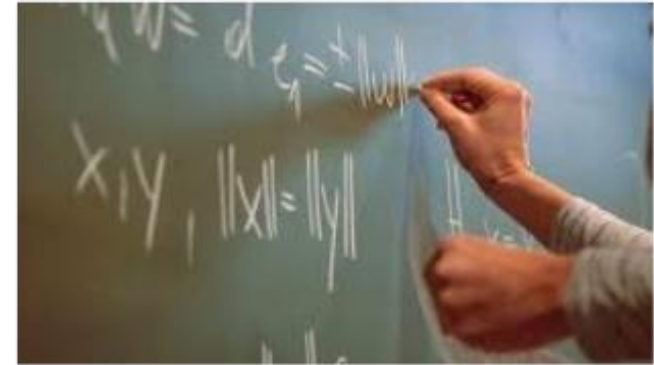


Lecture 8

Fair Allocation of Indivisible Goods



The Model

The Model



The Model

A

B



C

D




E






The Model

	(A)	(B)	(C)	(D)	(E)
	4	1	2	2	2
	1	0	5	1	1
	1	1	5	1	1

The Model

	(A)	(B)	(C)	(D)	(E)
	4	1	2	2	2
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The Model

	(A)	(B)	(C)	(D)	(E)
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Additive
valuations

$$\begin{aligned} \triangle \{ (B) (D) (E) \} &= \triangle \{ (B) \} + \triangle \{ (D) \} + \triangle \{ (E) \} \\ &= 0 + 1 + 1 = 2 \end{aligned}$$

Envy-Freeness



[Gamow and Stern, 1958; Foley, 1967]

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Each agent prefers its own bundle over that of any other agent.



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	(A)	(B)	(C)
	4	1	2
	1	1	5

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	(A)	(B)	(C)
	4	1	2
	1	1	5



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Each agent prefers its own bundle over that of any other agent.

	(A)	(B)	(C)
My bundle is the best	4	1	2
	1	1	5

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Each agent prefers its own bundle over that of any other agent.

	(A)	(B)	(C)
	4	1	2
 My bundle is the best	1	1	5

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Each agent prefers its own bundle over that of any other agent.

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My bundle is the best	4	1	2
My bundle is the best	1	1	5

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Each agent prefers its own bundle over that of any other agent.

	(A)	(B)	(C)
My bundle is the best	4	1	2
My bundle is the best	1	1	5

Allocation $A = (A_1, A_2, \dots, A_n)$ is EF if for every pair of agents i, k , we have $v_i(A_i) \geq v_i(A_k)$.

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Not guaranteed to exist (two agents, one good)

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Not guaranteed to exist (two agents, one good)



Checking whether an EF allocation exists is NP-complete



Envy-Freeness Up To One Good [Budish, 2011]

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Envy can be eliminated by removing some good in the envied bundle.

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	(A)	(B)	(C)
	4	1	2
	1	1	5

Envy-Freeness Up To One Good [Budish, 2011]

Envy can be eliminated by removing some good in the envied bundle.

	(A)	(B)	(C)
Red Triangle	4	1	2
Blue Triangle	1	1	5

Envy-Freeness Up To One Good [Budish, 2011]

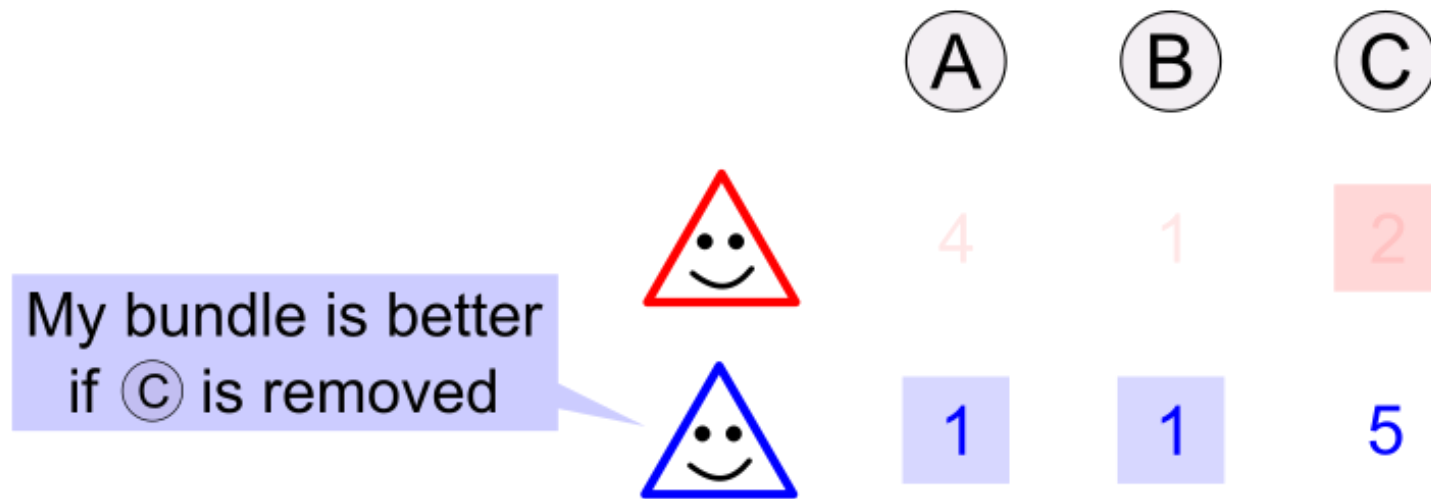
Envy can be eliminated by removing some good in the envied bundle.

My bundle is better
if (A) is removed

	(A)	(B)	(C)
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Envy can be eliminated by removing some good in the envied bundle.

My bundle is better
if (A) is removed



My bundle is better
if (C) is removed



(A)

(B)

(C)

4

1

2

1

1

5

Envy-Freeness Up To One Good [Budish, 2011]

Envy can be eliminated by removing some good in the envied bundle.

	(A)	(B)	(C)
My bundle is better if (A) is removed	4	1	2
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Allocation $A = (A_1, \dots, A_n)$ is EF1 if for every pair of agents i, k , there exists a good $j \in A_k$ such that $v_i(A_i) \geq v_i(A_k \setminus \{j\})$.

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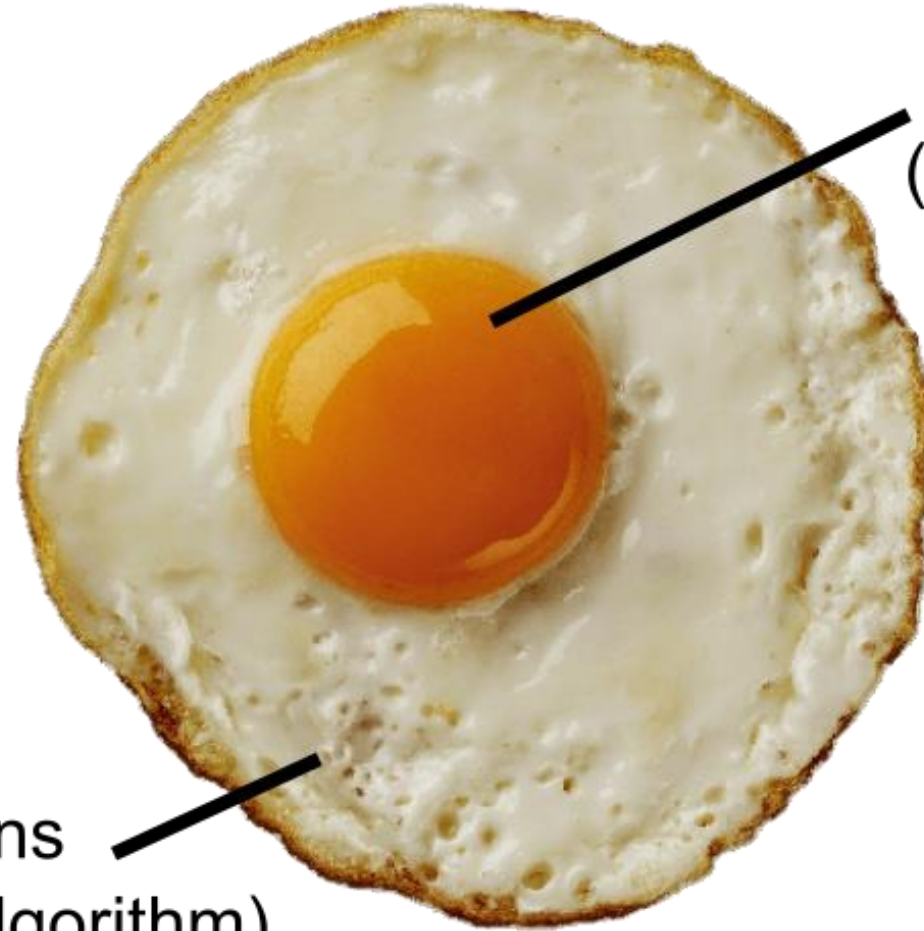
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Guaranteed to exist and efficiently computable

Coming Up

Algorithms for finding an EF1 allocation



Additive valuations
(Round-robin algorithm)

Monotone valuations
(Envy-cycle elimination algorithm)

Round-robin algorithm

Round-robin algorithm



- Fix an ordering of the agents, say $a_1, a_2, a_3, \dots, a_n$.

Round-robin algorithm

- Fix an ordering of the agents, say $a_1, a_2, a_3, \dots, a_n$.
- Agents take turns according to the ordering $(a_1, a_2, \dots, a_n, a_1, a_2, \dots, a_n, \dots)$ to pick their favorite item from the set of remaining items.




Round-robin algorithm

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For additive valuations, the allocation computed by round-robin algorithm satisfies EF1.

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a_1 a_2 a_3 \dots a_n


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a_1 a_2 a_3 \dots a_n

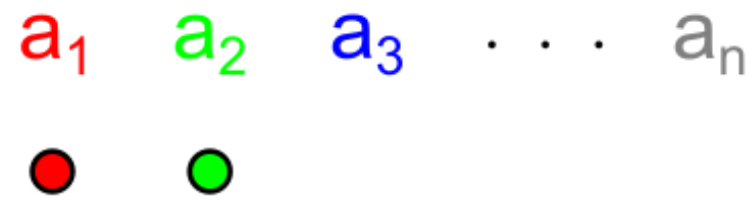
First round

For additive valuations, the allocation computed by round-robin algorithm satisfies EF1.


First round a_1 a_2 a_3 \dots a_n



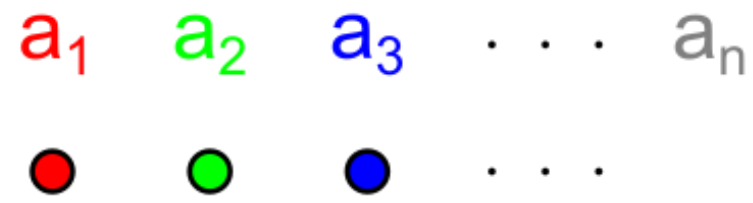
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First round a_1 a_2 a_3 \dots a_n


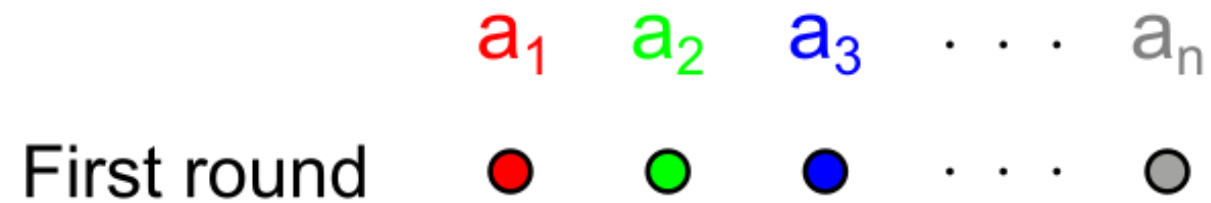
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First round a_1 a_2 a_3 \dots a_n


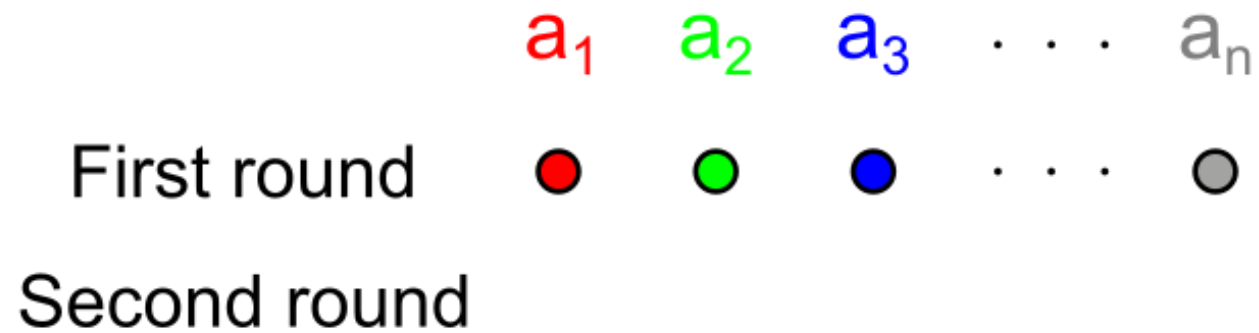
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First round a_1 a_2 a_3 \dots a_n


