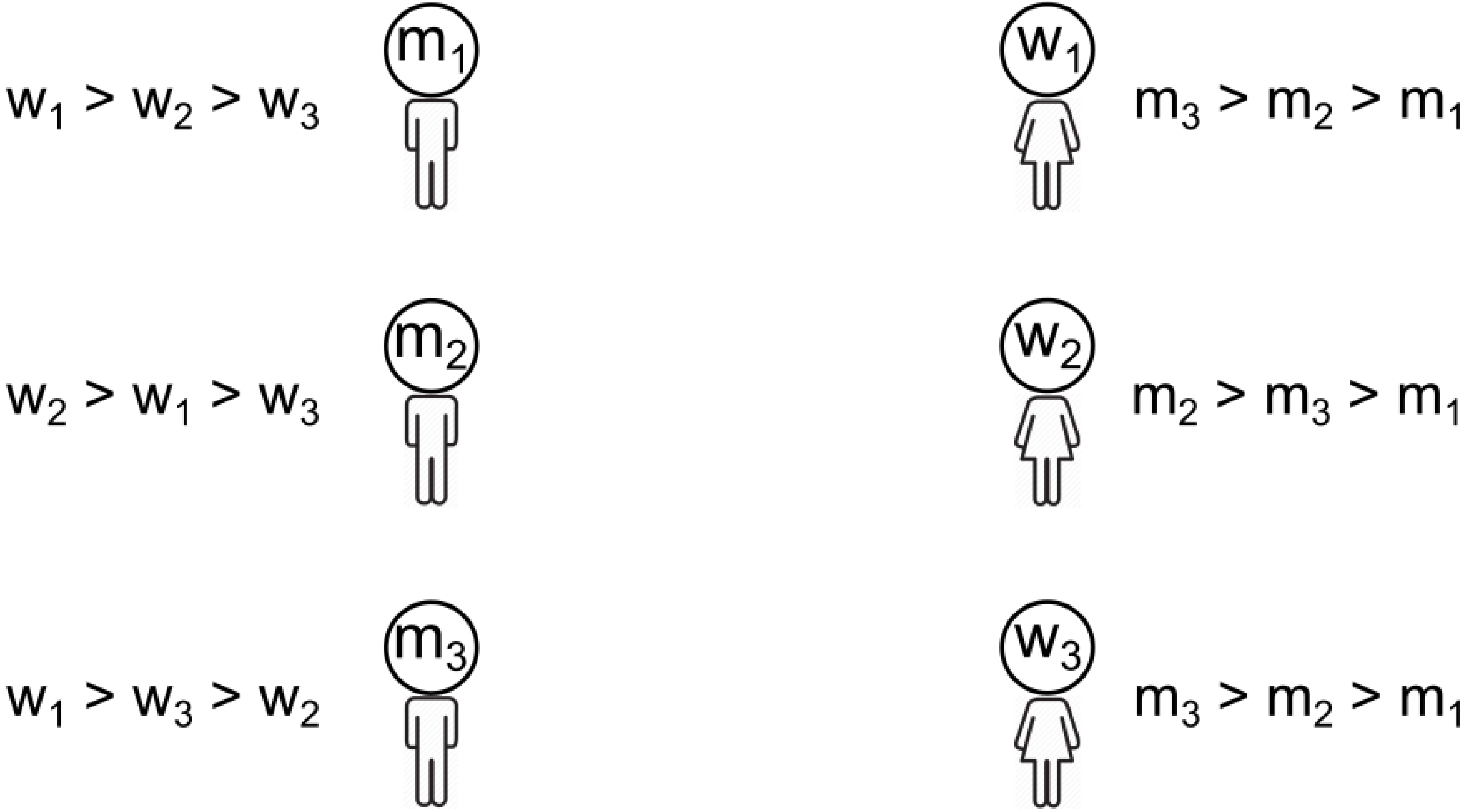
COL866: Special Topics in Algorithms

Lecture 4

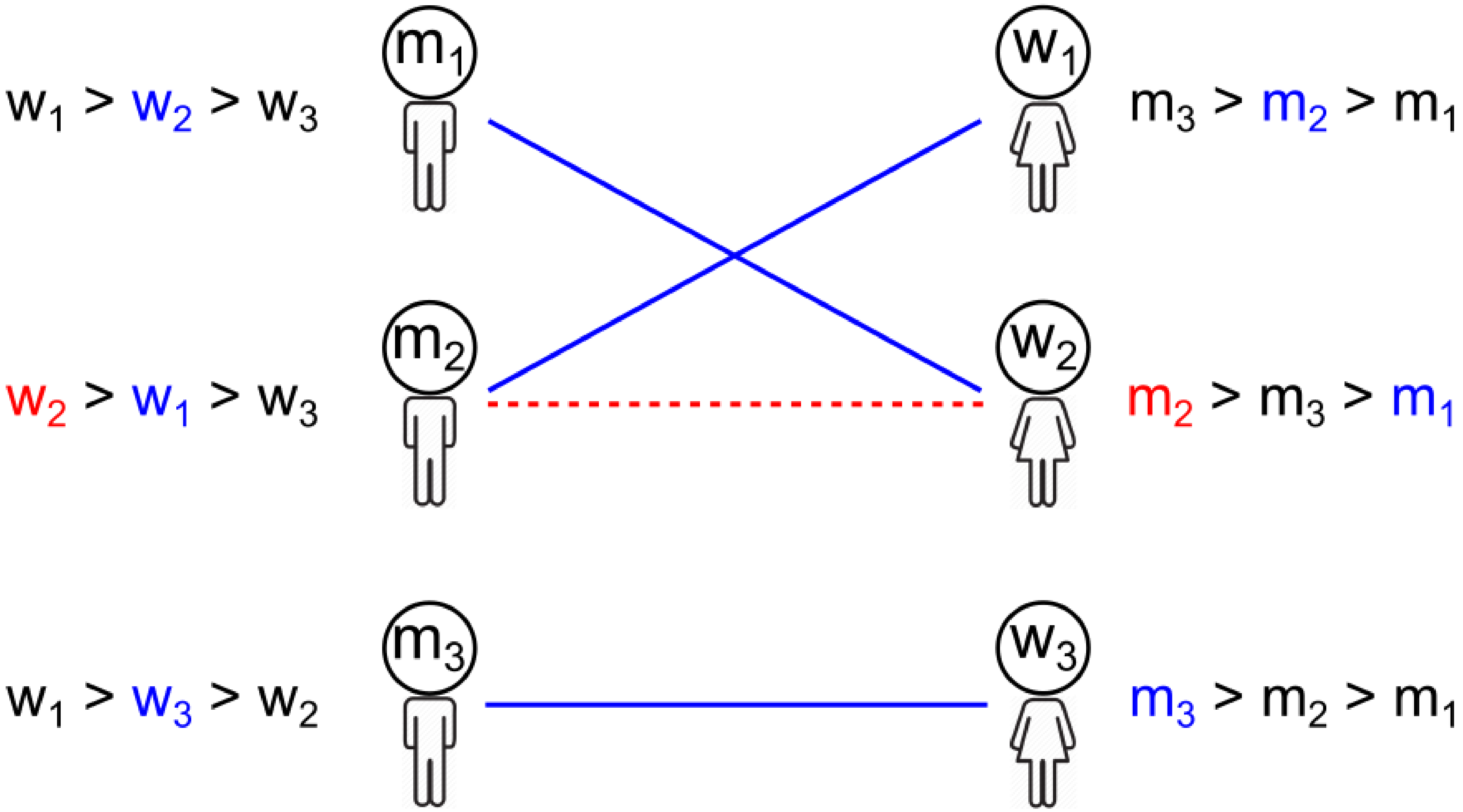
Fairness in Stable Matching Problem

Aug 04, 2023 | Rohit Vaish

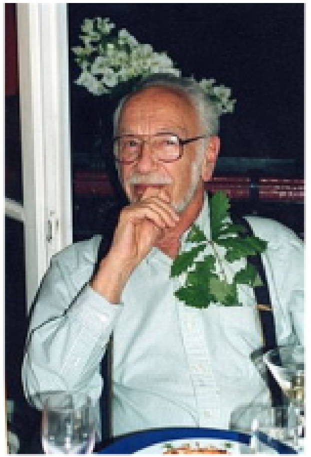
Stable Matching Problem



Stable Matching Problem



A matching is stable if there is no blocking pair.

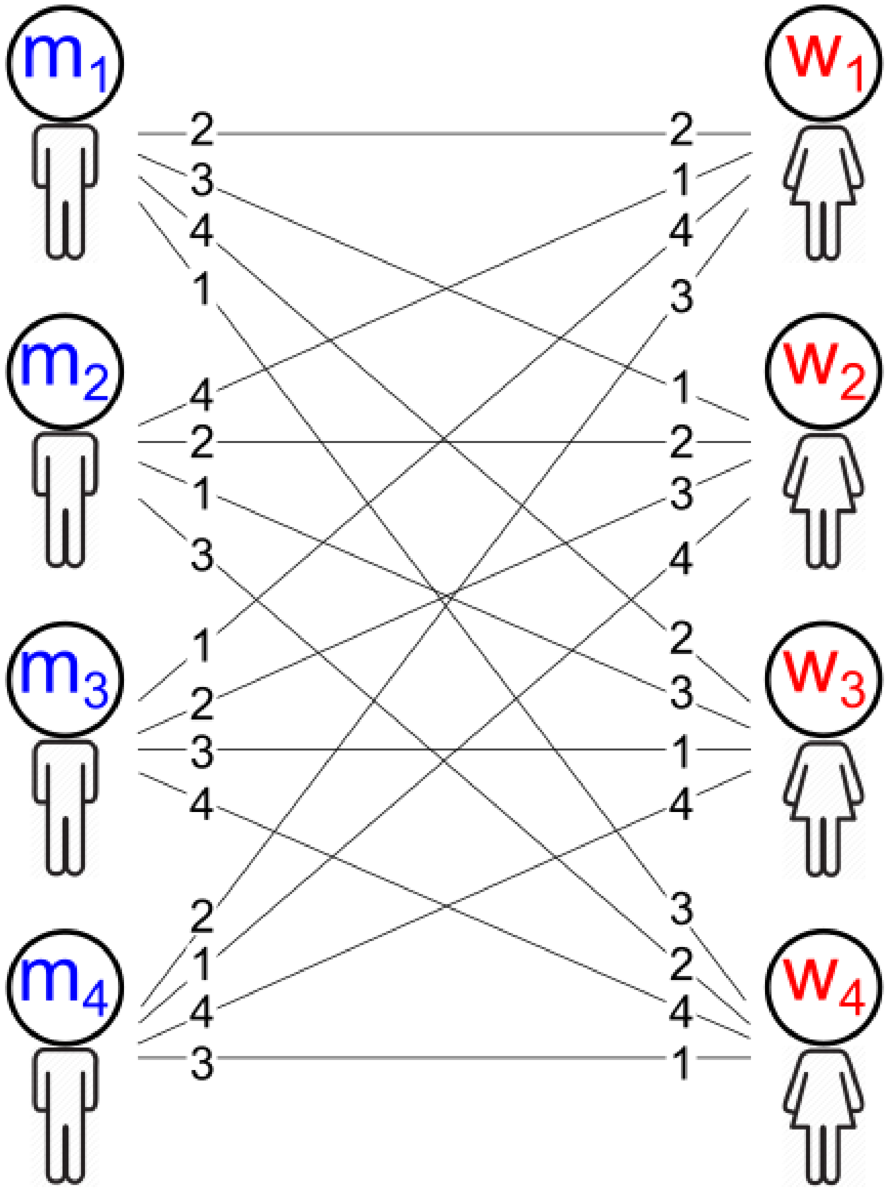


COLLEGE ADMISSIONS AND THE STABILITY OF MARRIAGE

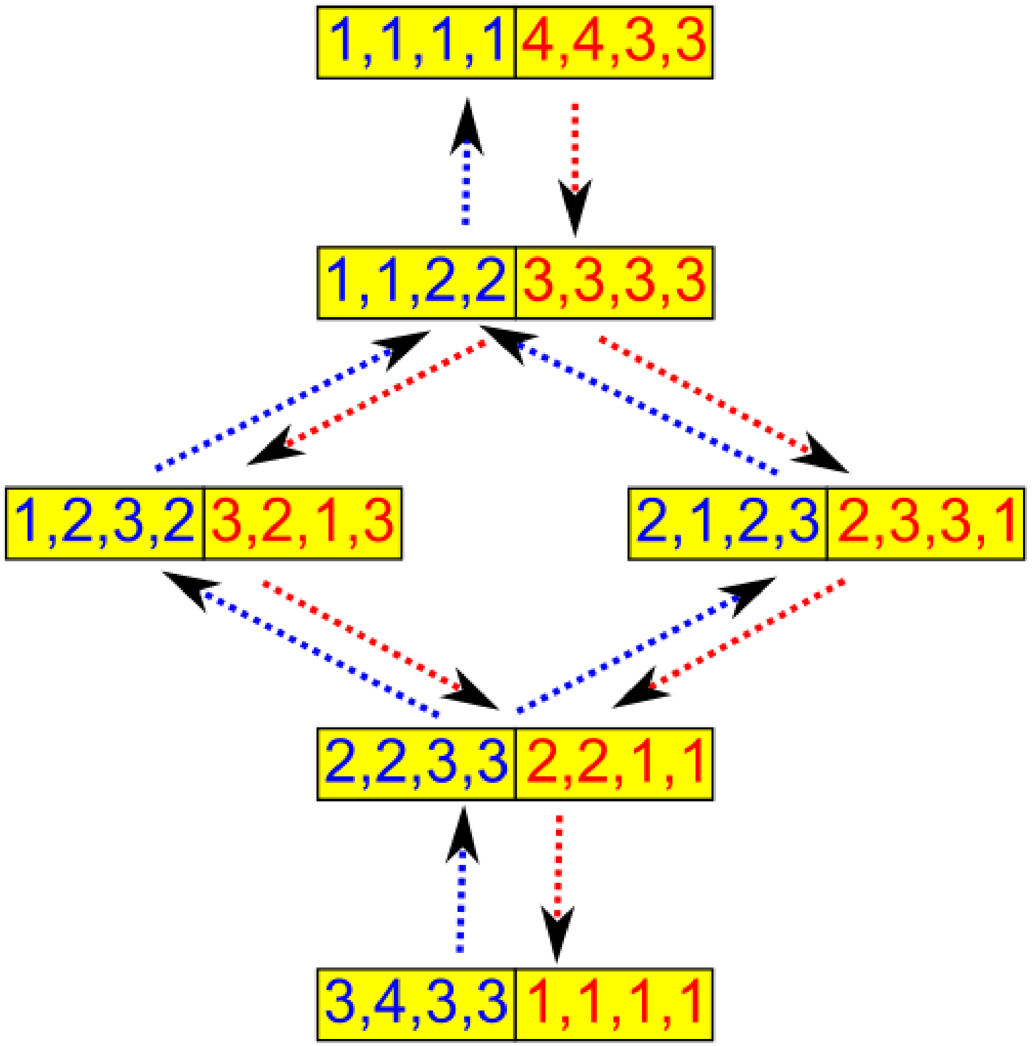
D. GALE* AND L. S. SHAPLEY, Brown University and the RAND Corporation

Source: *The American Mathematical Monthly*, Jan., 1962, Vol. 69, No. 1 (Jan., 1962), pp. 9–15

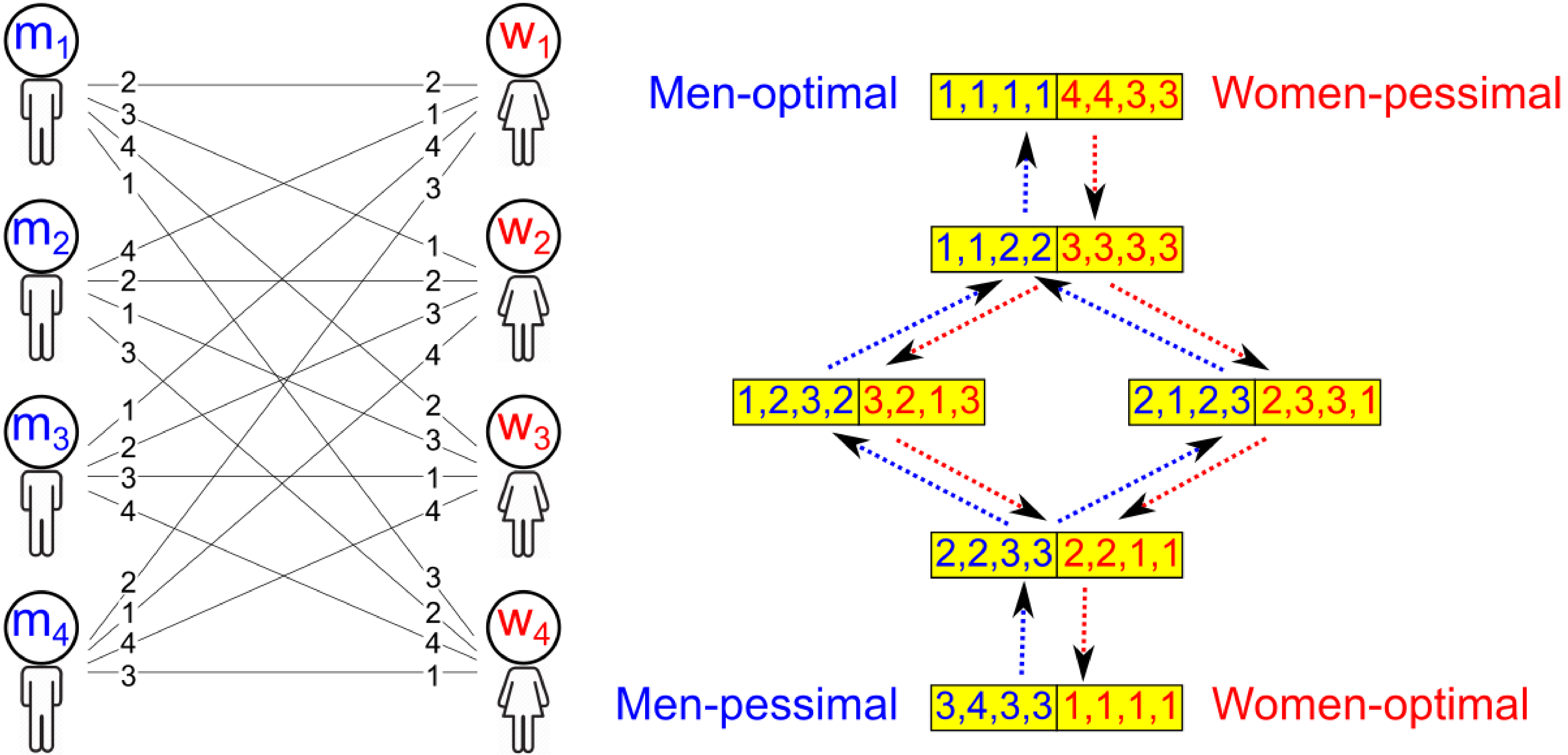
Given any preference profile, a stable matching for that profile always exists and can be computed in polynomial time.



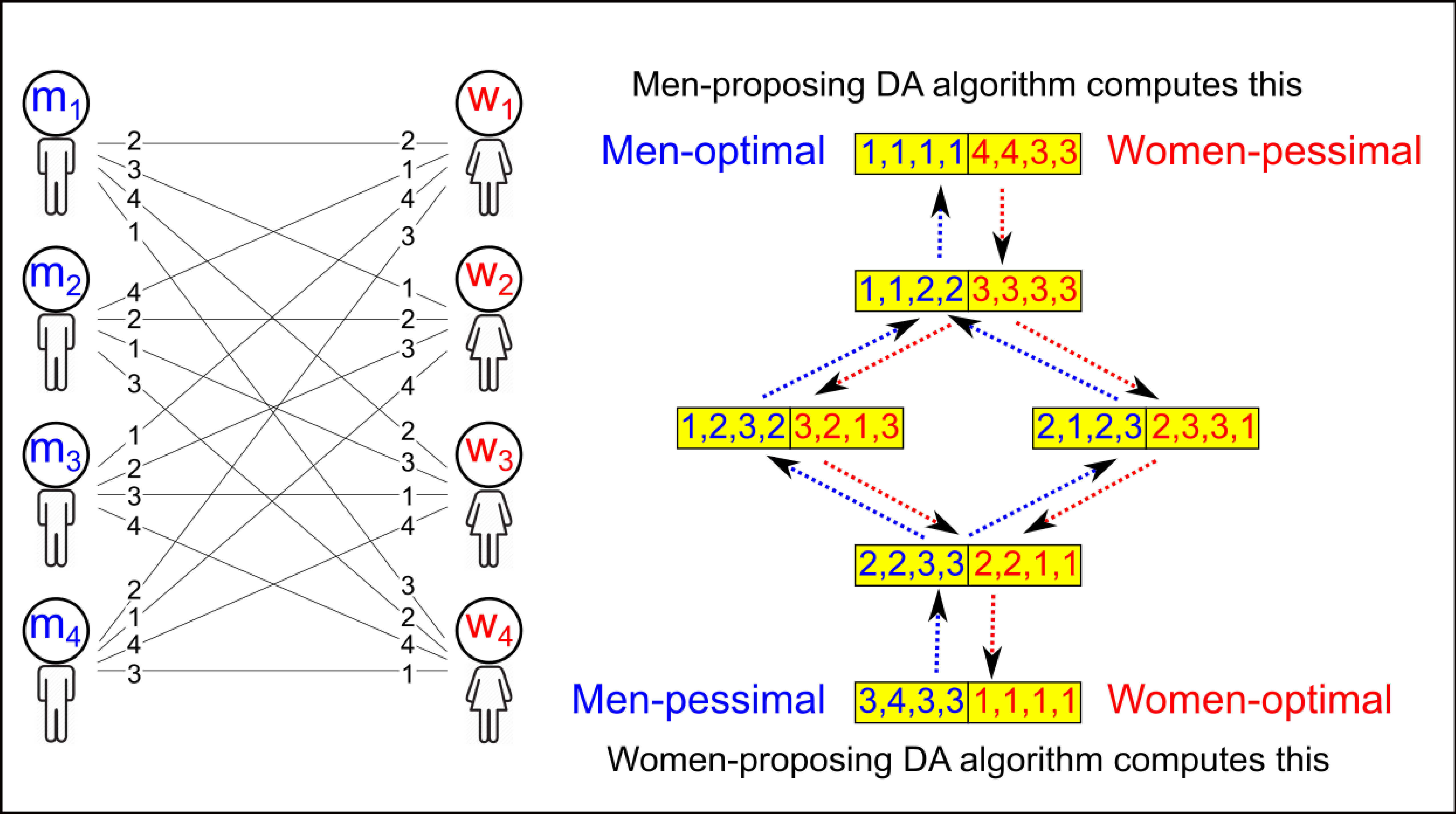
m1
2 3 4 1
m2
4 2 1 3
m3
1 2 3 4
m4
2 1 4 3
w1
2 1 4 3
w2
1 2 3 4
w3
2 3 1 4
w4
3 2 4 1



1,1,1,1 4,4,3,3
1,1,2,2 3,3,3,3
1,2,3,2 3,2,1,3
2,1,2,3 2,3,3,1
2,2,3,3 2,2,1,1
3,4,3,3 1,1,1,1



m1
m2
m3
m4
w1
w2
w3
w4
Men-optimal
1,1,1,1 4,4,3,3
Women-pessimal
1,1,2,2 3,3,3,3
1,2,3,2 3,2,1,3
2,1,2,3 2,3,3,1
2,2,3,3 2,2,1,1
Men-pessimal
3,4,3,3 1,1,1,1
Women-optimal


m1
m2
m3
m4
w1
w2
w3
w4
Men-proposing DA algorithm computes this
Men-optimal
1,1,1,1 4,4,3,3
Women-pessimal
1,1,2,2 3,3,3,3
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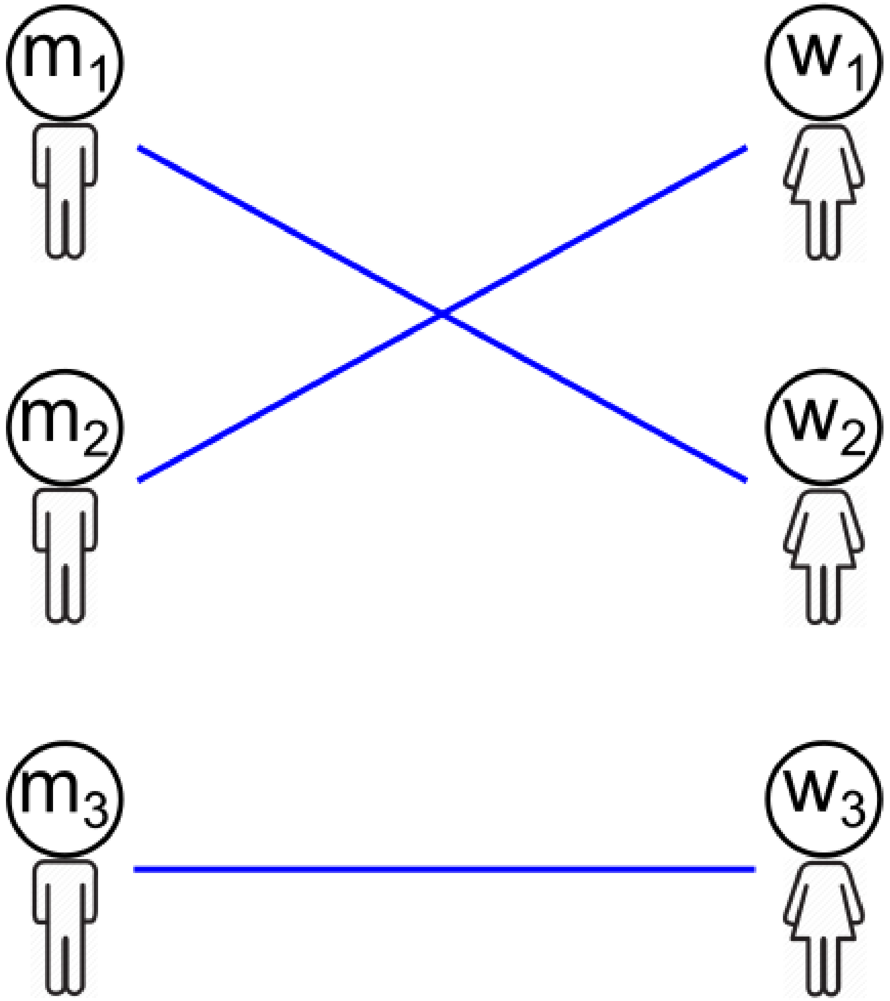
2,2,3,3 2,2,1,1
Men-pessimal
3,4,3,3 1,1,1,1
Women-optimal
Women-proposing DA algorithm computes this

Goal for Today

Understanding the structure of the set of stable matchings through linear programming.

