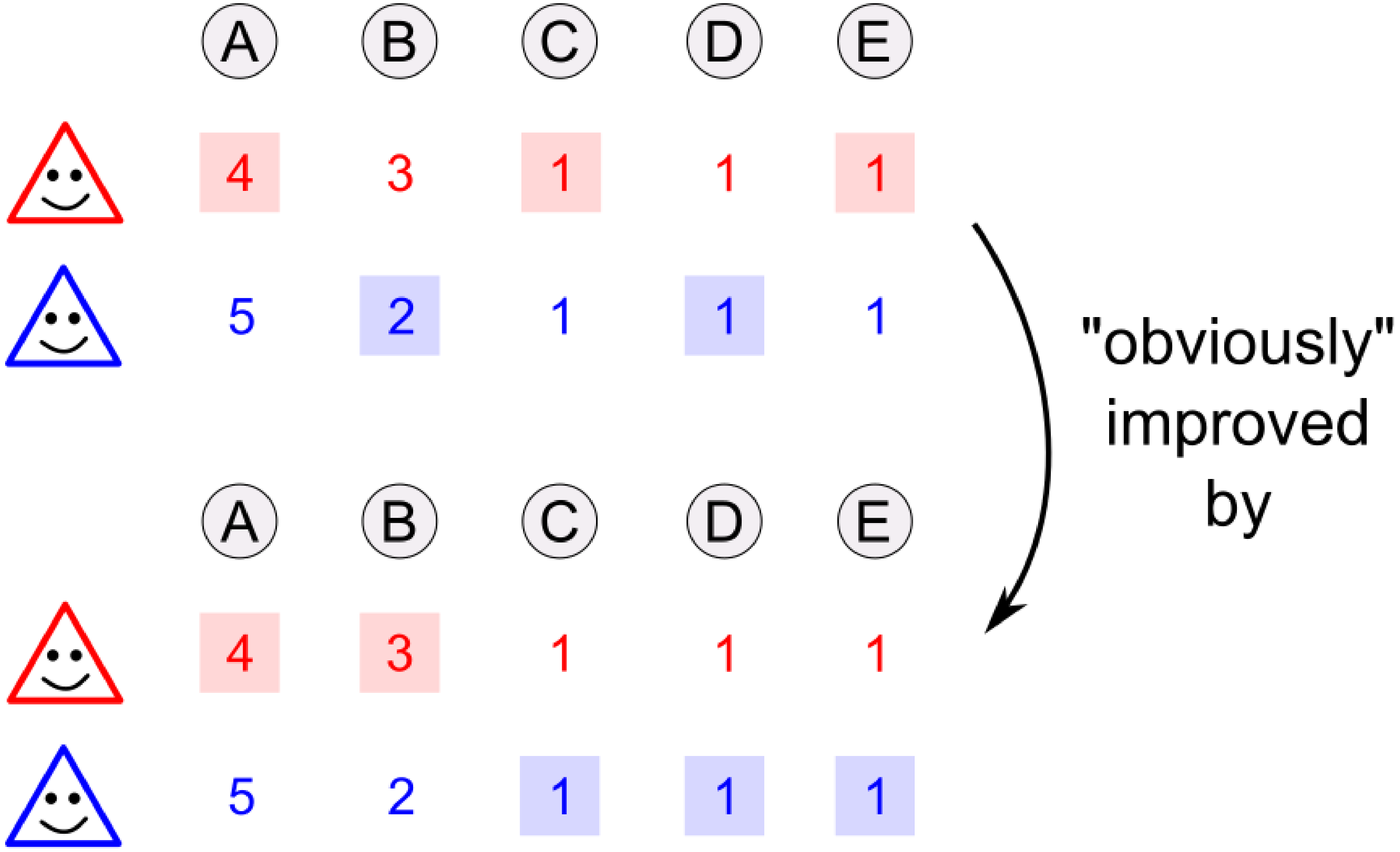

Strictly improving someone without hurting anyone else

Pareto Optimality

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To make someone better off, someone else must be made worse off.