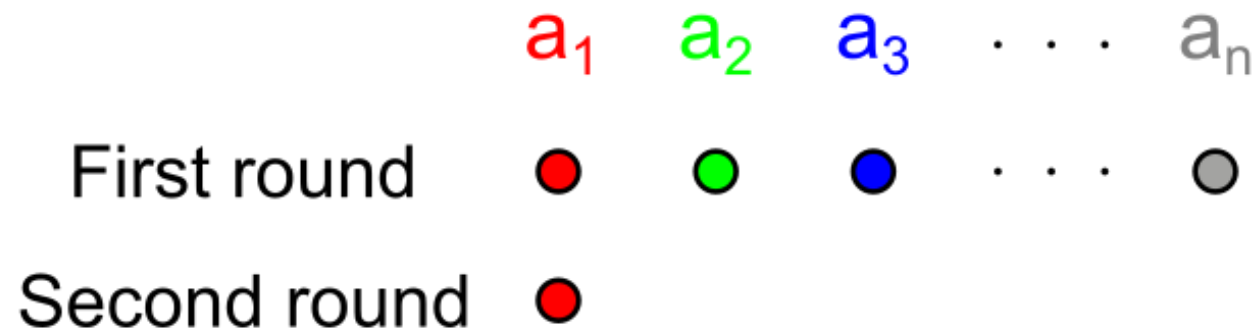
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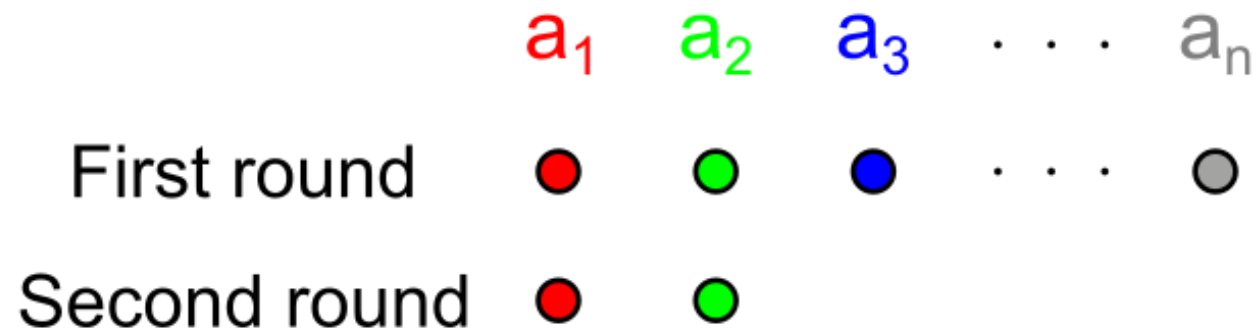
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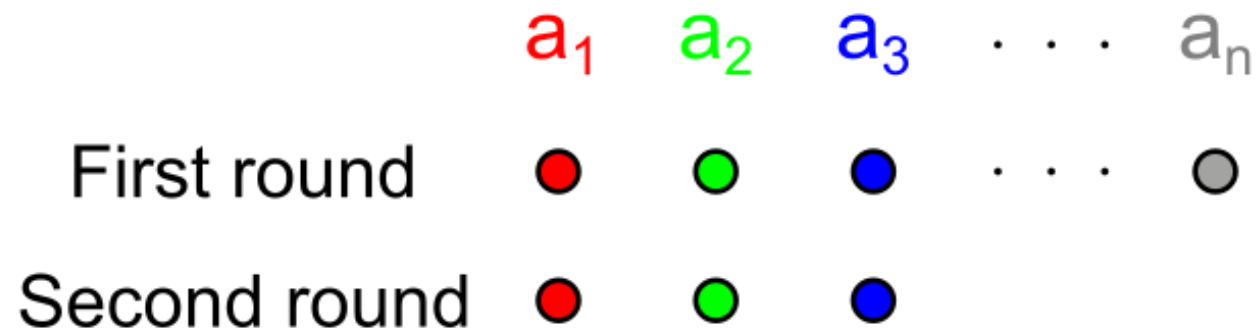
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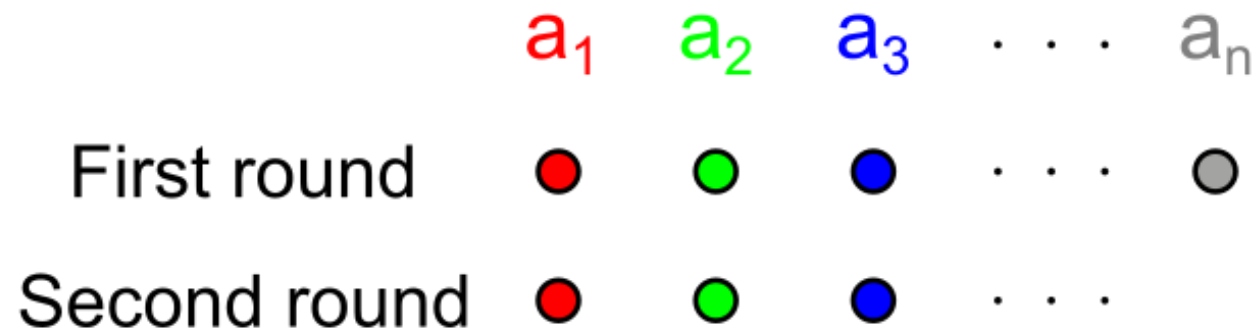
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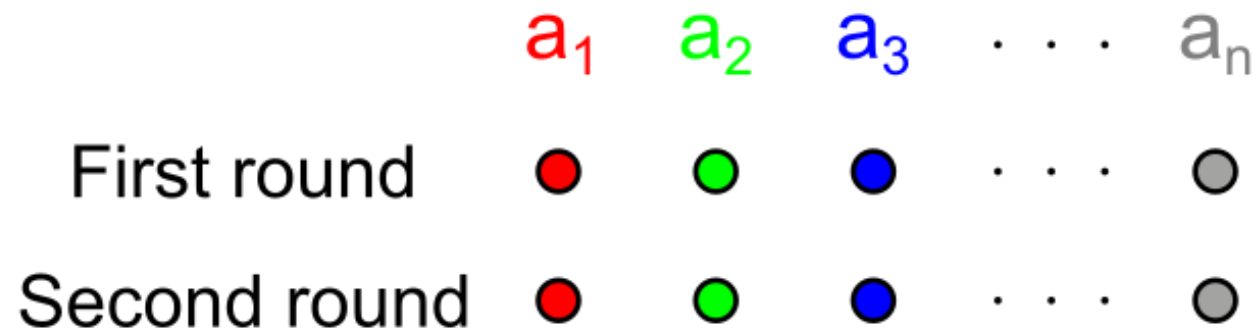
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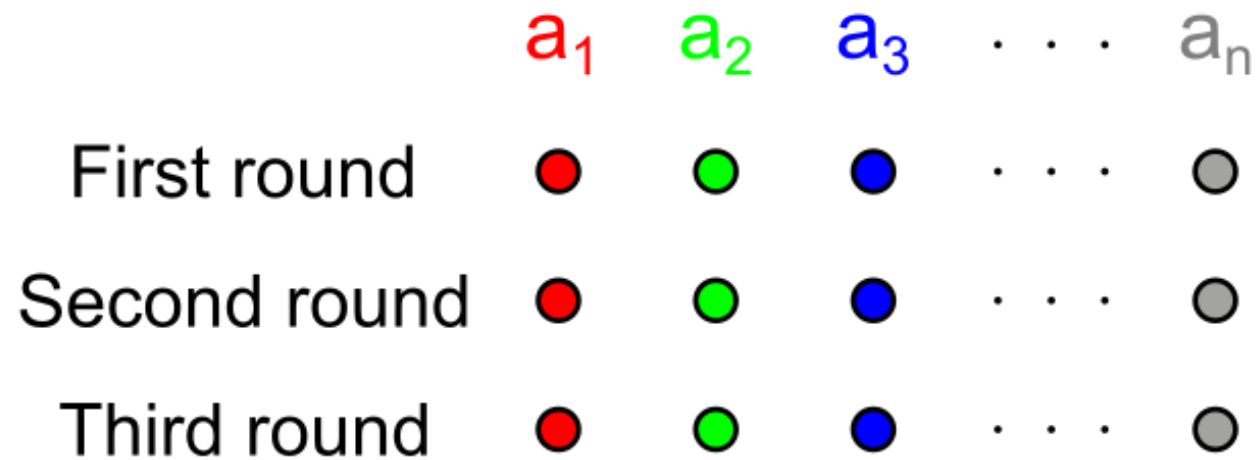
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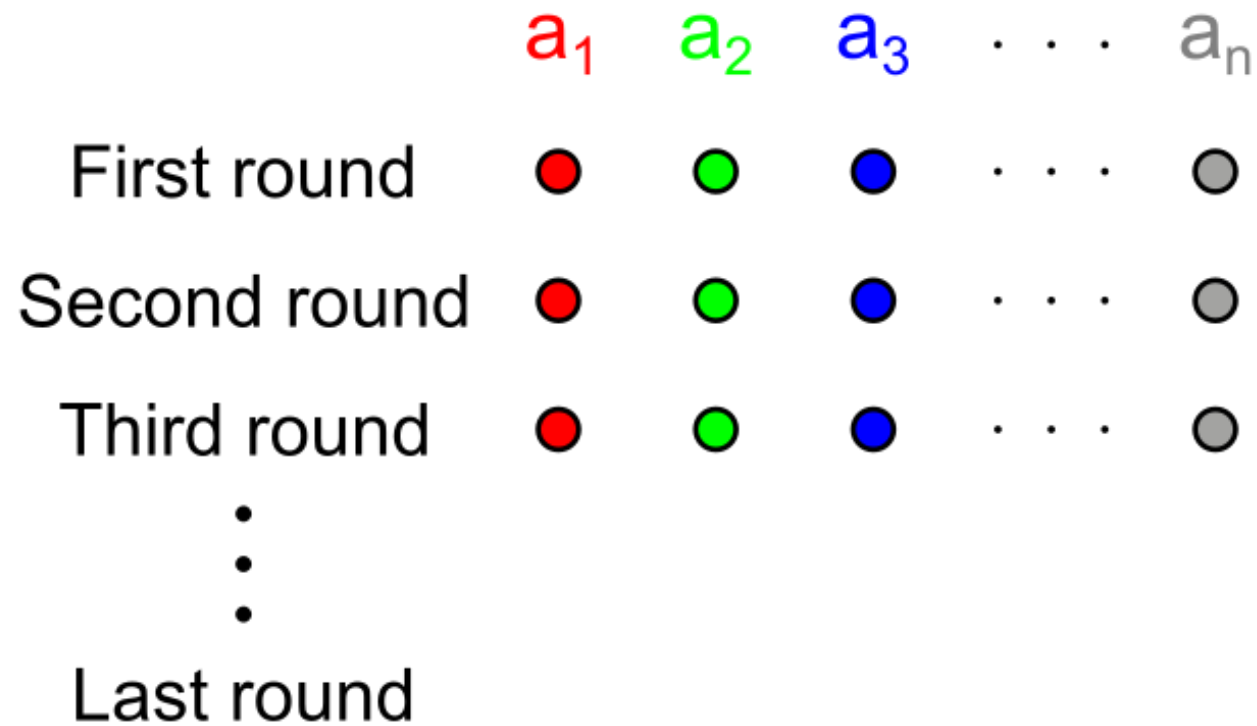
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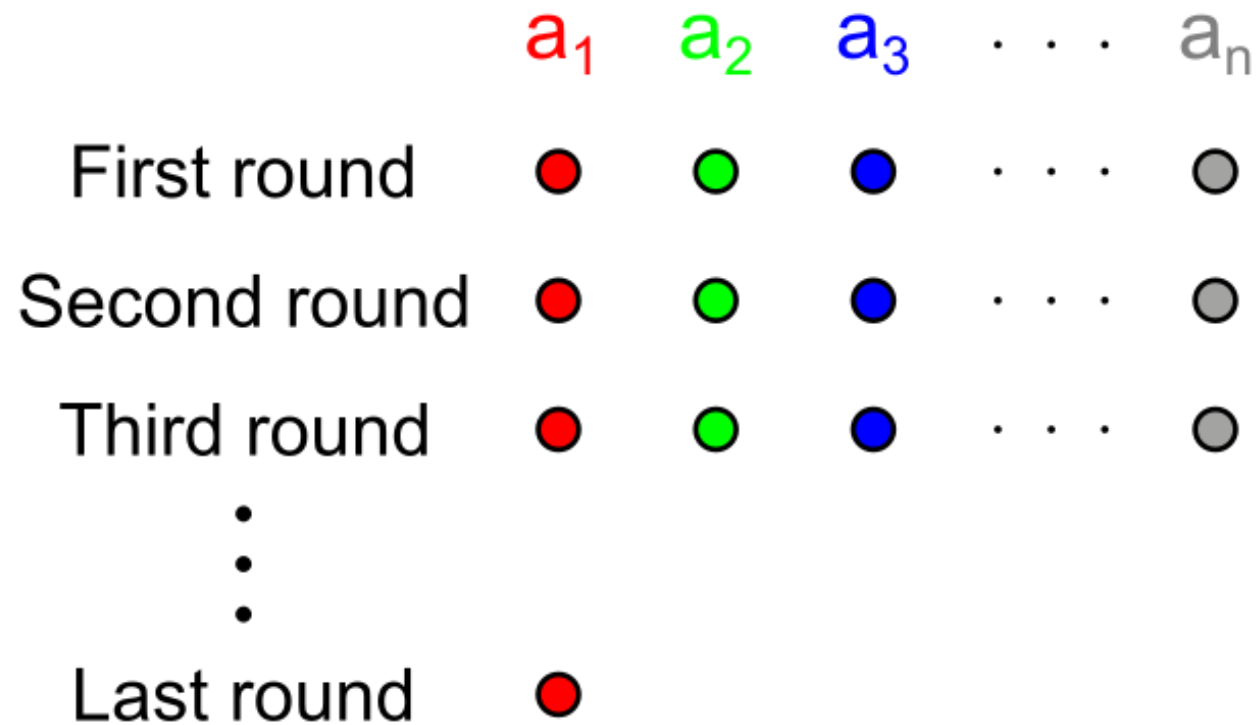
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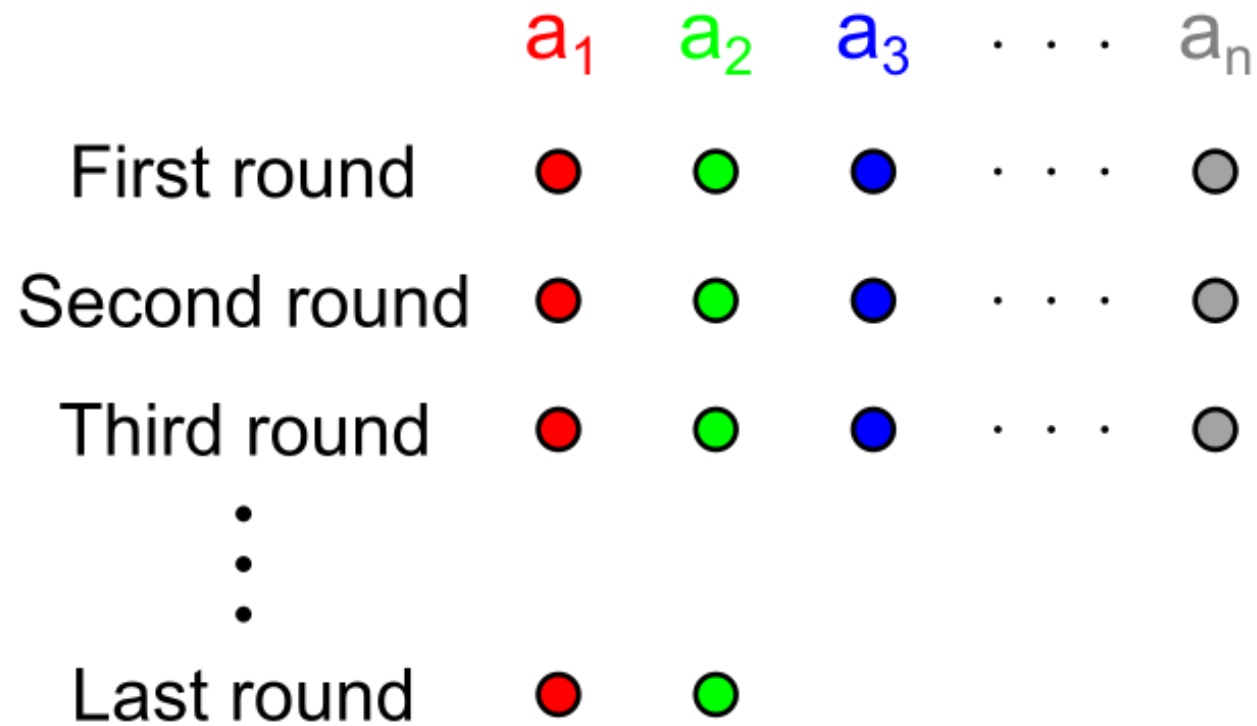
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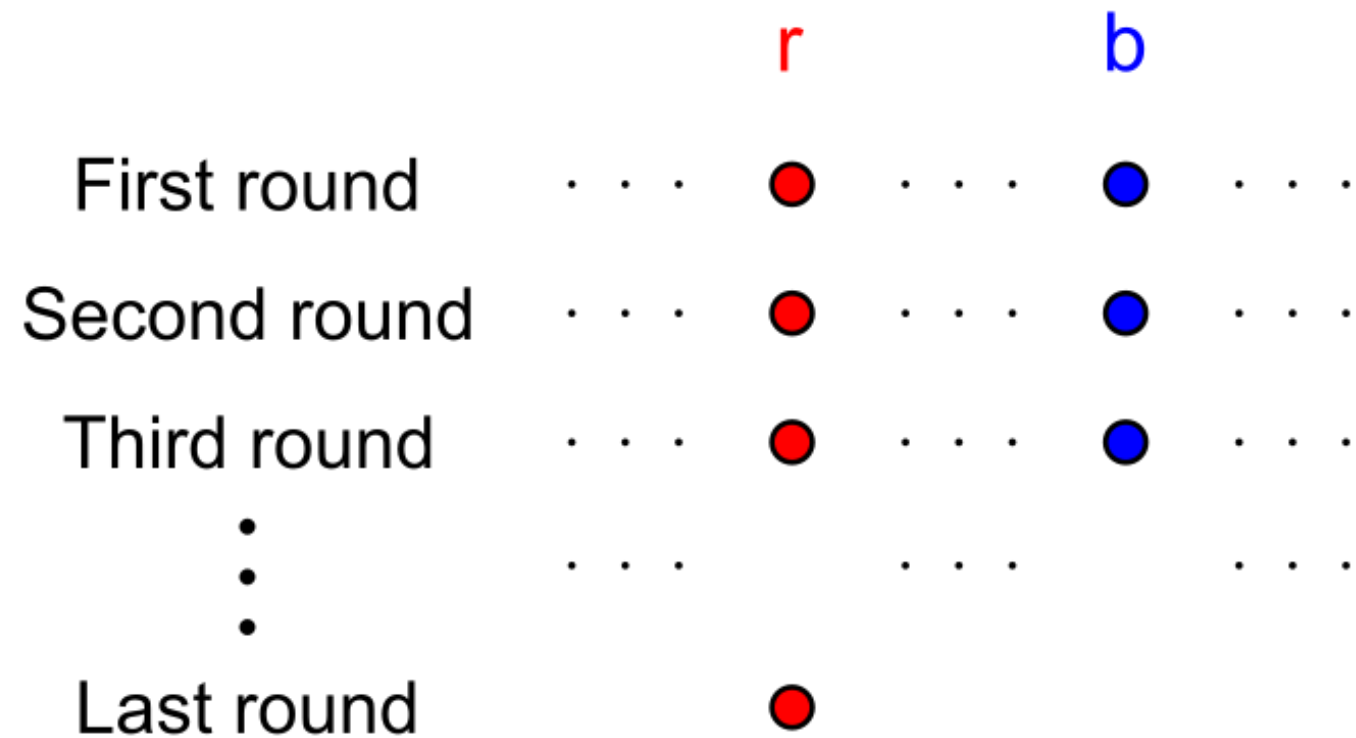


For additive valuations, the allocation computed by round-robin algorithm satisfies EF1.

Fix a pair of agents (r, b). Analyze envy of r towards b .

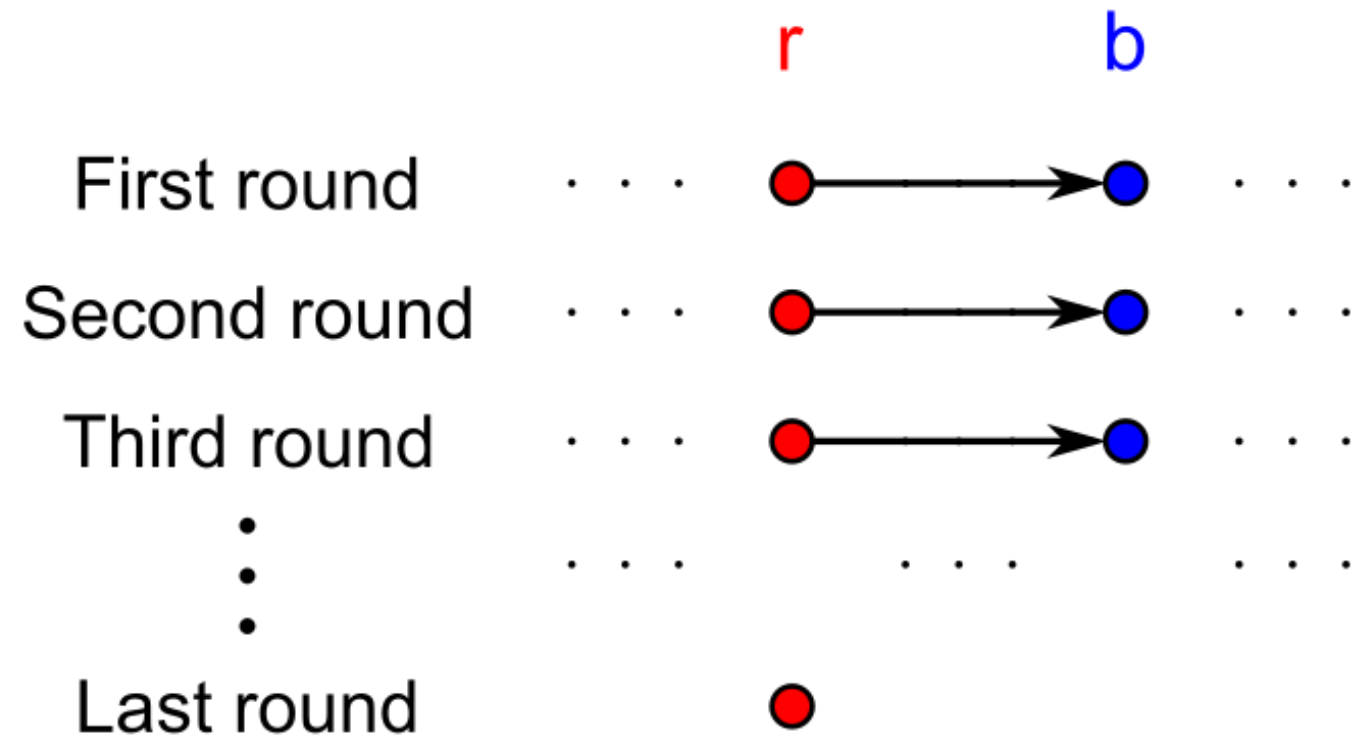
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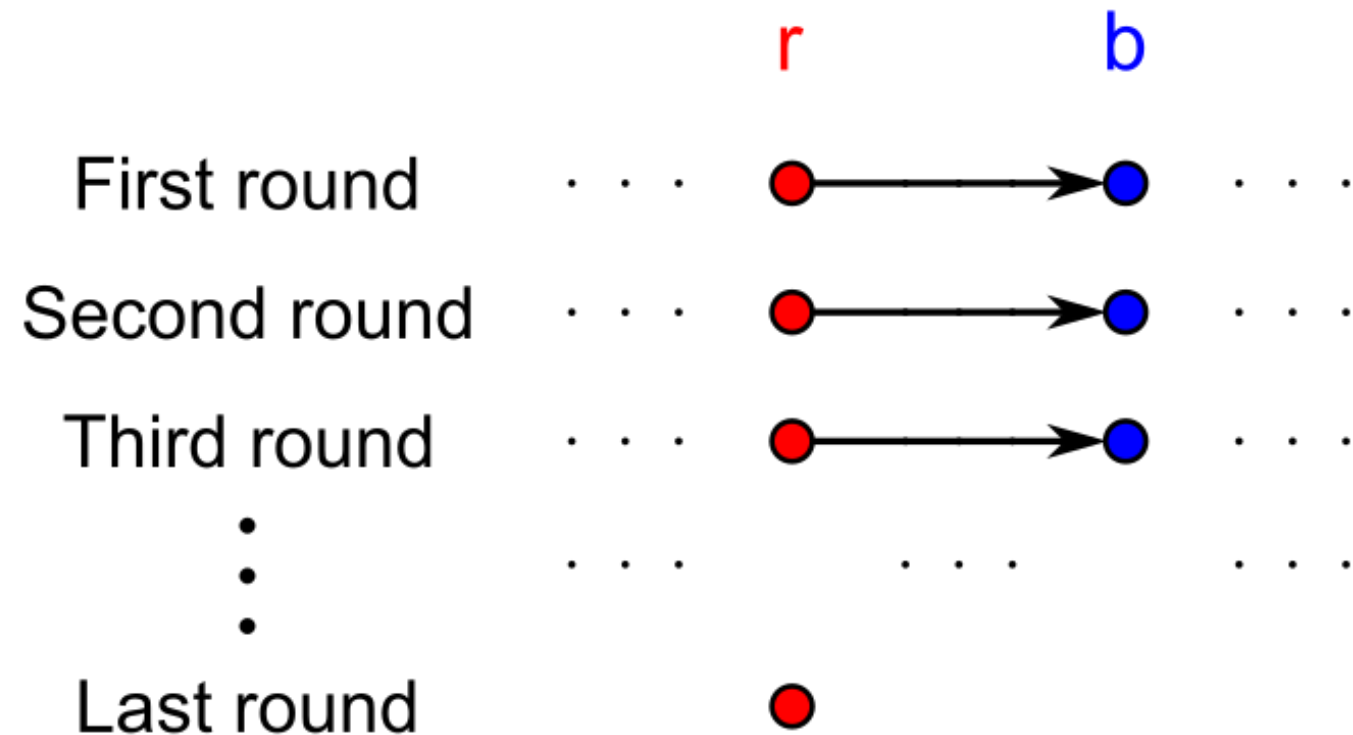
For additive valuations, the allocation computed by round-robin algorithm satisfies EF1.