(This will guide us towards fair stable matchings.)



$$P = \begin{array}{c} \\ m_1 \\ m_2 \\ m_3 \end{array} \begin{array}{c} \begin{array}{ccc} w_1 & w_2 & w_3 \end{array} \\ \left[\begin{array}{ccc} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{array} \right] \end{array}$$

Fractional Stable Matching

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Any non-negative nxn matrix X satisfying the following:

Fractional Stable Matching

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$X_{i,j} \geq 0 \text{ for all } i \in [n] \text{ and } j \in [n]$

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$\sum_j X_{i,j} = 1$ for all $i \in [n]$ *Every man is fully matched.*

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$\sum_i X_{i,j} = 1 \text{ for all } j \in [n]$ *Every woman is fully matched.*

$$X_{i,j} + \sum_{k:w_k \succ_{m_i} w_j} X_{i,k} + \sum_{k:m_k \succ_{w_j} m_i} X_{k,j} \geq 1 \text{ for all } i \in [n] \text{ and } j \in [n]$$ *Stability*

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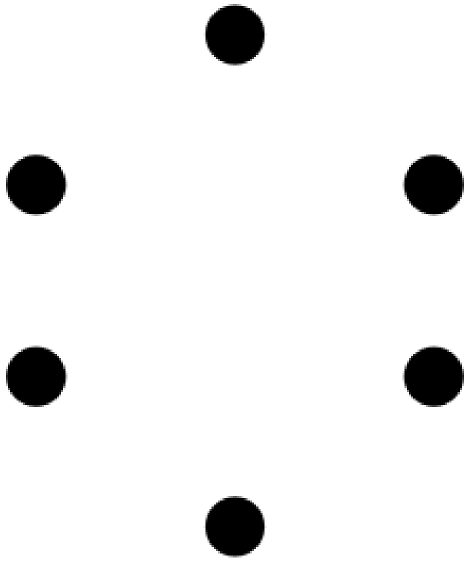
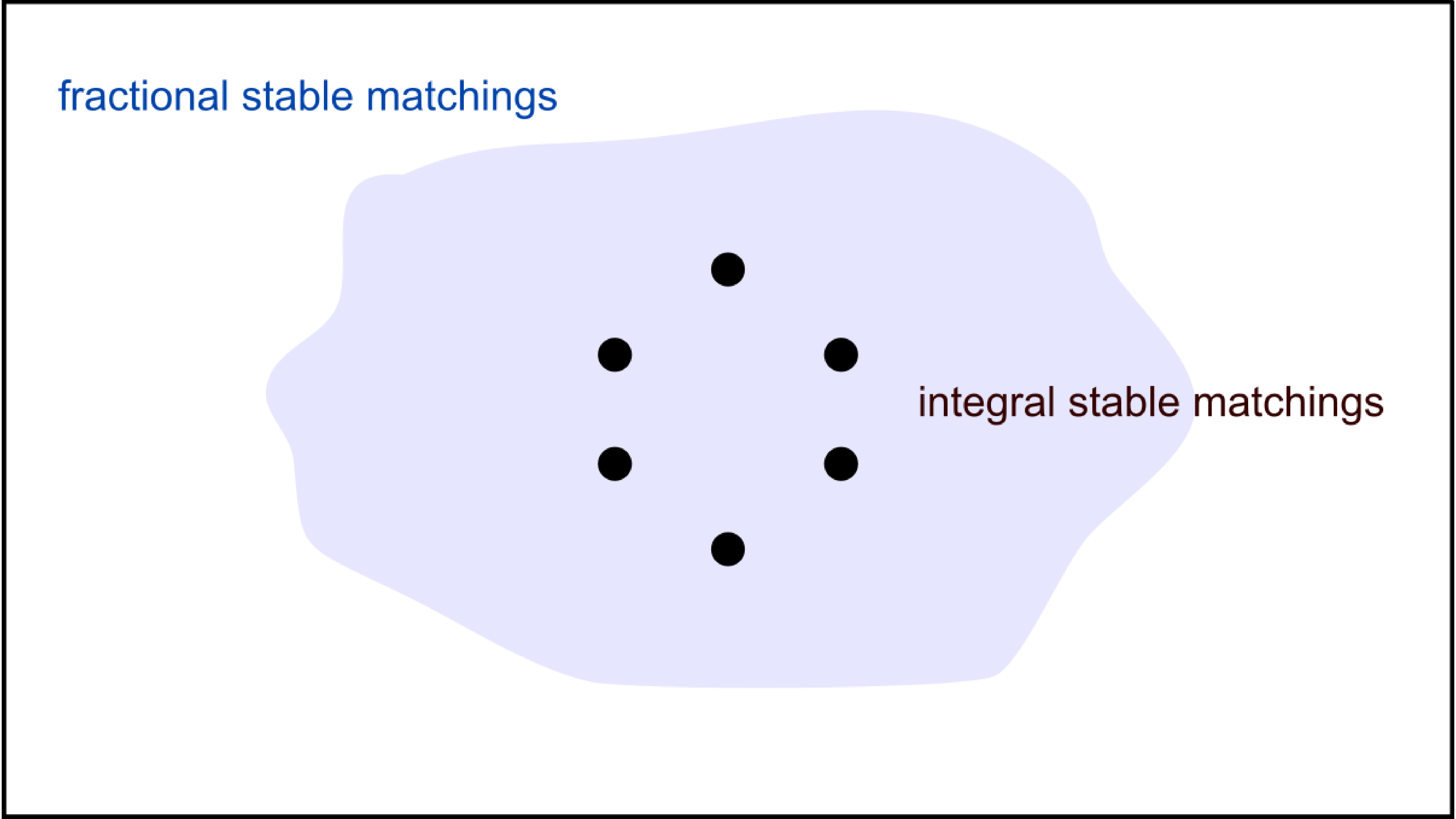
Any integral stable matching is also a fractional stable matching.



integral stable matchings


fractional stable matchings
integral stable matchings

Fractional Stable Matching

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Stability

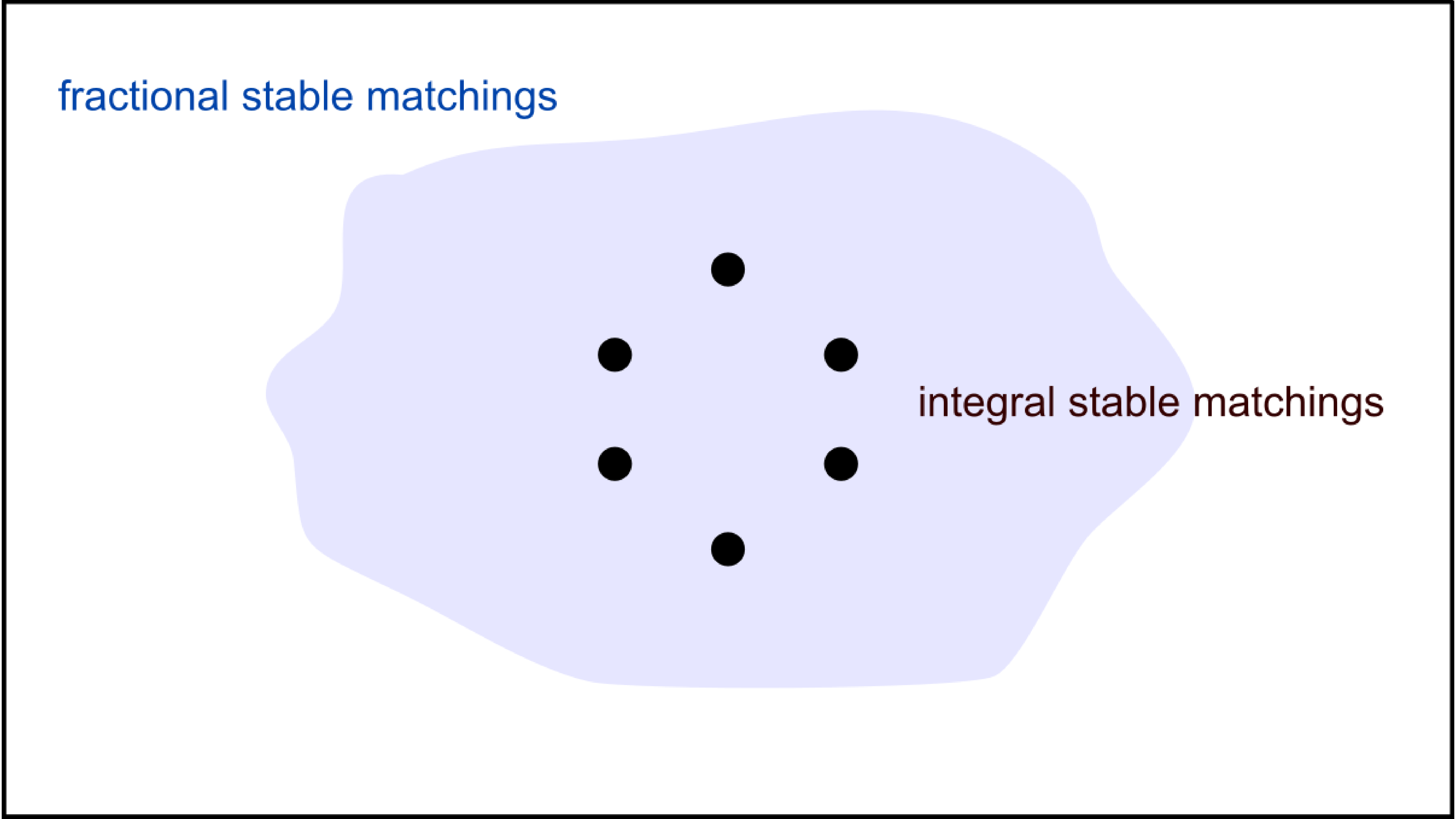
Any convex combination of integral stable matchings is also a fractional stable matching.

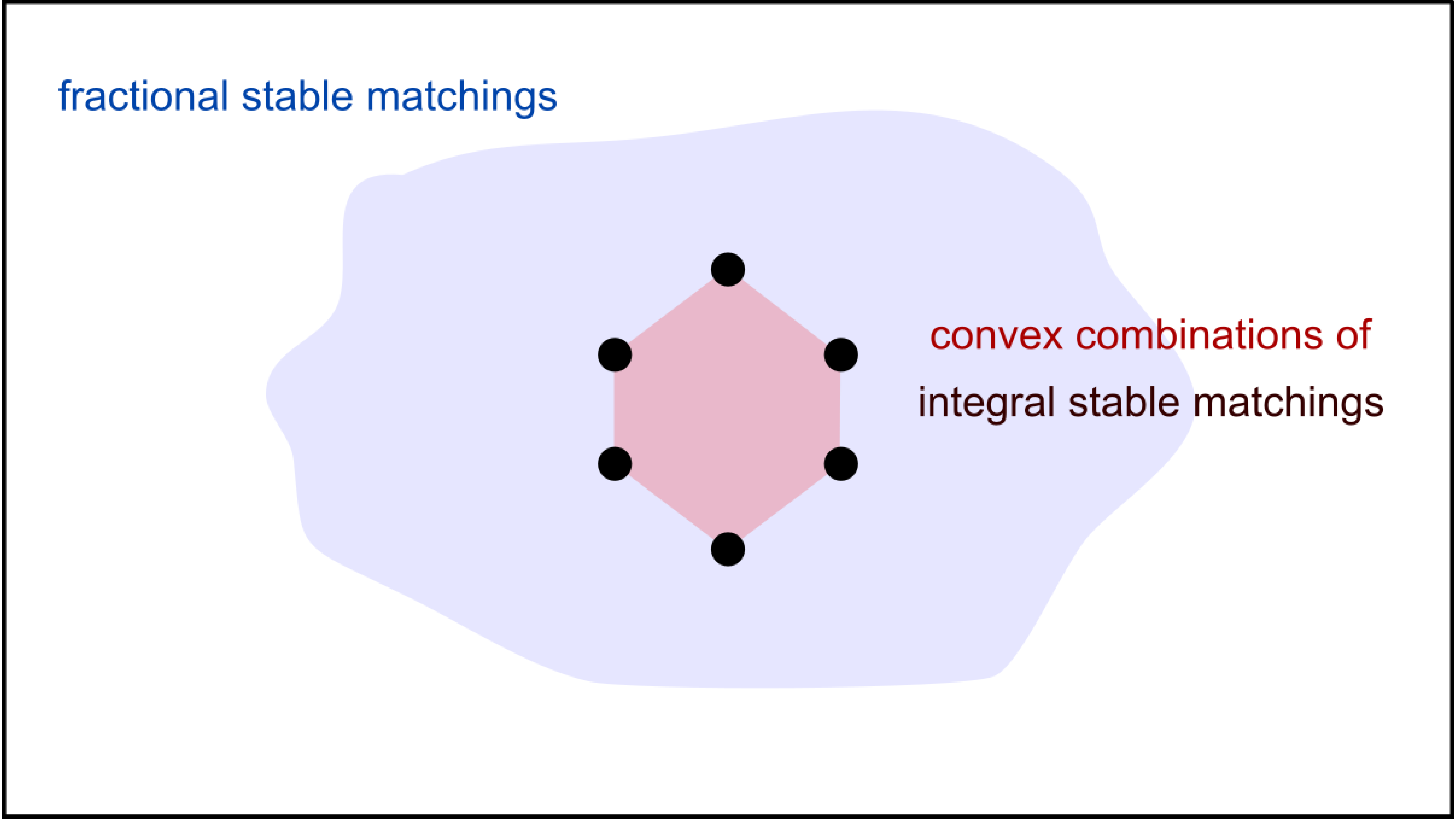
$$X = \sum_k \lambda_k P^k$$

P^k: integral stable matching

such that $\lambda_k \geq 0$ for all k and $\sum_k \lambda_k = 1$

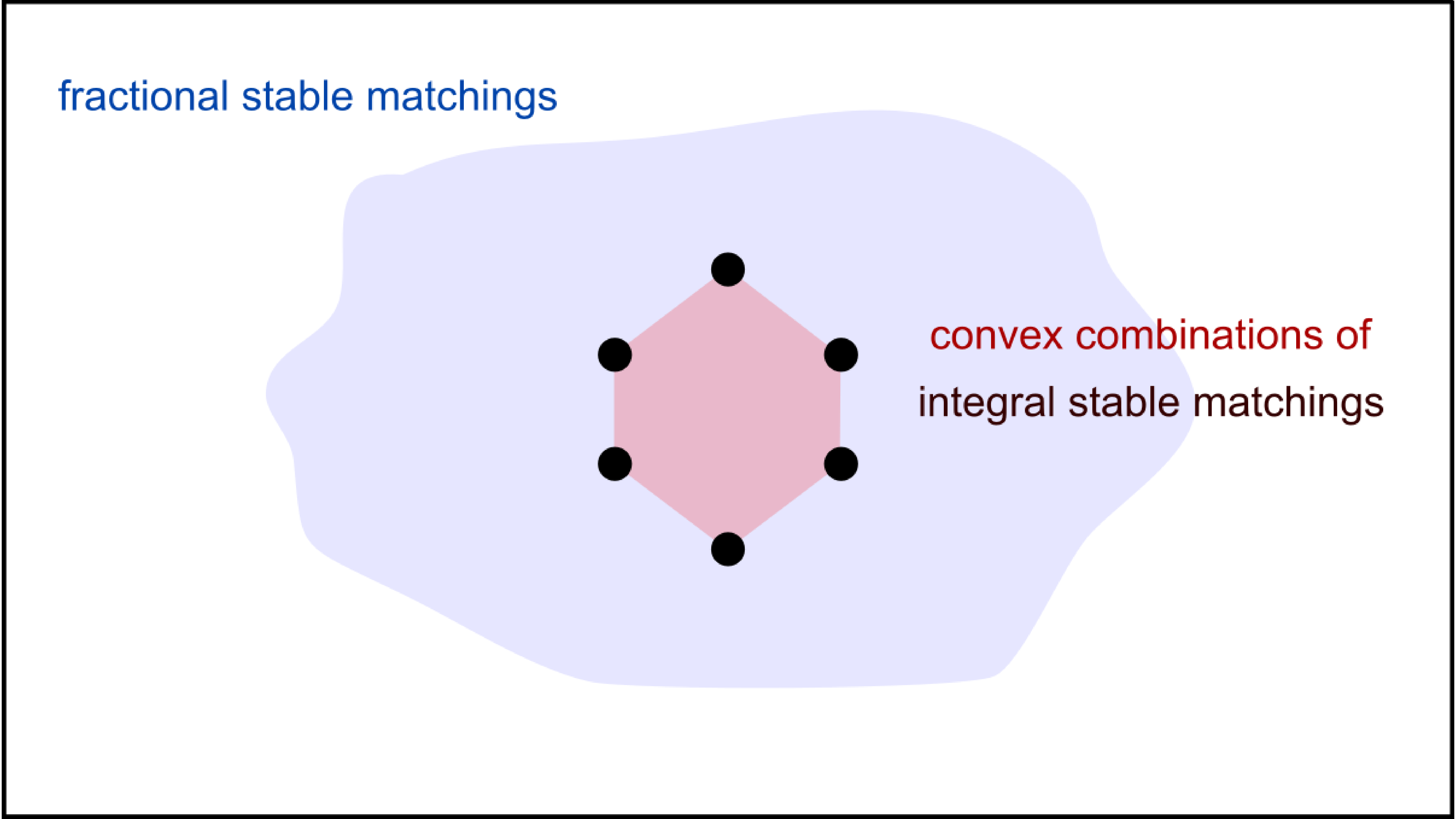
Any integral stable matching is also a fractional stable matching.


fractional stable matchings
integral stable matchings


fractional stable matchings
convex combinations of
integral stable matchings

[Vande Vate, *Oper. Res. Let.* 1989]

Any fractional stable matching can be expressed as a convex combination of integral stable matchings.


fractional stable matchings
convex combinations of
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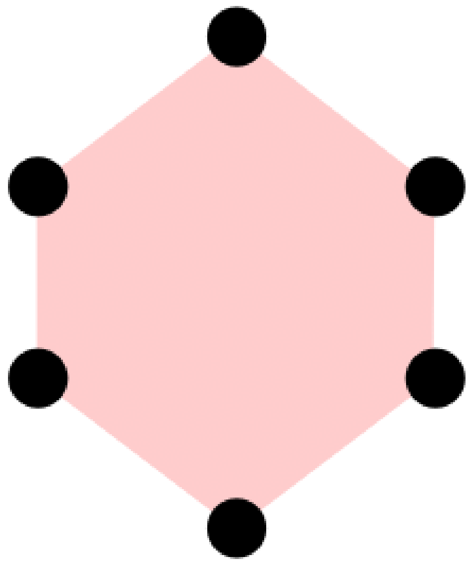


convex combinations of

integral stable matchings

=

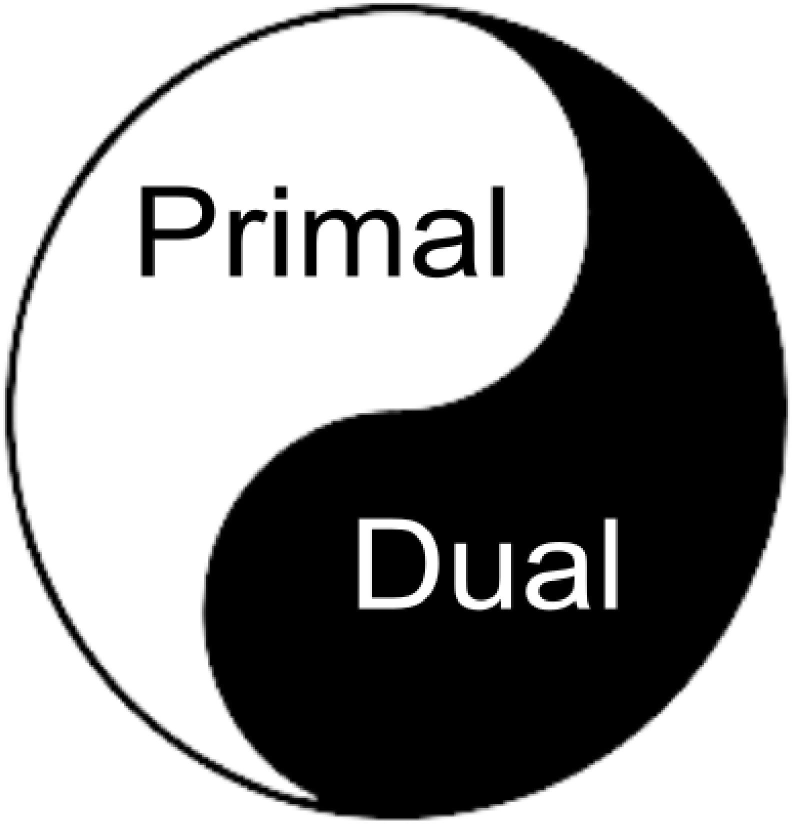
fractional stable matchings

[Vande Vate, *Oper. Res. Let.* 1989]

Any fractional stable matching can be expressed as a convex combination of integral stable matchings.

Coming up...

An elegant geometric proof that uses LP duality and its application in fair stable matchings.


Primal
Dual

Primal

$$\max \sum_{i,j} X_{i,j}$$

$$\sum_j X_{i,j} = 1 \; \forall i$$

$$\sum_i X_{i,j} = 1 \; \forall j$$

$$X_{i,j} + \sum_{k: w_k \succ_{m_i} w_j} X_{i,k} + \sum_{k: m_k \succ_{w_j} m_i} X_{k,j} \geq 1 \; \forall i, j$$

$$X_{i,j} \geq 0 \; \forall i, j$$

Primal

$$\max \sum_{i,j} X_{i,j}$$

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Primal

$$\max \sum_{i,j} X_{i,j}$$

$$\sum_j X_{i,j} = 1 \ \forall i$$

$$\sum_i X_{i,j} = 1 \ \forall j$$

$$-X_{i,j} - \sum_{k: w_k \succ_{m_i} w_j} X_{i,k} - \sum_{k: m_k \succ_{w_j} m_i} X_{k,j} \leq -1 \ \forall i, j$$

$$X_{i,j} \geq 0 \ \forall i, j$$

Primal

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Primal

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$$\sum_j X_{i,j} = 1 \ \forall i \qquad \alpha_i$$

$$\sum_i X_{i,j} = 1 \ \forall j \qquad \beta_j$$

$$\gamma_{i,j}$$

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$$X_{i,j} \geq 0 \ \forall i,j$$

Don't worry about us.

Primal

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Combine the constraints in order to construct an upper bound on the objective.

$$\gamma_{i,j}$$

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$$\sum_{i,j} \alpha_i X_{i,j} = \sum_i \alpha_i$$

$$\sum_{i,j} \beta_j X_{i,j} = \sum_j \beta_j$$

$$\gamma_{i,j}$$

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$$X_{i,j} \geq 0 \; \forall i,j$$

Primal

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$$\sum_{i,j} \beta_j X_{i,j} = \sum_j \beta_j$$

$$-\sum_{i,j} \left(\gamma_{i,j} X_{i,j} + \sum_{k: w_k \succ_{m_i} w_j} \gamma_{i,j} X_{i,k} + \sum_{k: m_k \succ_{w_j} m_i} \gamma_{i,j} X_{k,j} \right) \leq -\sum_{i,j} \gamma_{i,j}$$

$$X_{i,j} \geq 0 \ \forall i, j$$

Primal

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$$X_{i,j} \geq 0 \; \forall i, j$$

as long as

$\gamma_{i,j} \geq 0 \; \forall i, j$

Primal

Let's combine these.

$$\max \sum_{i,j} X_{i,j}$$

$$\sum_{i,j} \alpha_i X_{i,j} = \sum_i \alpha_i$$

$$\sum_{i,j} \beta_j X_{i,j} = \sum_j \beta_j$$

$$-\sum_{i,j} \left(\gamma_{i,j} X_{i,j} + \sum_{k:w_k \succ_{m_i} w_j} \gamma_{i,j} X_{i,k} + \sum_{k:m_k \succ_{w_j} m_i} \gamma_{i,j} X_{k,j} \right) \leq -\sum_{i,j} \gamma_{i,j}$$

$$X_{i,j} \geq 0 \ \forall i,j$$

as long as $\gamma_{i,j} \geq 0 \ \forall i,j$

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$\gamma_{i,j} \geq 0 \; \forall i, j$

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$$X_{i,j} \geq 0 \; \forall i,j$$

as long as
$\gamma_{i,j} \geq 0 \; \forall i,j$

$$\sum_{i,j} X_{i,j} \left(\alpha_i + \beta_j - \gamma_{i,j} - \sum_{k: w_j \succ_{m_i} w_k} \gamma_{i,k} - \sum_{k: m_i \succ_{w_j} m_k} \gamma_{k,j} \right)$$

$$\max \sum_{i,j} X_{i,j}$$

$$\sum_{i,j} \alpha_i X_{i,j} = \sum_i \alpha_i$$

$$\sum_{i,j} \beta_j X_{i,j} = \sum_j \beta_j$$

$$-\sum_{i,j} \left(\gamma_{i,j} X_{i,j} + \sum_{k: w_k \succ_{m_i} w_j} \gamma_{i,j} X_{i,k} + \sum_{k: m_k \succ_{w_j} m_i} \gamma_{i,j} X_{k,j} \right) \leq -\sum_{i,j} \gamma_{i,j}$$

$$X_{i,j} \geq 0 \; \forall i, j$$

as long as $\gamma_{i,j} \geq 0 \; \forall i, j$

$$\sum_{i,j} X_{i,j} \left(\alpha_i + \beta_j - \gamma_{i,j} - \sum_{k: w_j \succ_{m_i} w_k} \gamma_{i,k} - \sum_{k: m_i \succ_{w_j} m_k} \gamma_{k,j} \right)$$

$$\leq \sum_i \alpha_i + \sum_j \beta_j - \sum_{i,j} \gamma_{i,j}$$

$$\sum_{i,j} X_{i,j} \left(\alpha_i + \beta_j - \gamma_{i,j} - \sum_{k: w_j \succ_{m_i} w_k} \gamma_{i,k} - \sum_{k: m_i \succ_{w_j} m_k} \gamma_{k,j} \right) \le \sum_i \alpha_i + \sum_j \beta_j - \sum_{i,j} \gamma_{i,j}$$

subject to this being at least 1

minimize this

$$\textstyle\sum_{i,j} X_{i,j} \left(\alpha_i + \beta_j - \gamma_{i,j} - \sum_{k: w_j \succ_{m_i} w_k} \gamma_{i,k} - \sum_{k: m_i \succ_{w_j} m_k} \gamma_{k,j} \right)$$

$$\leq \sum_i \alpha_i + \sum_j \beta_j - \sum_{i,j} \gamma_{i,j}$$

as long as

$\gamma_{i,j} \geq 0 \; \forall i, j$

subject to this being at least 1

minimize this

$$\sum_{i,j} X_{i,j} \left(\alpha_i + \beta_j - \gamma_{i,j} - \sum_{k: w_j \succ_{m_i} w_k} \gamma_{i,k} - \sum_{k: m_i \succ_{w_j} m_k} \gamma_{k,j} \right)$$

$$\leq \sum_i \alpha_i + \sum_j \beta_j - \sum_{i,j} \gamma_{i,j}$$

Dual

$$\min \sum_i \alpha_i + \sum_j \beta_j - \sum_{i,j} \gamma_{i,j}$$

$$\alpha_i + \beta_j - \gamma_{i,j} - \sum_{k: w_j \succ_{m_i} w_k} \gamma_{i,k} - \sum_{k: m_i \succ_{w_j} m_k} \gamma_{k,j} \geq 1 \; \forall i, j$$

$$\gamma_{i,j} \geq 0 \; \forall i, j$$

Recall the Primal

$$\max \sum_{i,j} X_{i,j}$$

$$\sum_j X_{i,j} = 1 \ \forall i$$

$$\sum_i X_{i,j} = 1 \ \forall j$$

$$X_{i,j} + \sum_{k: w_k \succ_{m_i} w_j} X_{i,k} + \sum_{k: m_k \succ_{w_j} m_i} X_{k,j} \geq 1 \ \forall i, j$$

$$X_{i,j} \geq 0 \ \forall i, j$$

Recall the Primal

$$\max \sum_{i,j} X_{i,j}$$

$$\sum_j X_{i,j} = 1 \; \forall i$$

$$\sum_i X_{i,j} = 1 \; \forall j$$

$$X_{i,j} + \sum_{k: w_k \succ_{m_i} w_j} X_{i,k} + \sum_{k: m_k \succ_{w_j} m_i} X_{k,j} \geq 1 \; \forall i,j$$

$$X_{i,j} \geq 0 \; \forall i,j$$

By Gale and Shapley's result, primal is always feasible!

Let $X_{i,j}$ be a feasible primal solution.