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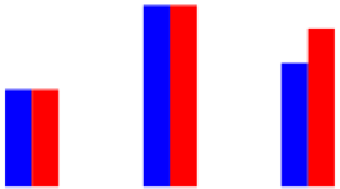
Allocation B *Pareto improves* A if 

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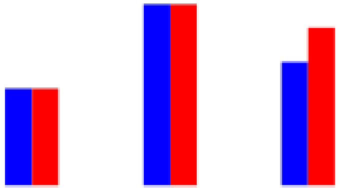
Allocation B *Pareto improves* A if 

Allocation A is *Pareto optimal* (PO) if no other allocation B Pareto improves it.

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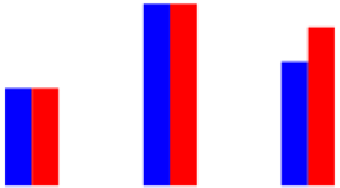
Allocation B *Pareto improves* A if 

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Is EF1 compatible with Pareto optimality?

Round Robin Fails Pareto Optimality

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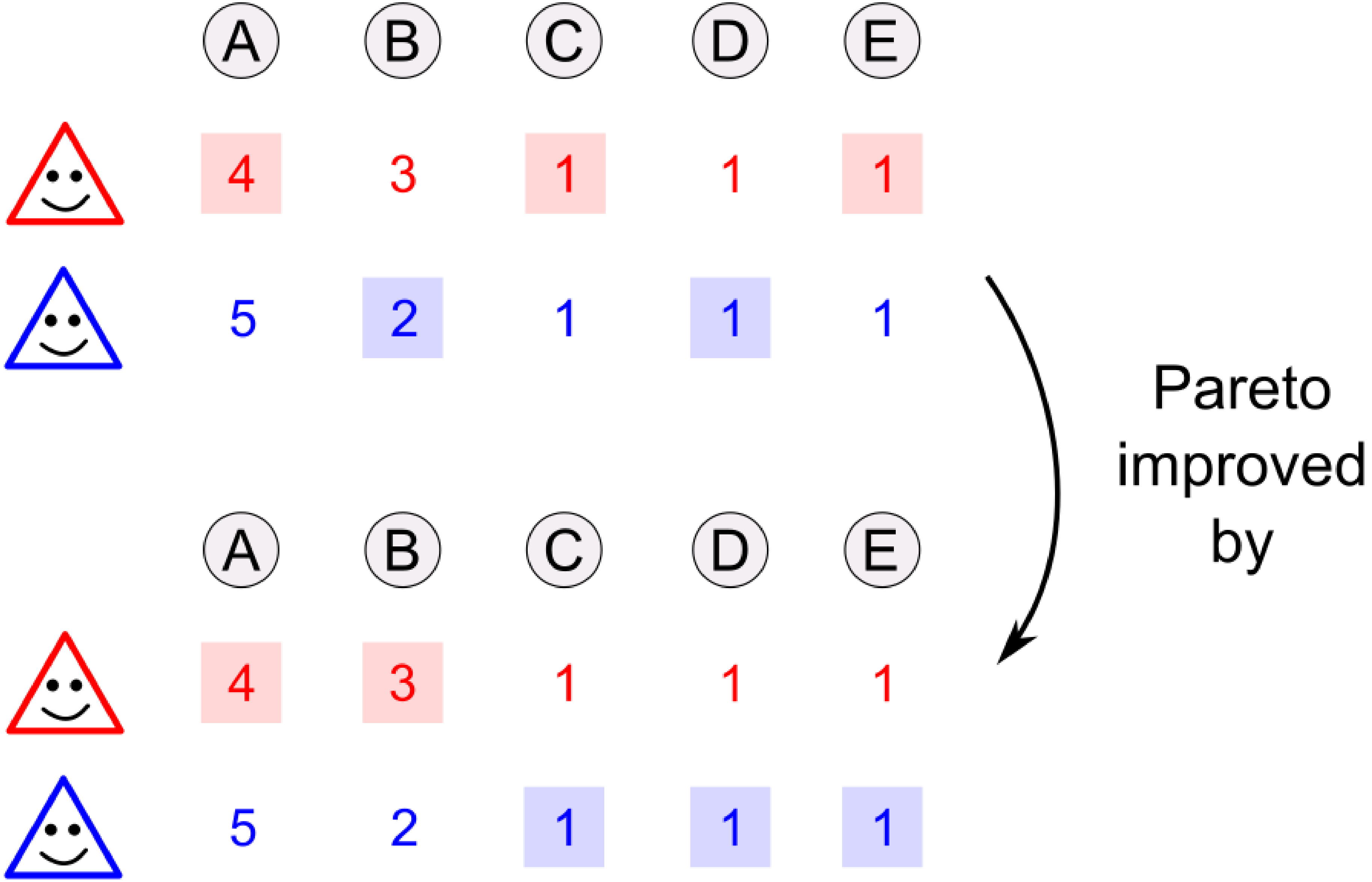