Fix a pair of agents (r, b). Analyze envy of r towards b .



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Fix a pair of agents (r, b). Analyze envy of r towards b .



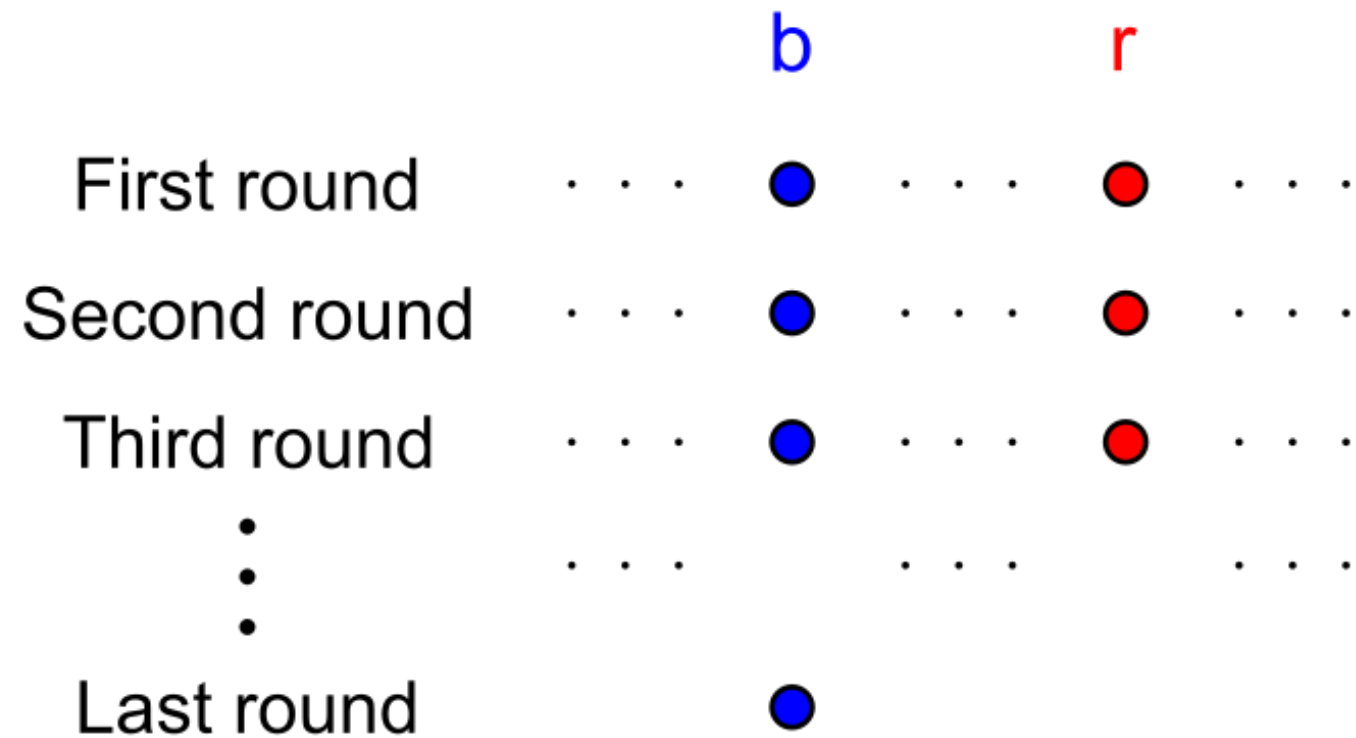
If r precedes b : Then, by additivity, $v_r(A_r) \geq v_r(A_b)$.

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Fix a pair of agents (r, b). Analyze envy of r towards b .

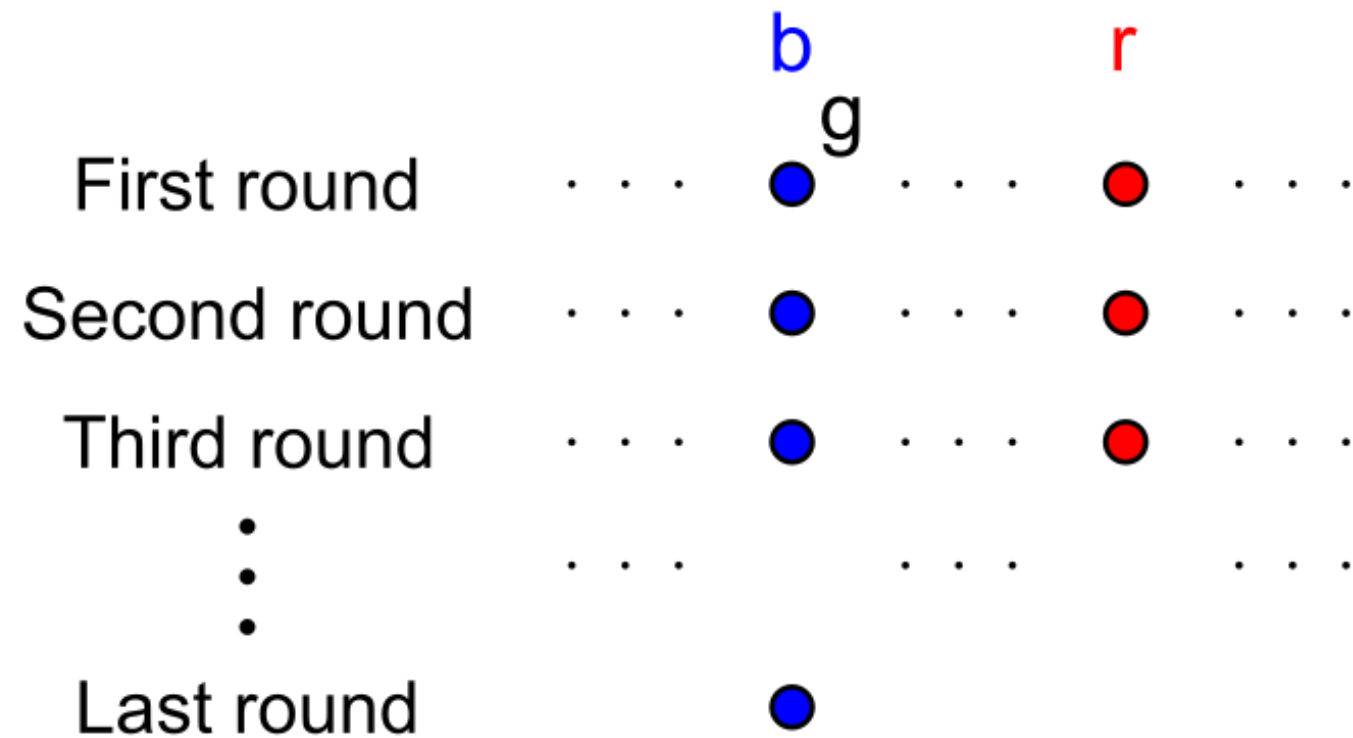
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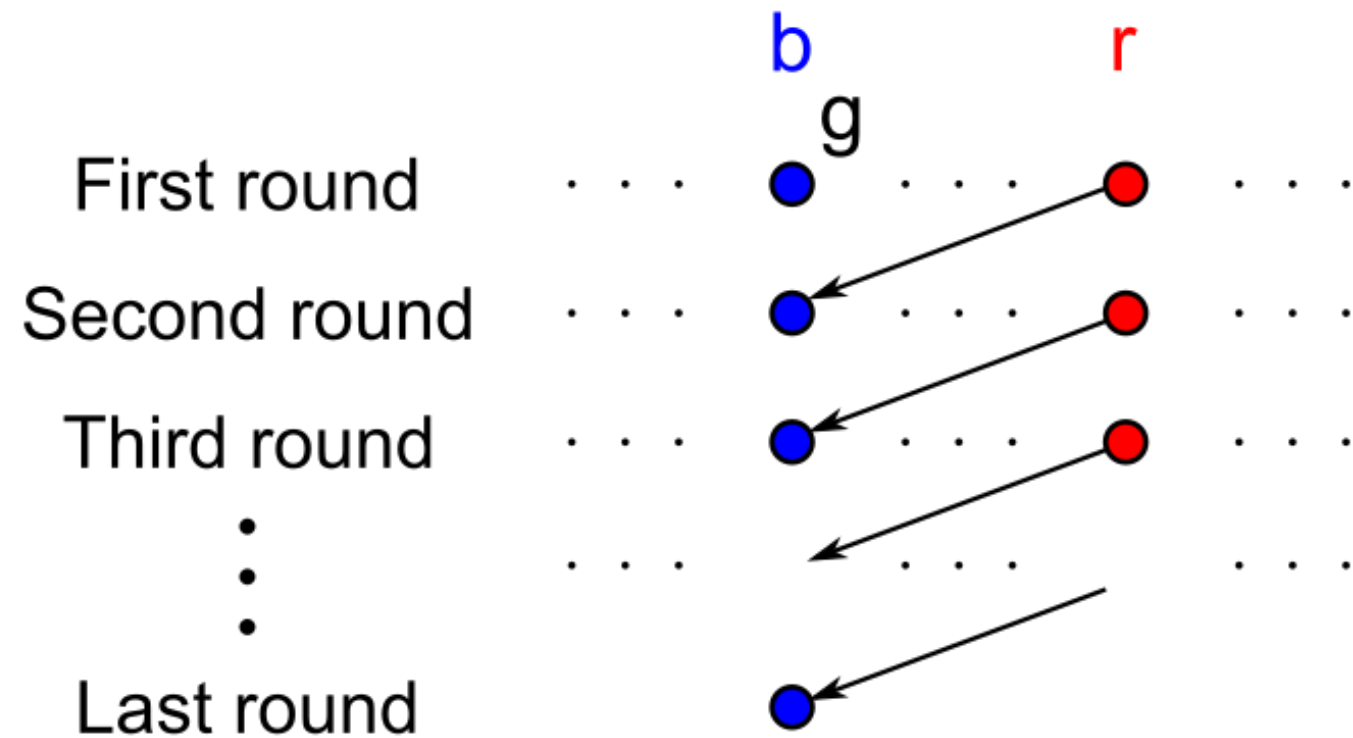
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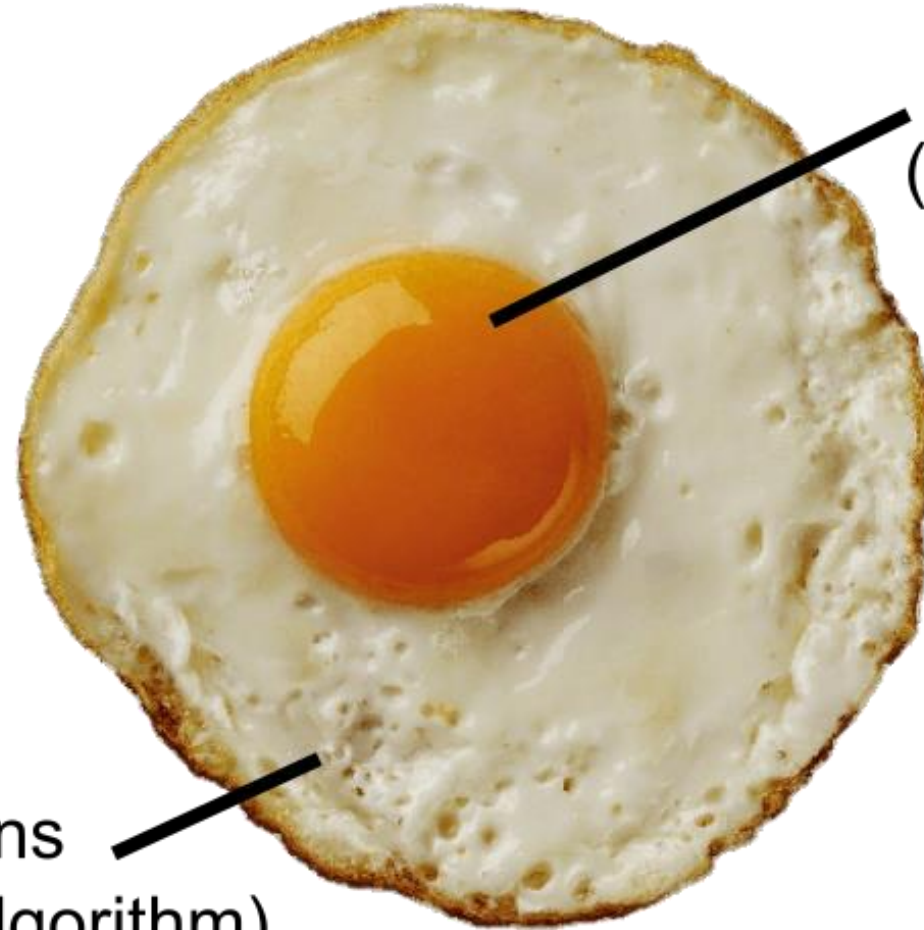


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Fix a pair of agents (r, b). Analyze envy of r towards b .



Algorithms for EF1



Additive valuations
(Round-robin algorithm)

Monotone valuations
(Envy-cycle elimination algorithm)




Envy graph of an allocation

Envy graph of an allocation

- Vertices = agents
- Edge from vertex i to vertex k if agent i envies agent k in the given allocation.




Envy graph of an allocation

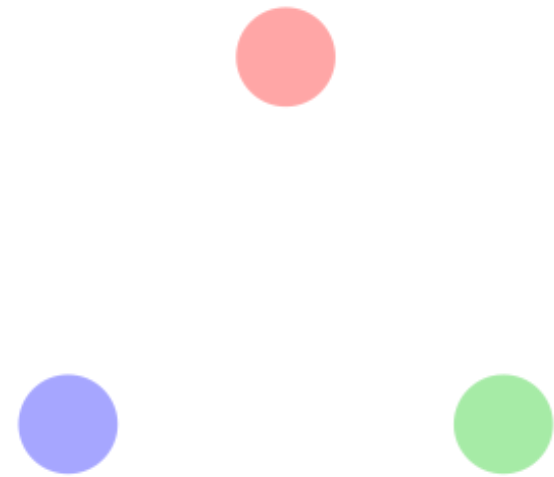
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	(A)	(B)	(C)	(D)	(E)
	4	1	2	4	2
	1	0	5	1	1
	1	1	5	1	1

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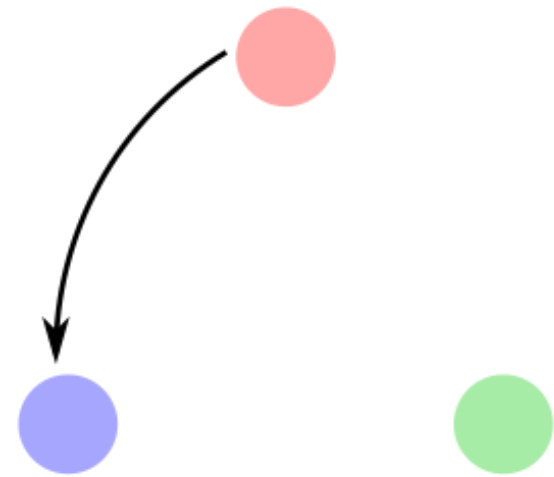
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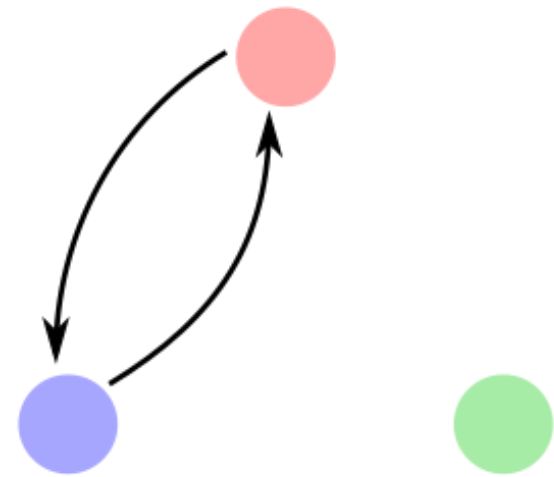
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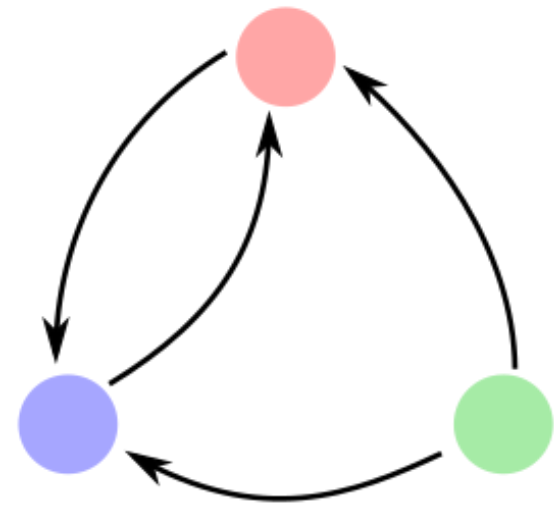
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	1	0	5	1	1
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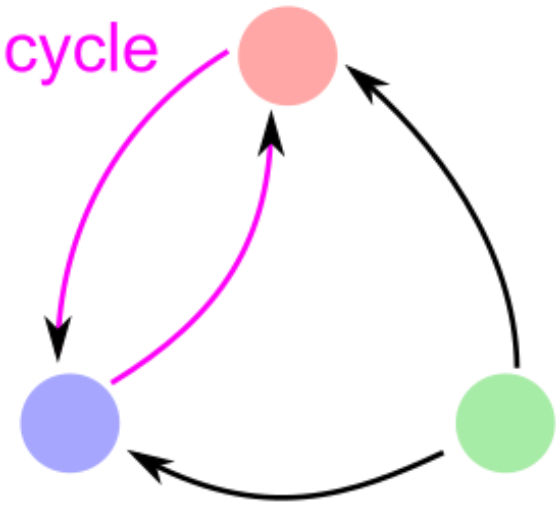


Envy graph of an allocation

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	(A)	(B)	(C)	(D)	(E)
	4	1	2	4	2
	1	0	5	1	1
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Envy cycle



Envy-cycle elimination algorithm

[Lipton, Markakis, Mossel, and Saberi, *EC* 2004]

Envy-cycle elimination algorithm

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While there is an unallocated good

Envy-cycle elimination algorithm

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While there is an unallocated good

- If the envy graph has a source vertex, assign the good to that agent.

Envy-cycle elimination algorithm

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- If the envy graph has a source vertex, assign the good to that agent.
- Otherwise, resolve envy cycles until a source vertex shows up, and then assign the good to it.