Dual

$$\min \sum_i \alpha_i + \sum_j \beta_j - \sum_{i,j} \gamma_{i,j}$$

$$\alpha_i + \beta_j - \gamma_{i,j} - \sum_{k: w_j \succ_{m_i} w_k} \gamma_{i,k} - \sum_{k: m_i \succ_{w_j} m_k} \gamma_{k,j} \geq 1 \ \forall i, j$$

$$\gamma_{i,j} \geq 0 \ \forall i, j$$

Dual

$$\min \sum_i \alpha_i + \sum_j \beta_j - \sum_{i,j} \gamma_{i,j}$$

$$\alpha_i + \beta_j - \gamma_{i,j} - \sum_{k: w_j \succ_{m_i} w_k} \gamma_{i,k} - \sum_{k: m_i \succ_{w_j} m_k} \gamma_{k,j} \geq 1 \; \forall i, j$$

$$\gamma_{i,j} \geq 0 \; \forall i, j$$

Substitute $\alpha_i = \sum_j X_{i,j}$, $\beta_j = \sum_i X_{i,j}$, $\gamma_{i,j} = X_{i,j}$ for all i, j

Dual

$$\min \sum_i \alpha_i + \sum_j \beta_j - \sum_{i,j} \gamma_{i,j}$$

$$\alpha_i + \beta_j - \gamma_{i,j} - \sum_{k:w_j \succ_{m_i} w_k} \gamma_{i,k} - \sum_{k:m_i \succ_{w_j} m_k} \gamma_{k,j} \geq 1 \ \forall i, j$$

$$\gamma_{i,j} \geq 0 \ \forall i, j$$

Substitute $\alpha_i = \sum_j X_{i,j}$, $\beta_j = \sum_i X_{i,j}$, $\gamma_{i,j} = X_{i,j}$ for all i, j

Dual

$$\sum_{i,j} X_{i,j}$$

$$\alpha_i + \beta_j - \gamma_{i,j} - \sum_{k:w_j \succ_{m_i} w_k} \gamma_{i,k} - \sum_{k:m_i \succ_{w_j} m_k} \gamma_{k,j} \geq 1 \; \forall i,j$$

$$\gamma_{i,j} \geq 0 \; \forall i,j$$

Substitute $\alpha_i = \sum_j X_{i,j}$, $\beta_j = \sum_i X_{i,j}$, $\gamma_{i,j} = X_{i,j}$ for all i,j

Dual

$$\sum_{i,j} X_{i,j}$$

$$\alpha_i + \beta_j - \gamma_{i,j} - \sum_{k:w_j \succ_{m_i} w_k} \gamma_{i,k} - \sum_{k:m_i \succ_{w_j} m_k} \gamma_{k,j} \geq 1 \ \forall i, j$$

$$\gamma_{i,j} \geq 0 \ \forall i, j$$

Substitute $\alpha_i = \sum_j X_{i,j}$, $\beta_j = \sum_i X_{i,j}$, $\gamma_{i,j} = X_{i,j}$ for all i, j

Dual

$$\sum_{i,j} X_{i,j}$$

$$\sum_j X_{i,j} + \sum_i X_{i,j} - X_{i,j} - \sum_{k:w_j \succ_{m_i} w_k} X_{i,k} - \sum_{k:m_i \succ_{w_j} m_k} X_{k,j} \geq 1 \ \forall i, j$$

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Substitute $\alpha_i = \sum_j X_{i,j}$, $\beta_j = \sum_i X_{i,j}$, $\gamma_{i,j} = X_{i,j}$ for all i, j

Dual

$$\sum_{i,j} X_{i,j}$$

$$\sum_j X_{i,j} + \sum_i X_{i,j} - X_{i,j} - \sum_{k:w_j \succ_{m_i} w_k} X_{i,k} - \sum_{k:m_i \succ_{w_j} m_k} X_{k,j} \geq 1 \; \forall i,j$$

$$\gamma_{i,j} \geq 0 \; \forall i,j$$

Substitute $\alpha_i = \sum_j X_{i,j}$, $\beta_j = \sum_i X_{i,j}$, $\gamma_{i,j} = X_{i,j}$ for all i, j

Dual

$$\sum_{i,j} X_{i,j}$$

$$\sum_{k: w_k \succ_{m_i} w_j} X_{i,k} + \sum_i X_{i,j} - \sum_{k: m_i \succ_{w_j} m_k} X_{k,j} \geq 1 \; \forall i, j$$

$$\gamma_{i,j} \geq 0 \; \forall i, j$$

Substitute $\alpha_i = \sum_j X_{i,j}$, $\beta_j = \sum_i X_{i,j}$, $\gamma_{i,j} = X_{i,j}$ for all i, j

Dual

$$\sum_{i,j} X_{i,j}$$

$$\sum_{k:w_k \succ_{m_i} w_j} X_{i,k} + \sum_{i} X_{i,j} - \sum_{k:m_i \succ_{w_j} m_k} X_{k,j} \geq 1 \ \forall i,j$$

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Substitute $\alpha_i = \sum_j X_{i,j}$, $\beta_j = \sum_i X_{i,j}$, $\gamma_{i,j} = X_{i,j}$ for all i, j

Dual

$$\sum_{i,j} X_{i,j}$$

$$\sum_{k:w_k \succ_{m_i} w_j} X_{i,k} + \boxed{\sum_i X_{i,j} - \sum_{k:m_i \succ_{w_j} m_k} X_{k,j}} \geq 1 \ \forall i, j$$

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Substitute $\alpha_i = \sum_j X_{i,j}$, $\beta_j = \sum_i X_{i,j}$, $\gamma_{i,j} = X_{i,j}$ for all i, j

Dual

$$\sum_{i,j} X_{i,j}$$

$$\sum_{k:w_k \succ_{m_i} w_j} X_{i,k} + \boxed{\sum_{k:m_k \succ_{w_j} m_i} X_{k,j} + X_{i,j}} \geq 1 \ \forall i,j$$

$$\gamma_{i,j} \geq 0 \ \forall i,j$$

Substitute $\alpha_i = \sum_j X_{i,j}$, $\beta_j = \sum_i X_{i,j}$, $\gamma_{i,j} = X_{i,j}$ for all i, j

Dual

$$\sum_{i,j} X_{i,j}$$

$$\sum_{k:w_k \succ_{m_i} w_j} X_{i,k} + \sum_{k:m_k \succ_{w_j} m_i} X_{k,j} + X_{i,j} \geq 1 \ \forall i,j$$

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Dual

$$\sum_{i,j} X_{i,j}$$

$$\sum_{k:w_k \succ_{m_i} w_j} X_{i,k} + \sum_{k:m_k \succ_{w_j} m_i} X_{k,j} + X_{i,j} \geq 1 \ \forall i,j$$

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Dual

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$$\sum_{k:w_k \succ_{m_i} w_j} X_{i,k} + \sum_{k:m_k \succ_{w_j} m_i} X_{k,j} + X_{i,j} \geq 1 \ \forall i, j$$

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Dual

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$$\sum_{k: w_k \succ_{m_i} w_j} X_{i,k} + \sum_{k: m_k \succ_{w_j} m_i} X_{k,j} + X_{i,j} \geq 1 \ \forall i, j$$

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Substitute $\alpha_i = \sum_j X_{i,j}$, $\beta_j = \sum_i X_{i,j}$, $\gamma_{i,j} = X_{i,j}$ for all i, j

Dual

Stability constraint from primal

$$\sum_{i,j} X_{i,j}$$

$$\sum_{k:w_k \succ_{m_i} w_j} X_{i,k} + \sum_{k:m_k \succ_{w_j} m_i} X_{k,j} + X_{i,j} \geq 1 \ \forall i,j$$

$$X_{i,j} \geq 0 \ \forall i,j$$

Substitute $\alpha_i = \sum_j X_{i,j}$, $\beta_j = \sum_i X_{i,j}$, $\gamma_{i,j} = X_{i,j}$ for all i,j

Dual

Stability constraint from primal

$$\sum_{i,j} X_{i,j}$$

$$\sum_{k:w_k \succ_{m_i} w_j} X_{i,k} + \sum_{k:m_k \succ_{w_j} m_i} X_{k,j} + X_{i,j} \geq 1 \;\forall i,j$$

$$X_{i,j} \geq 0 \;\forall i,j$$

Dual feasible!

Substitute $\alpha_i = \sum_j X_{i,j}$, $\beta_j = \sum_i X_{i,j}$, $\gamma_{i,j} = X_{i,j}$ for all i, j

Dual

Stability constraint from primal

Objective is equal to that in the primal

$$\sum_{i,j} X_{i,j}$$

$$\sum_{k:w_k \succ_{m_i} w_j} X_{i,k} + \sum_{k:m_k \succ_{w_j} m_i} X_{k,j} + X_{i,j} \geq 1 \ \forall i,j$$

$$X_{i,j} \geq 0 \ \forall i,j$$

Dual feasible!

Substitute $\alpha_i = \sum_j X_{i,j}$, $\beta_j = \sum_i X_{i,j}$, $\gamma_{i,j} = X_{i,j}$ for all i, j

Dual

By strong duality:

$X_{i,j}$ must be *primal optimal*, and

$\alpha_i = \sum_j X_{i,j}$, $\beta_j = \sum_i X_{i,j}$, $\gamma_{i,j} = X_{i,j}$ must be *dual optimal*.

Stability constraint from primal

Objective is equal to that in the primal

$$\sum_{i,j} X_{i,j}$$

$$\sum_{k: w_k \succ_{m_i} w_j} X_{i,k} + \sum_{k: m_k \succ_{w_j} m_i} X_{k,j} + X_{i,j} \geq 1 \;\forall i,j$$

$$X_{i,j} \geq 0 \;\forall i,j$$

Dual feasible!

Substitute $\alpha_i = \sum_j X_{i,j}$, $\beta_j = \sum_i X_{i,j}$, $\gamma_{i,j} = X_{i,j}$ for all i, j

Dual

Stability constraint from primal

Objective is equal to that in the primal

$$\sum_{i,j} X_{i,j}$$

$$\sum_{k:w_k \succ_{m_i} w_j} X_{i,k} + \sum_{k:m_k \succ_{w_j} m_i} X_{k,j} + X_{i,j} \geq 1 \ \forall i, j$$

$$X_{i,j} \geq 0 \ \forall i, j$$

Dual feasible!

By complementary slackness:

For any primal feasible X,

$$X_{i,j} > 0 \Rightarrow X_{i,j} + \sum_{k:w_k \succ_{m_i} w_j} X_{i,k} + \sum_{k:m_k \succ_{w_j} m_i} X_{k,j} = 1.$$

Substitute $\alpha_i = \sum_j X_{i,j}$, $\beta_j = \sum_i X_{i,j}$, $\gamma_{i,j} = X_{i,j}$ for all i, j

[Vande Vate, *Oper. Res. Let.* 1989]

Any fractional stable matching can be expressed as a convex combination of integral stable matchings.

[Vande Vate, *Oper. Res. Let.* 1989]

Any fractional stable matching can be expressed as a convex combination of integral stable matchings.

Proof by picture (and LP duality)

[Teo and Sethuraman, *MOR* 1998].

[Vande Vate, *Oper. Res. Let.* 1989]

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Any fractional stable matching can be expressed as a convex combination of integral stable matchings.

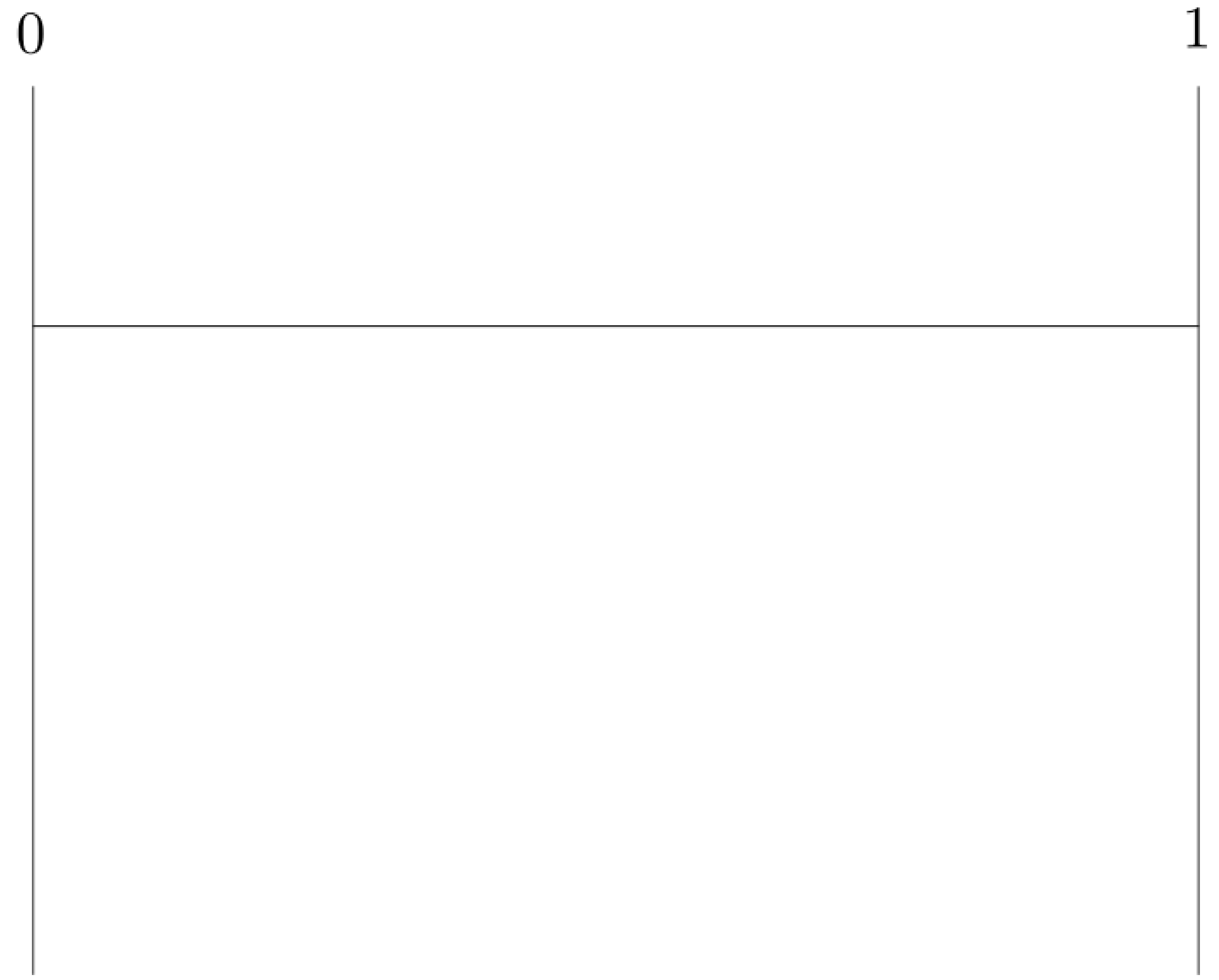
Let X be any fractional stable matching.

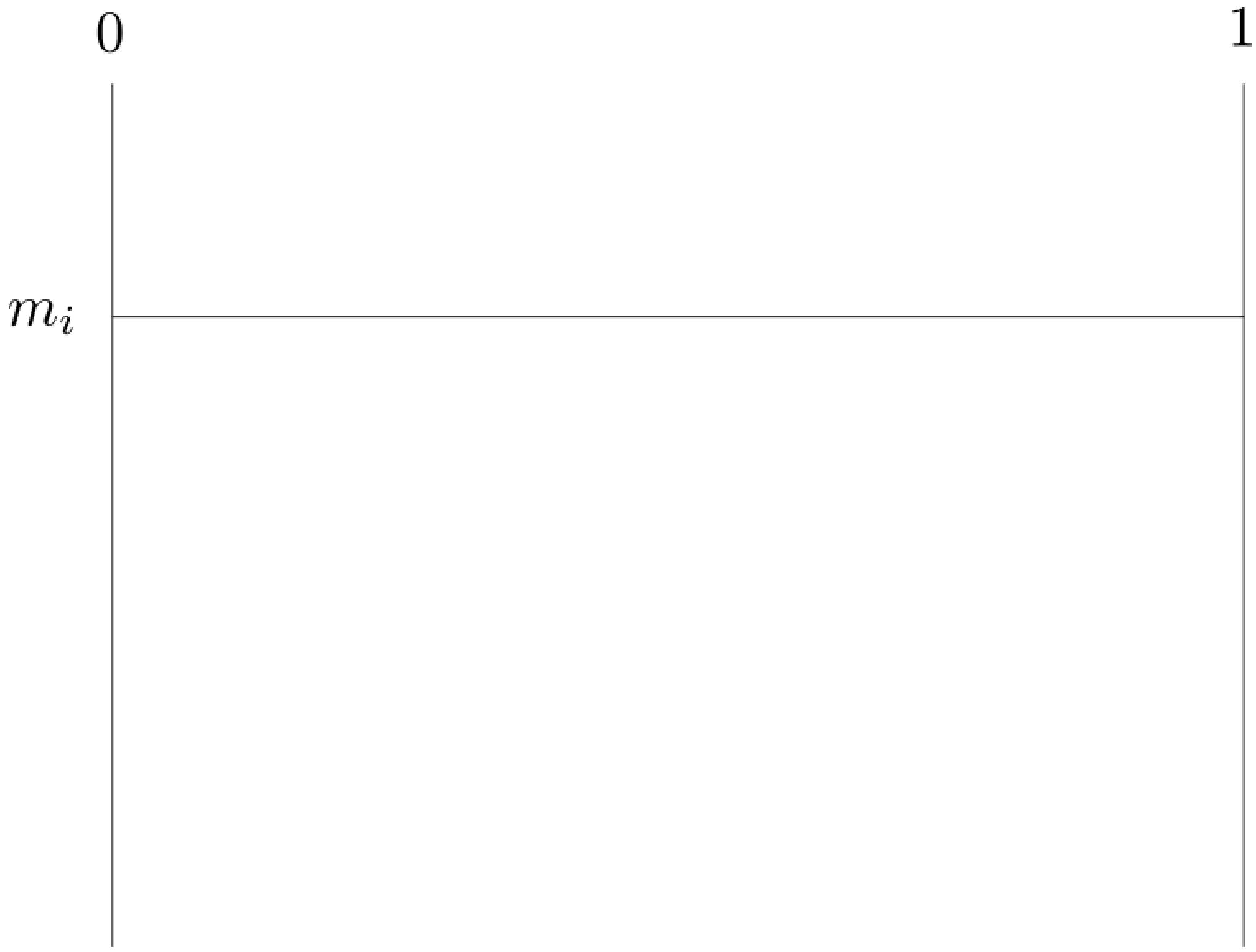
Then, X is primal feasible.

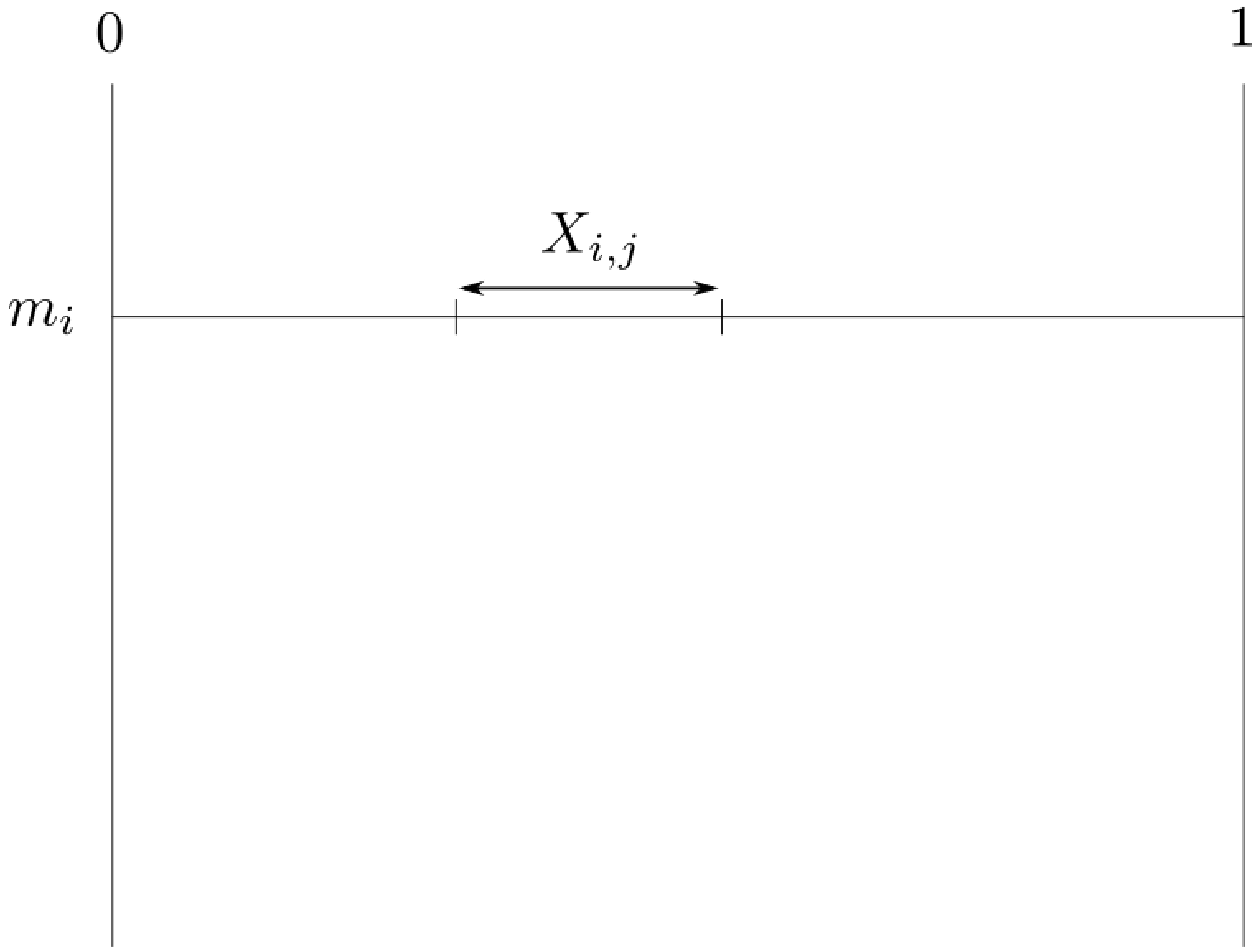
Recall complementary slackness:

For any primal feasible X,

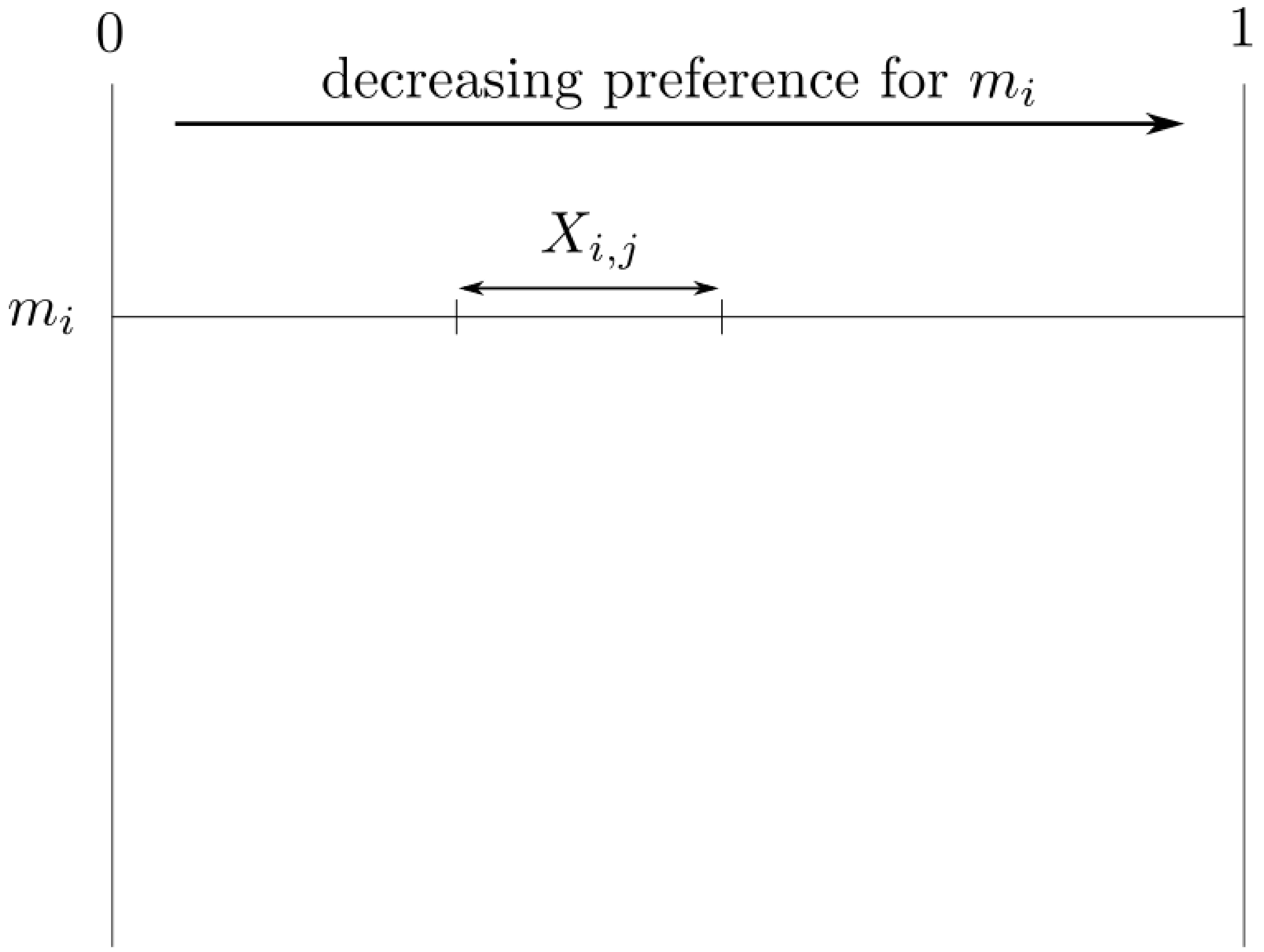
$$X_{i,j} > 0 \Rightarrow X_{i,j} + \sum_{k: w_k \succ_{m_i} w_j} X_{i,k} + \sum_{k: m_k \succ_{w_j} m_i} X_{k,j} = 1.$$