Envy-Cycle Elimination Fails Pareto Optimality

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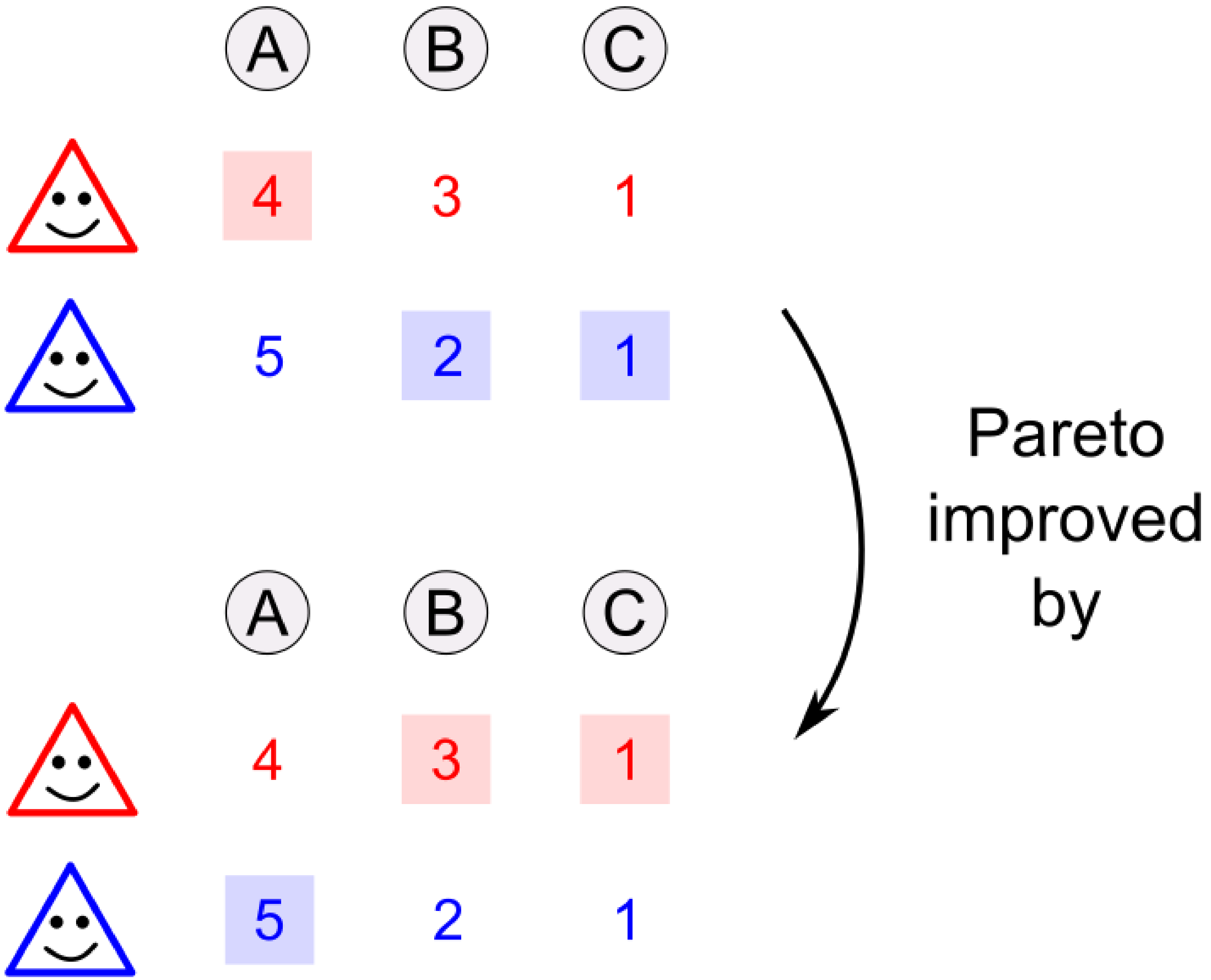

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Start with an EF1 allocation, and repeatedly make Pareto improvements to it.