Envy-cycle elimination algorithm

[Lipton, Markakis, Mossel, and Saberi, *EC* 2004]

While there is an unallocated good

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- Otherwise, **resolve envy cycles** until a source vertex shows up, and then assign the good to it.






each agent in the cycle gets the bundle
that it is pointing to

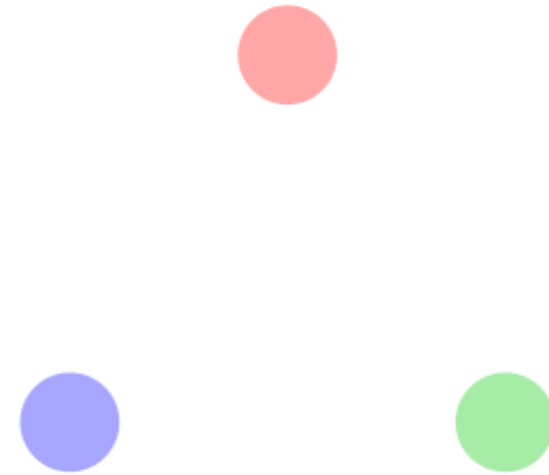
Envy-cycle elimination algorithm

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	(A)	(B)	(C)	(D)
	0	2	0	1
	1	2	5	10
	1	4	2	10



Envy-cycle elimination algorithm

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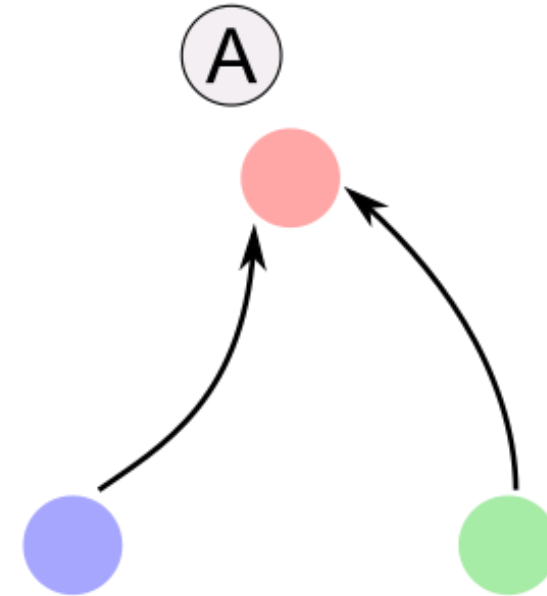
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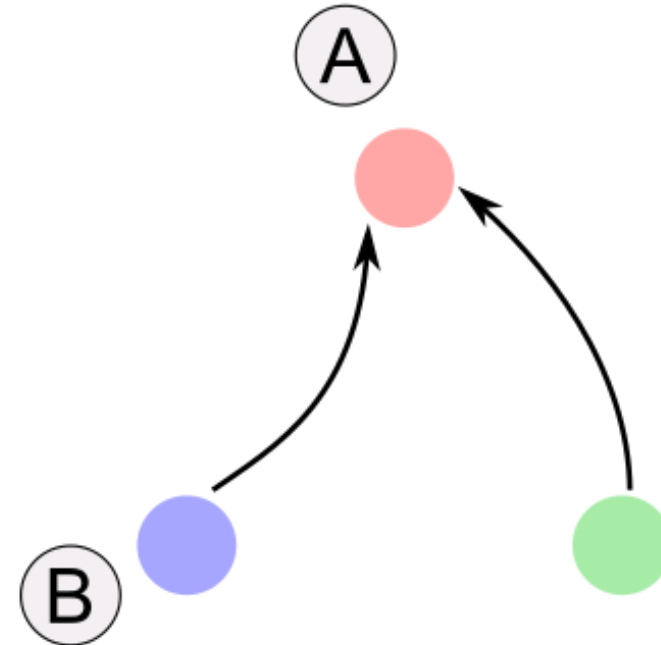
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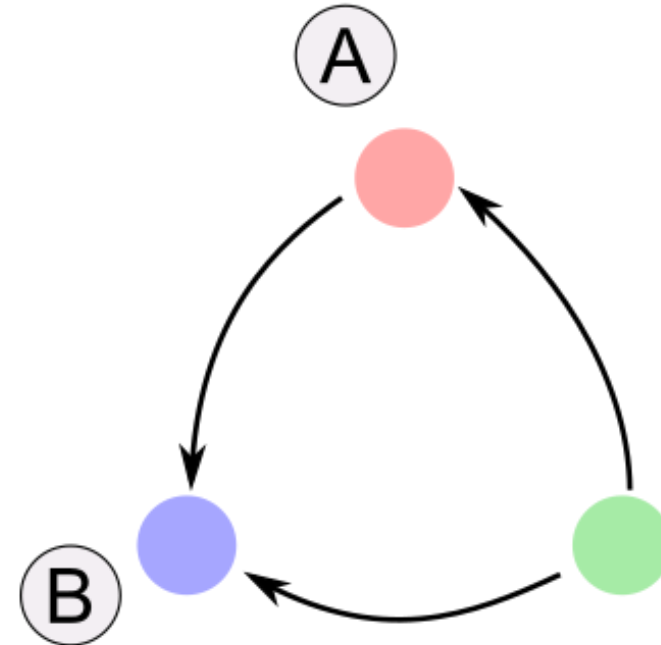
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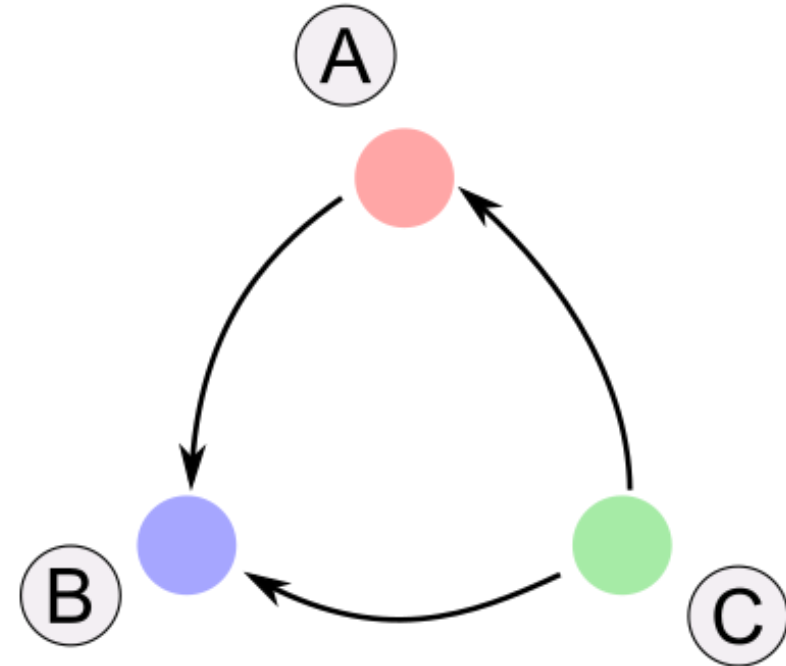
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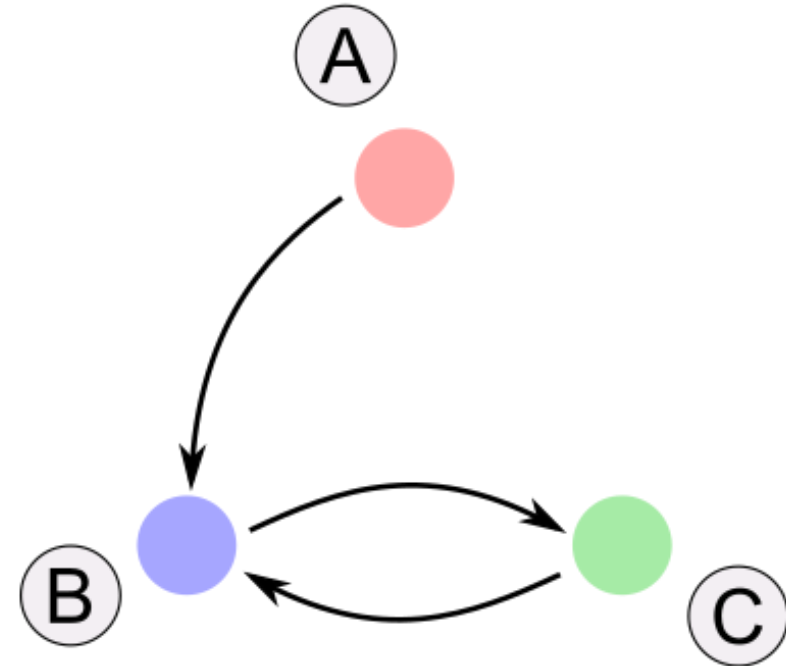
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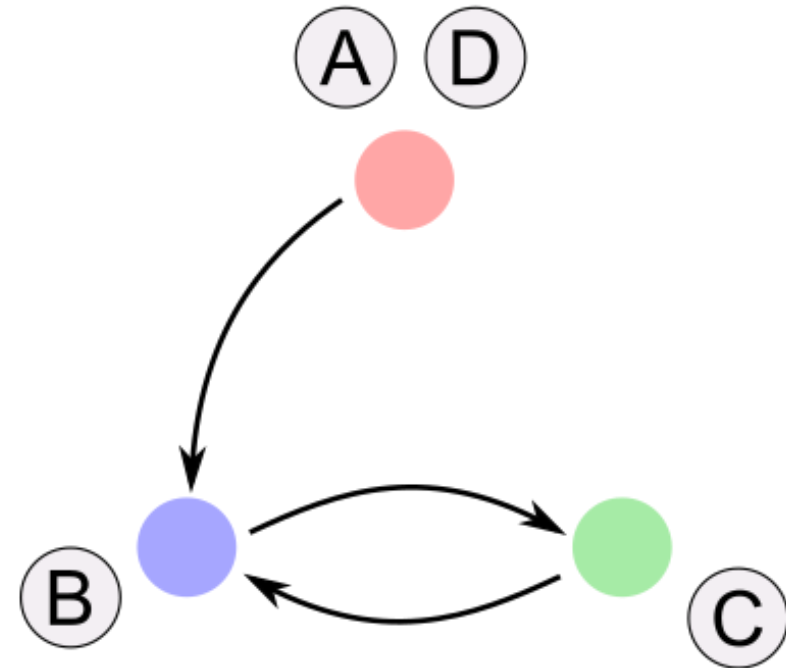
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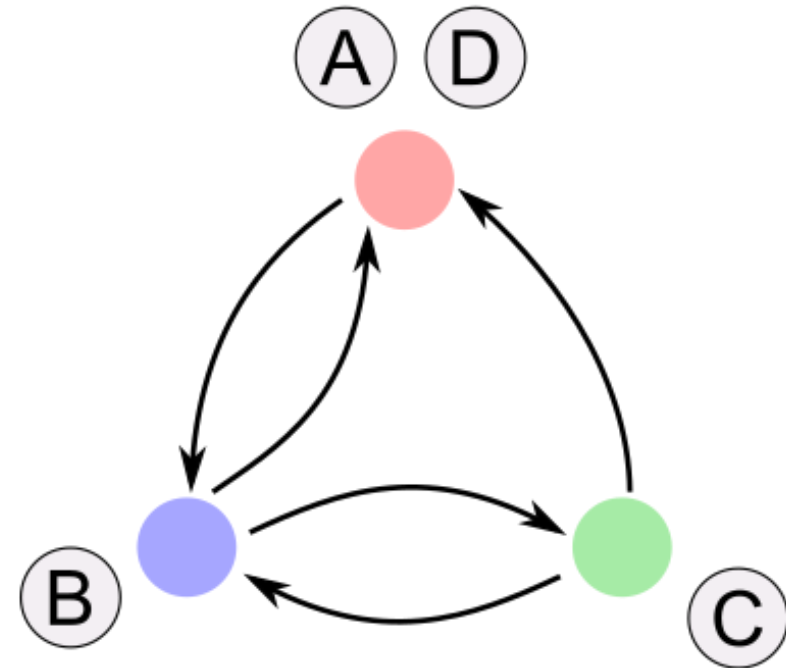
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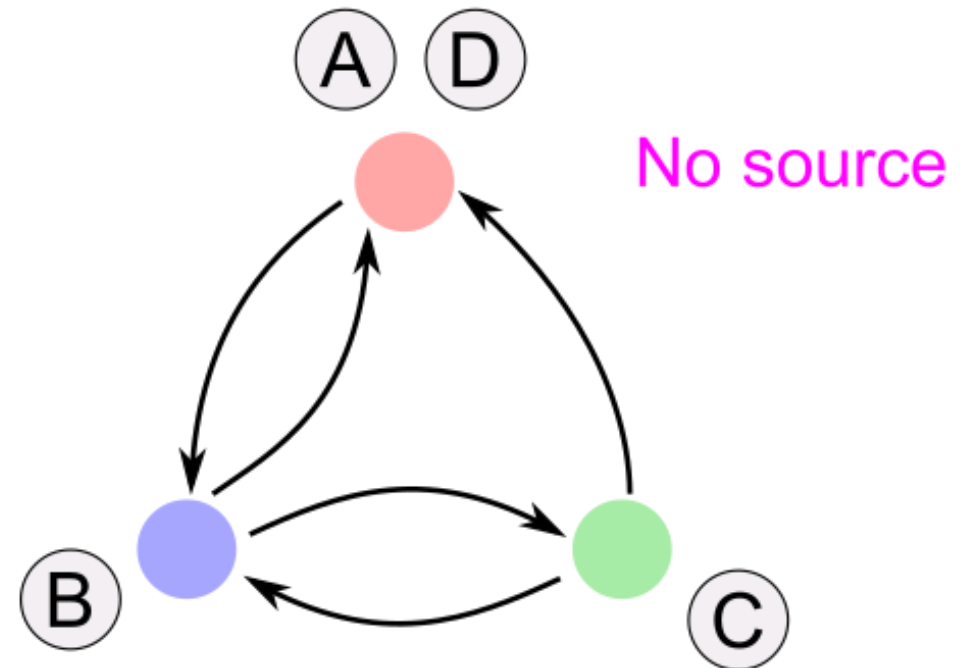
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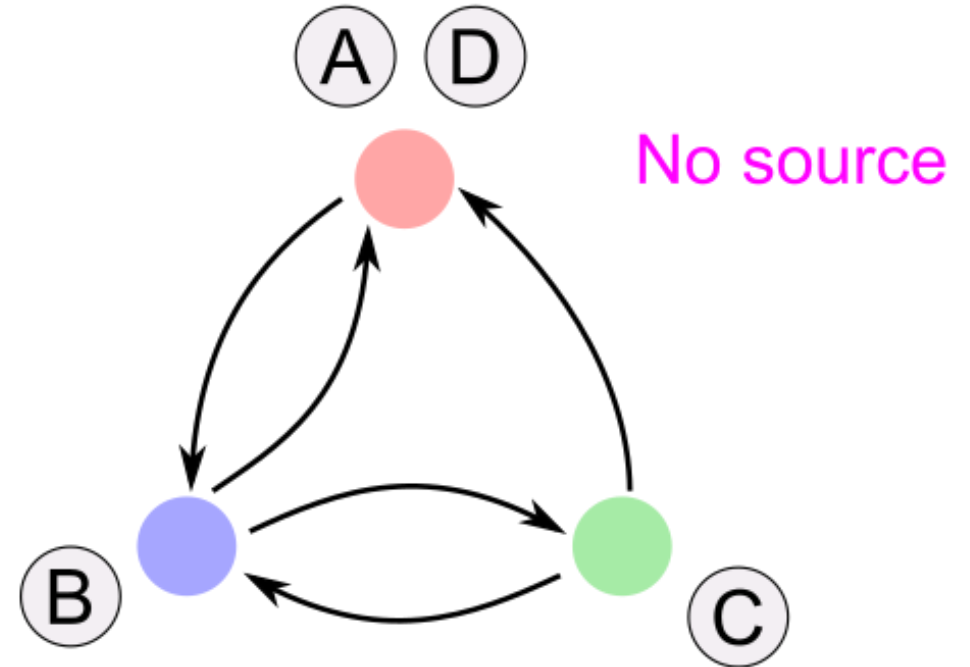
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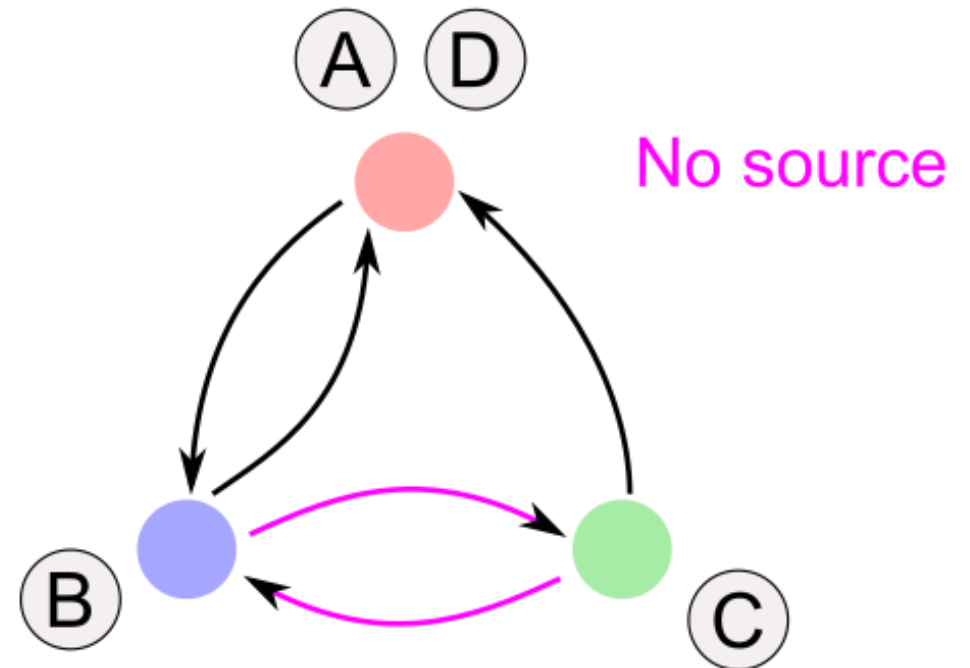
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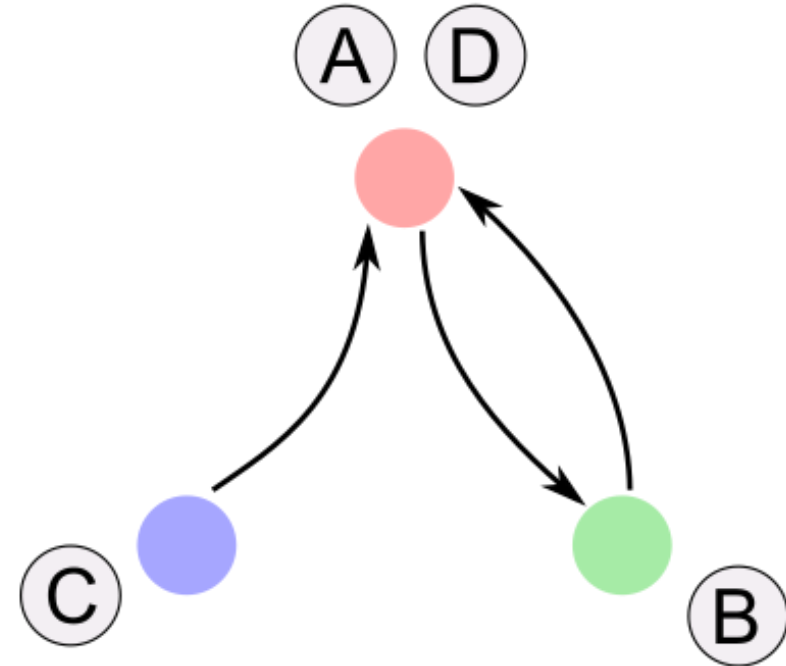
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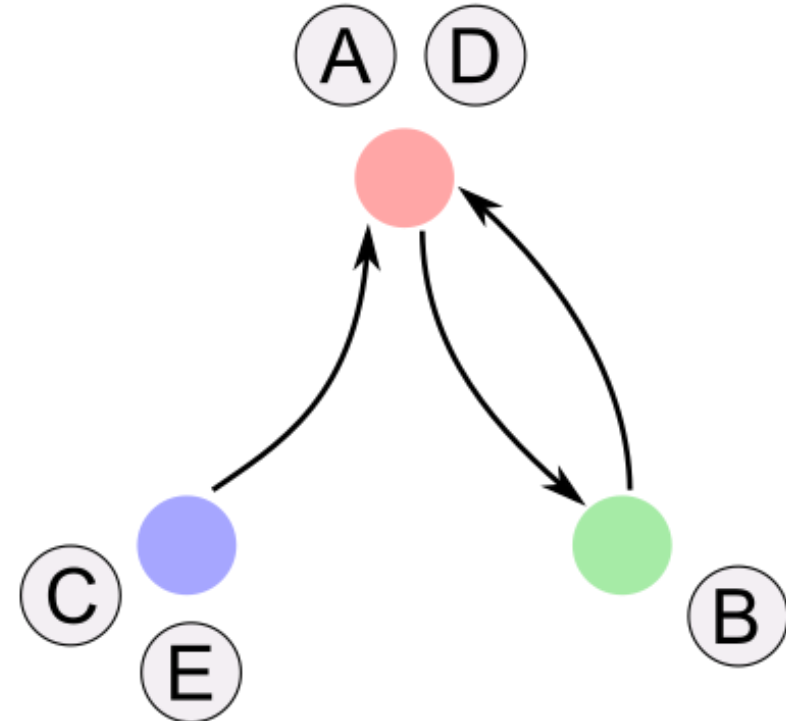
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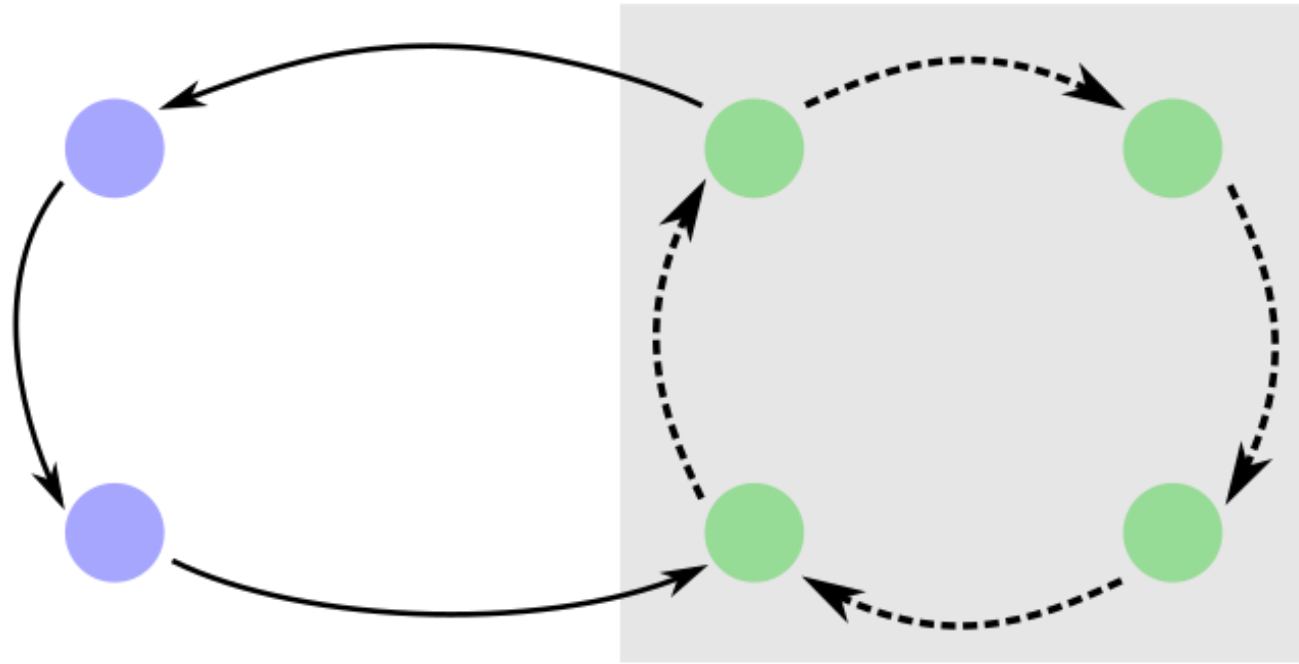
- With n agents, at most $O(n^2)$ cycle resolutions required to create a source.
- Polynomial running time!