0
1

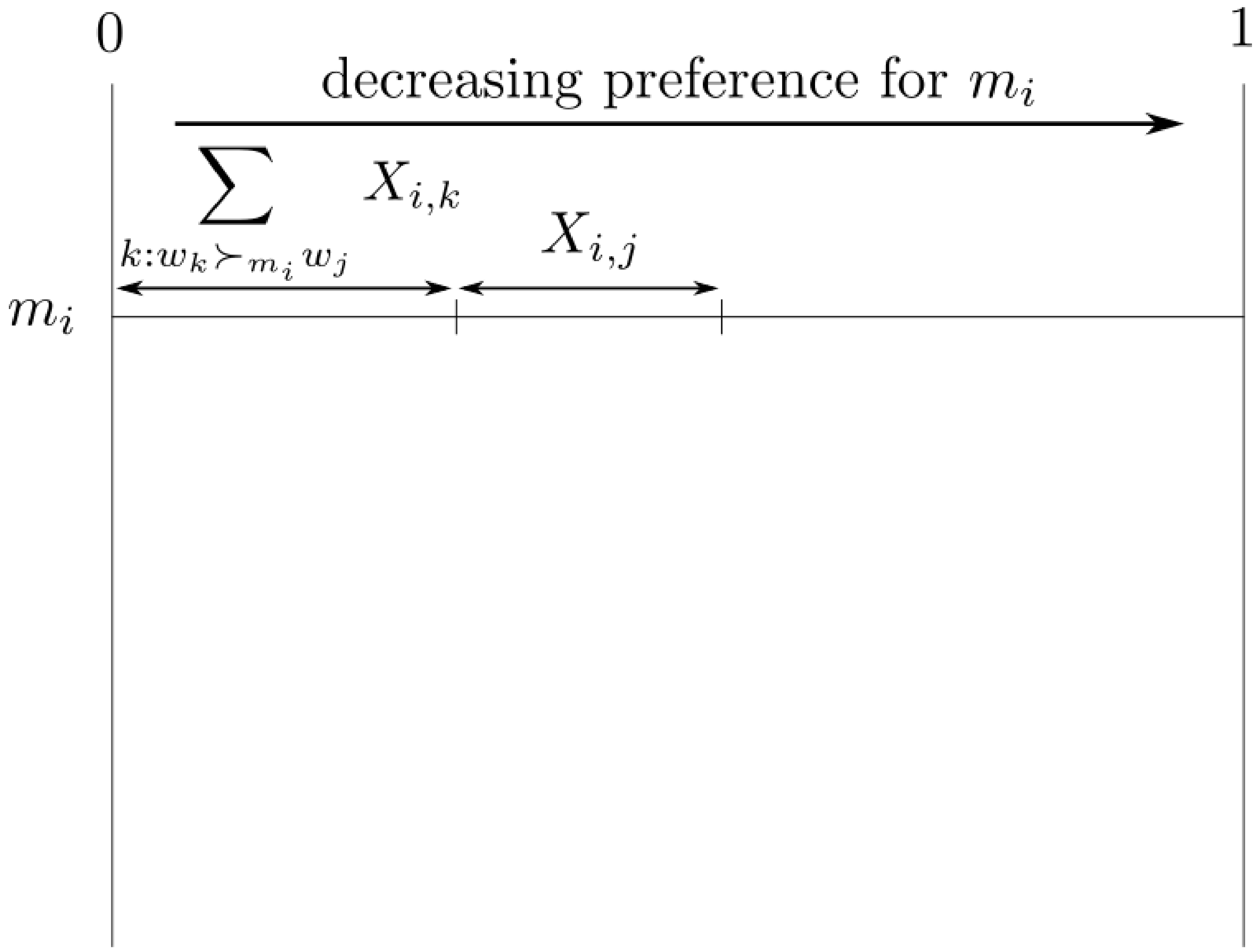

0
1
m_i



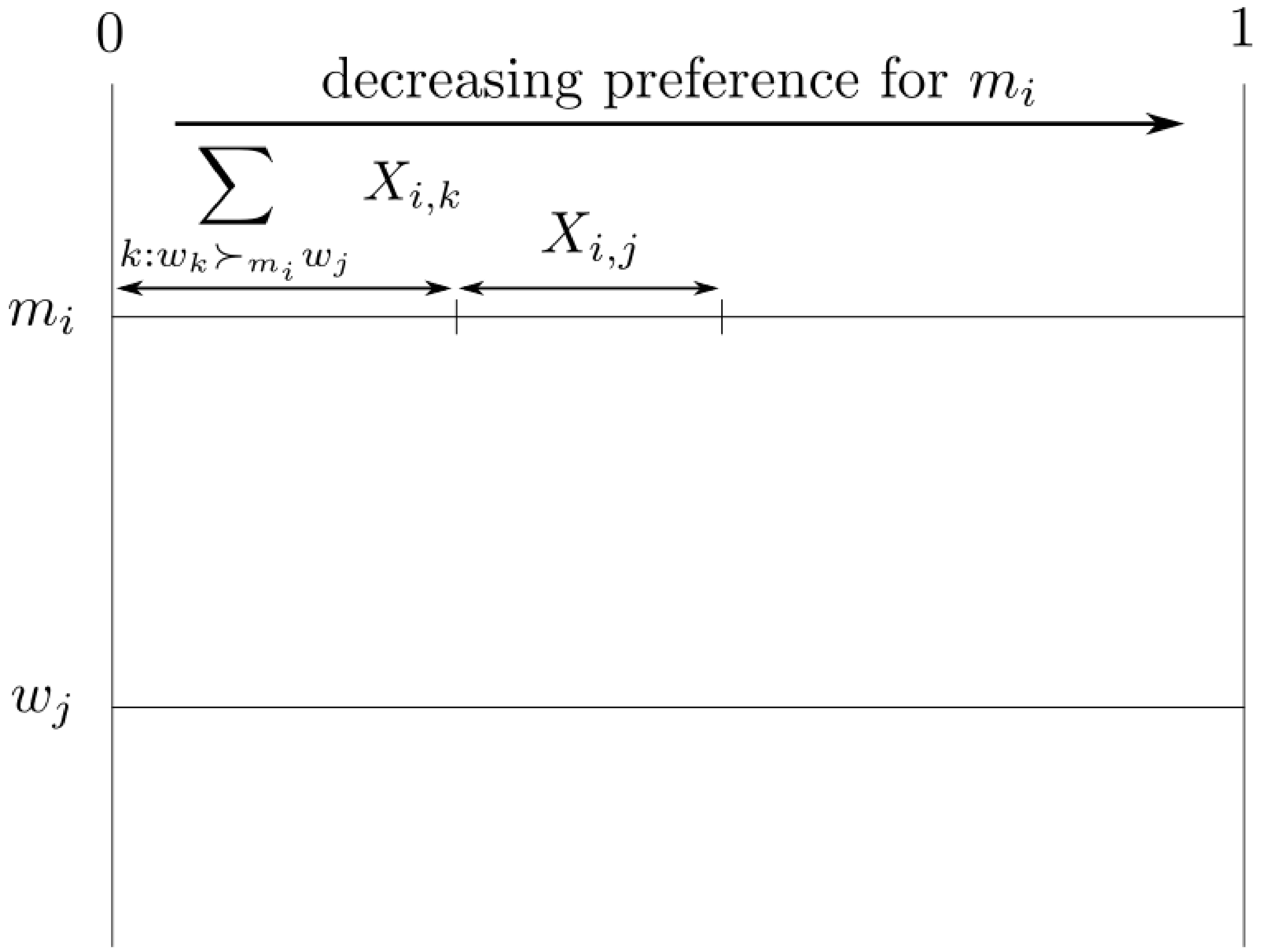
0
1
$X_{i,j}$
m_i

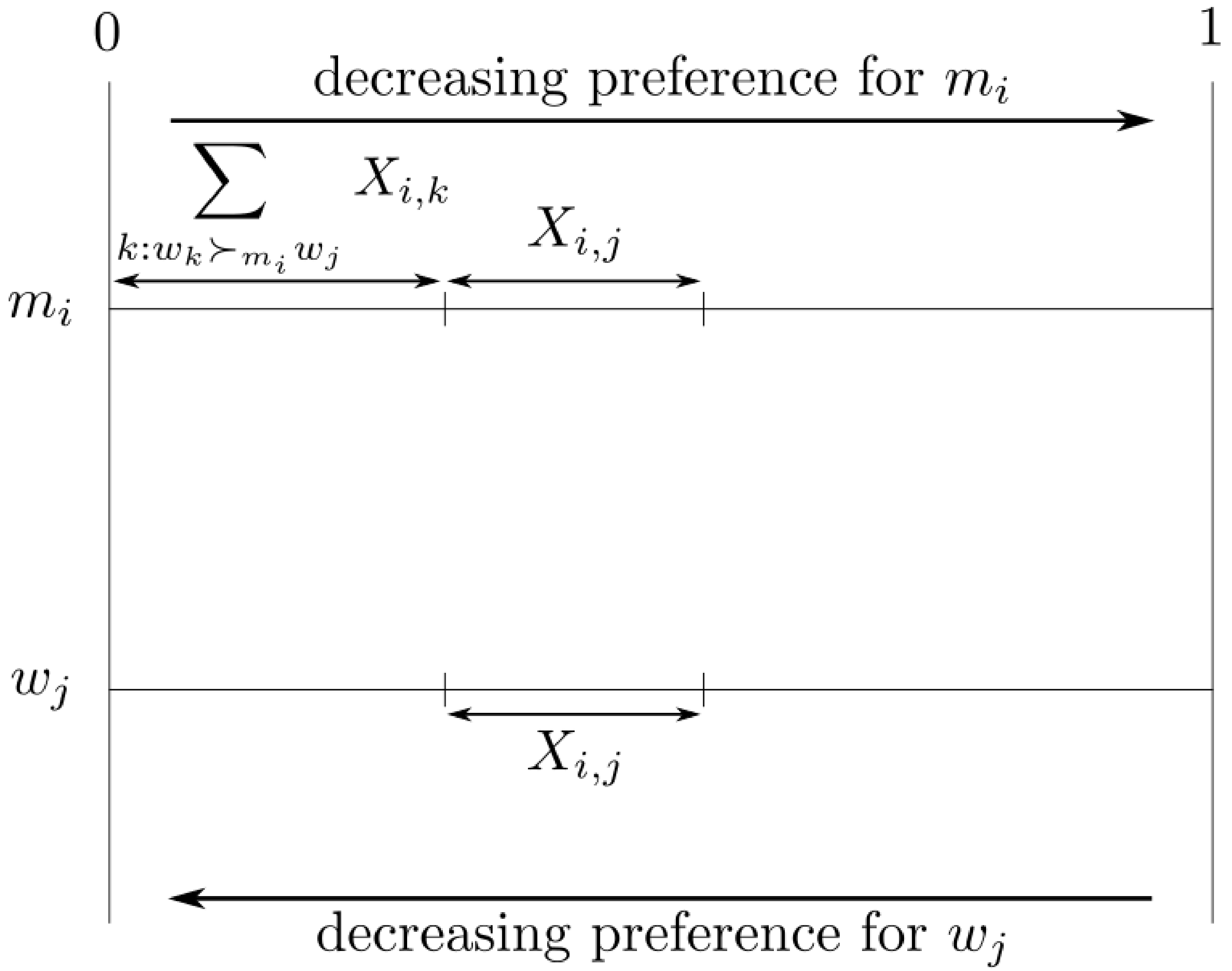


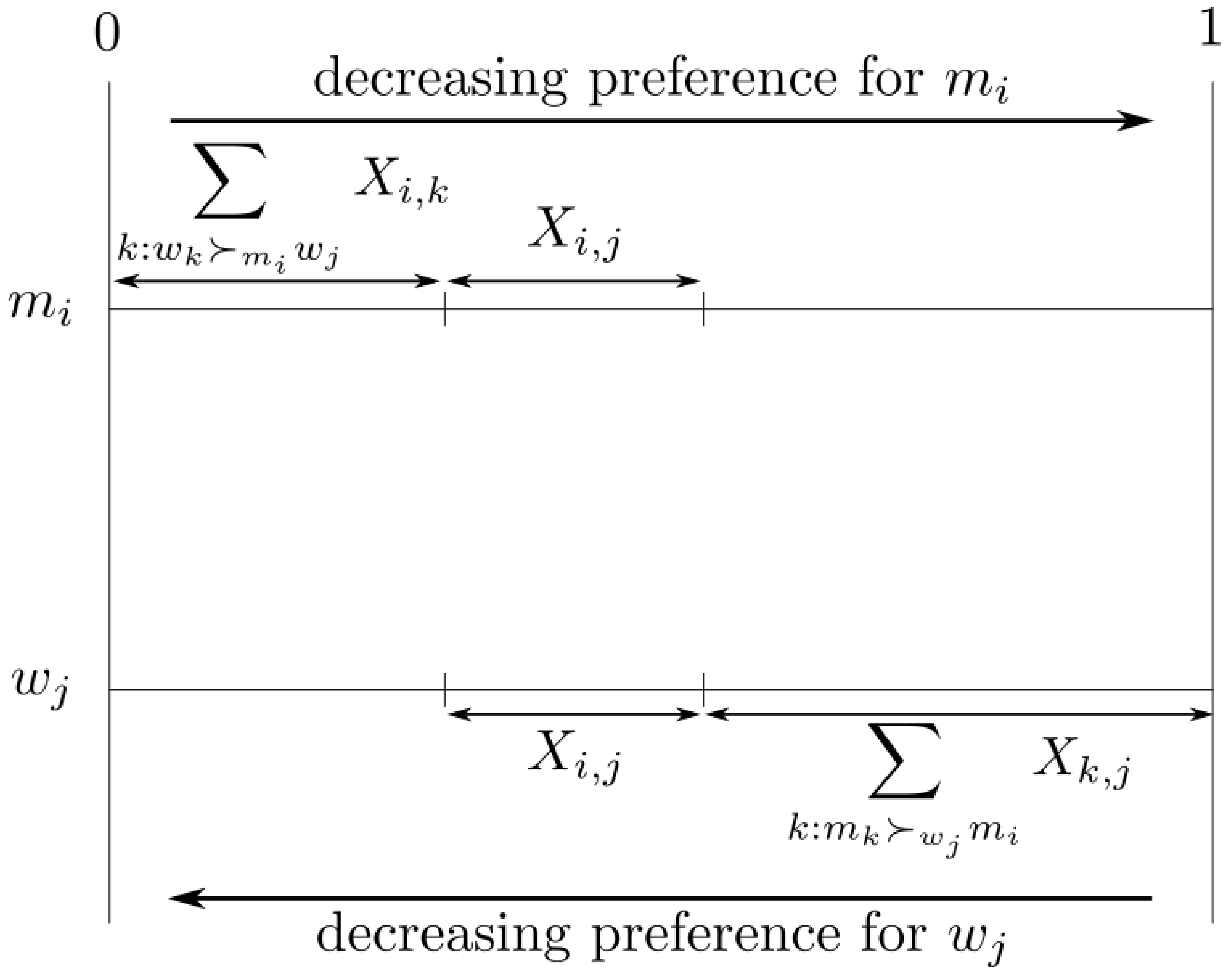
0
1
decreasing preference for m_i
$X_{i,j}$
m_i



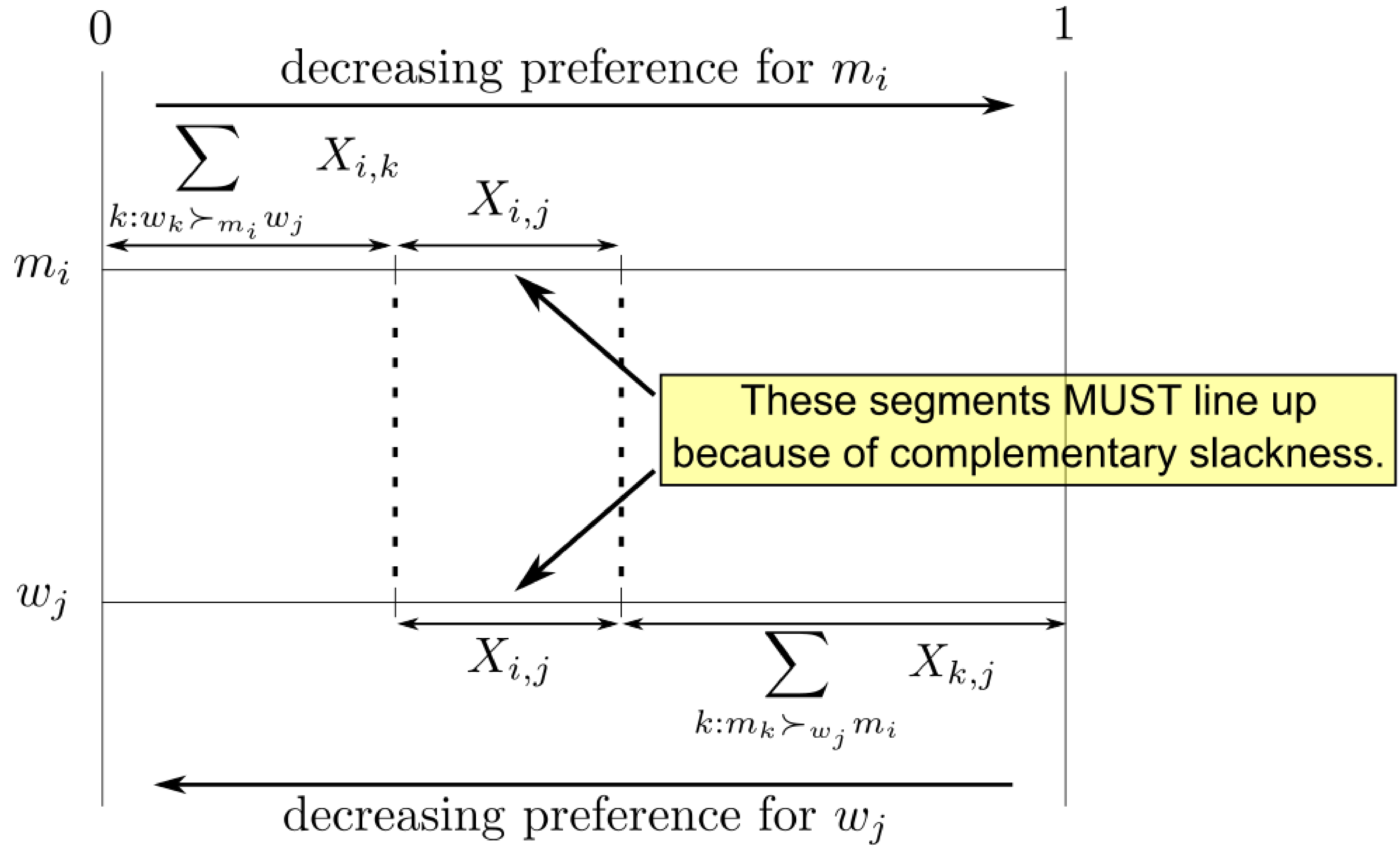
0
1
decreasing preference for m_i
$\sum_{k: w_k \succ_{m_i} w_j} X_{i,k}$
$X_{i,j}$
m_i

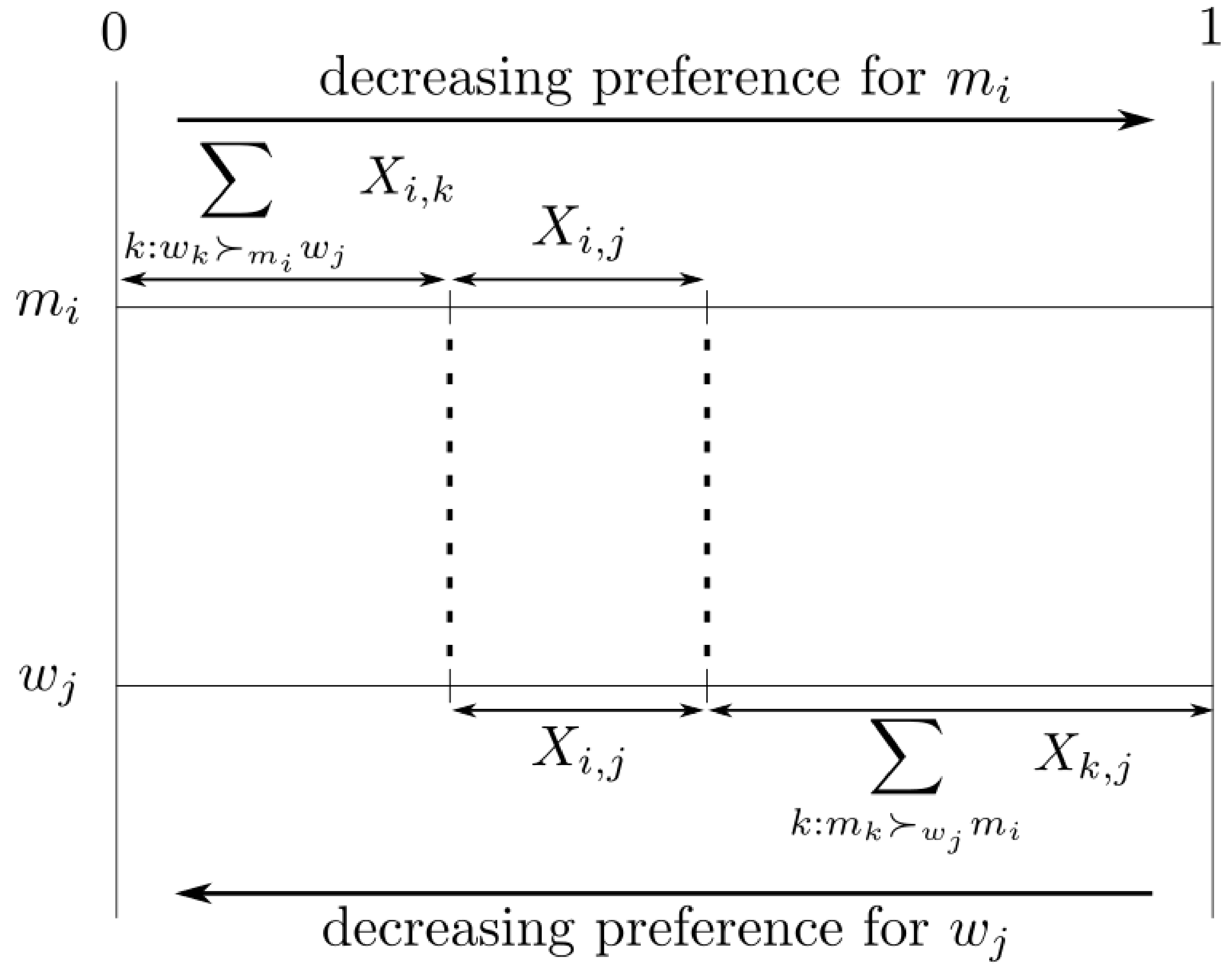

0
1
decreasing preference for m_i
$\sum_{k:w_k \succ_{m_i} w_j} X_{i,k}$
$X_{i,j}$
m_i
w_j

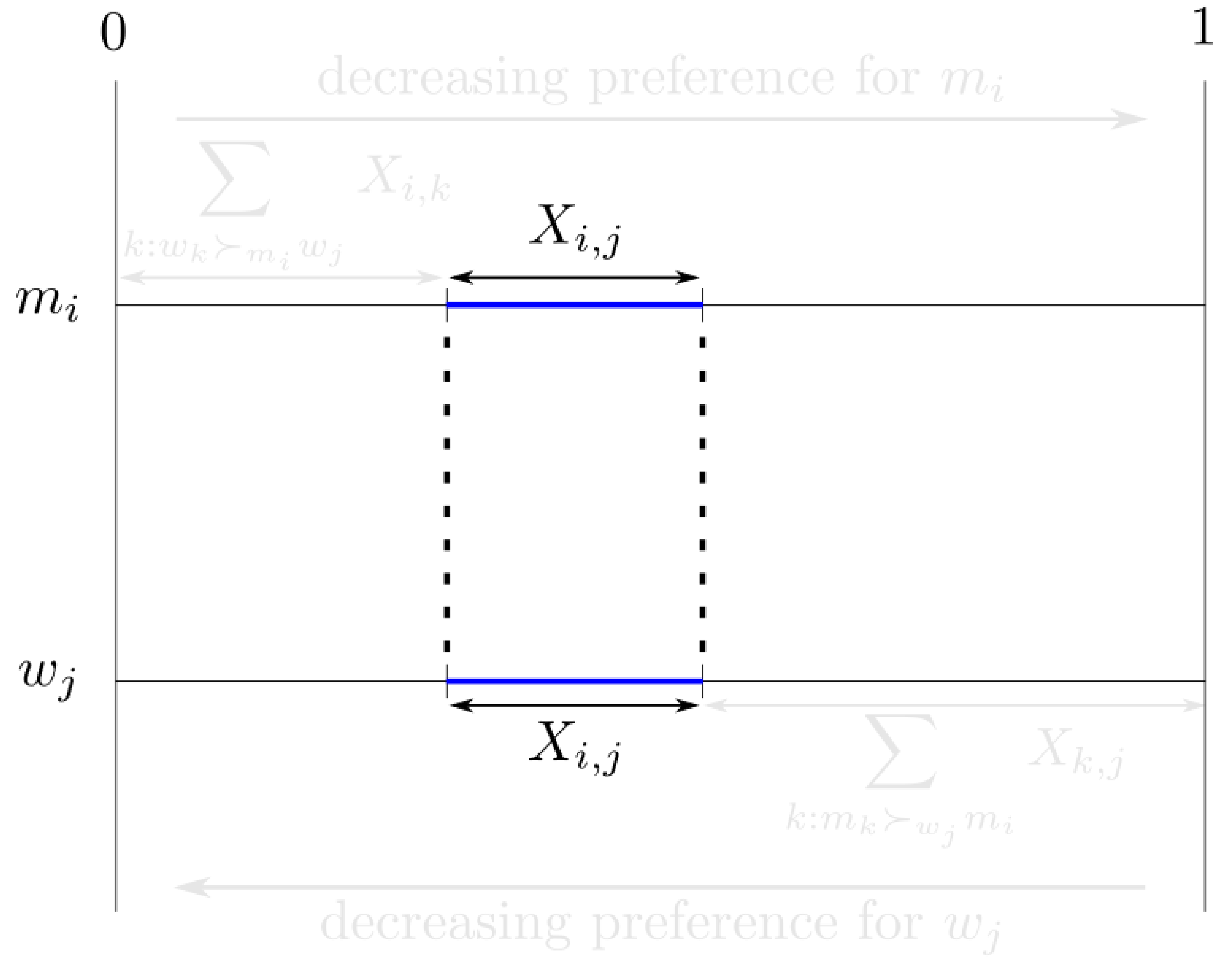

0
1
decreasing preference for m_i
$\sum_{k:w_k \succ_{m_i} w_j} X_{i,k}$
$X_{i,j}$
m_i
w_j
$X_{i,j}$
decreasing preference for w_j

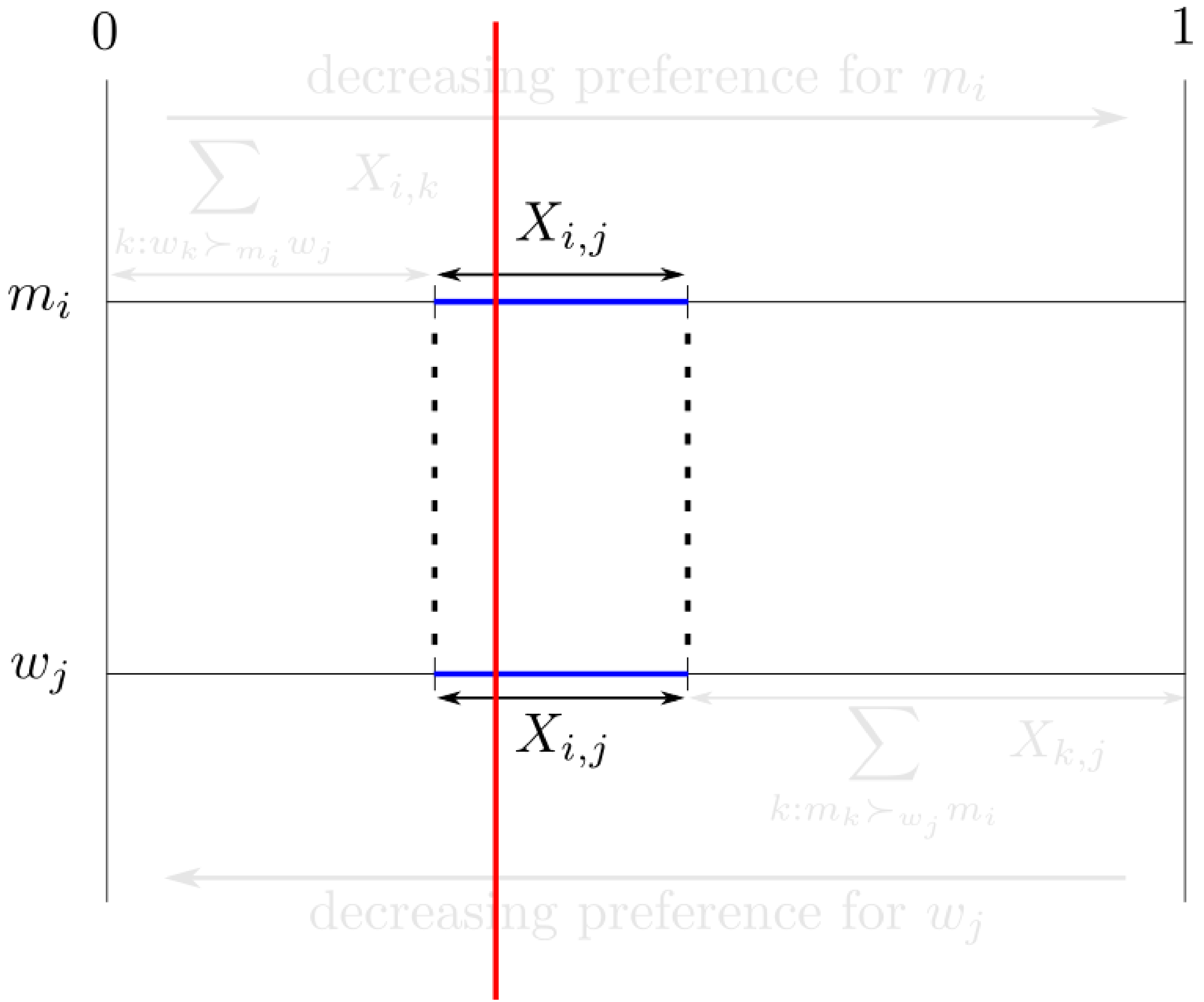


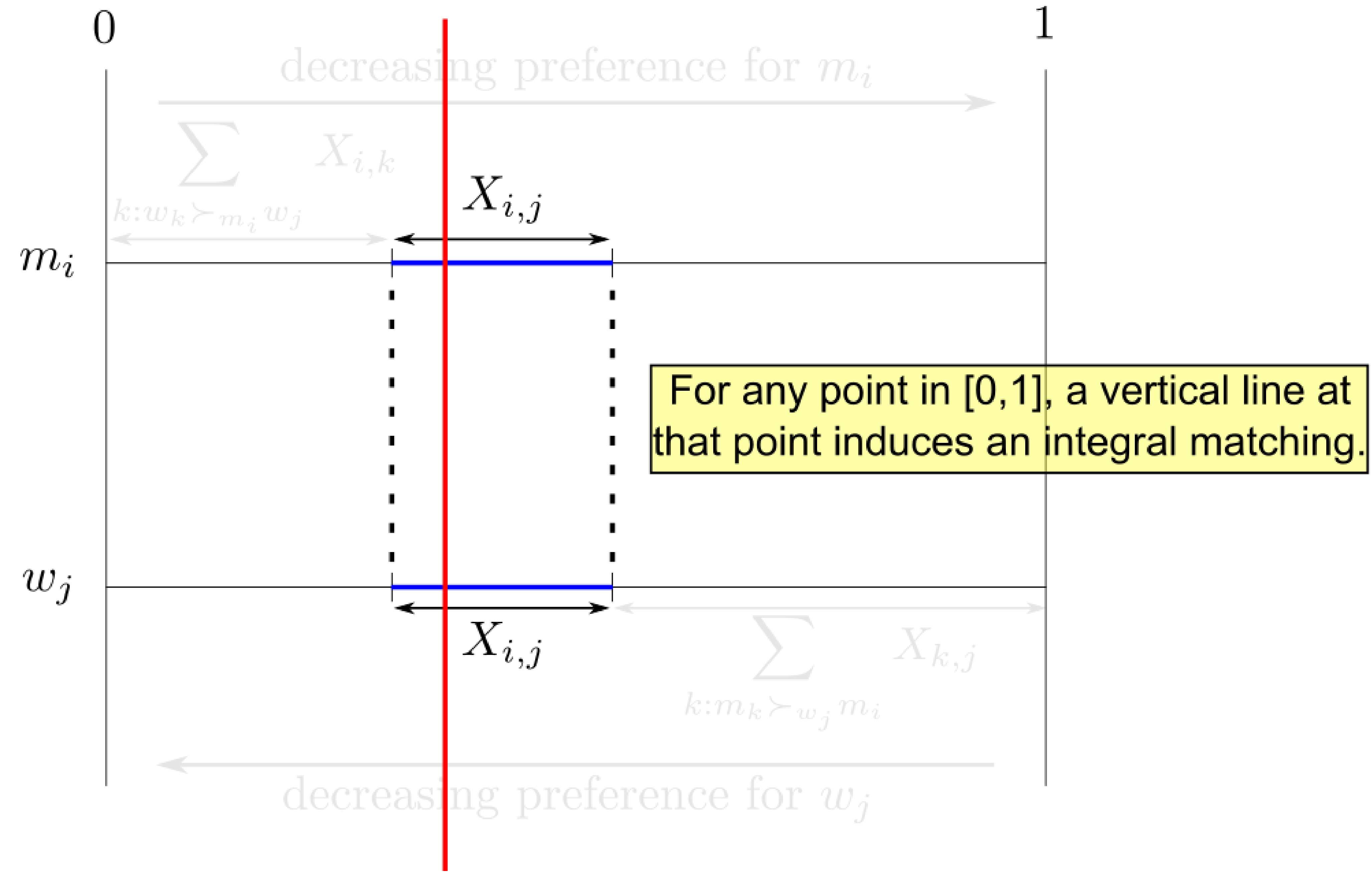
0
1
decreasing preference for m_i
$\sum_{k:w_k \succ_{m_i} w_j} X_{i,k}$
$X_{i,j}$
m_i
w_j
$X_{i,j}$
$\sum_{k:m_k \succ_{w_j} m_i} X_{k,j}$
decreasing preference for w_j