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Exercise: Pareto improvement can fail to preserve EF1.

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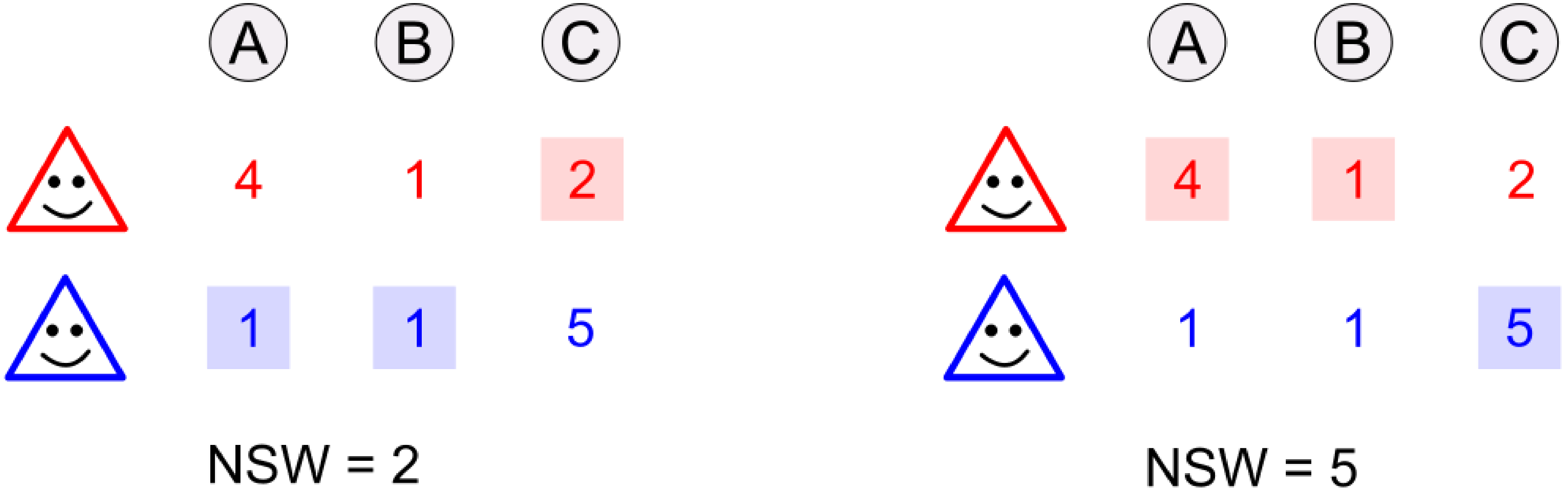

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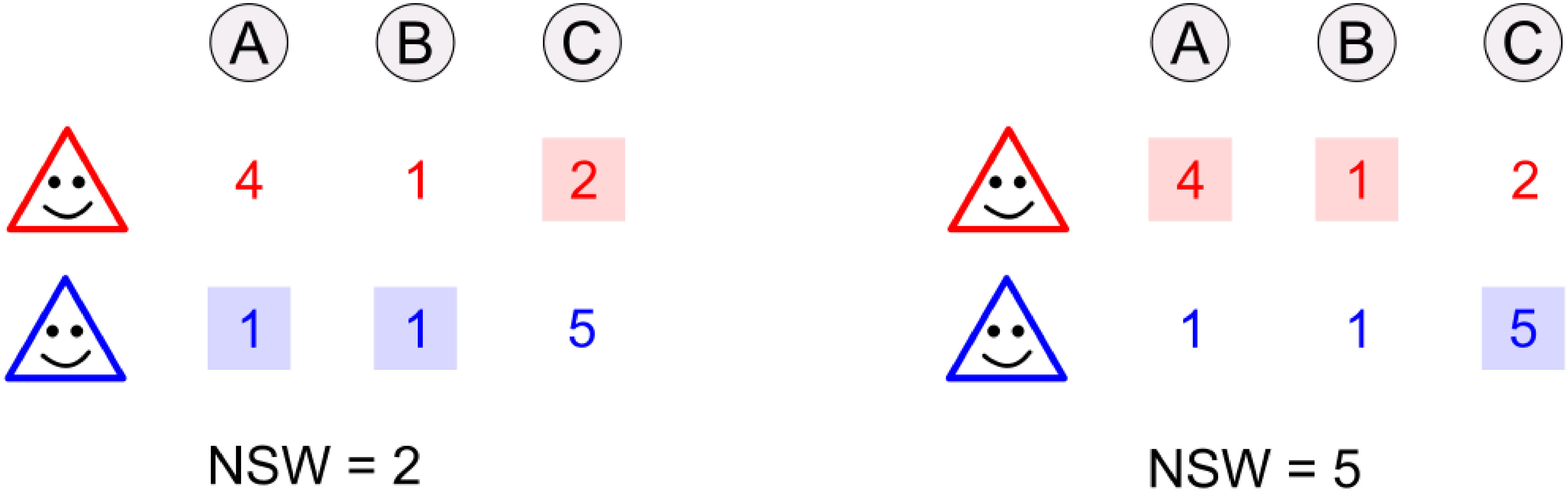