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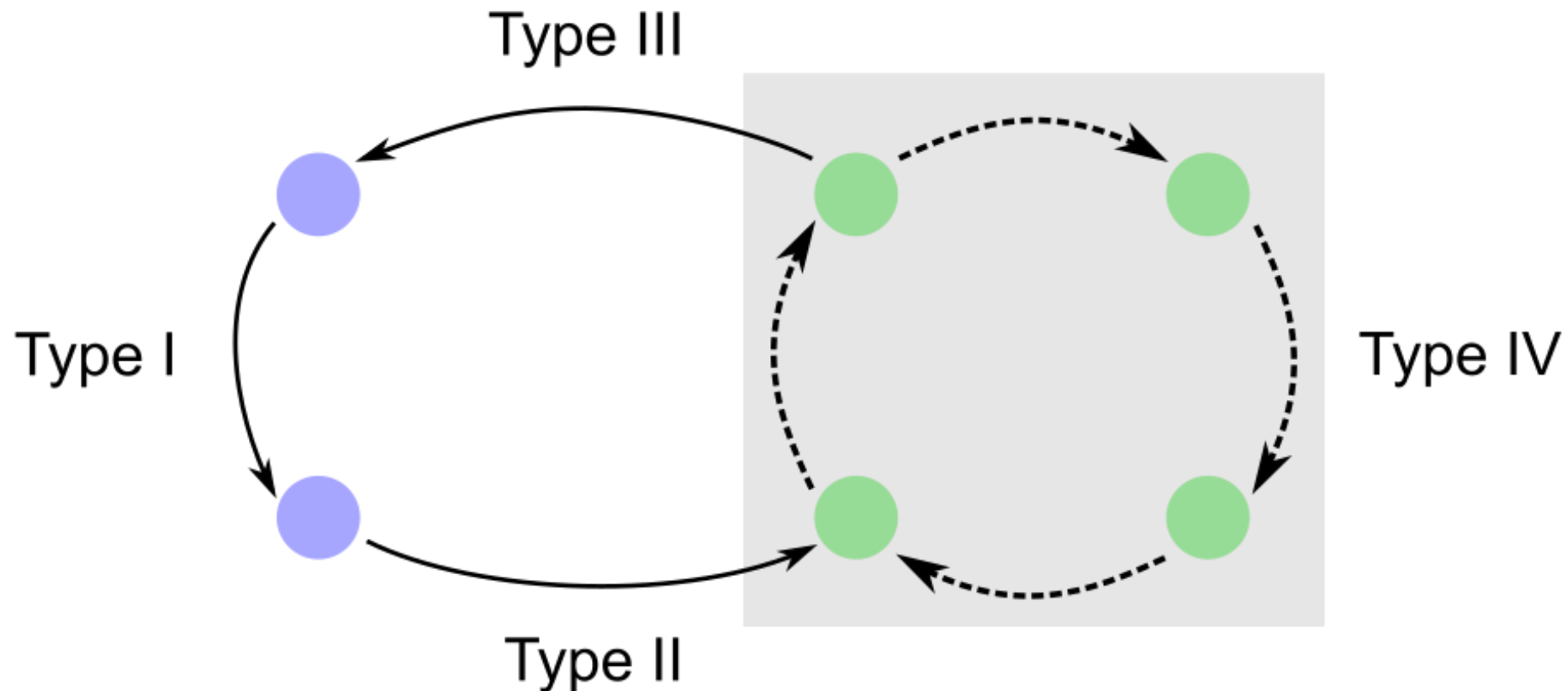
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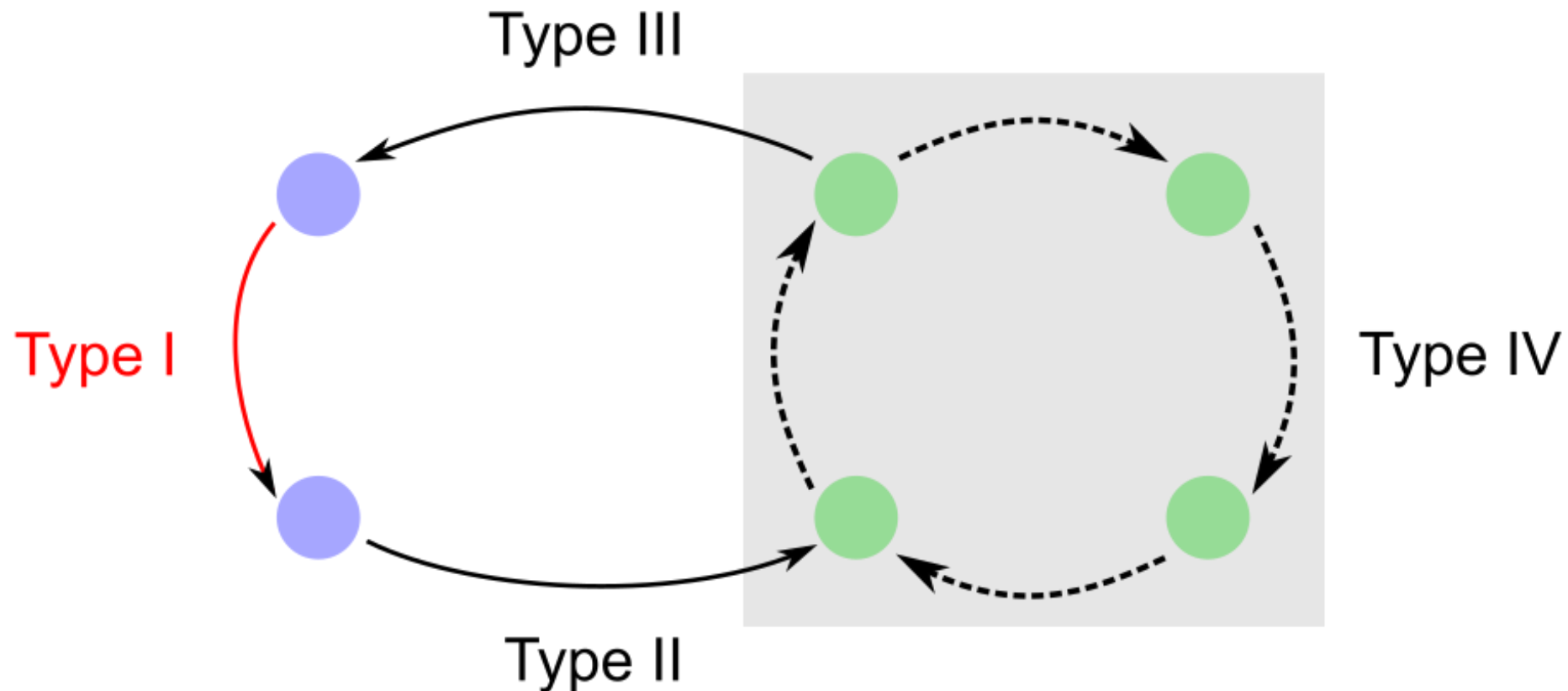
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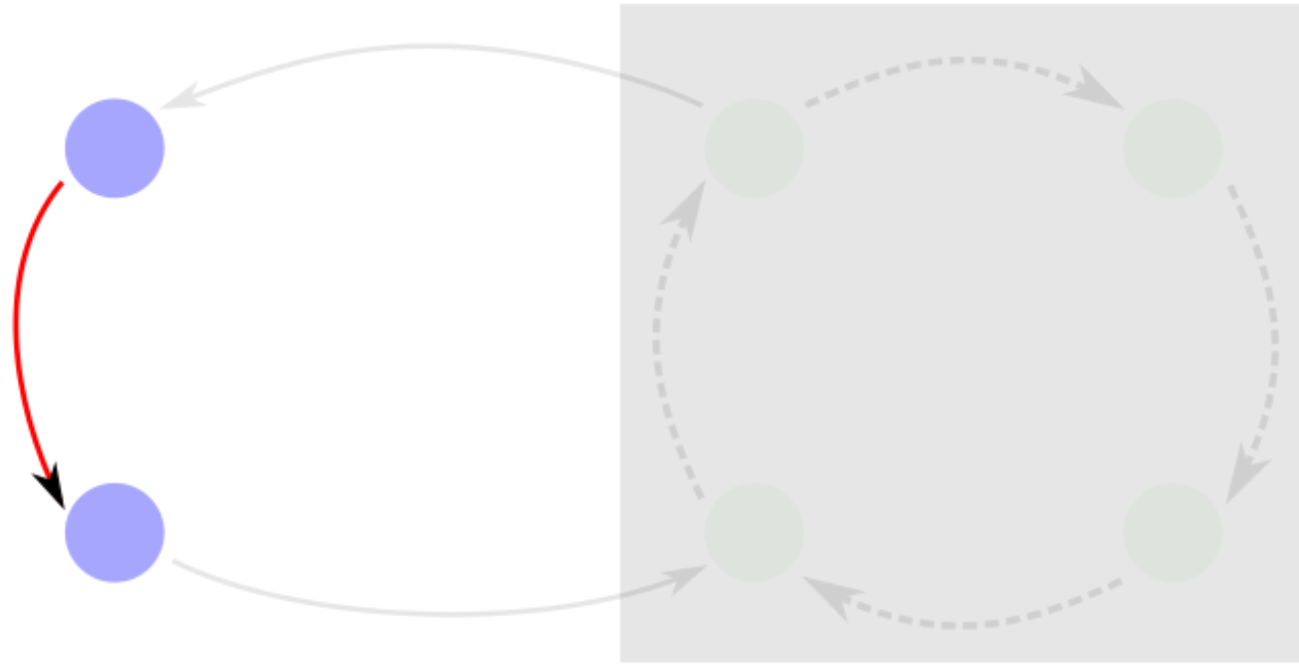
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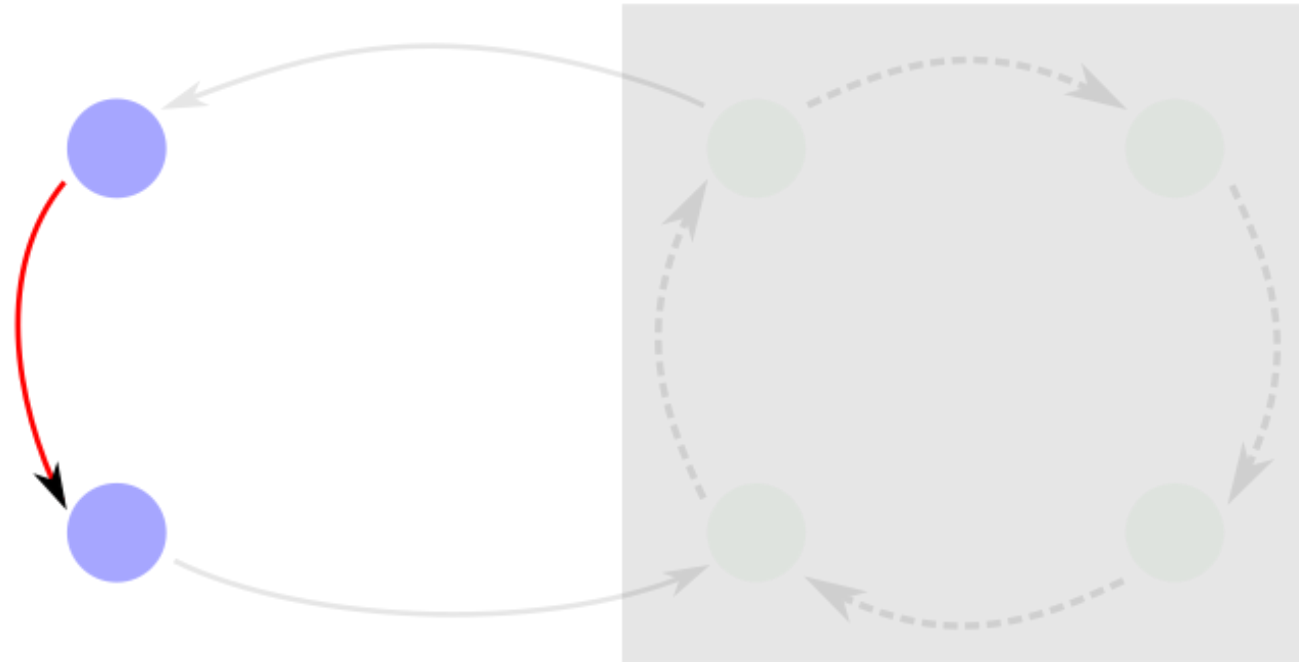
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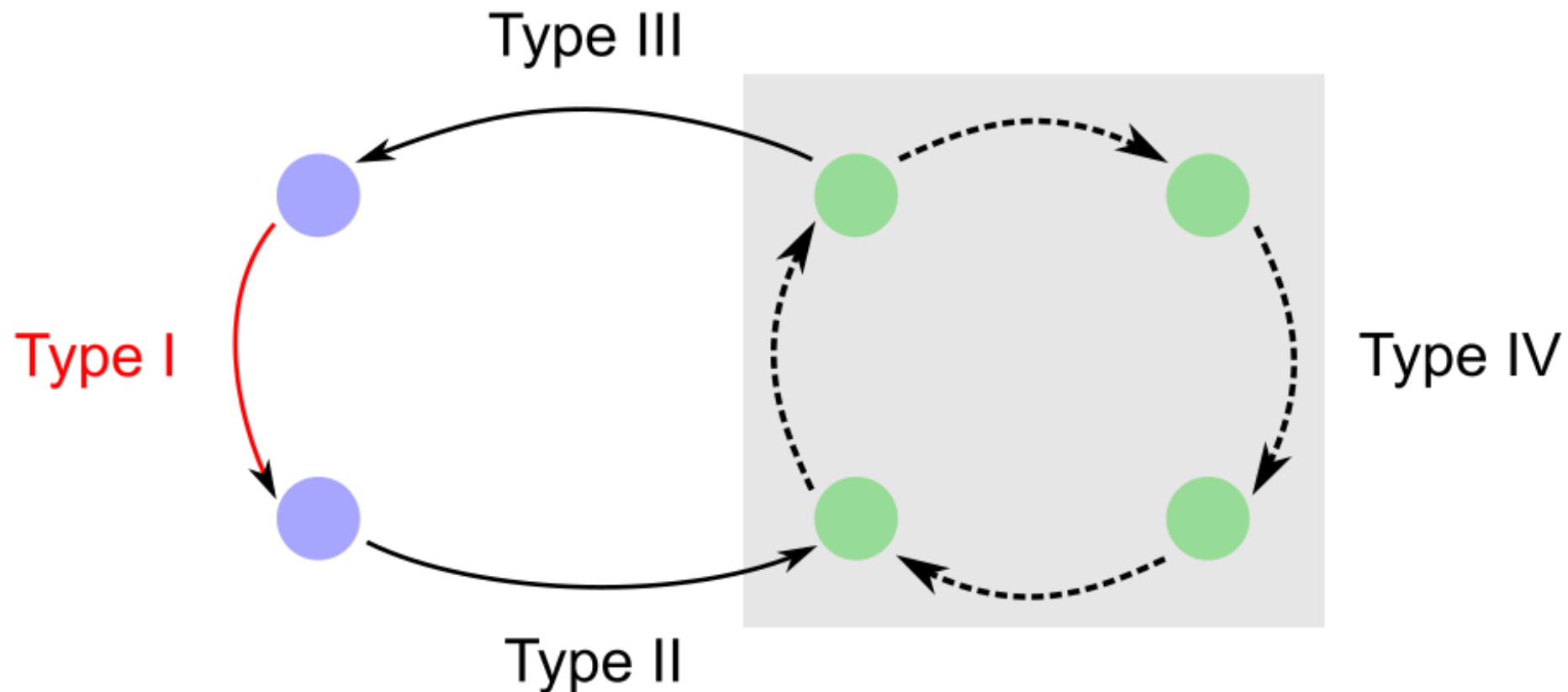
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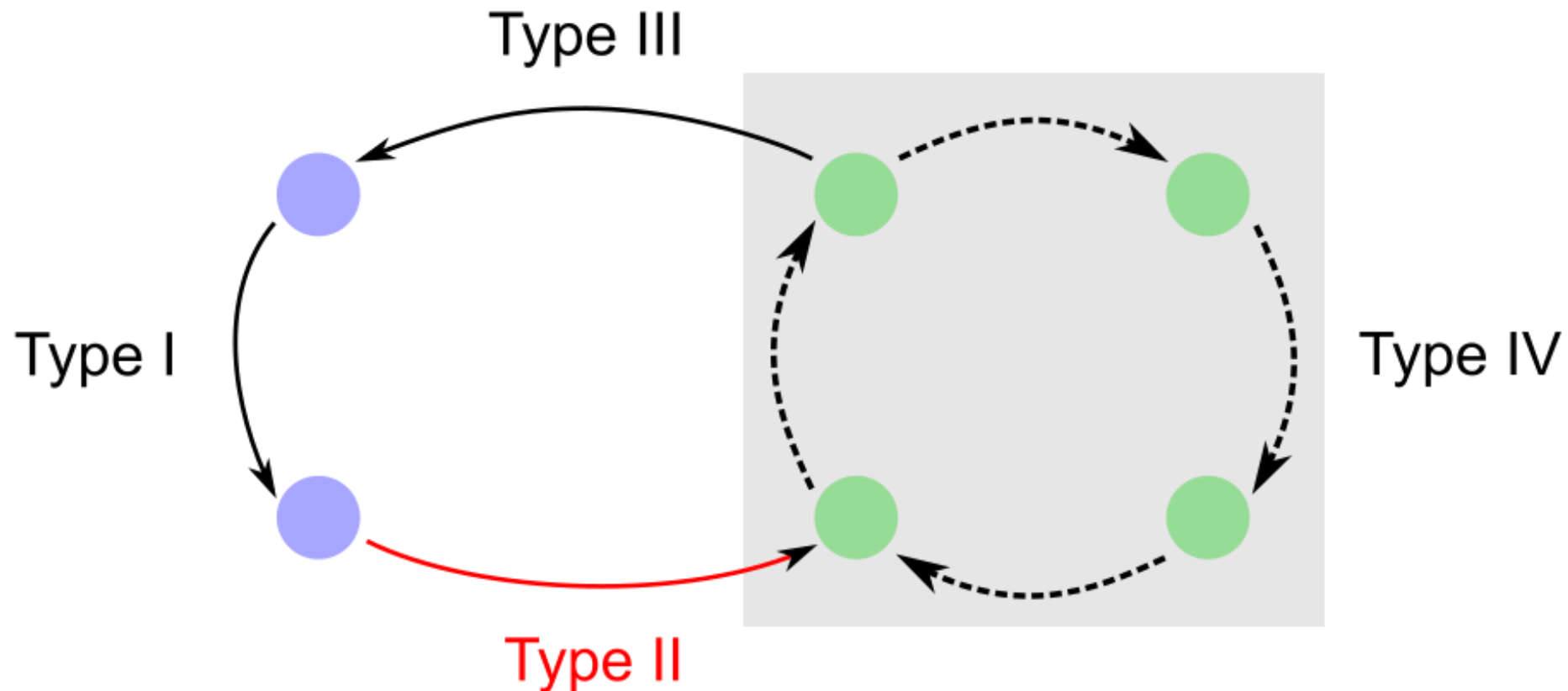
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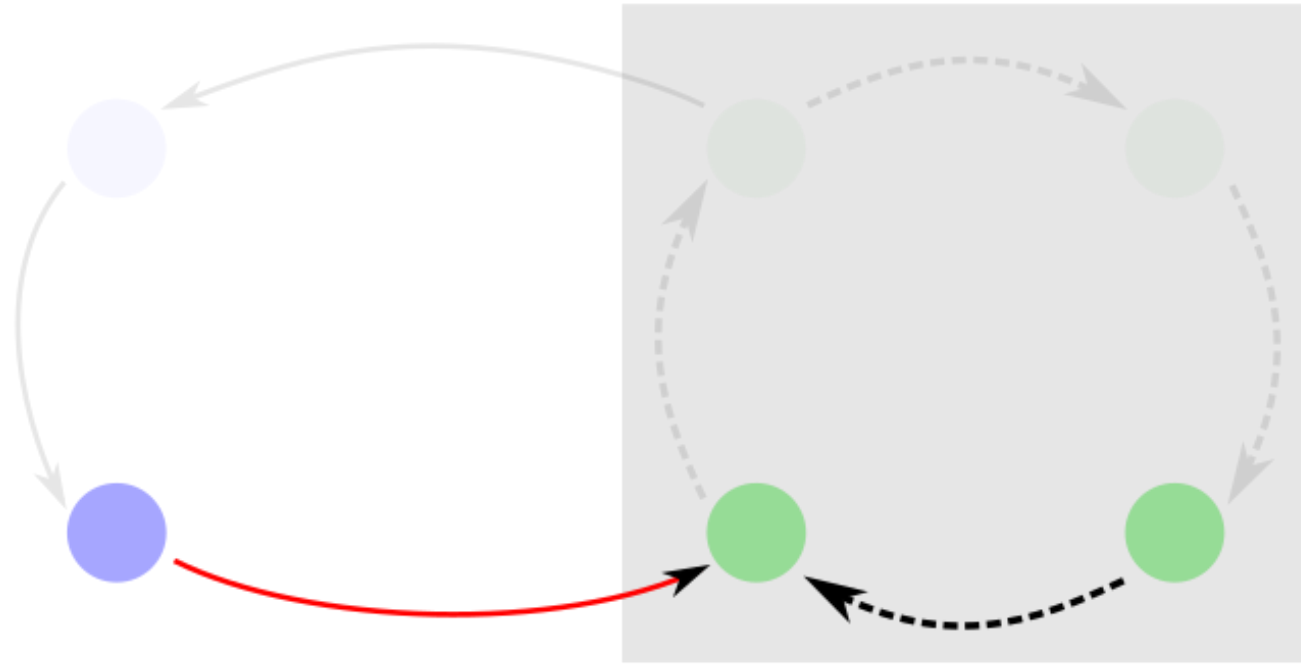