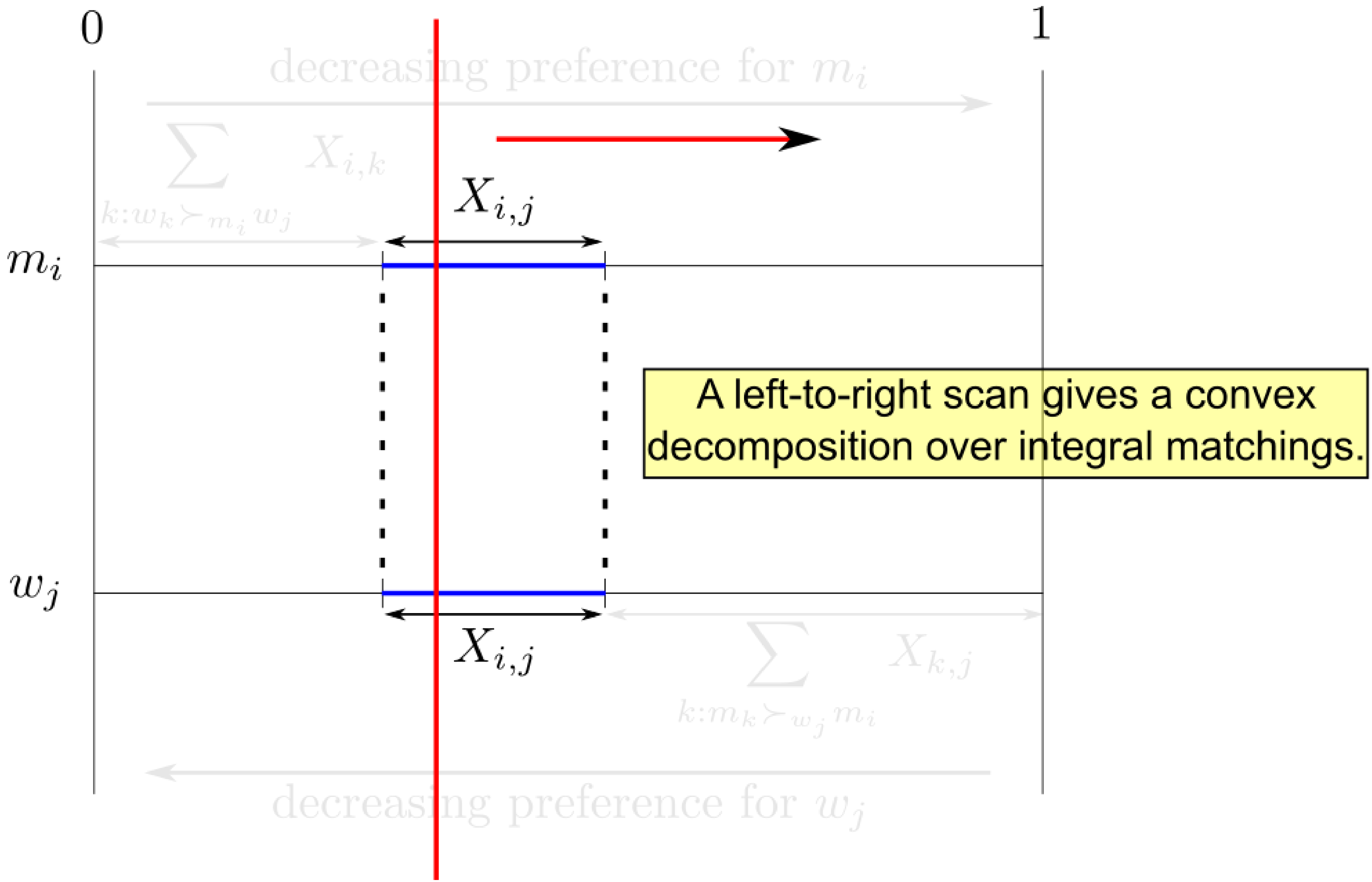

0
1
decreasing preference for m_i
$\sum_{k:w_k \succ_{m_i} w_j} X_{i,k}$
$X_{i,j}$
m_i
These segments MUST line up because of complementary slackness.
w_j
$X_{i,j}$
$\sum_{k:m_k \succ_{w_j} m_i} X_{k,j}$
decreasing preference for w_j

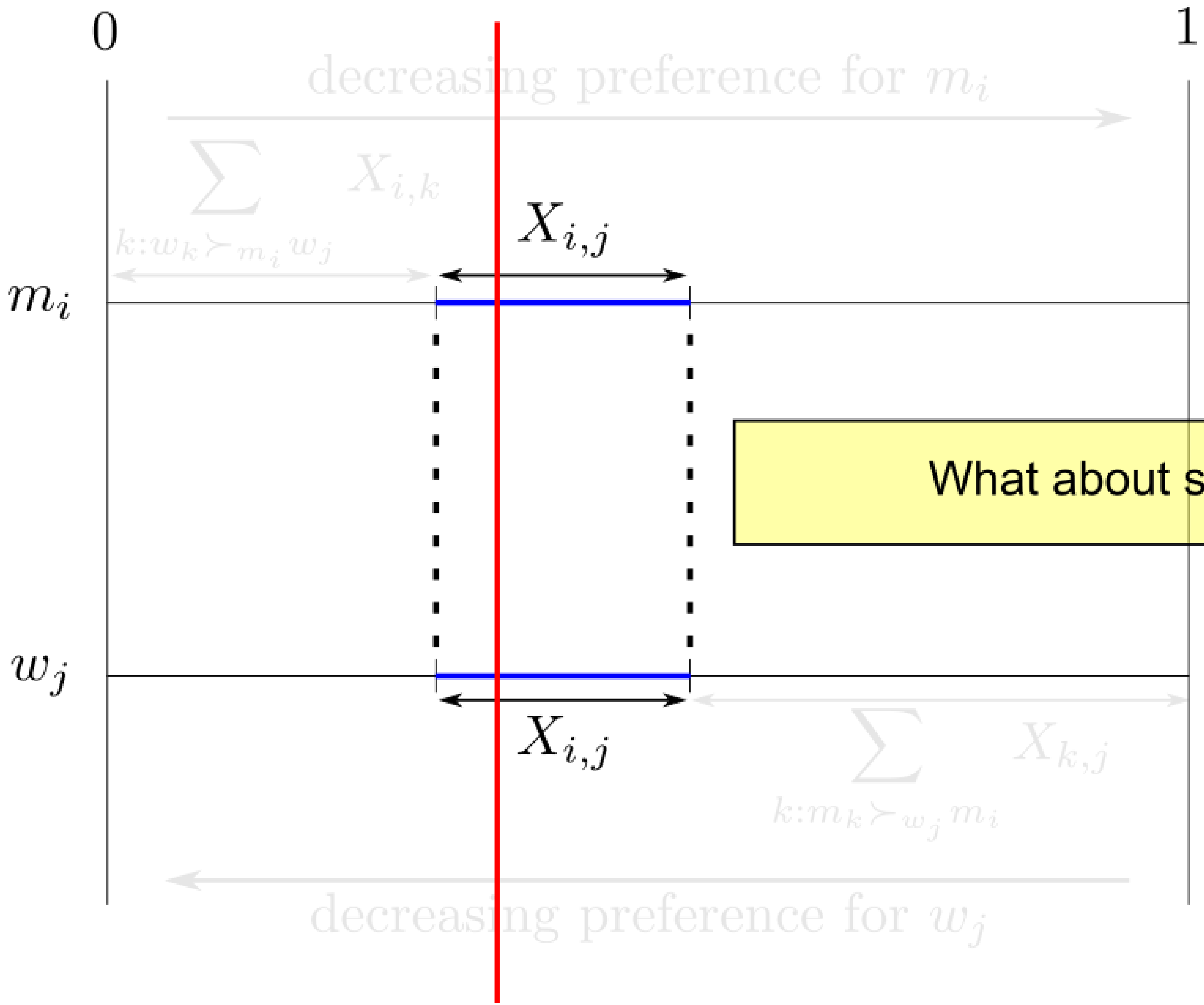

0
1
decreasing preference for m_i
$\sum_{k: w_k \succ_{m_i} w_j} X_{i,k}$
$X_{i,j}$
m_i
w_j
$X_{i,j}$
$\sum_{k: m_k \succ_{w_j} m_i} X_{k,j}$
decreasing preference for w_j


0
1
decreasing preference for m_i
$\sum_{k:w_k \succ_{m_i} w_j} X_{i,k}$
$X_{i,j}$
m_i
w_j
$X_{i,j}$
$\sum_{k:m_k \succ_{w_j} m_i} X_{k,j}$
decreasing preference for w_j

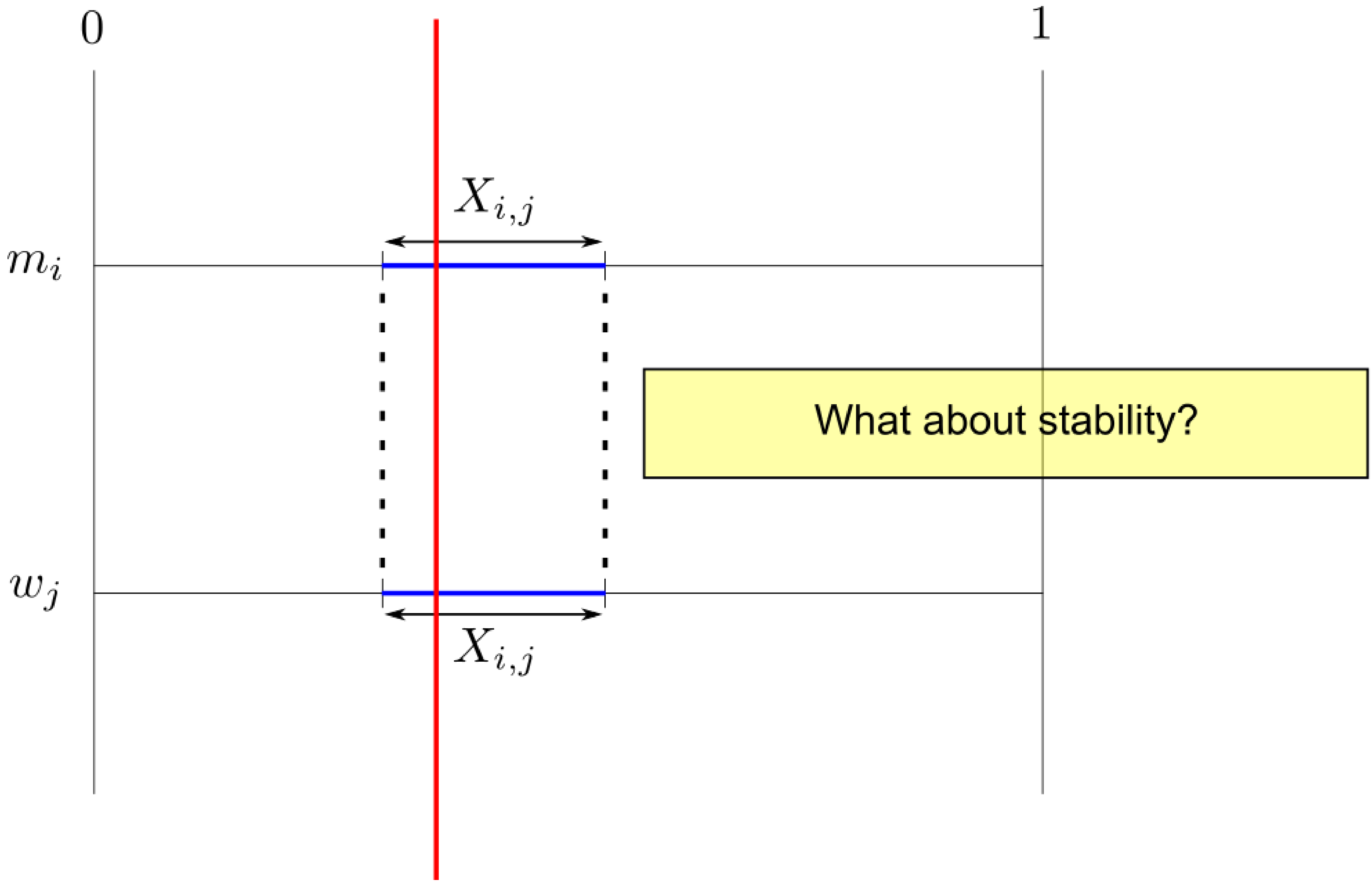

0
1
decreasing preference for m_i
$\sum_{k:w_k \succ_{m_i} w_j} X_{i,k}$
$X_{i,j}$
m_i
w_j
$X_{i,j}$
$\sum_{k:m_k \succ_{w_j} m_i} X_{k,j}$
decreasing preference for w_j

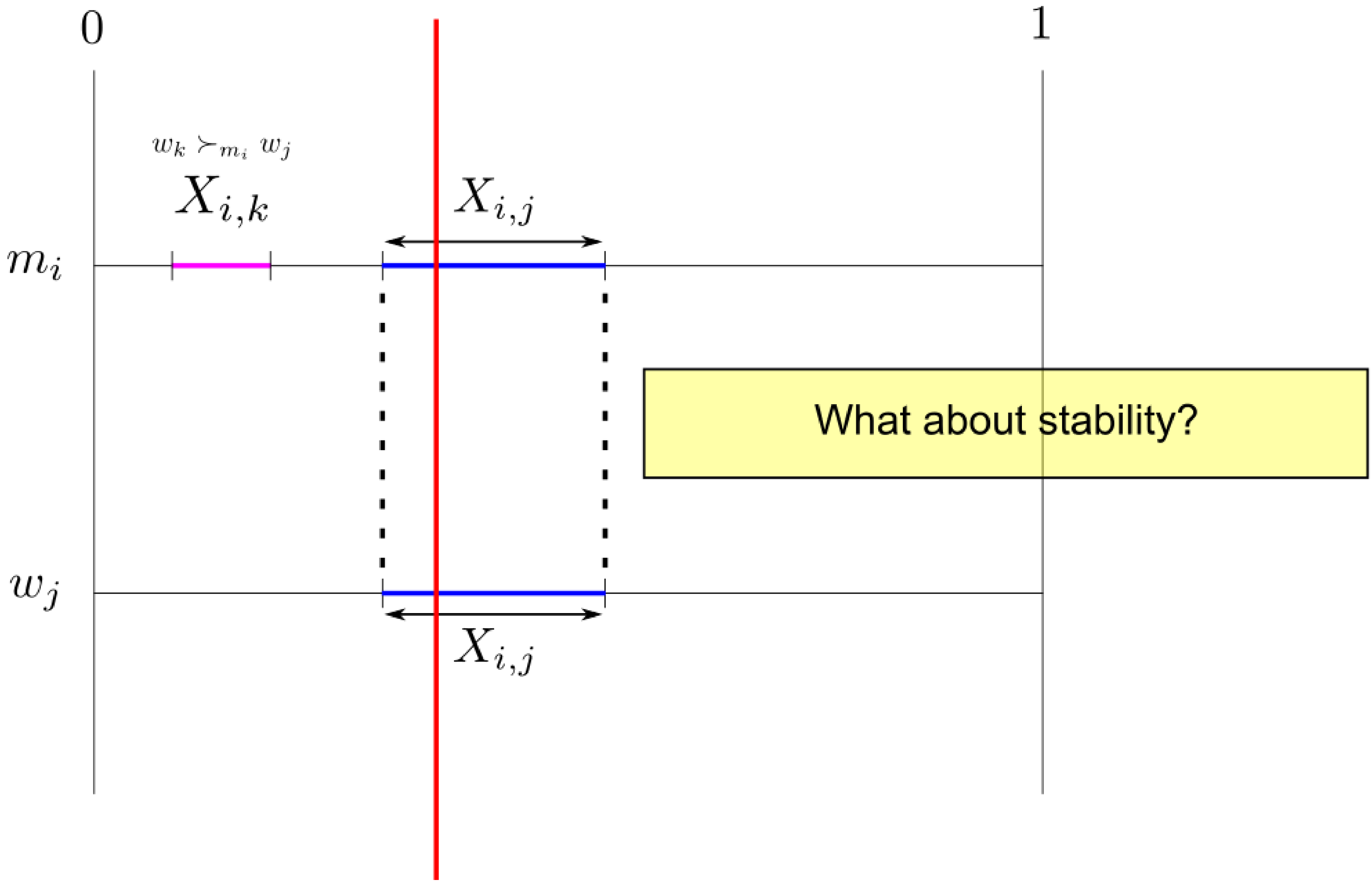

0
1
decreasing preference for m_i
$\sum_{k: w_k \succ_{m_i} w_j} X_{i,k}$
$X_{i,j}$
m_i
w_j
$X_{i,j}$
$\sum_{k: m_k \succ_{w_j} m_i} X_{k,j}$
decreasing preference for w_j
For any point in [0,1], a vertical line at that point induces an integral matching.

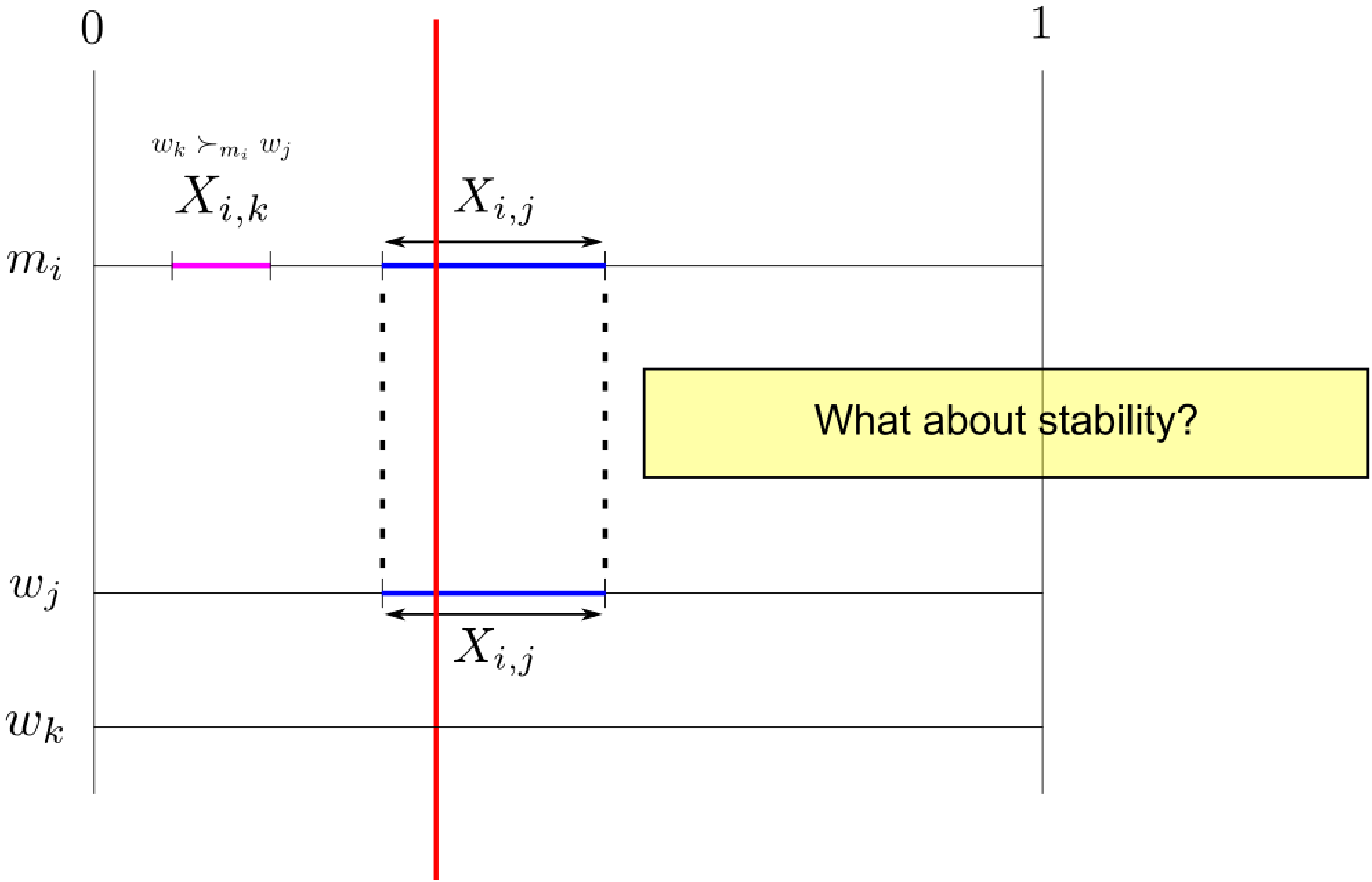

0
1
decreasing preference for m_i
$\sum_{k:w_k \succ_{m_i} w_j} X_{i,k}$
$X_{i,j}$
m_i
A left-to-right scan gives a convex decomposition over integral matchings.
w_j
$X_{i,j}$
$\sum_{k:m_k \succ_{w_j} m_i} X_{k,j}$
decreasing preference for w_j



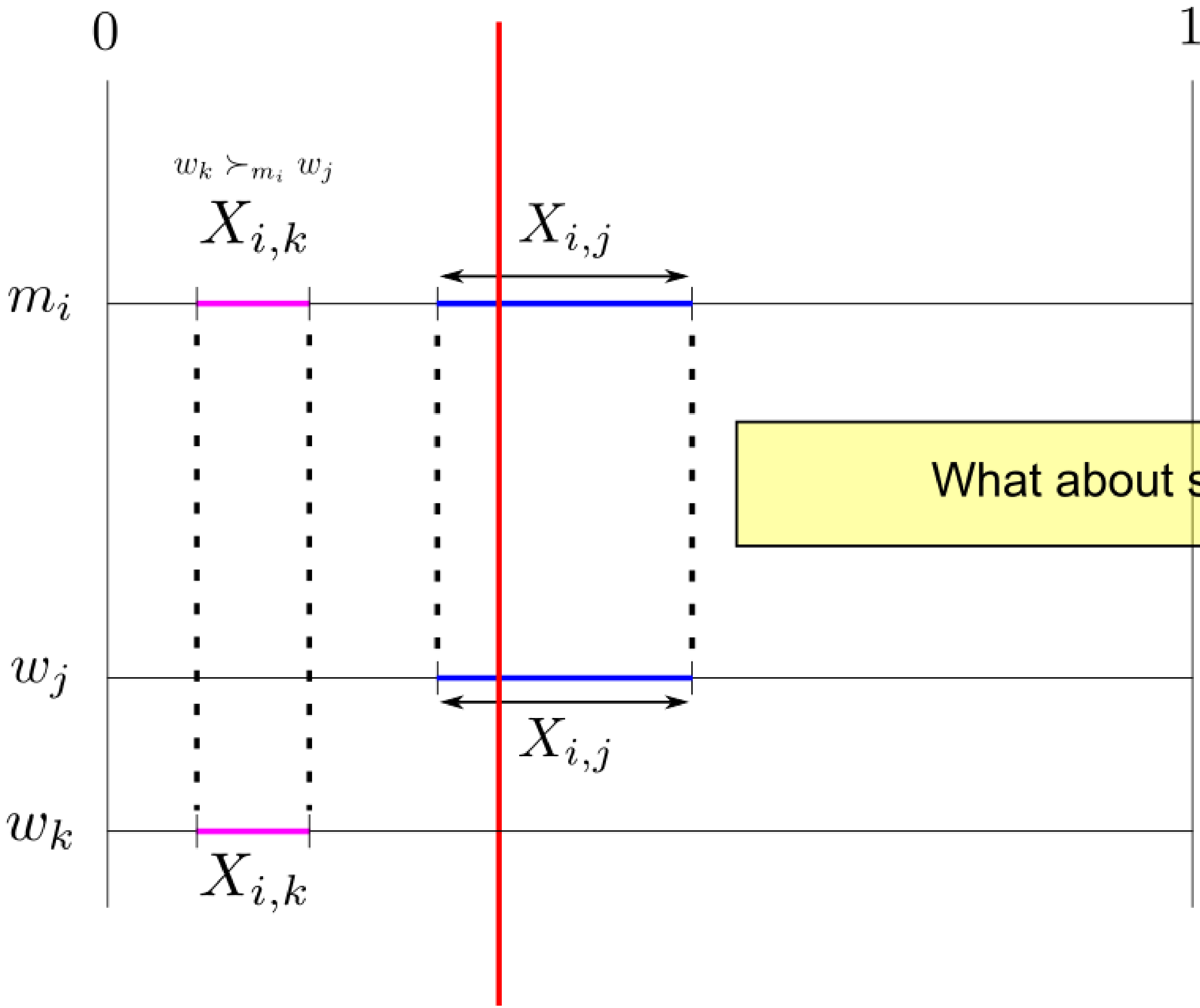
What about stability?


0
1
$X_{i,j}$
m_i
w_j
$X_{i,j}$
What about stability?

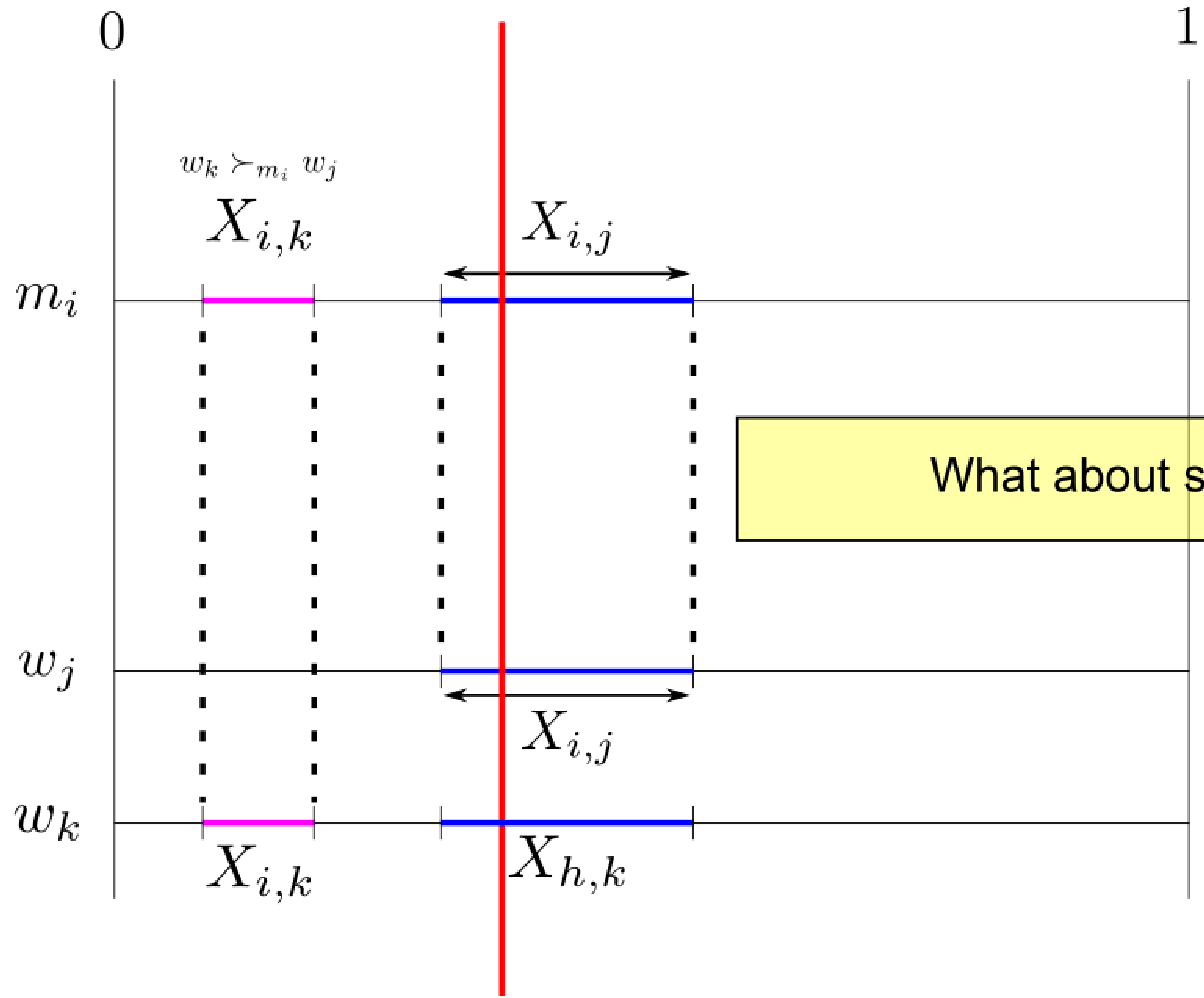

0
1
$w_k \succ_{m_i} w_j$
$X_{i,k}$
$X_{i,j}$
m_i
What about stability?
w_j
$X_{i,j}$



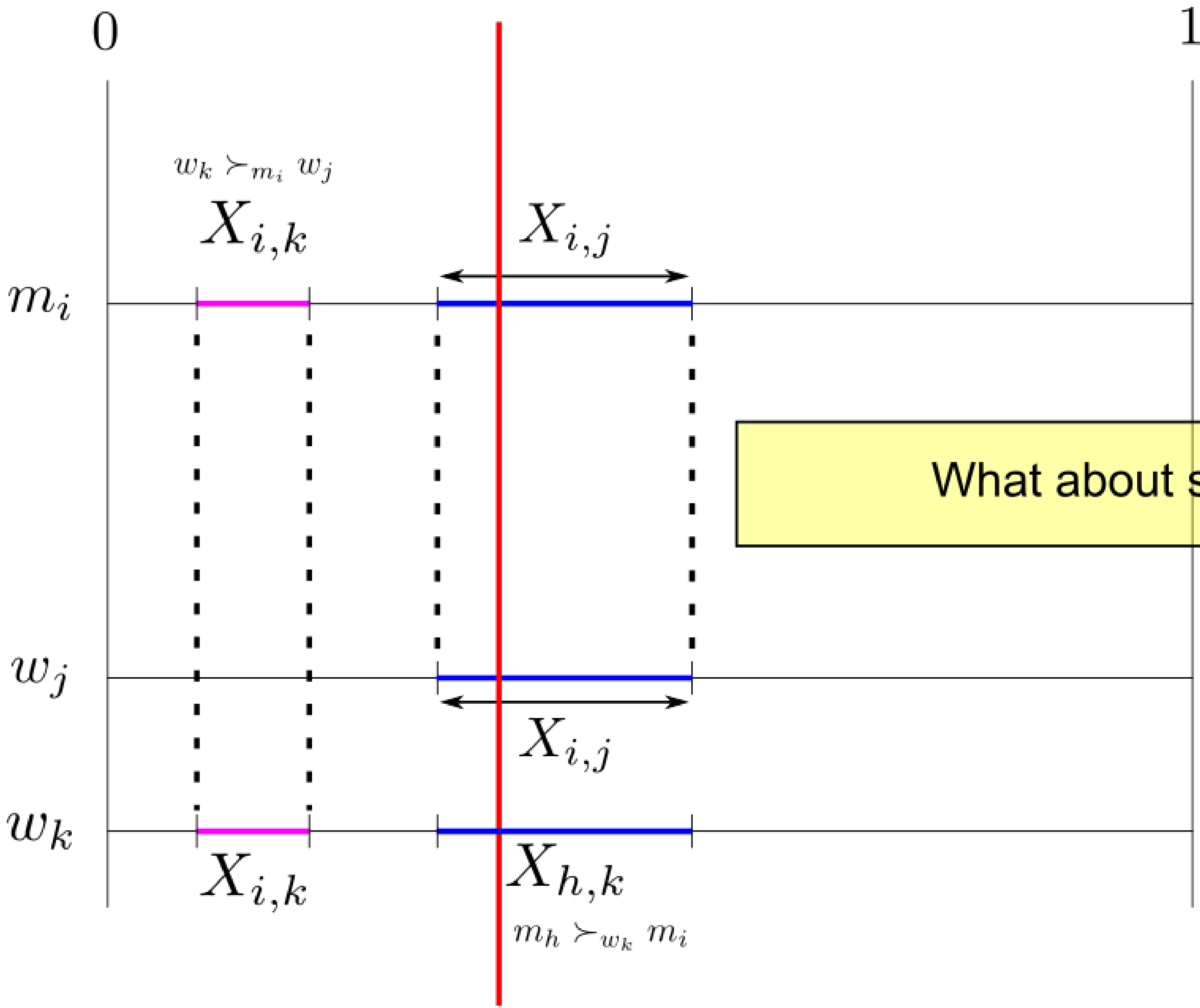
0
1
$w_k \succ_{m_i} w_j$
$X_{i,k}$
$X_{i,j}$
m_i
What about stability?
w_j
$X_{i,j}$
w_k