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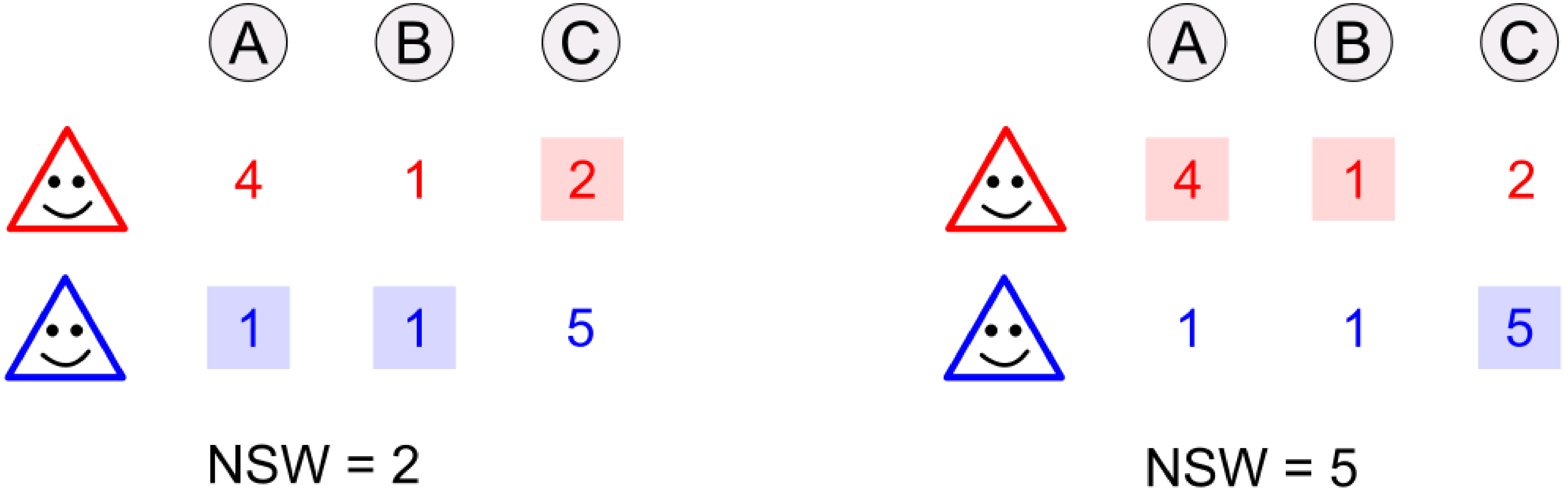

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*If optimal is 0, then find *any* largest set of agents who can simultaneously be given positive utility and maximize the geometric mean with respect to only those agents.

[Caragiannis, Kurokawa, Moulin, Procaccia, Shah, and Wang, *TEAC* 2019]

Any Nash optimal allocation satisfies Pareto optimality (PO) and envy-freeness up to one good (EF1).

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Why PO?

Pareto improvement strictly improves NSW.

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Does such a good g* always exist?

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We will show that transferring g* from agent k to agent i improves NSW.