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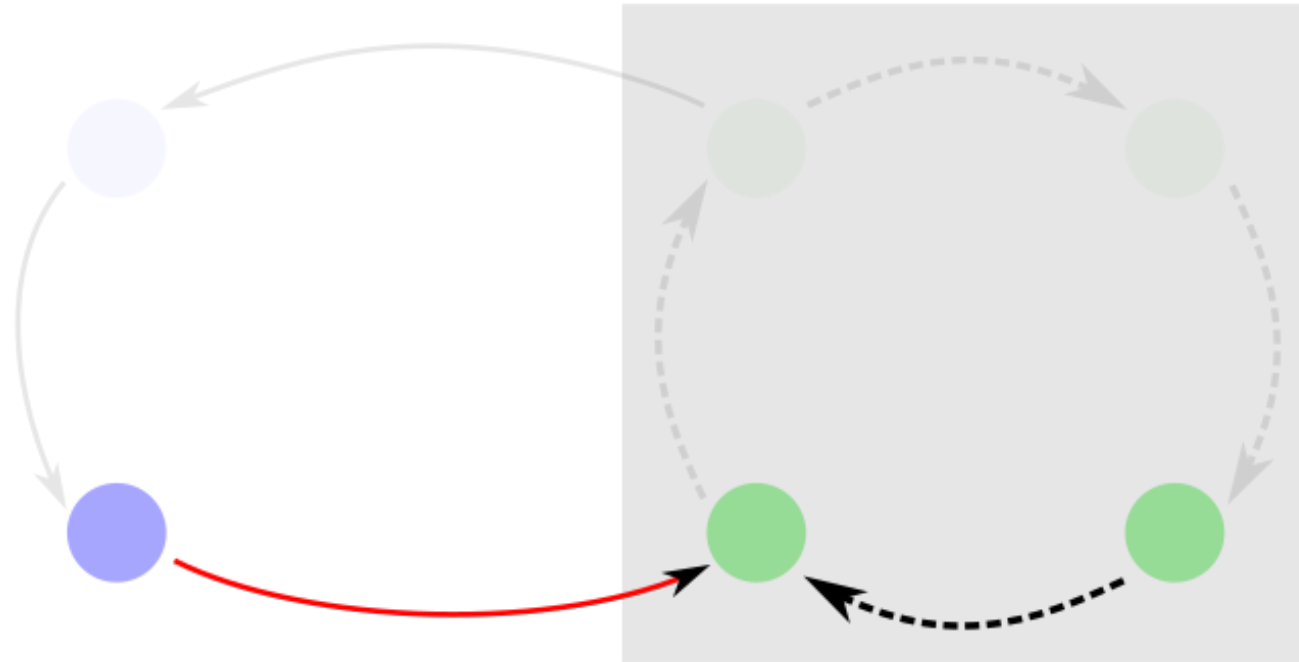
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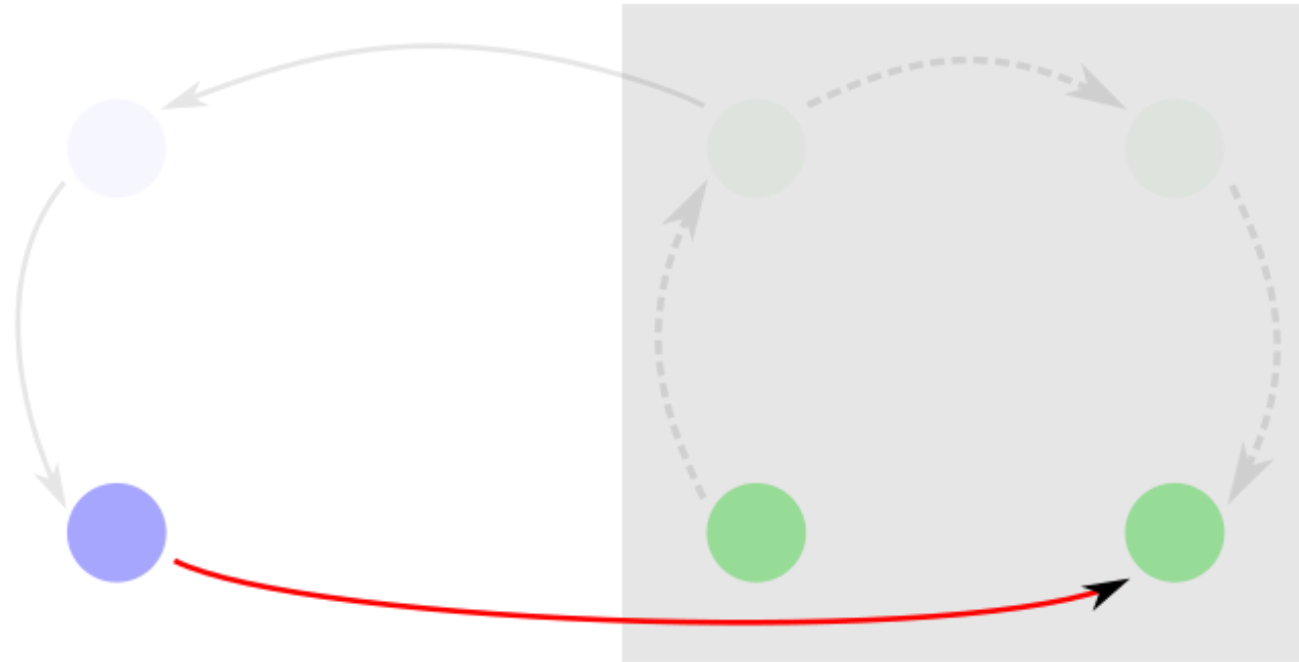
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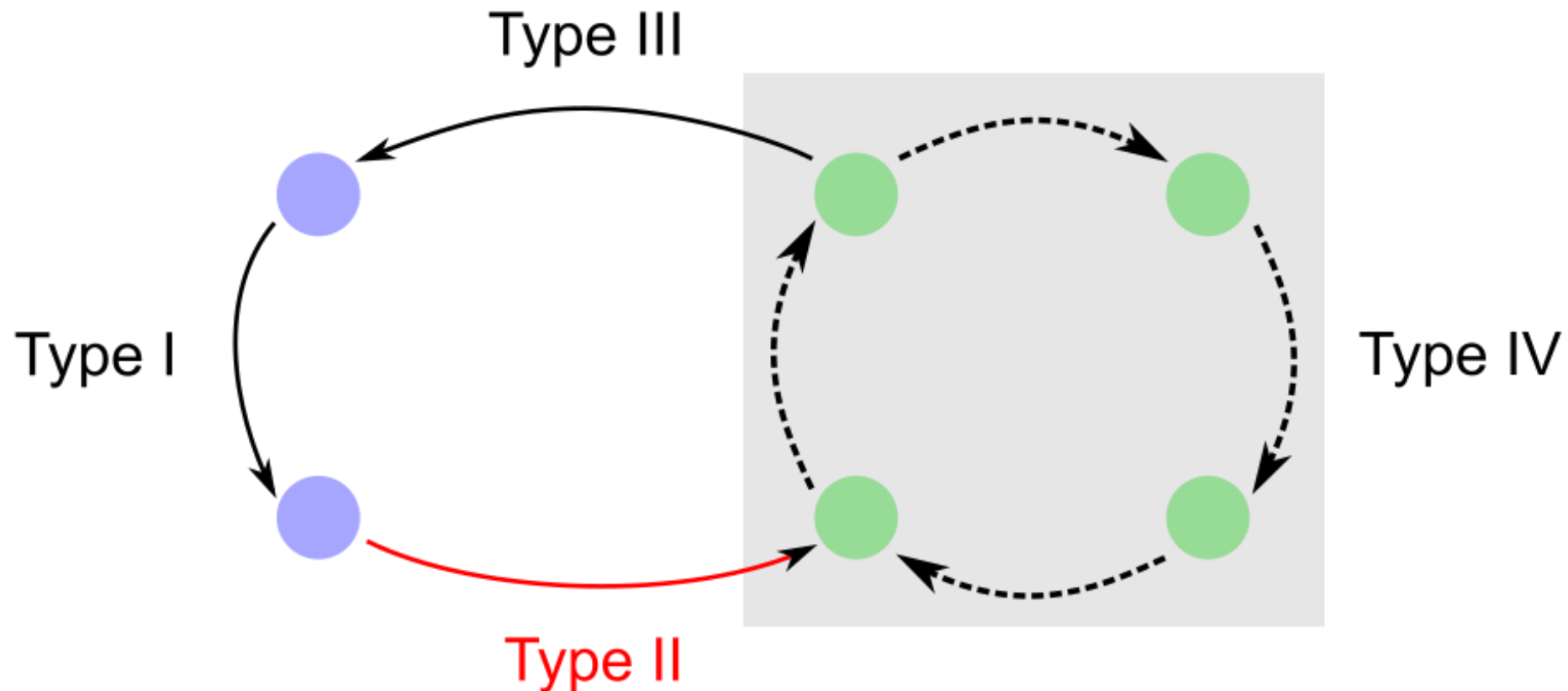
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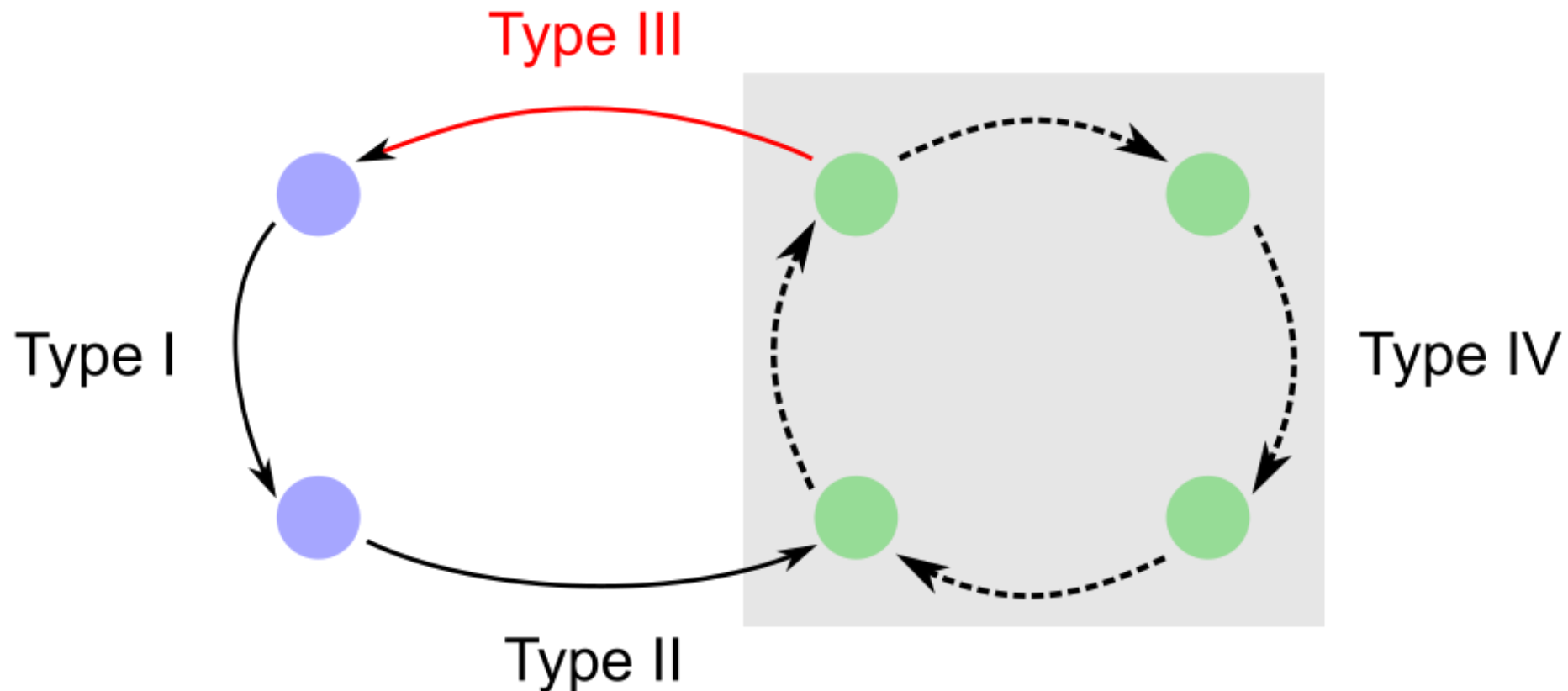
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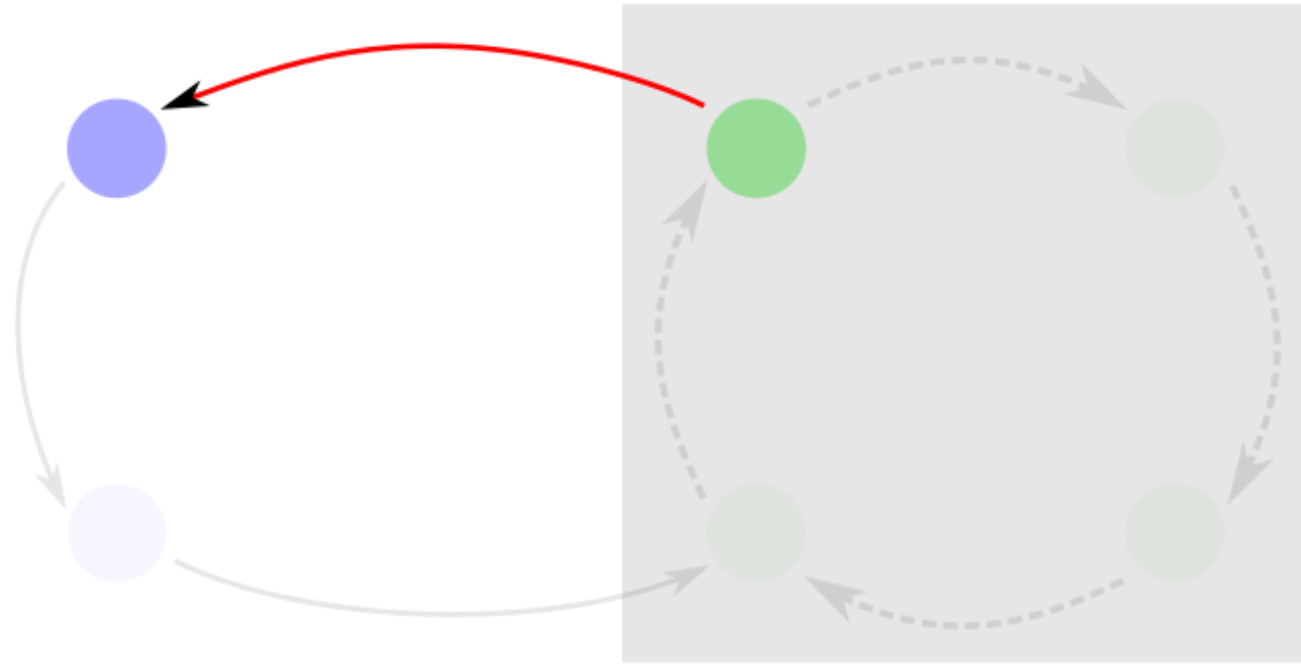
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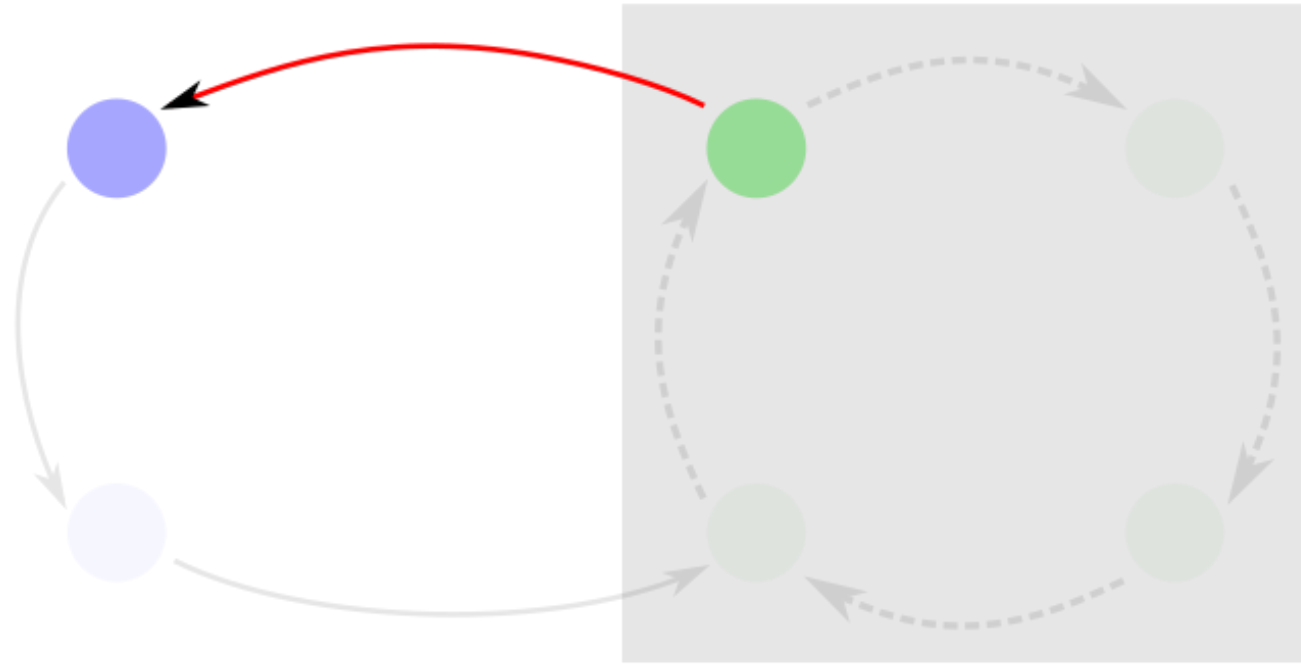
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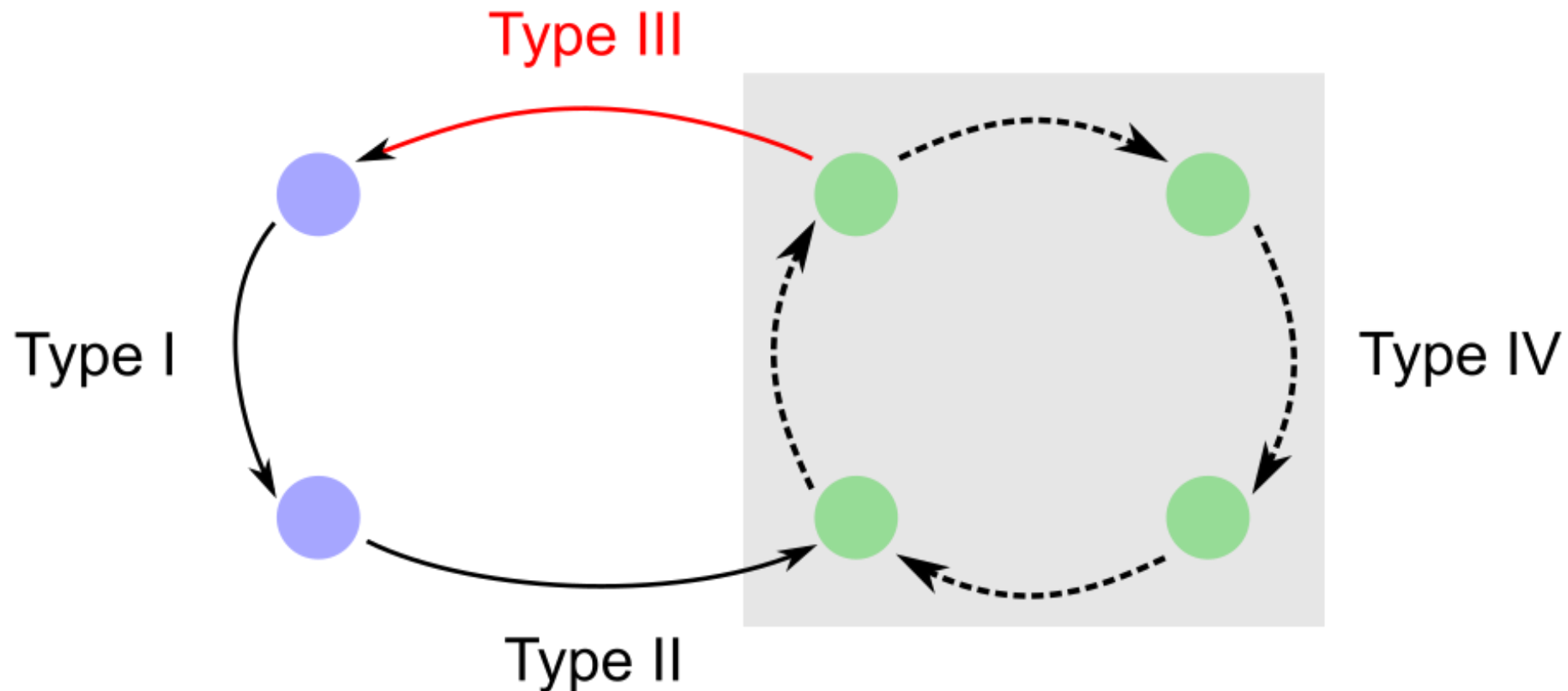
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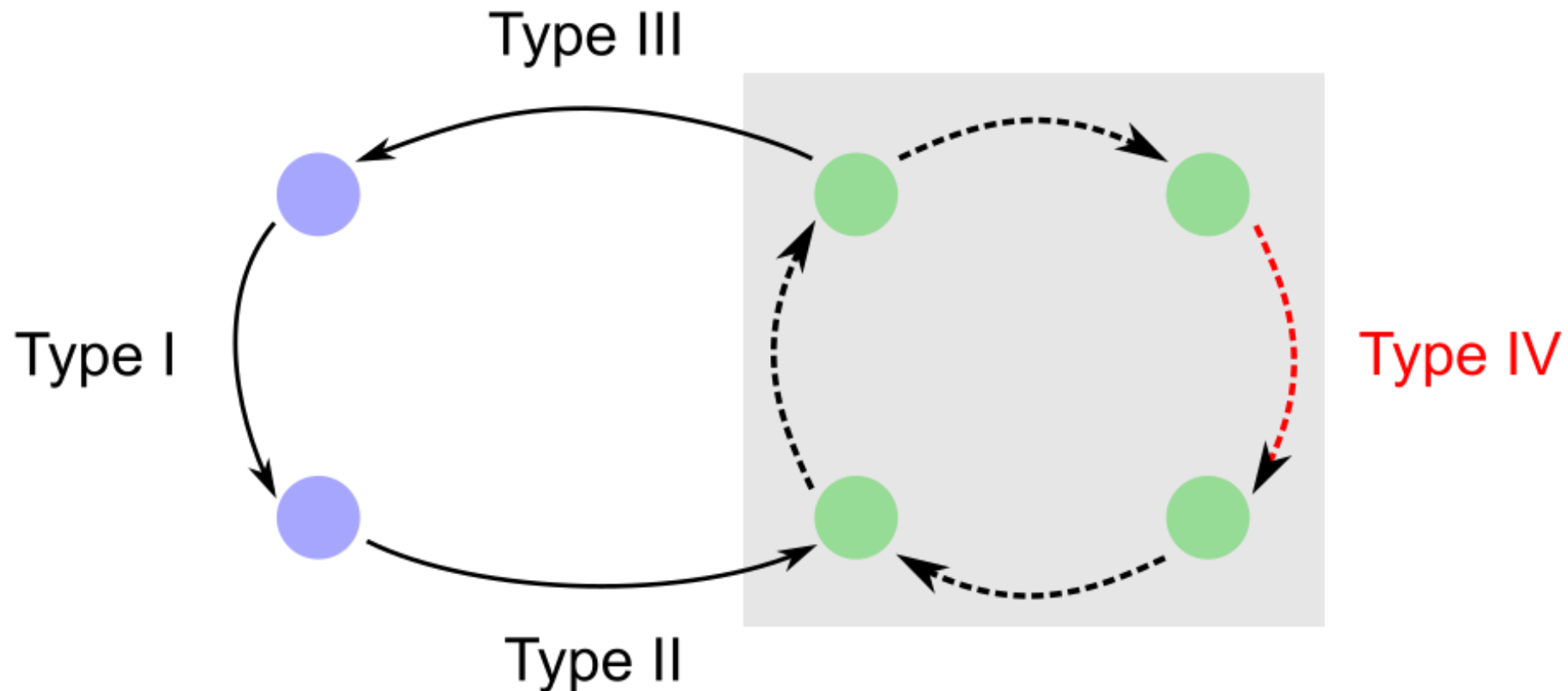