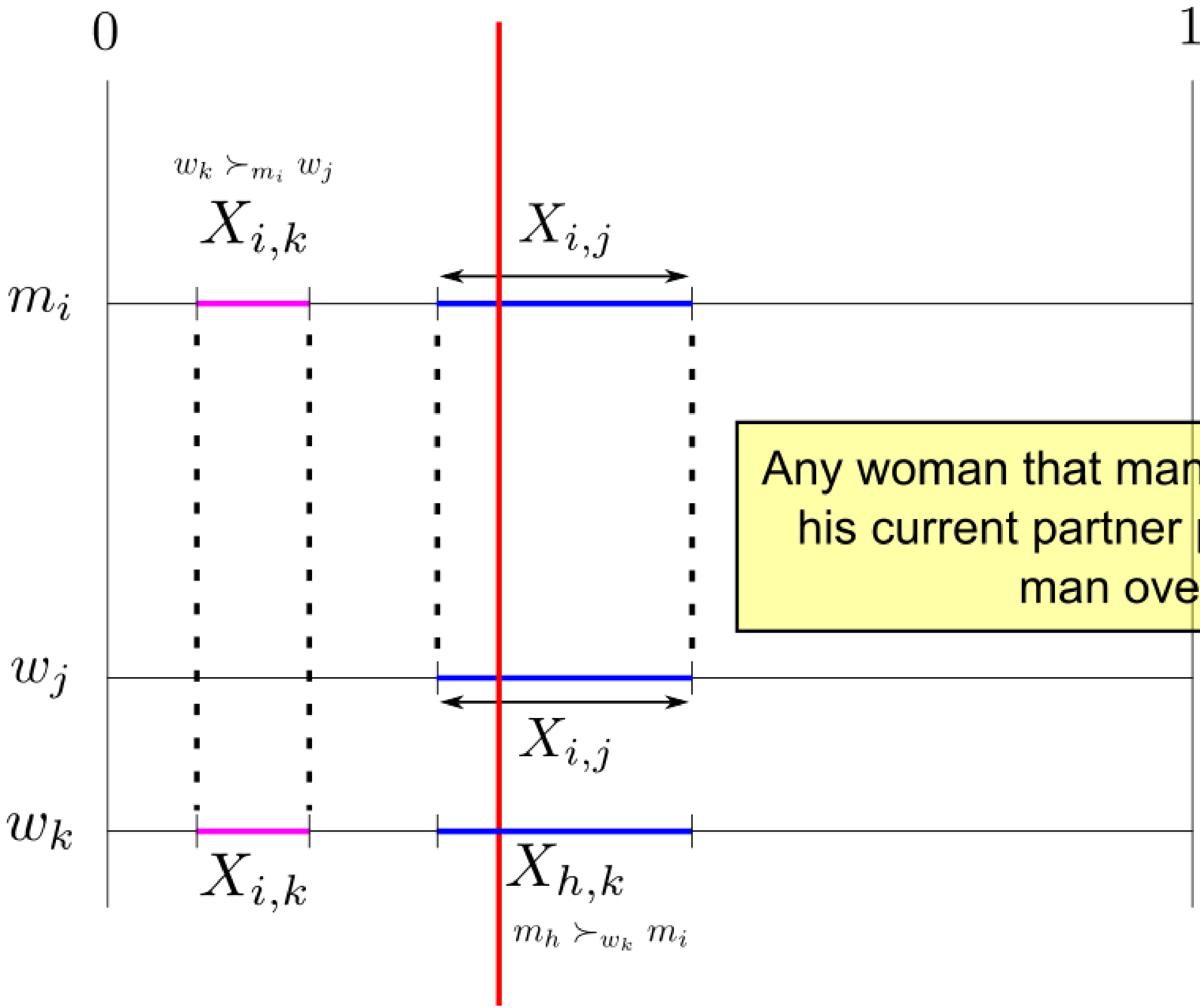
What about stability?



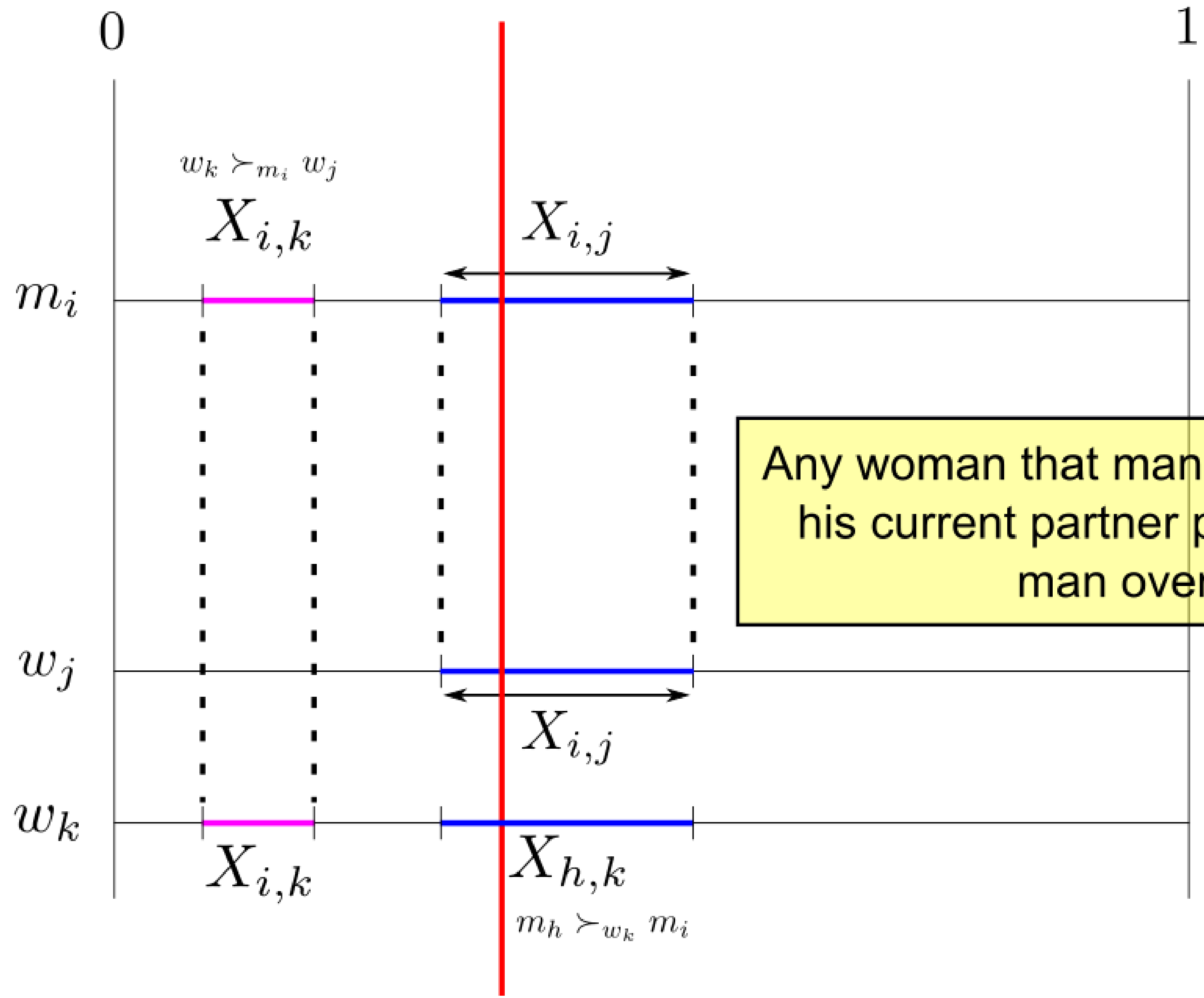
What about stability?



What about stability?



Any woman that man m_i prefers over his current partner prefers another man over m_i.



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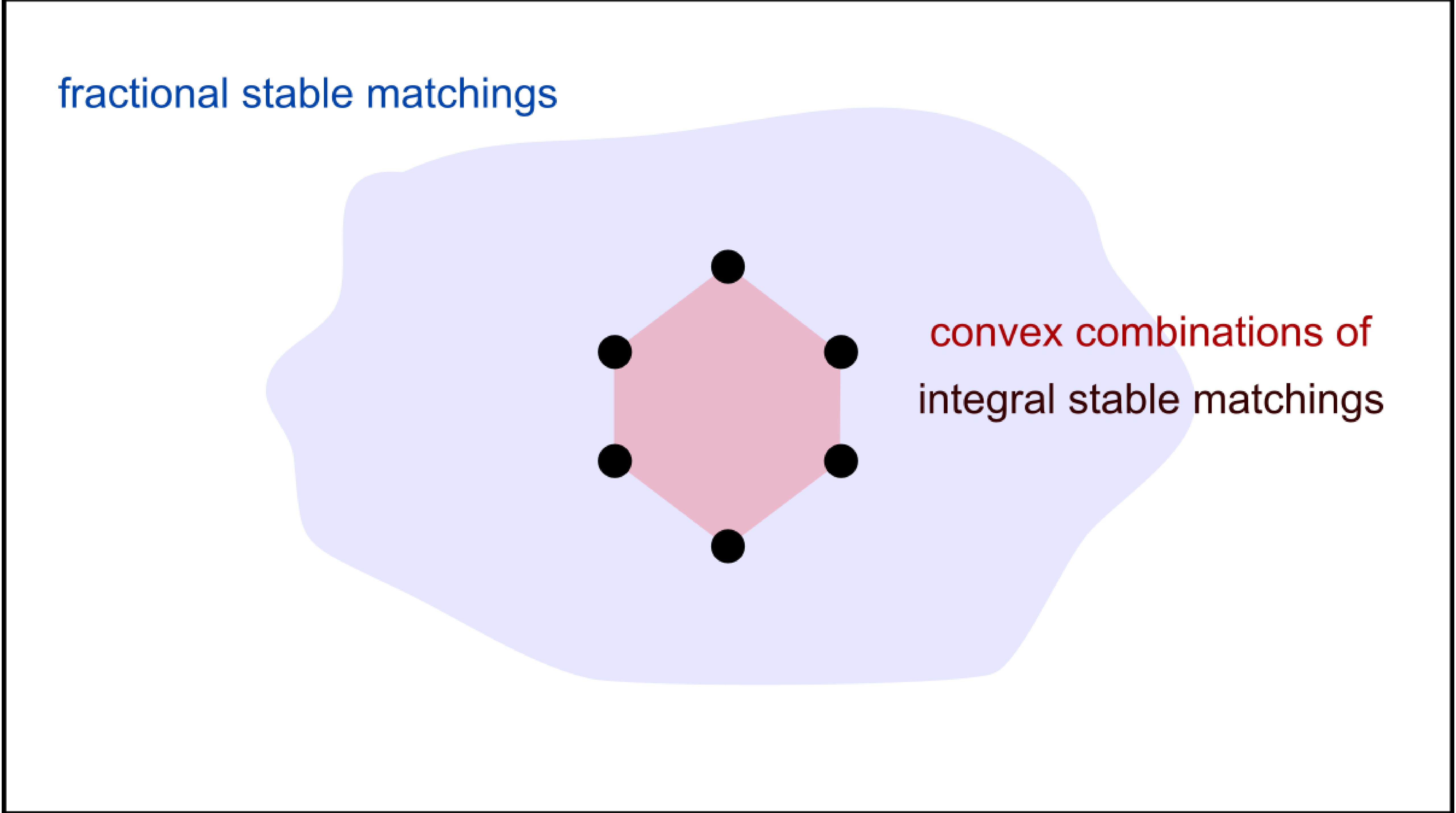
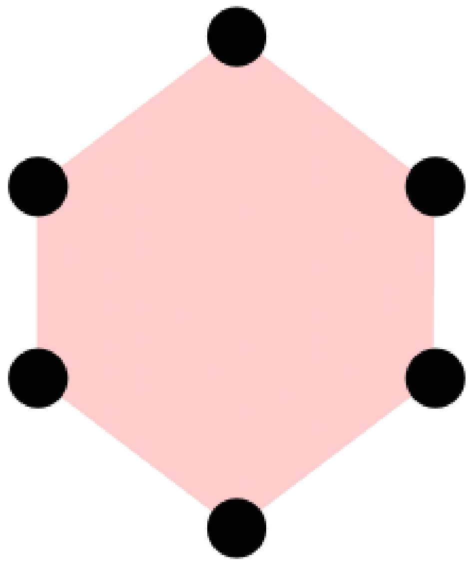

fractional stable matchings
convex combinations of
integral stable matchings



convex combinations of
integral stable matchings
=
fractional stable matchings

Let us use the decomposition technique to show the existence of a "fair" matching.

A "Fair" Stable Matching

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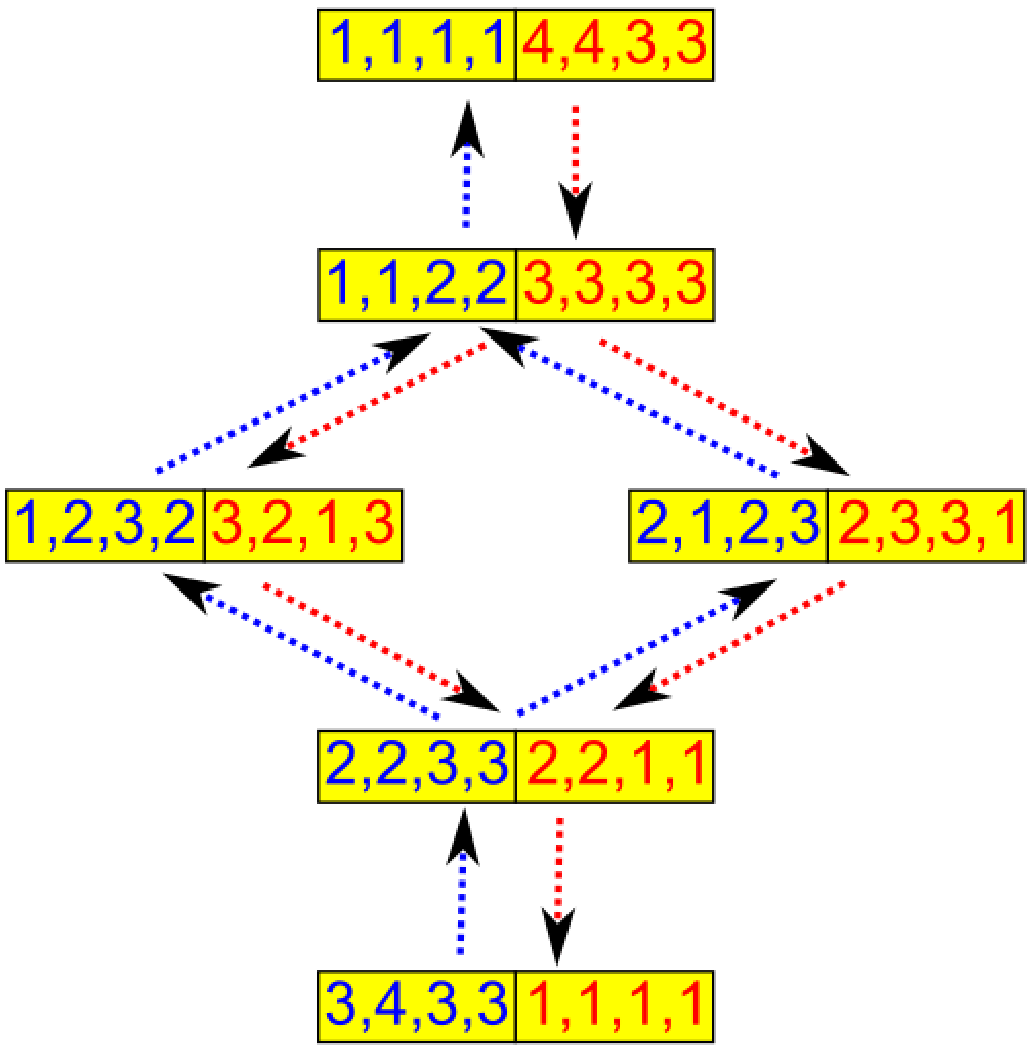

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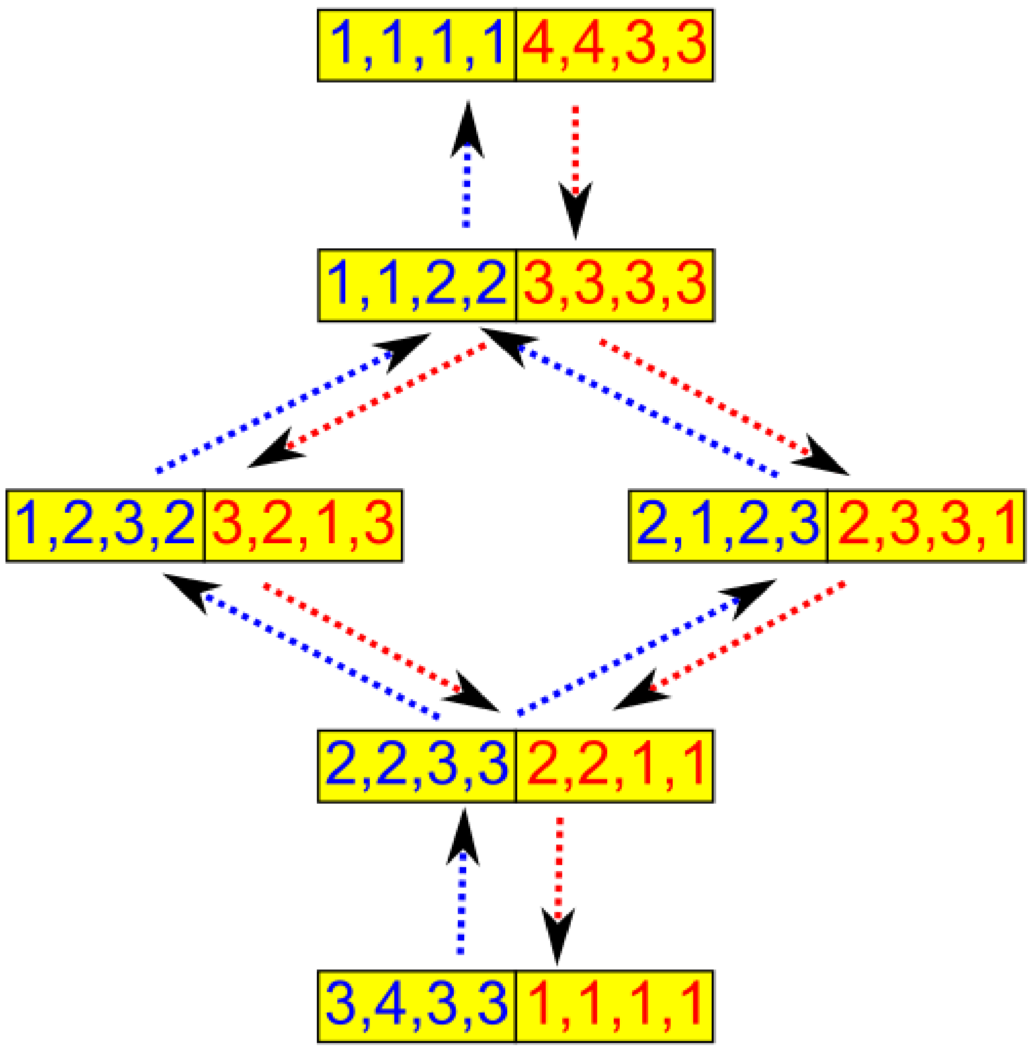
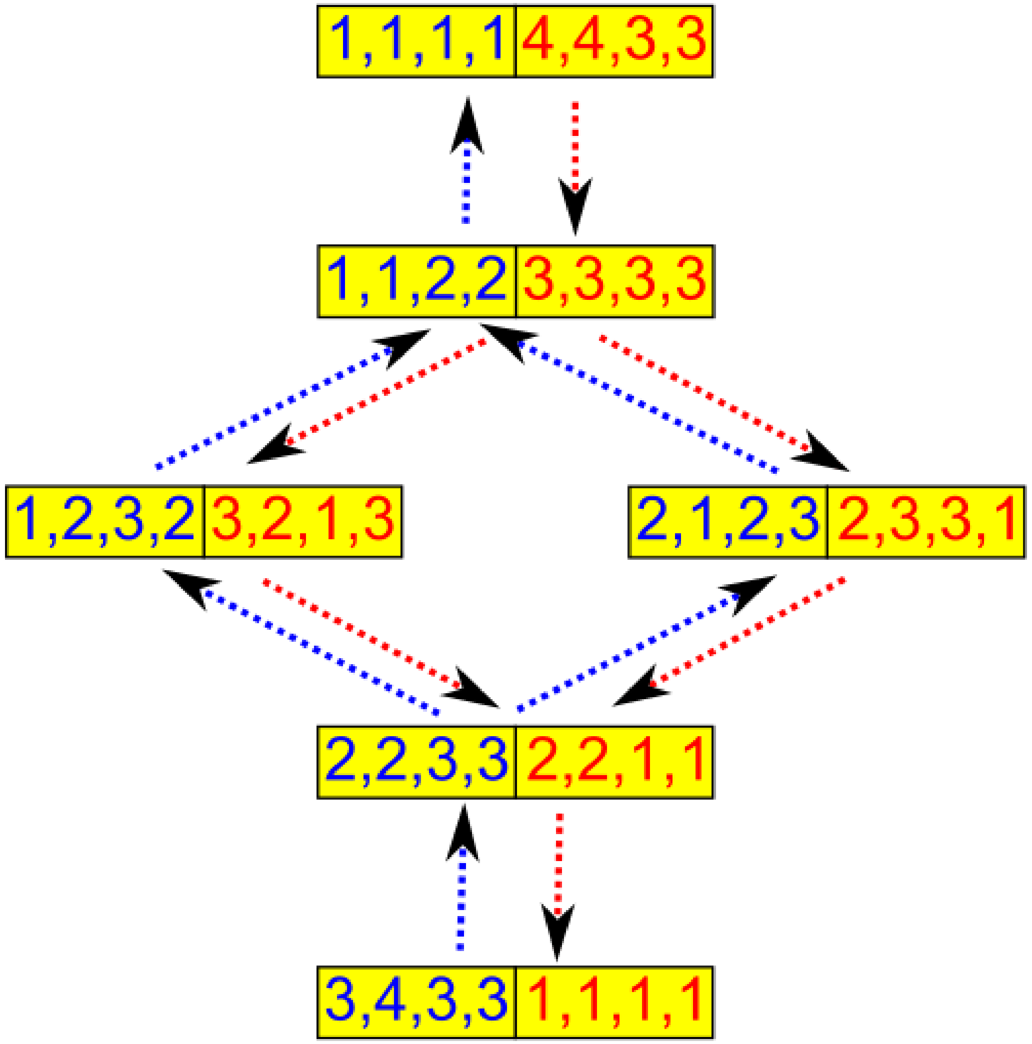
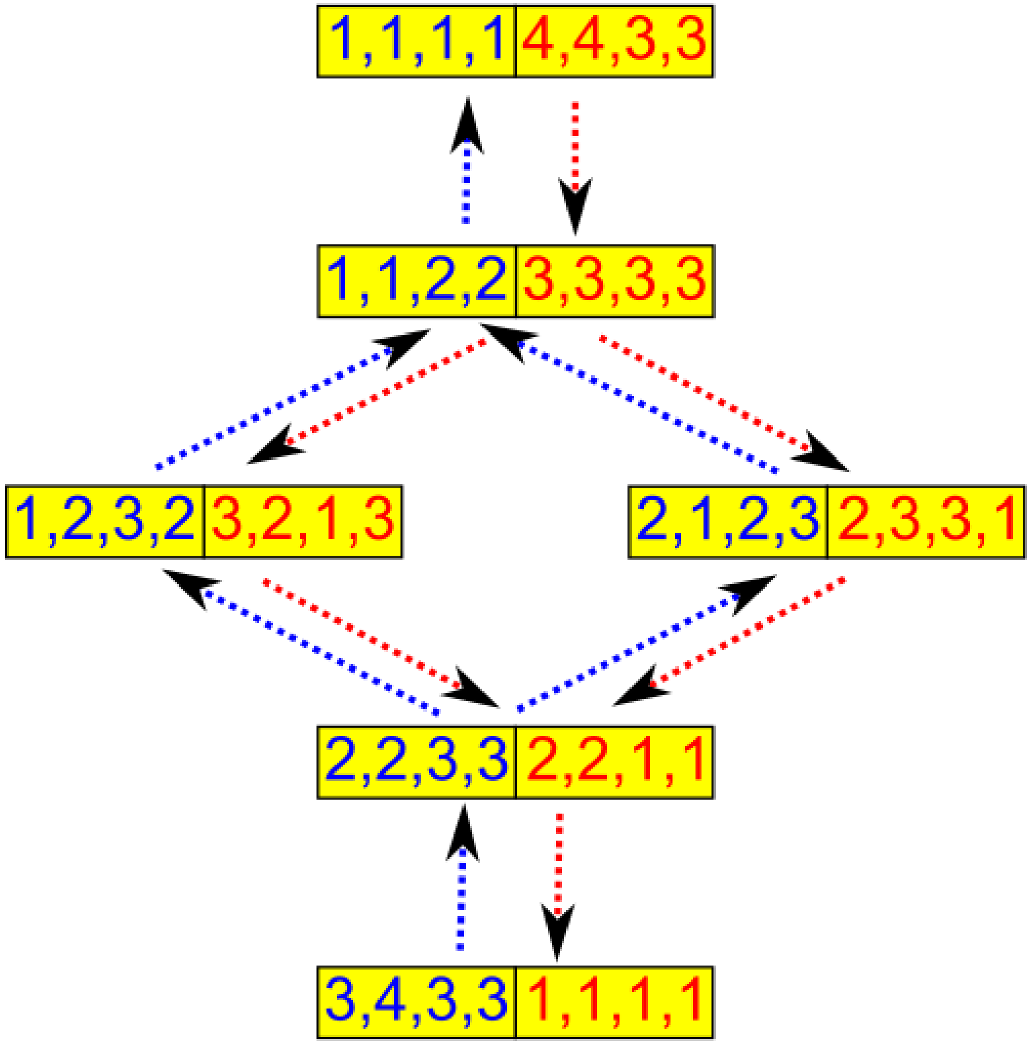