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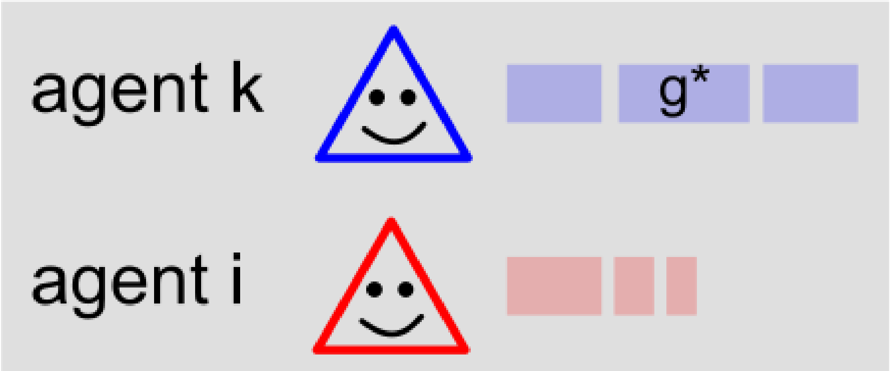

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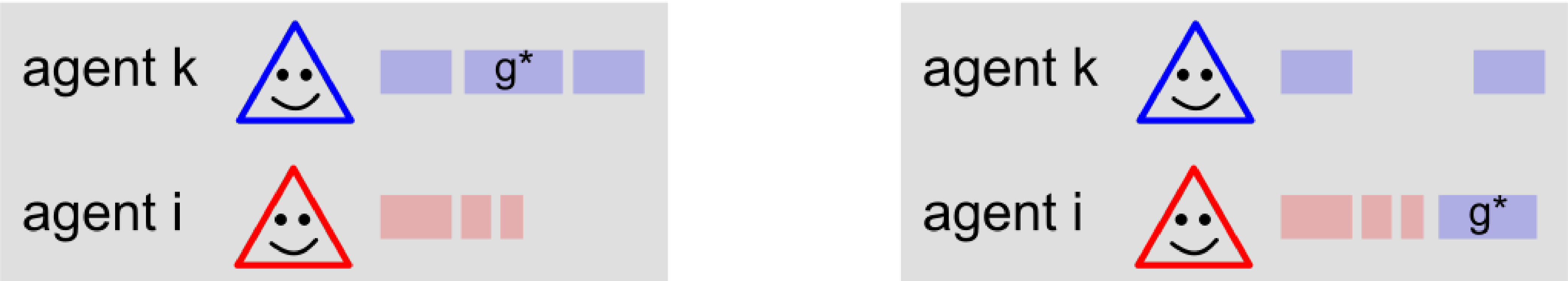

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$$\Leftrightarrow \left(1 - \frac{v_k(g^*)}{v_k(A_k)}\right) \cdot \left(1 + \frac{v_i(g^*)}{v_i(A_i)}\right) > 1$$

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$$\frac{\text{NSW}(B)}{\text{NSW}(A)} > 1 \Leftrightarrow v_k(B_k) \cdot v_i(B_i) > v_k(A_k) \cdot v_i(A_i)$$

$$\Leftrightarrow (v_k(A_k) - v_k(g^*)) \cdot (v_i(A_i) + v_i(g^*)) > v_k(A_k) \cdot v_i(A_i)$$

$$\Leftrightarrow \left(1 - \frac{v_k(g^*)}{v_k(A_k)}\right) \cdot \left(1 + \frac{v_i(g^*)}{v_i(A_i)}\right) > 1$$

$$\Leftrightarrow v_k(A_k) > \frac{v_k(g^*)}{v_i(g^*)} [v_i(A_i) + v_i(g^*)]$$

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combining these

■

Ok, so an EF1+PO allocation always exists.

But what about computation?

Can a Nash optimal allocation be efficiently computed?

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[Nguyen, Nguyen, Roos, and Rothe, *JAAMAS 2014*; Lee, *IPL. 2017*]

Maximizing Nash social welfare is APX-hard.

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An EF1+PO allocation can be computed in pseudopolynomial time.