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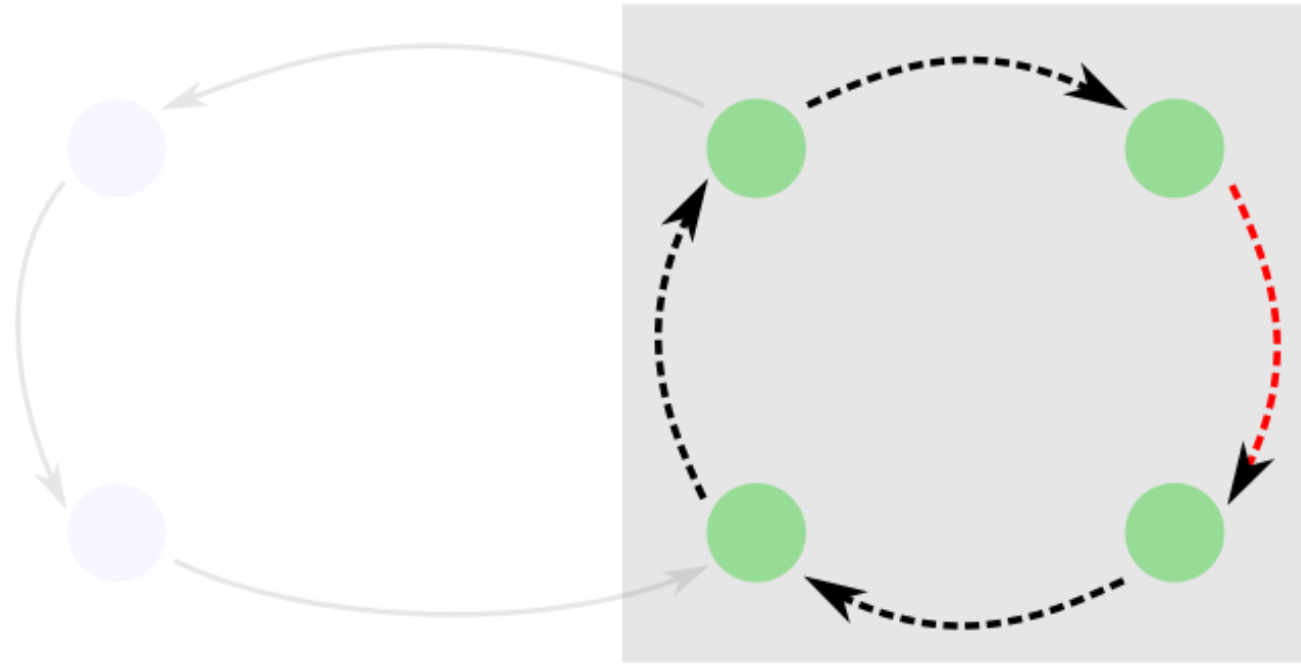
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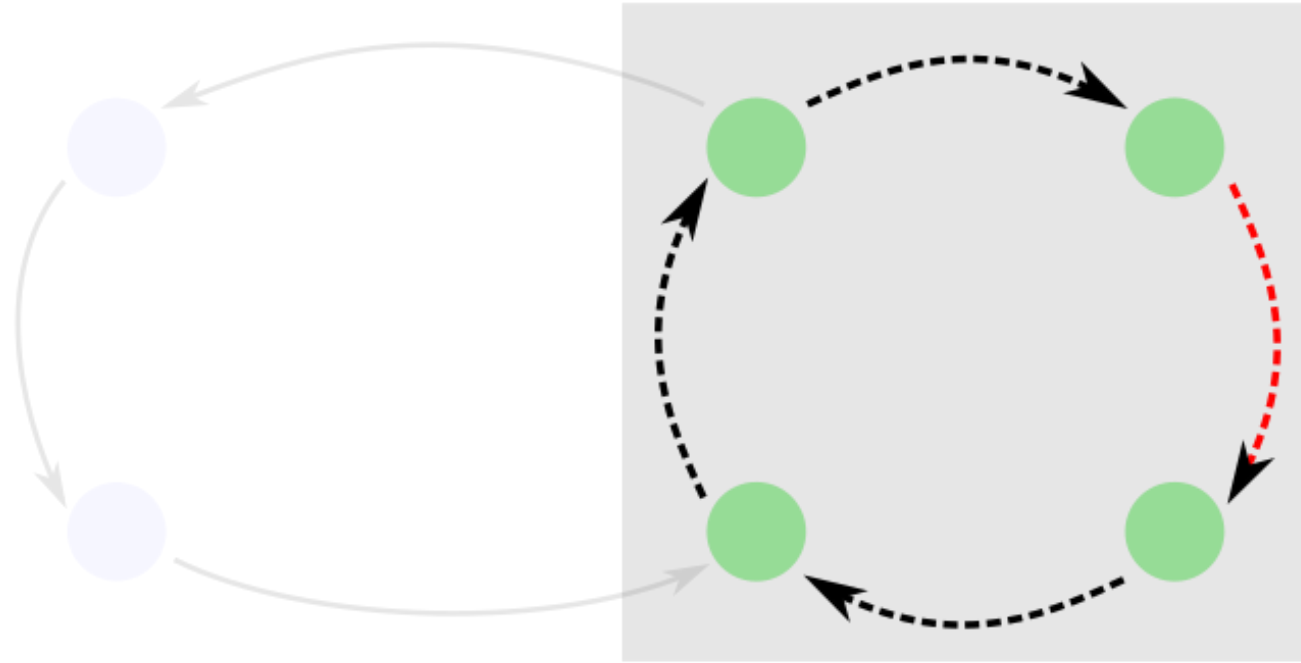
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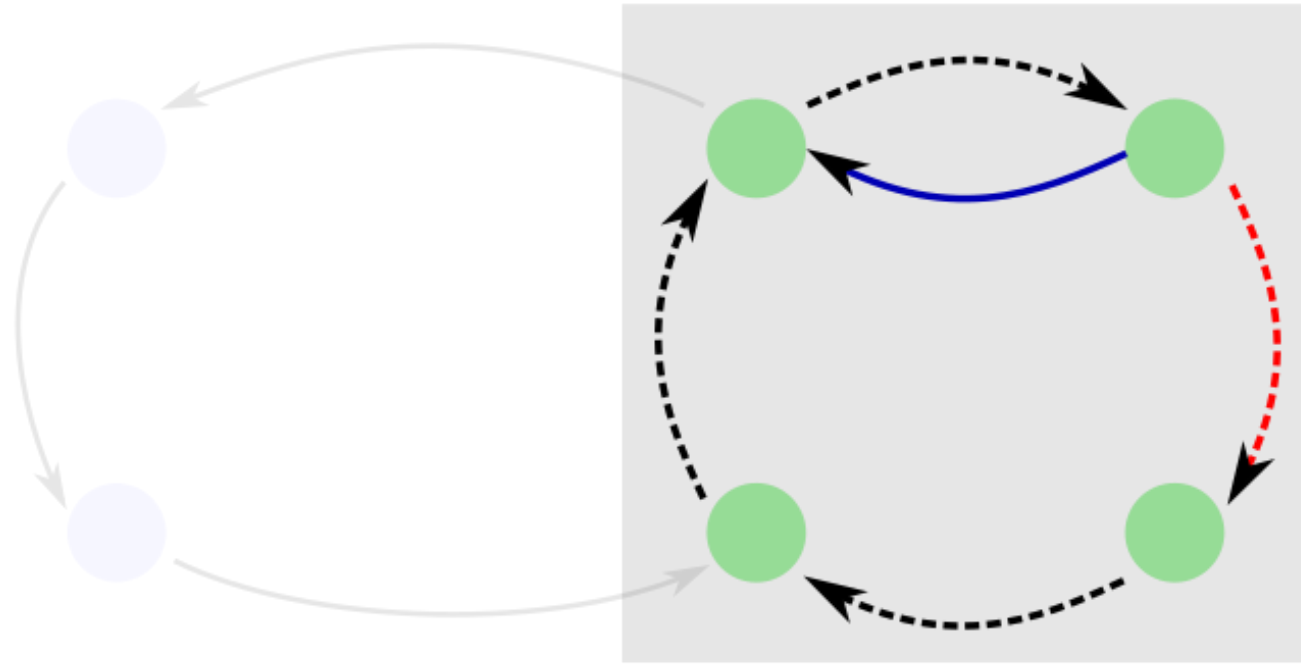
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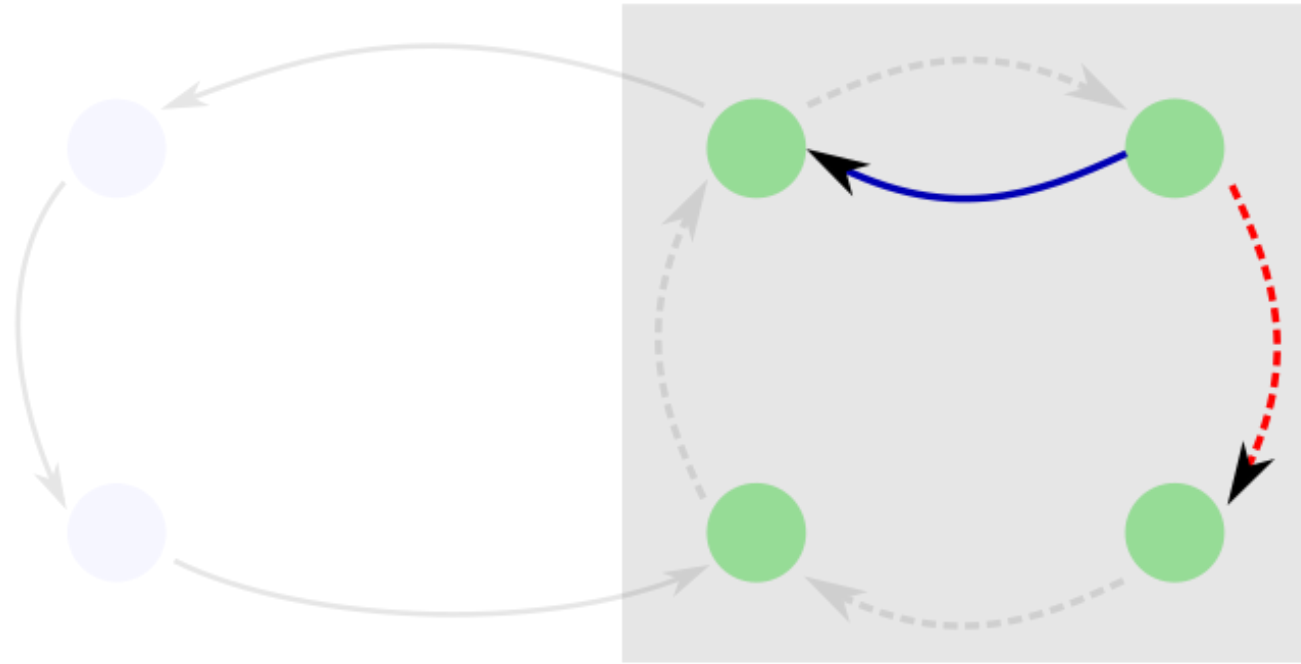
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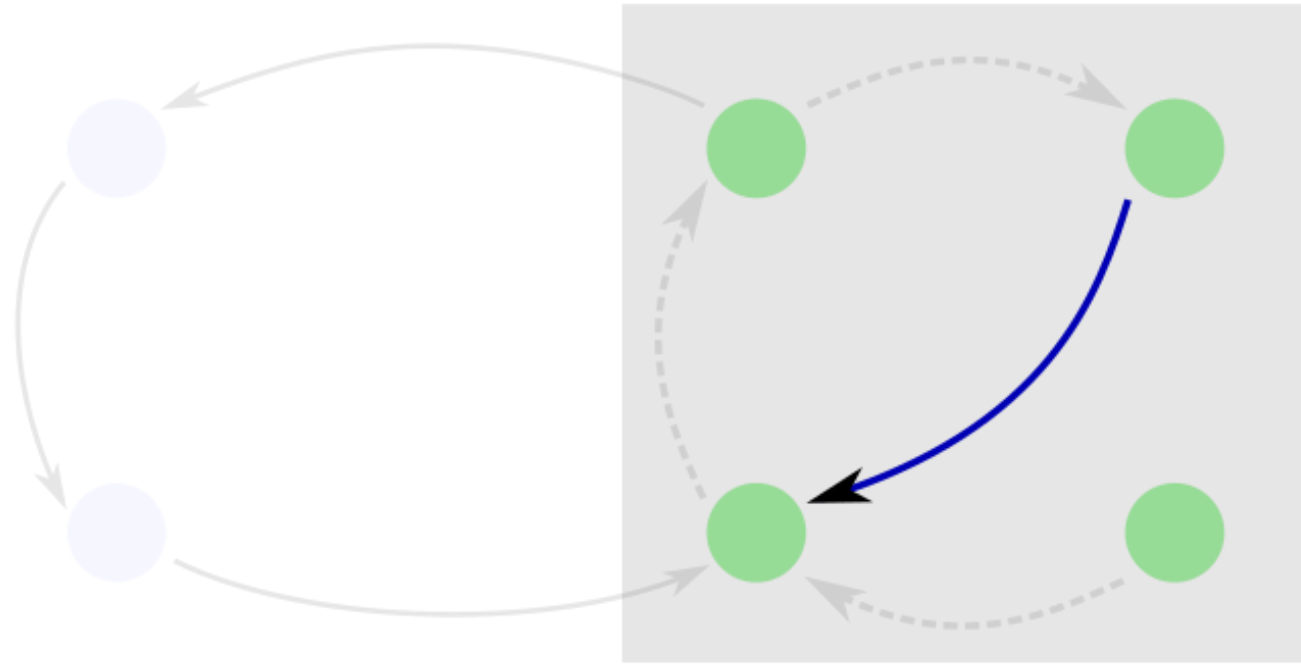
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Allocation A is EF1 if for every pair of agents i, k , there exists a good $j \in A_k$ such that $v_i(A_i) \geq v_i(A_k \setminus \{j\})$.