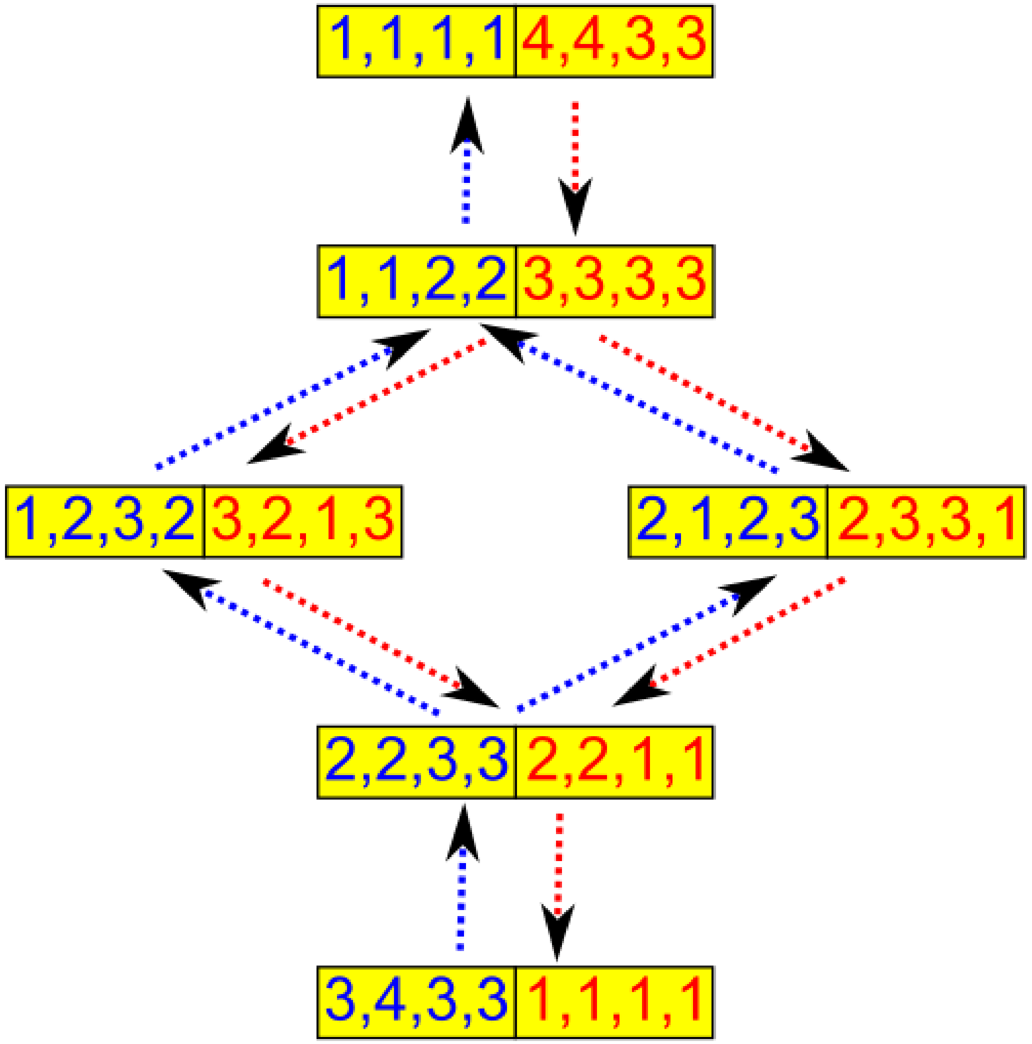
m_1 1 1 1 2 2 3



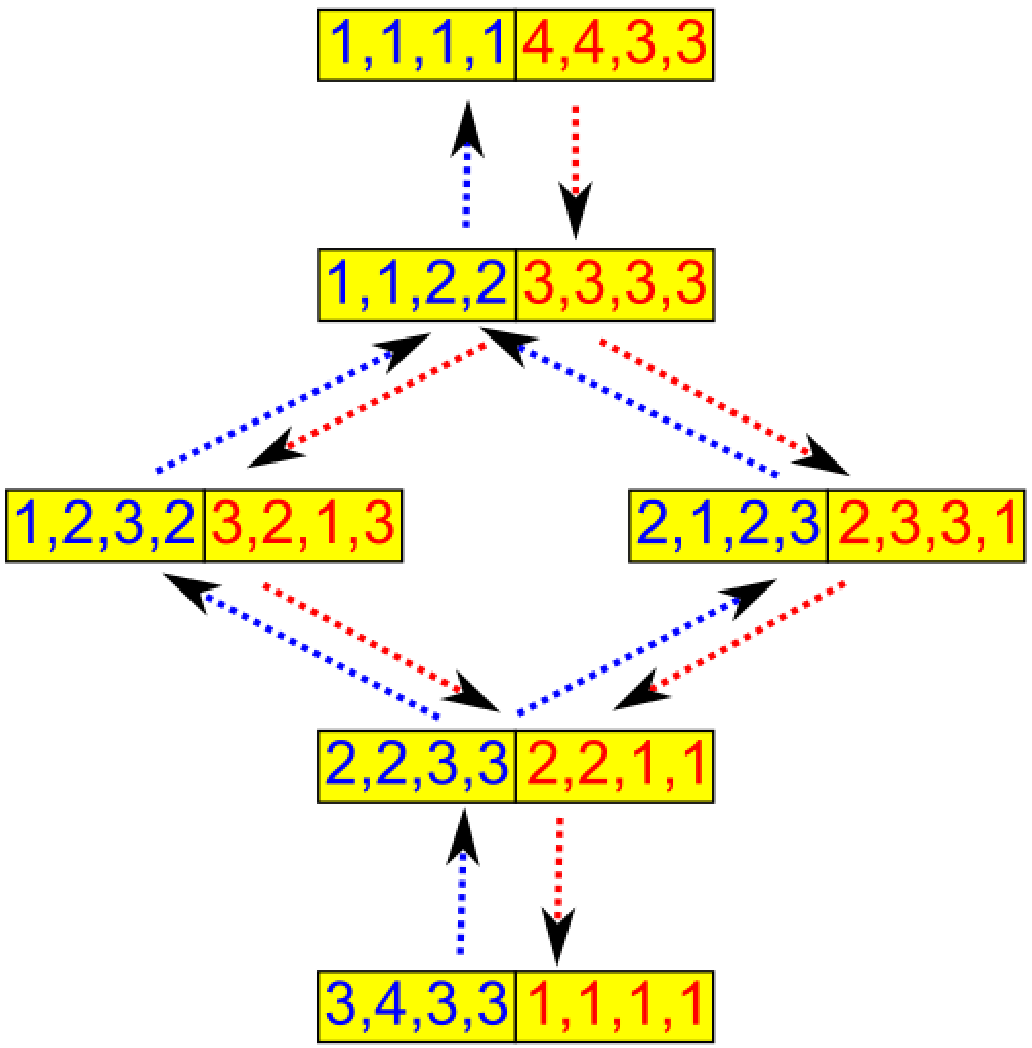
m_1	1	1	1	2	2	3
m_2	1	1	1	2	2	4



m_1	1	1	1	2	2	3
m_2	1	1	1	2	2	4
m_3	1	2	2	3	3	3
m_4	1	2	2	3	3	3



m_1	1	1	1	2	2	3
m_2	1	1	1	2	2	4
m_3	1	2	2	3	3	3
m_4	1	2	2	3	3	3
w_1	4	3	3	2	2	1
w_2	4	3	3	2	2	1
w_3	3	3	3	1	1	1
w_4	3	3	3	1	1	1



m_1	1	1	1	2	2	3
m_2	1	1	1	2	2	4
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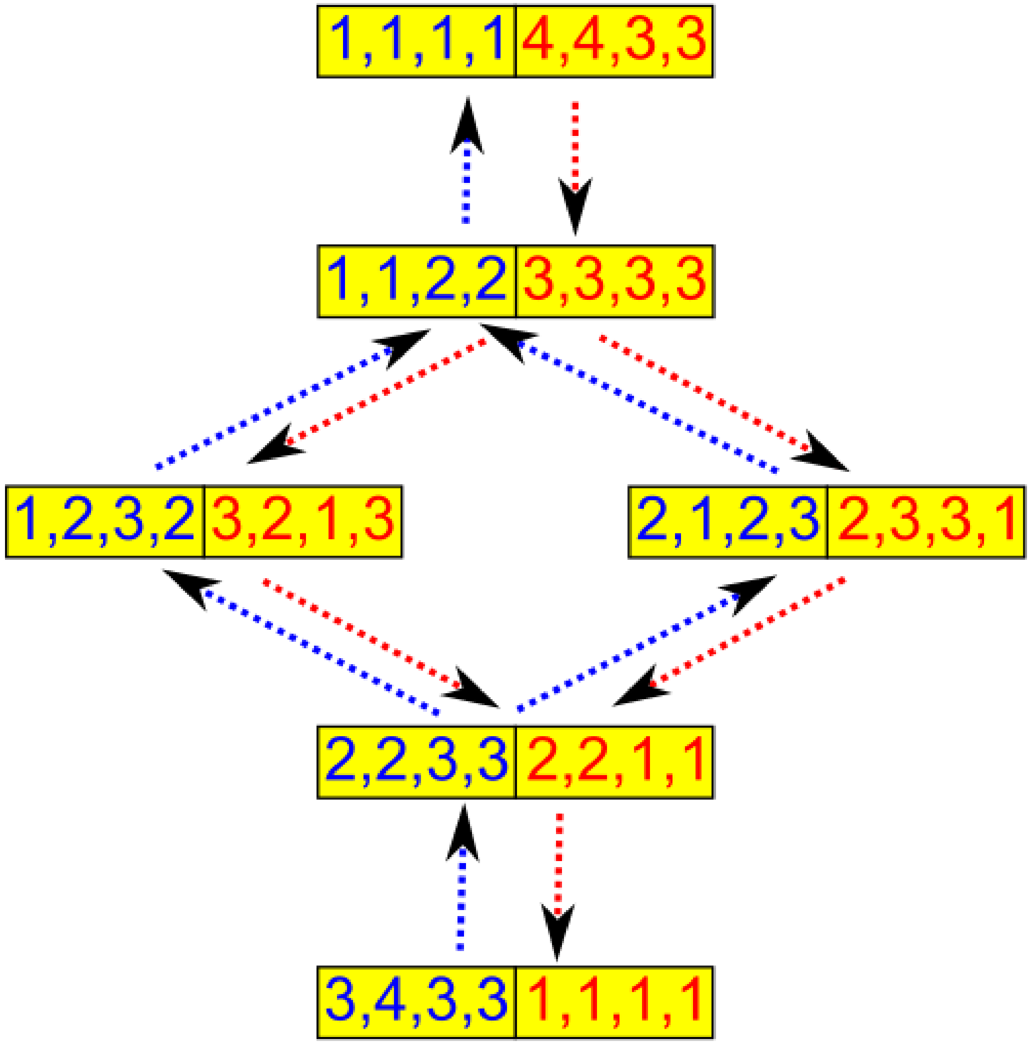
w_1	4	3	3	2	2	1
w_2	4	3	3	2	2	1
w_3	3	3	3	1	1	1
w_4	3	3	3	1	1	1



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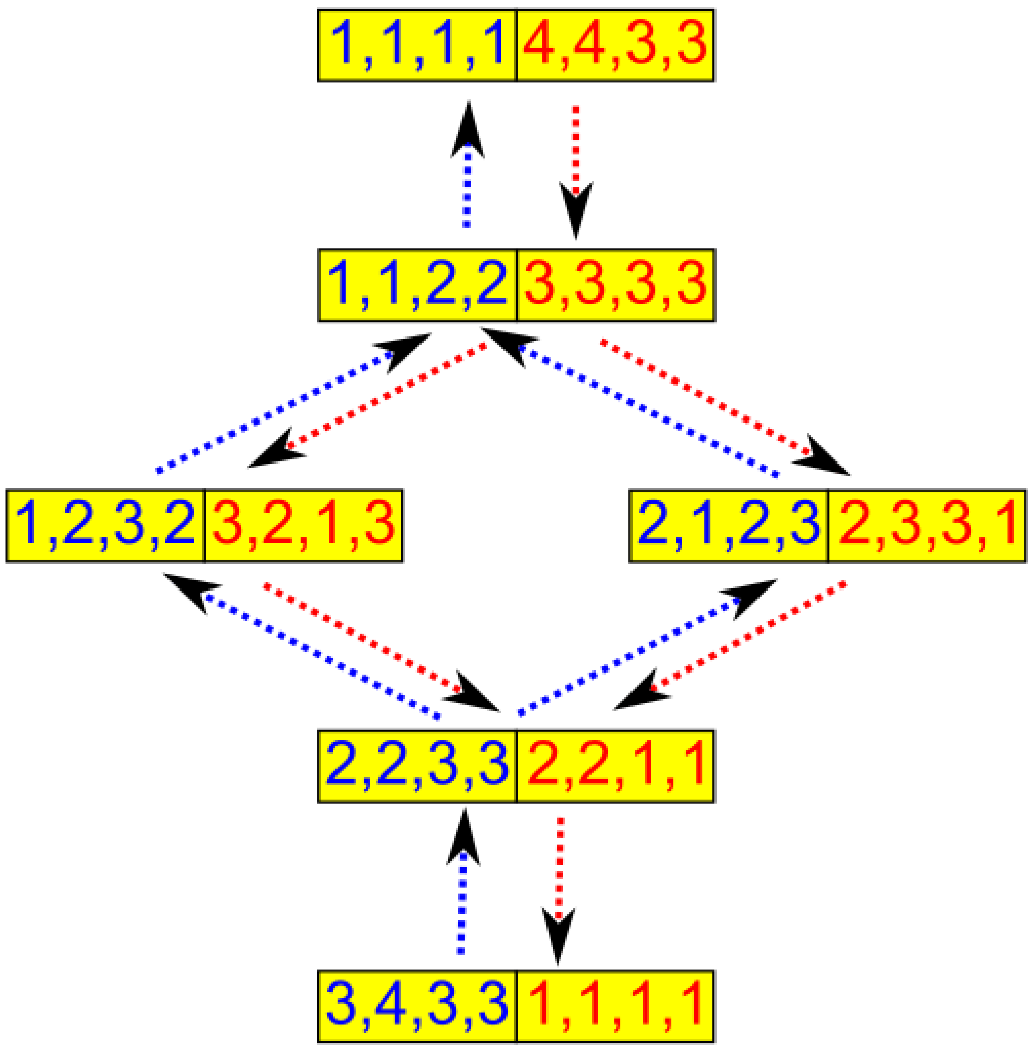
Proof by example.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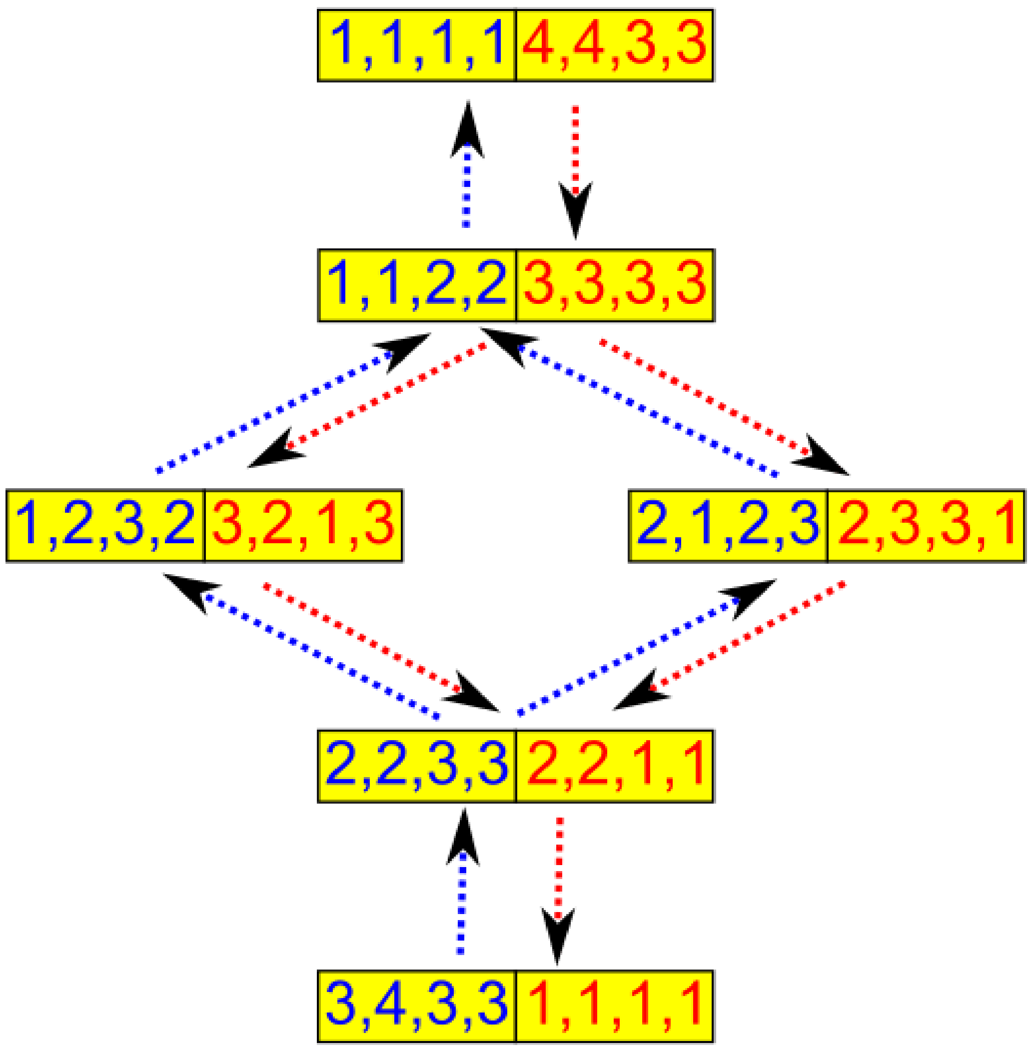
Consider a uniform combination of all integral stable matchings.



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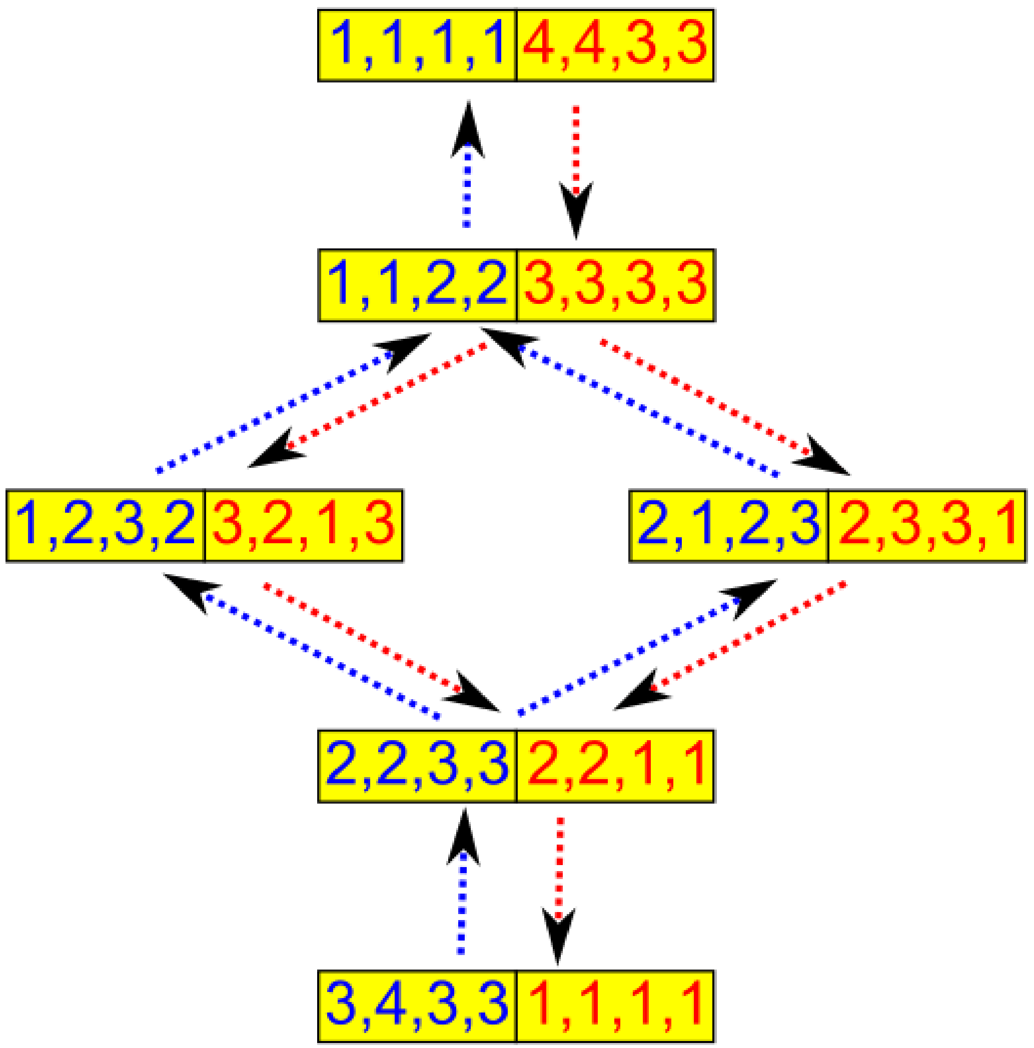

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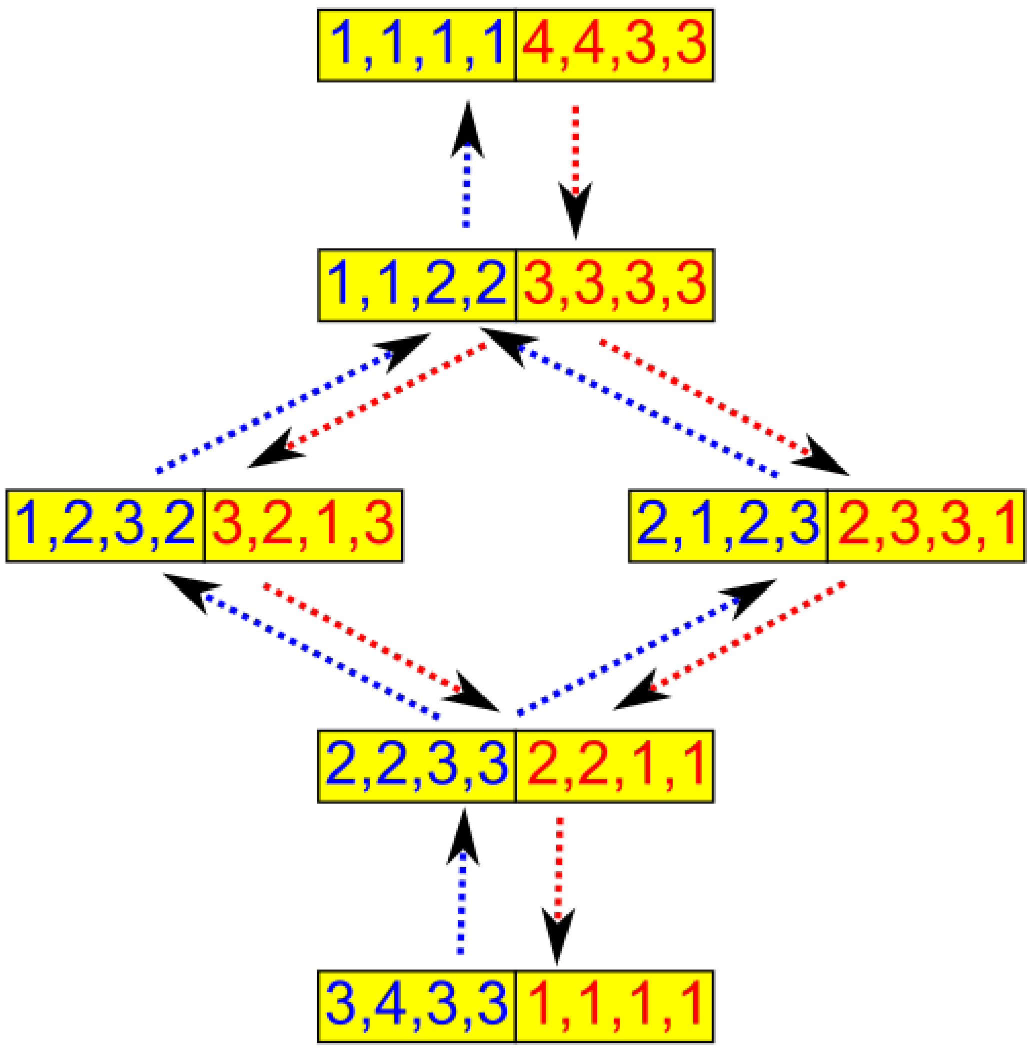

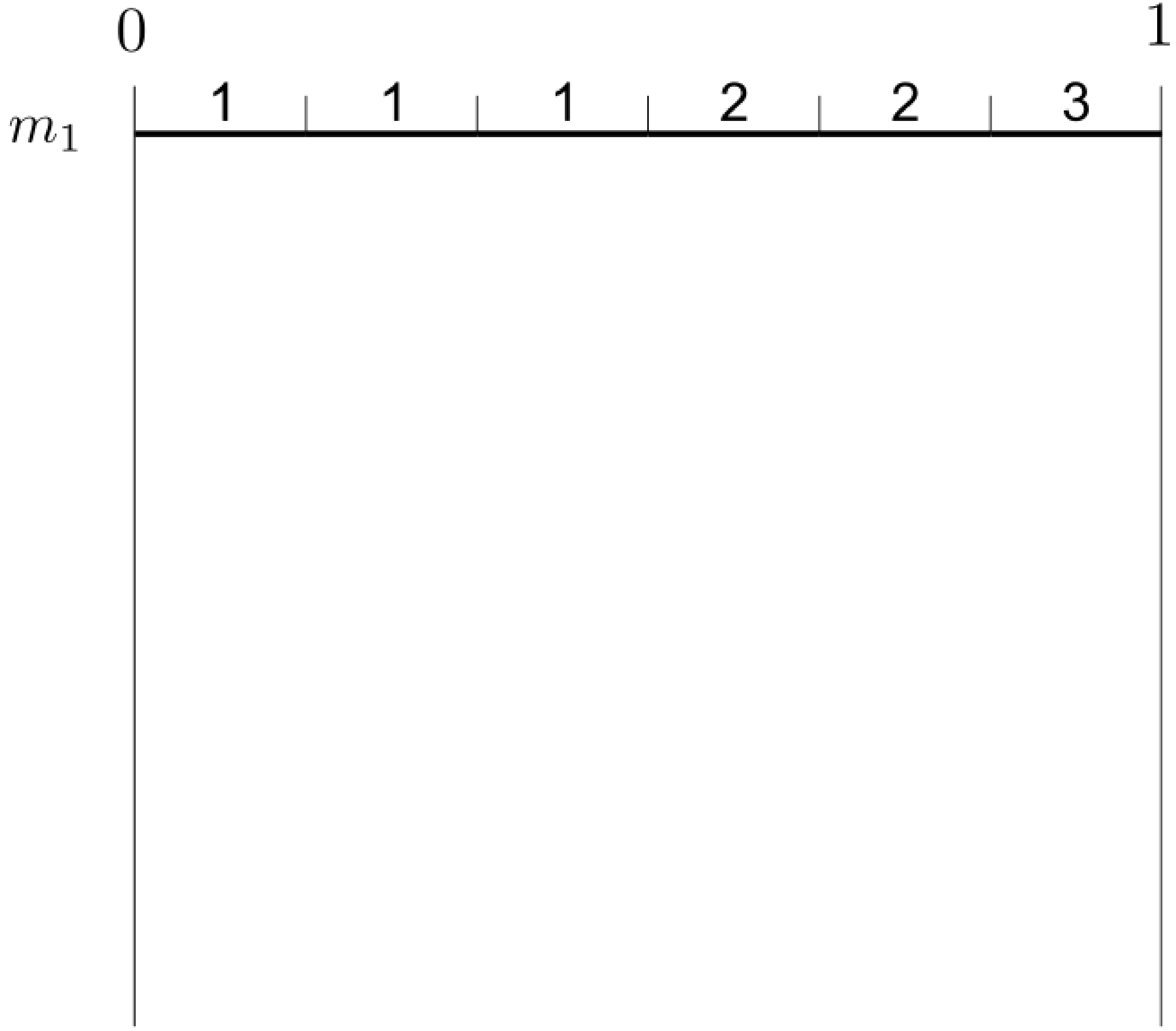

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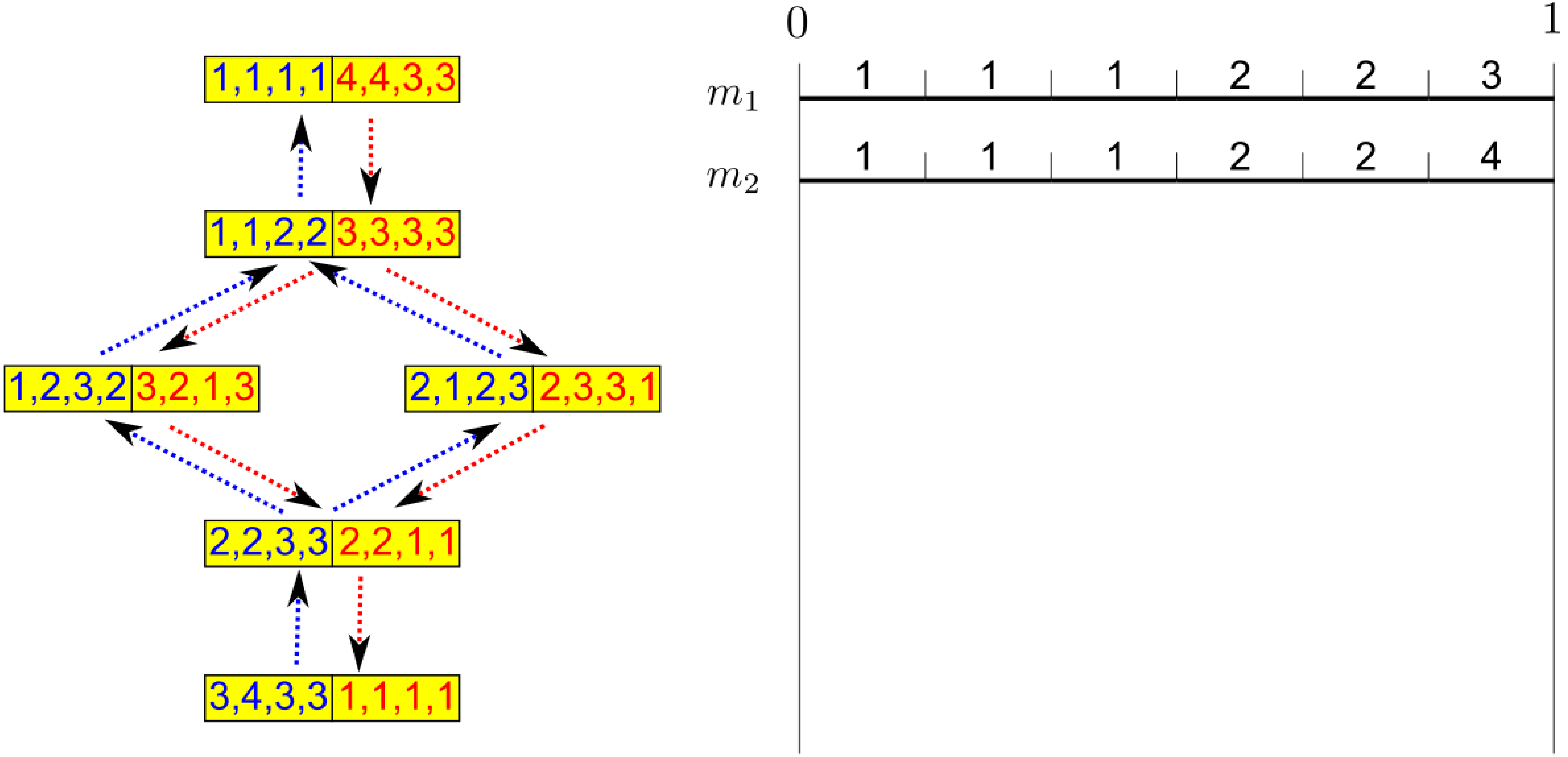