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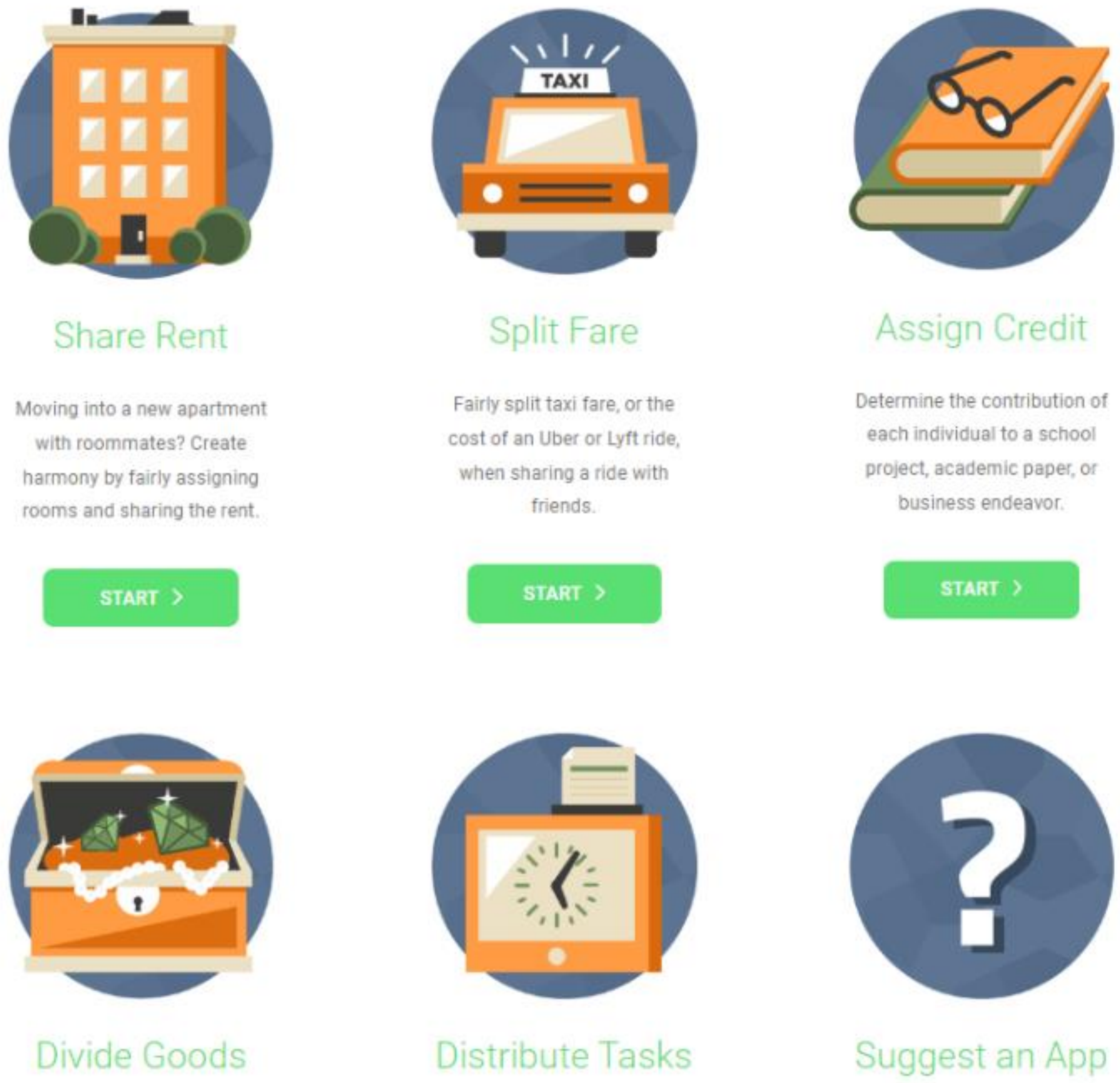
- A 0.69-approximation to Nash social welfare objective

splíddít


TAXI
Share Rent
Moving into a new apartment with roommates? Create harmony by fairly assigning rooms and sharing the rent.
START >
Split Fare
Fairly split taxi fare, or the cost of an Uber or Lyft ride, when sharing a ride with friends.
START >
Assign Credit
Determine the contribution of each individual to a school project, academic paper, or business endeavor.
START >
Divide Goods
Distribute Tasks
Suggest an App

Next Time

Envy-freeness up to *any* good (EFX)

Quiz

Quiz

Construct an instance where no Nash optimal allocation can be achieved as the outcome of round robin algorithm (for any ordering of agents, any tie-breaking choices, etc.).

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