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We will argue that each iteration of the algorithm "preserves" EF1.

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If the partial allocation at the beginning of an iteration is EF1, then the partial allocation at the end of that iteration is also EF1.

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Suppose good g is assigned to the source agent s . Then,

$$v_i(A_i) \geq v_i(A_s \cup \{g\} \setminus \{g\})$$

which means that EF1 is preserved.

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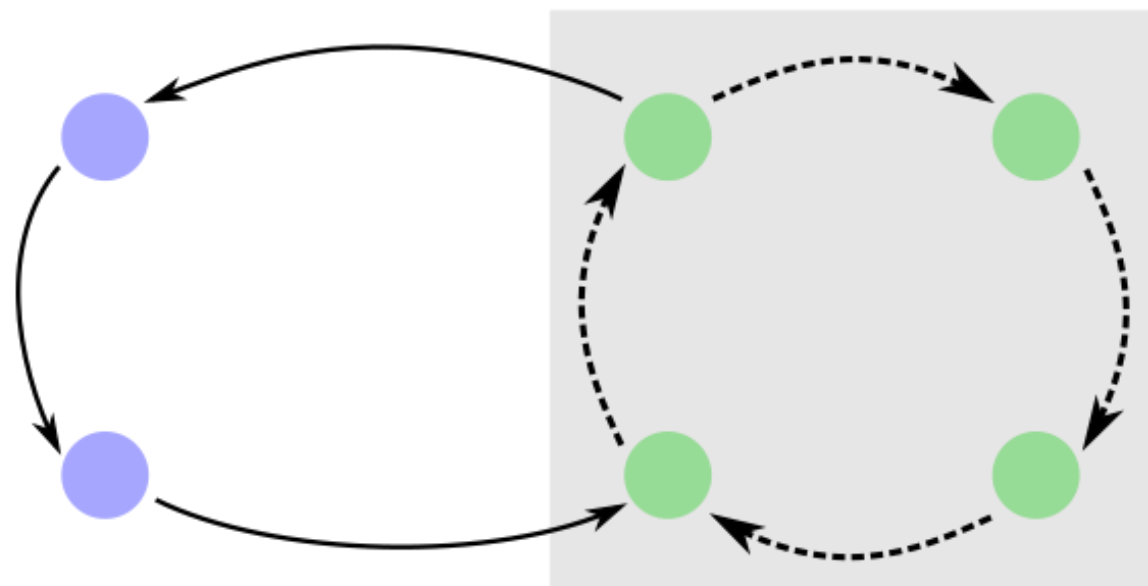
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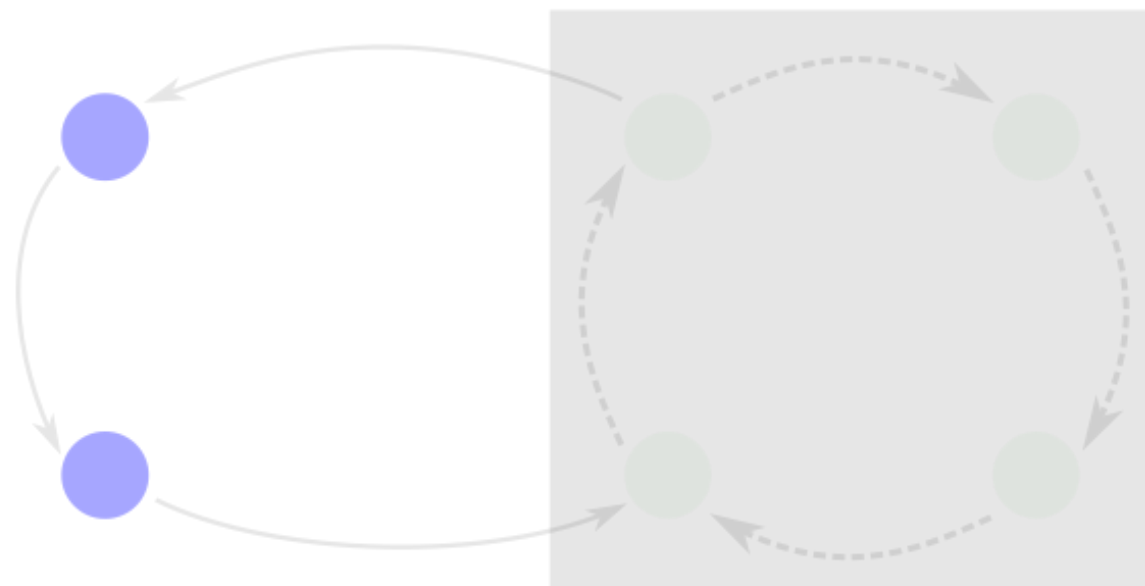


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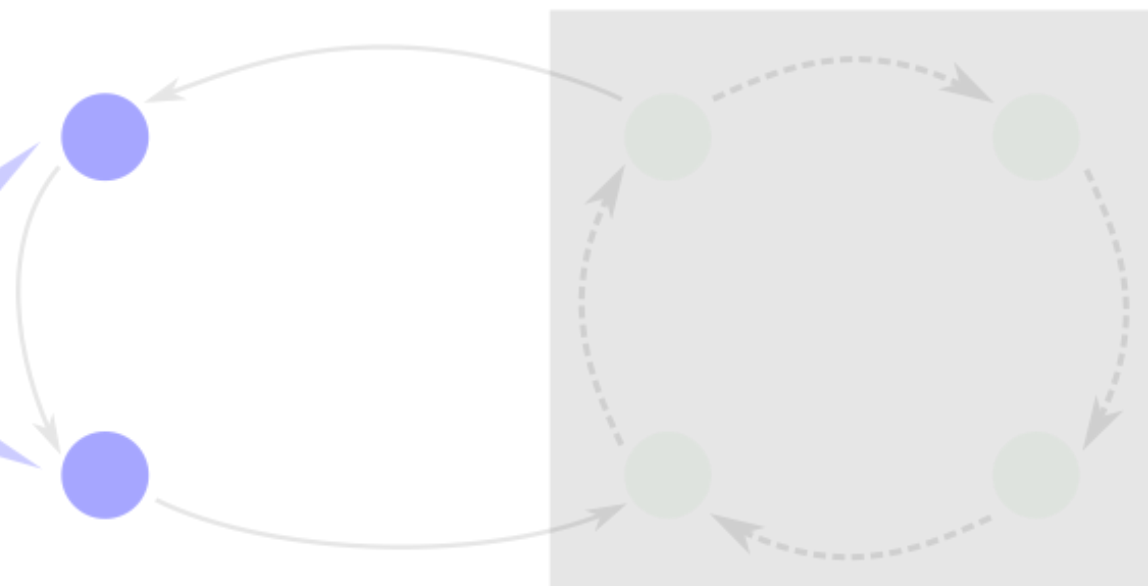
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From their perspective, the bundles in the cycle are only shifted around. So, EF1 relations are the same as before.

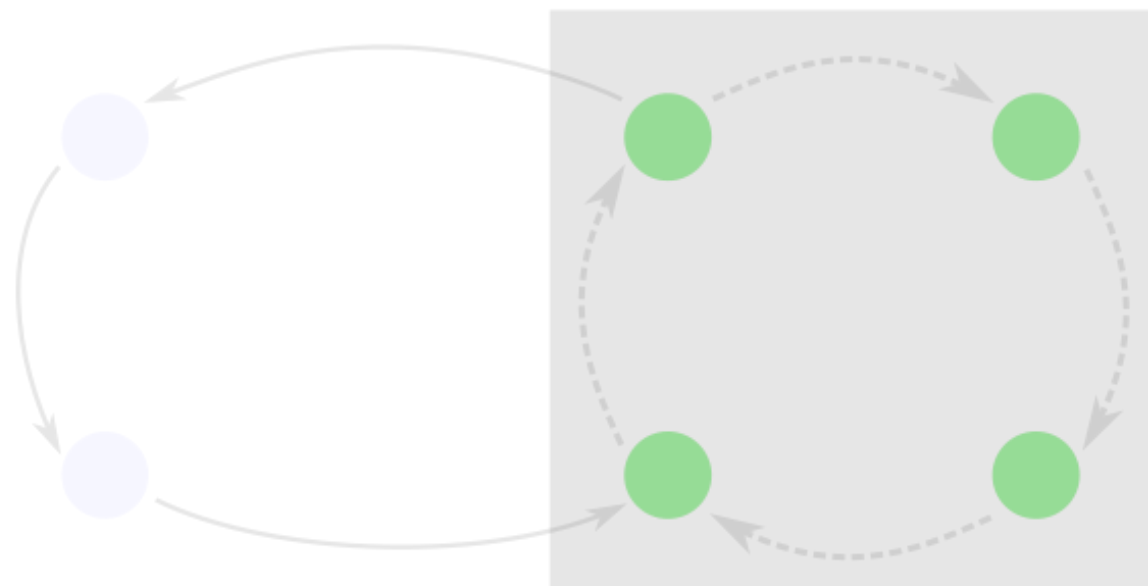


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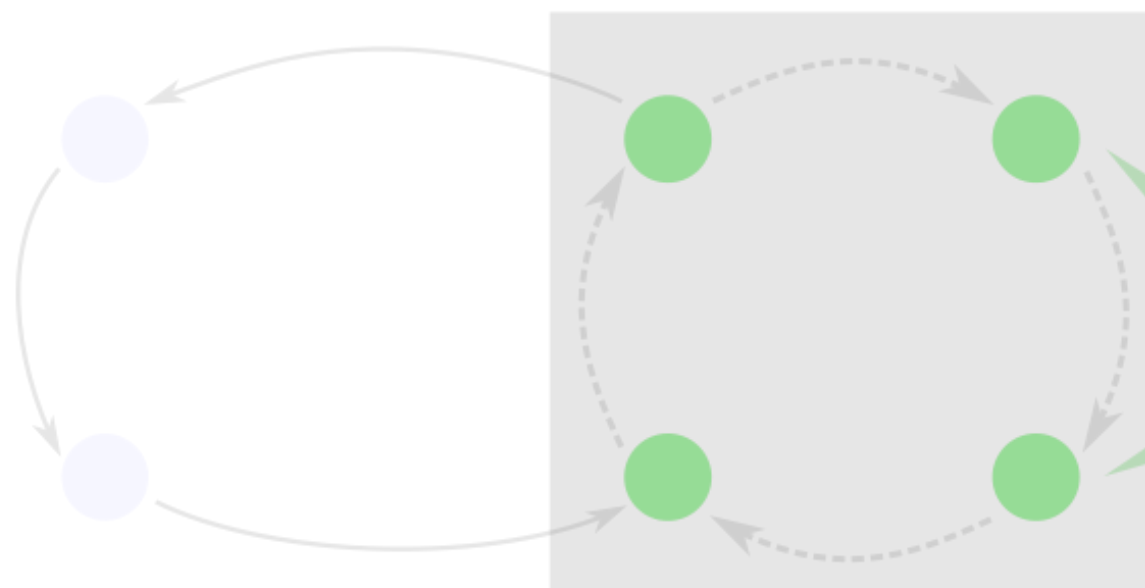


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These agents are strictly better off, and any envied bundles are only shifted around. So, again, EF1 is maintained.

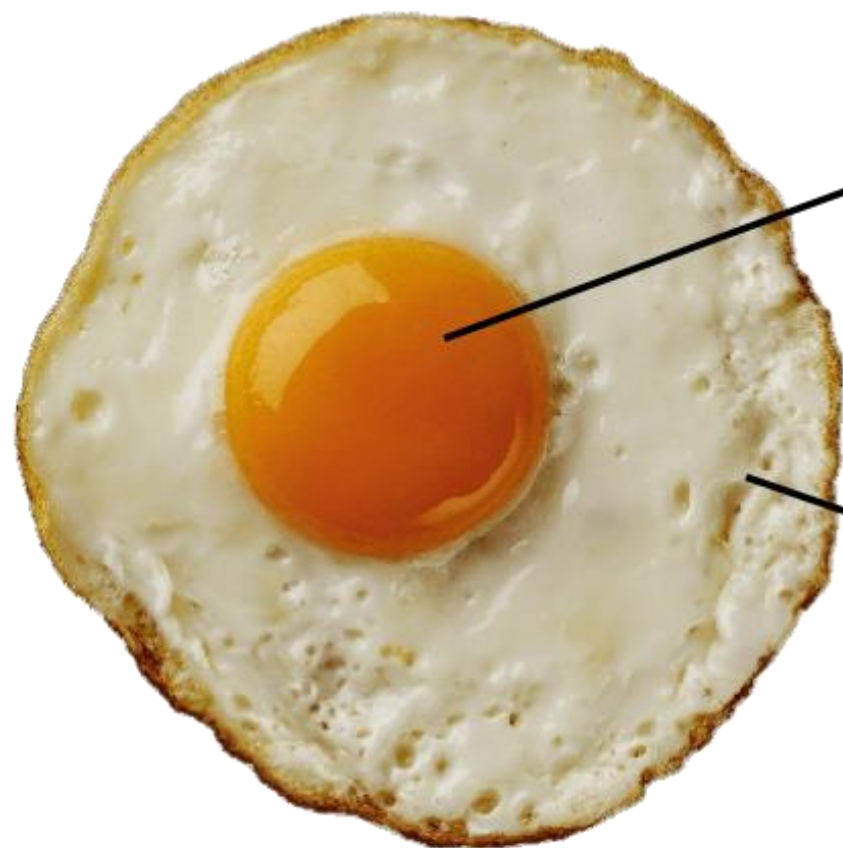
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Additive: $v_i(S) = \sum_{j \in S} v_i(\{j\})$

Monotone: $S \subseteq T \Rightarrow v_i(S) \leq v_i(T)$

Next Time

Fairness and Efficiency



Quiz

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Prove or disprove:

For two agents, the round robin allocation is Pareto optimal.

Quiz

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For two agents, the round robin allocation is Pareto optimal.

An allocation A is Pareto optimal if there is no other allocation B such that:

- every agent is weakly better off under B , and
- some agent is strictly better off under B .

References

- Envy-cycle elimination algorithm

Richard Lipton, Evangelos Markakis, Elchanan Mossel, and Amin Saberi
“On Approximately Fair Allocations of Indivisible Goods”

EC 2004, pg 125-131

<https://dl.acm.org/doi/10.1145/988772.988792>