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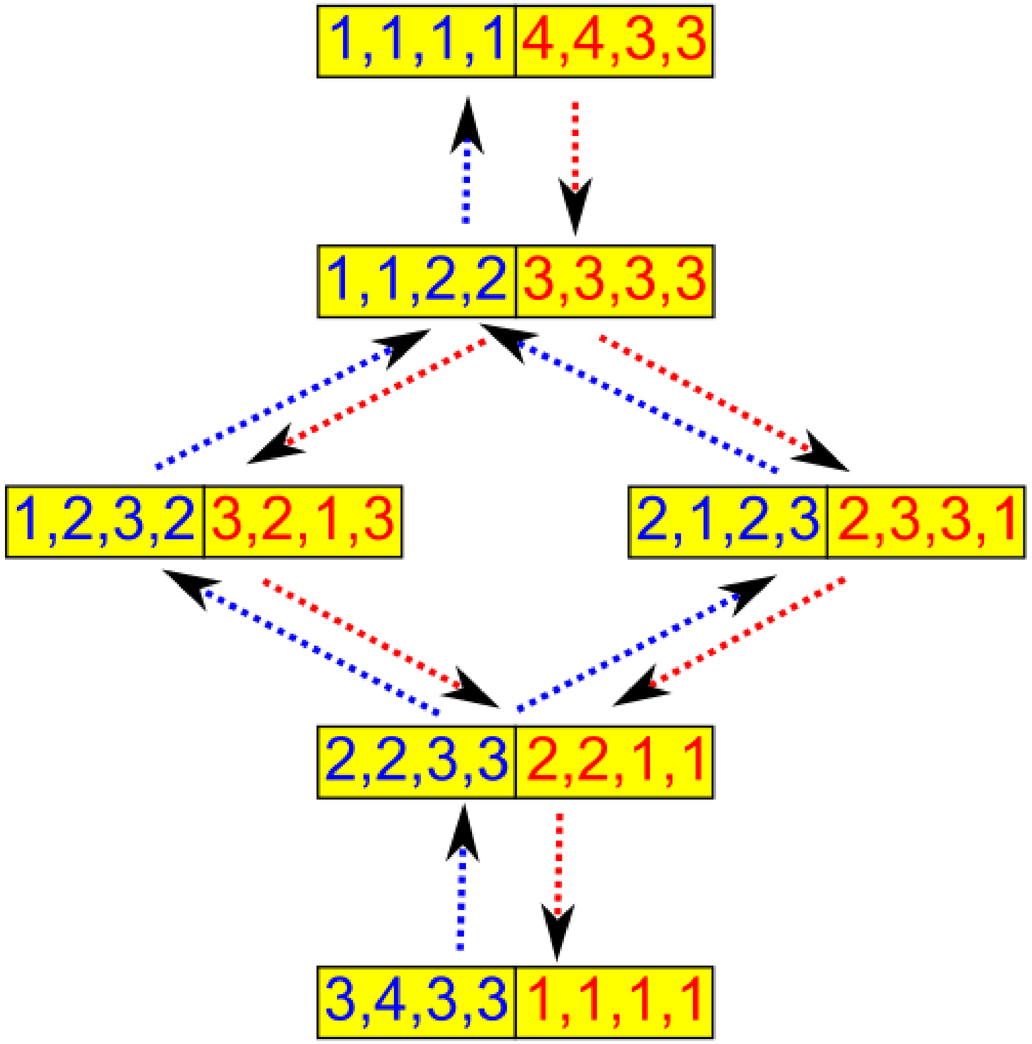

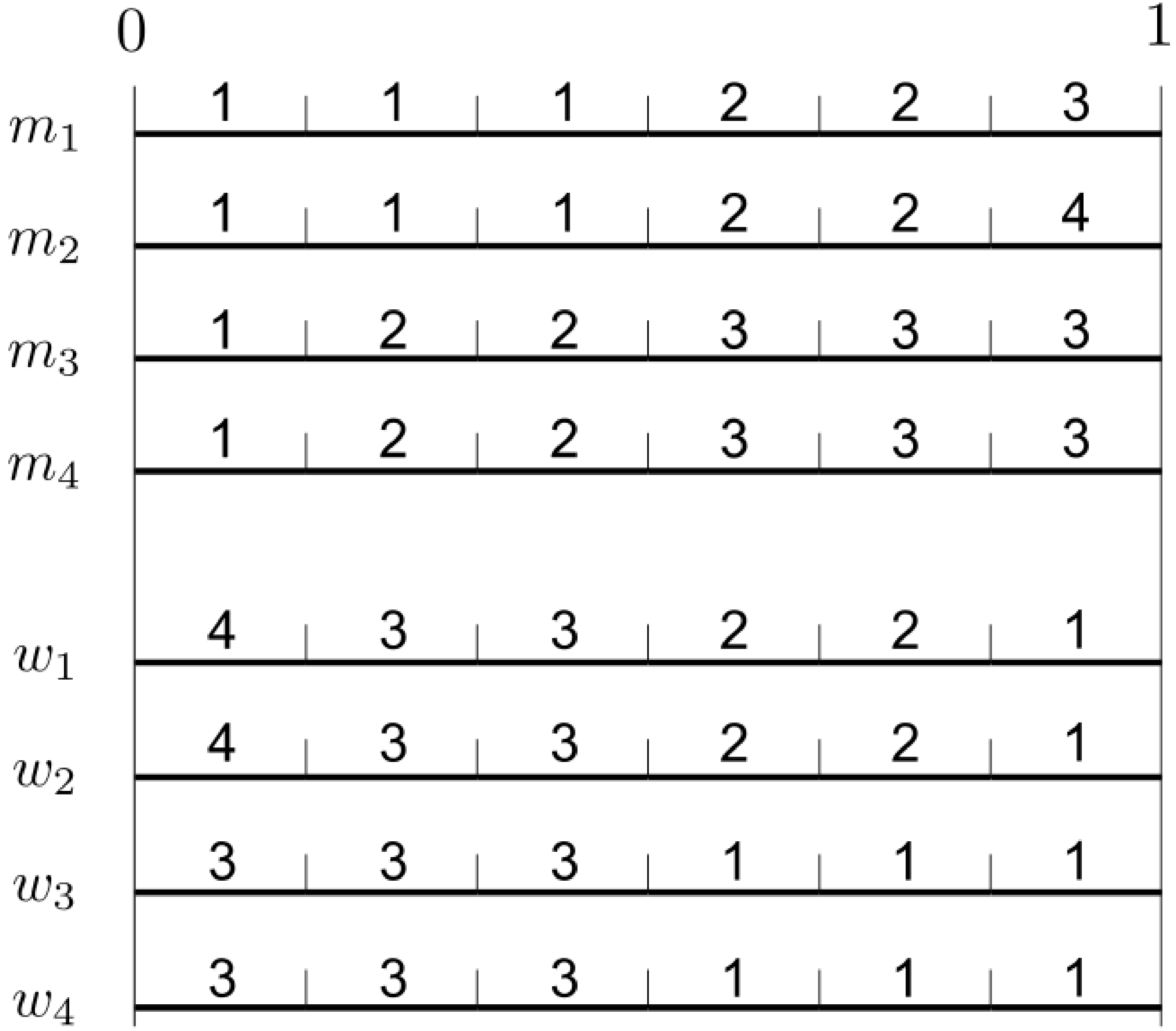

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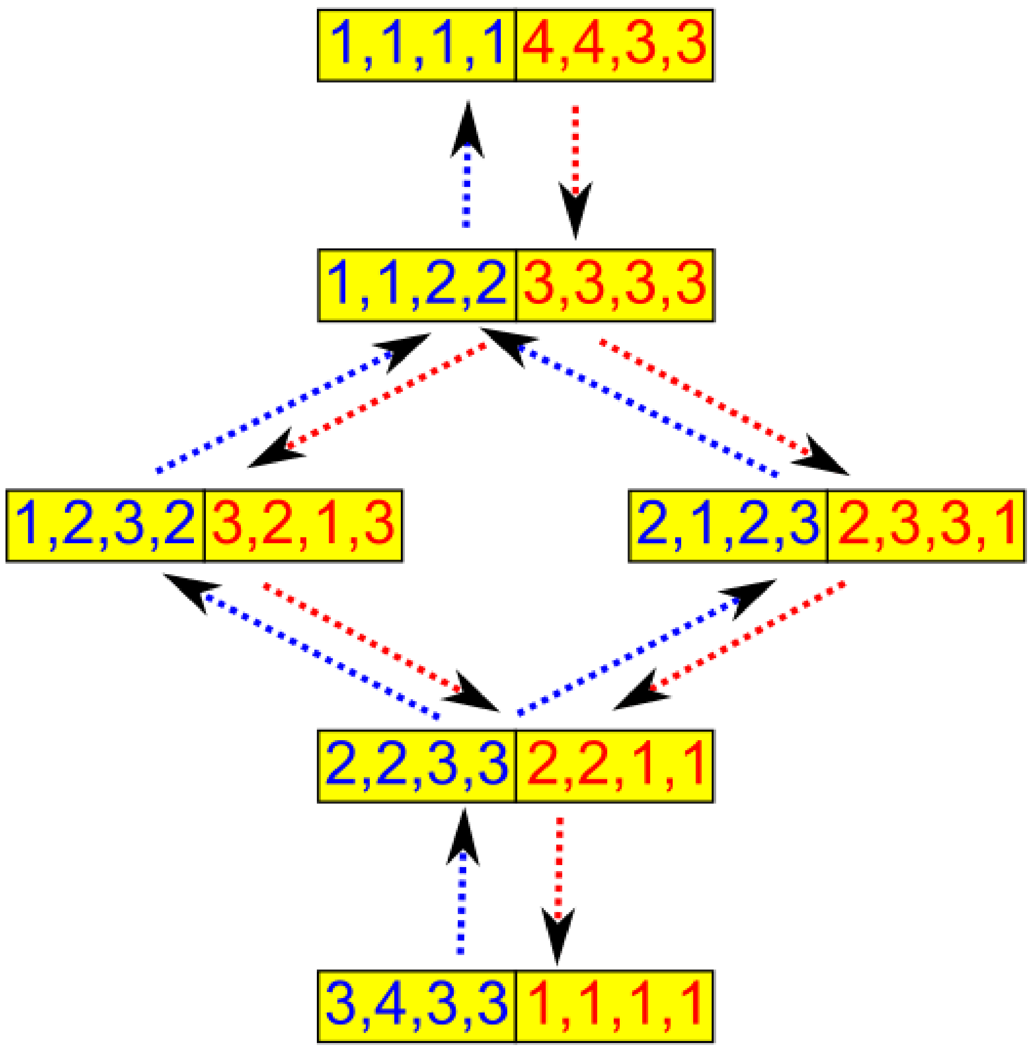

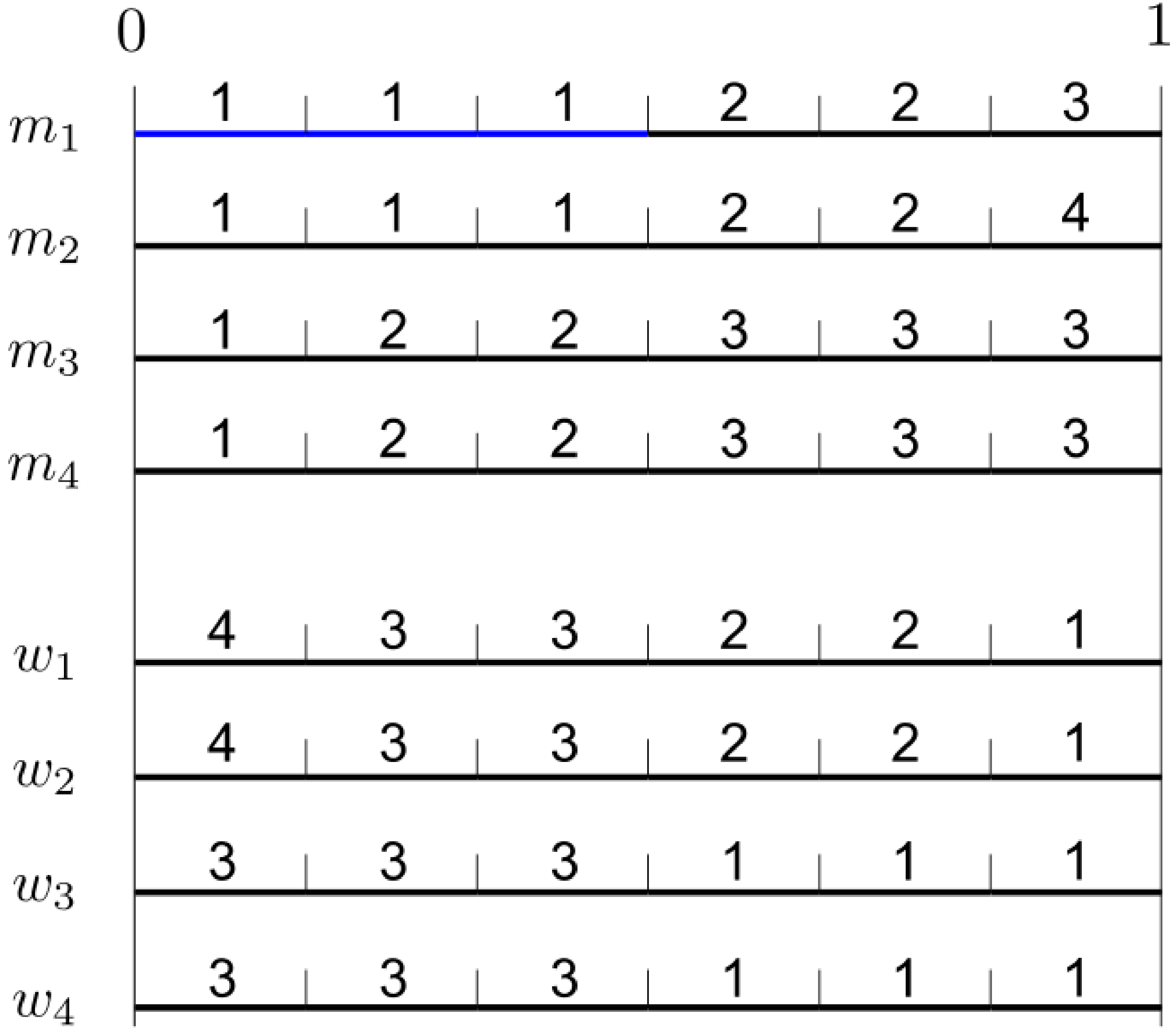

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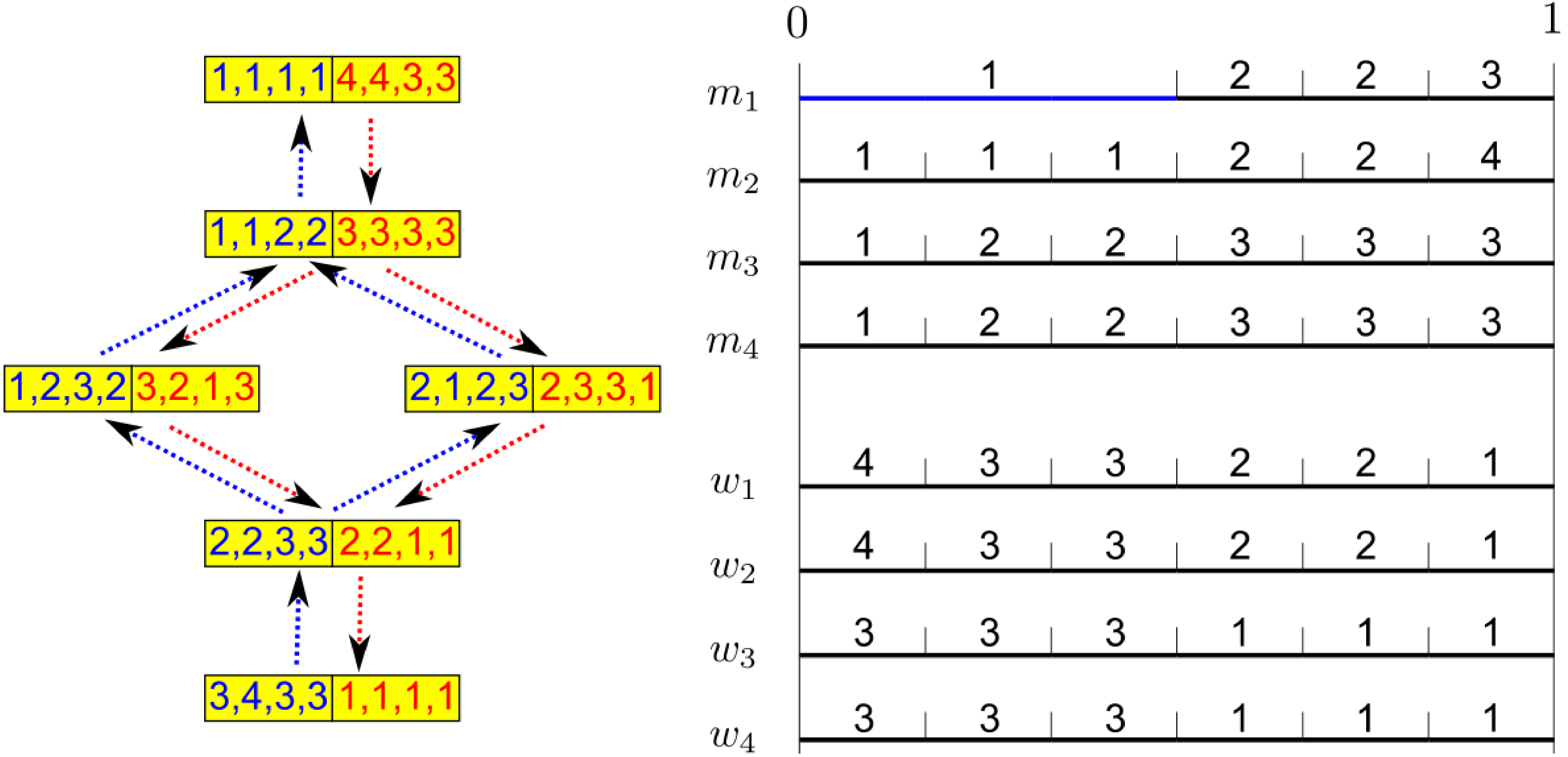

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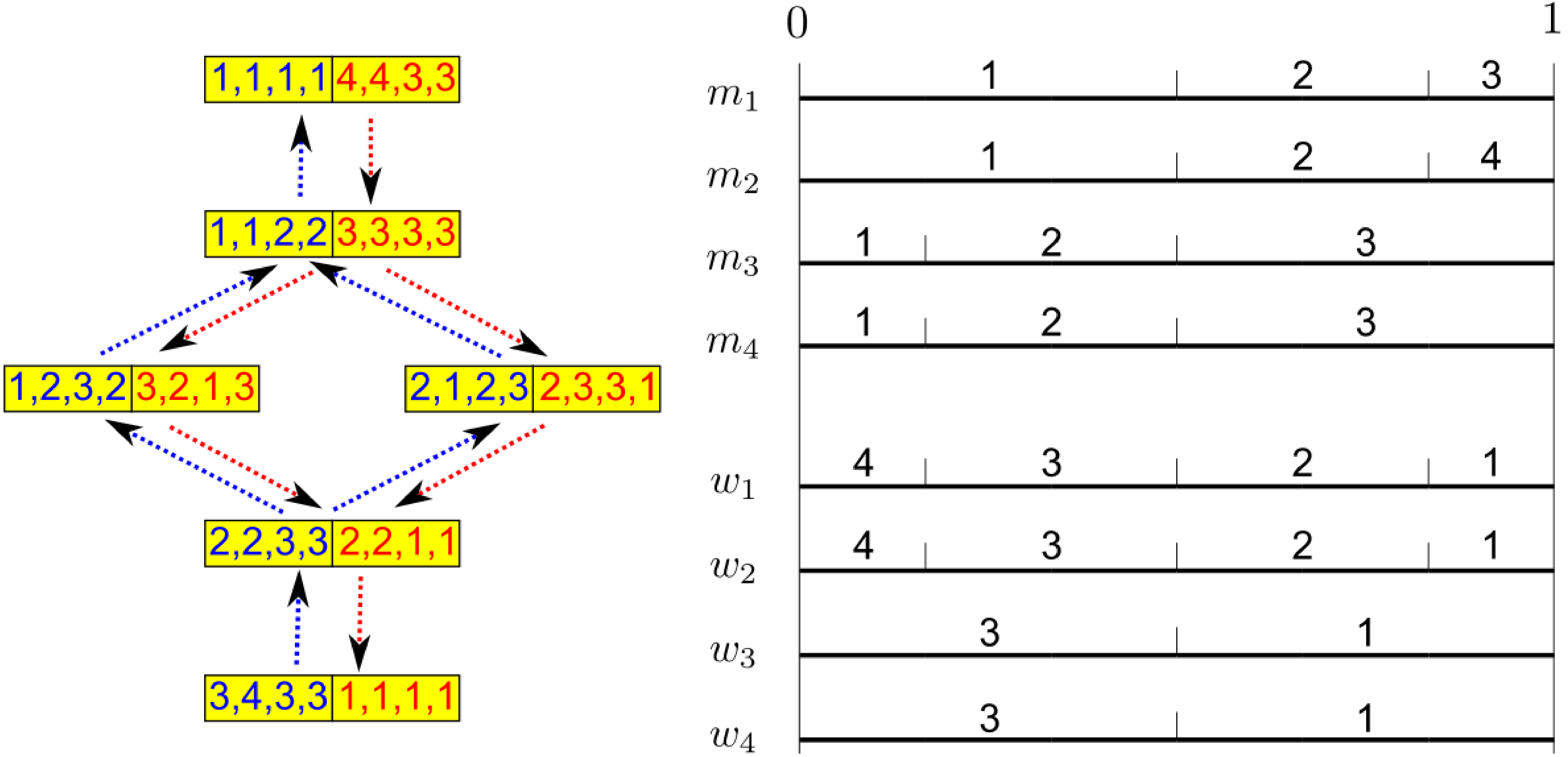

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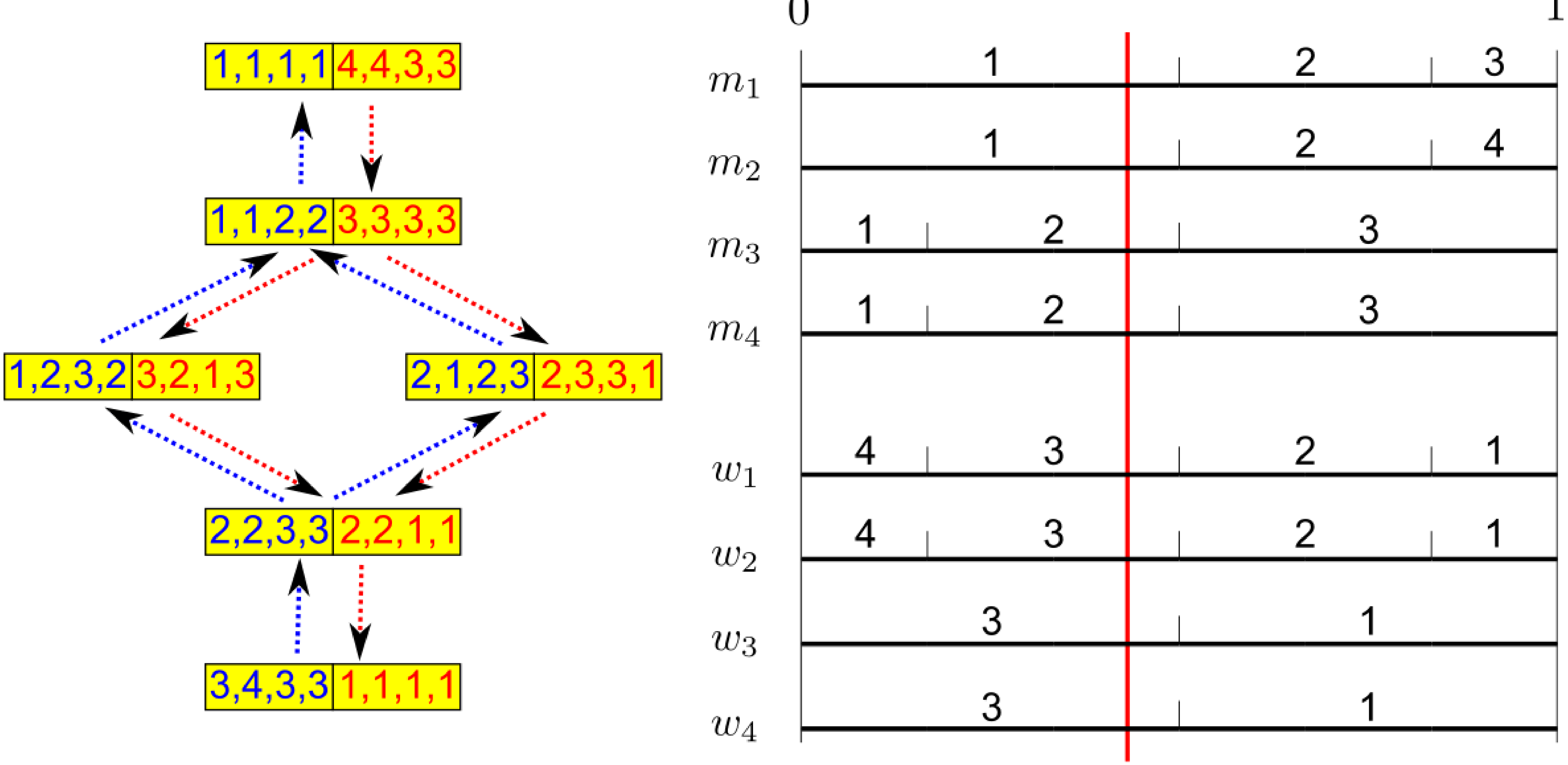

A vertical cut at any point in [0,1] induces a stable matching.

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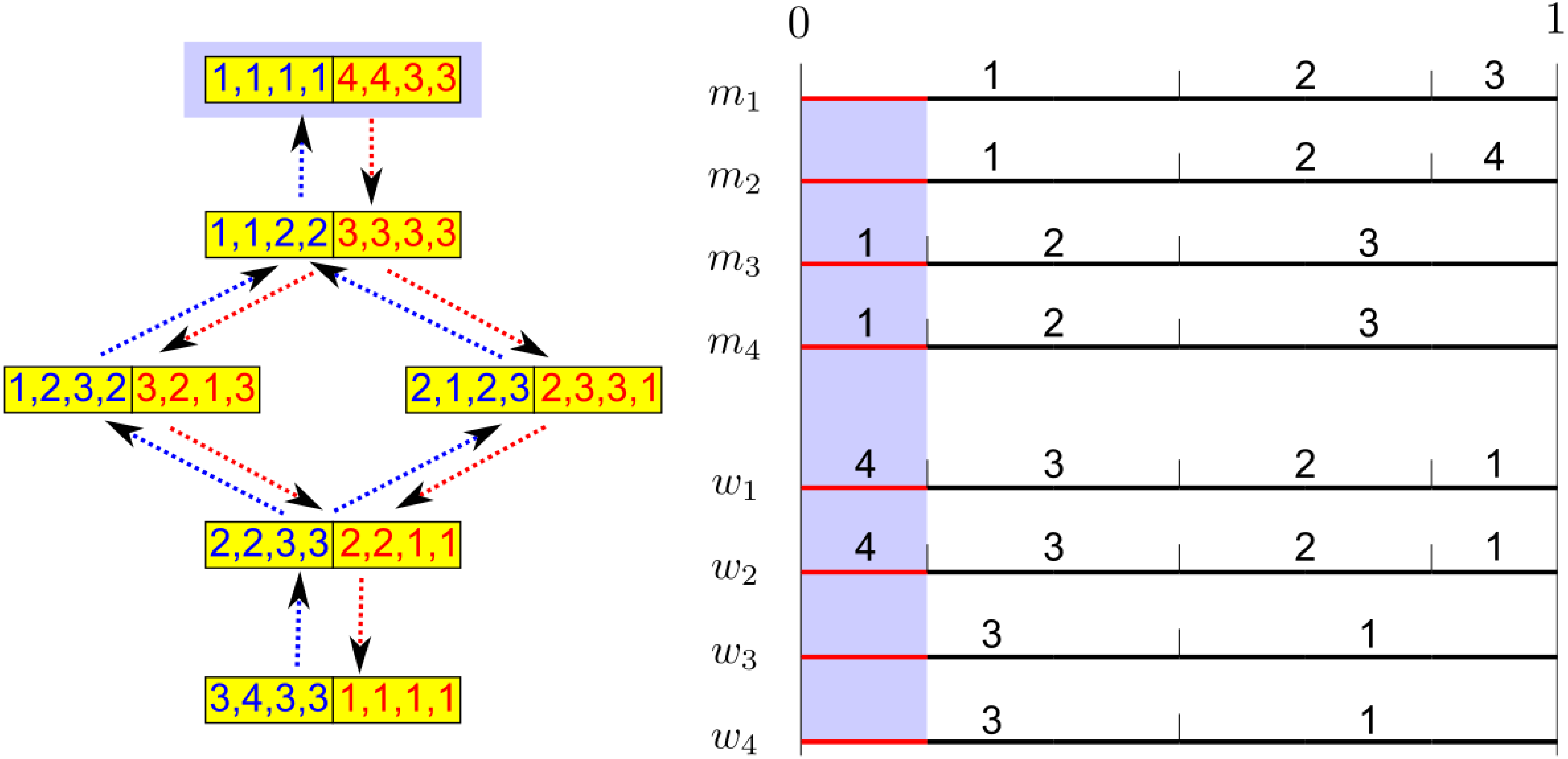

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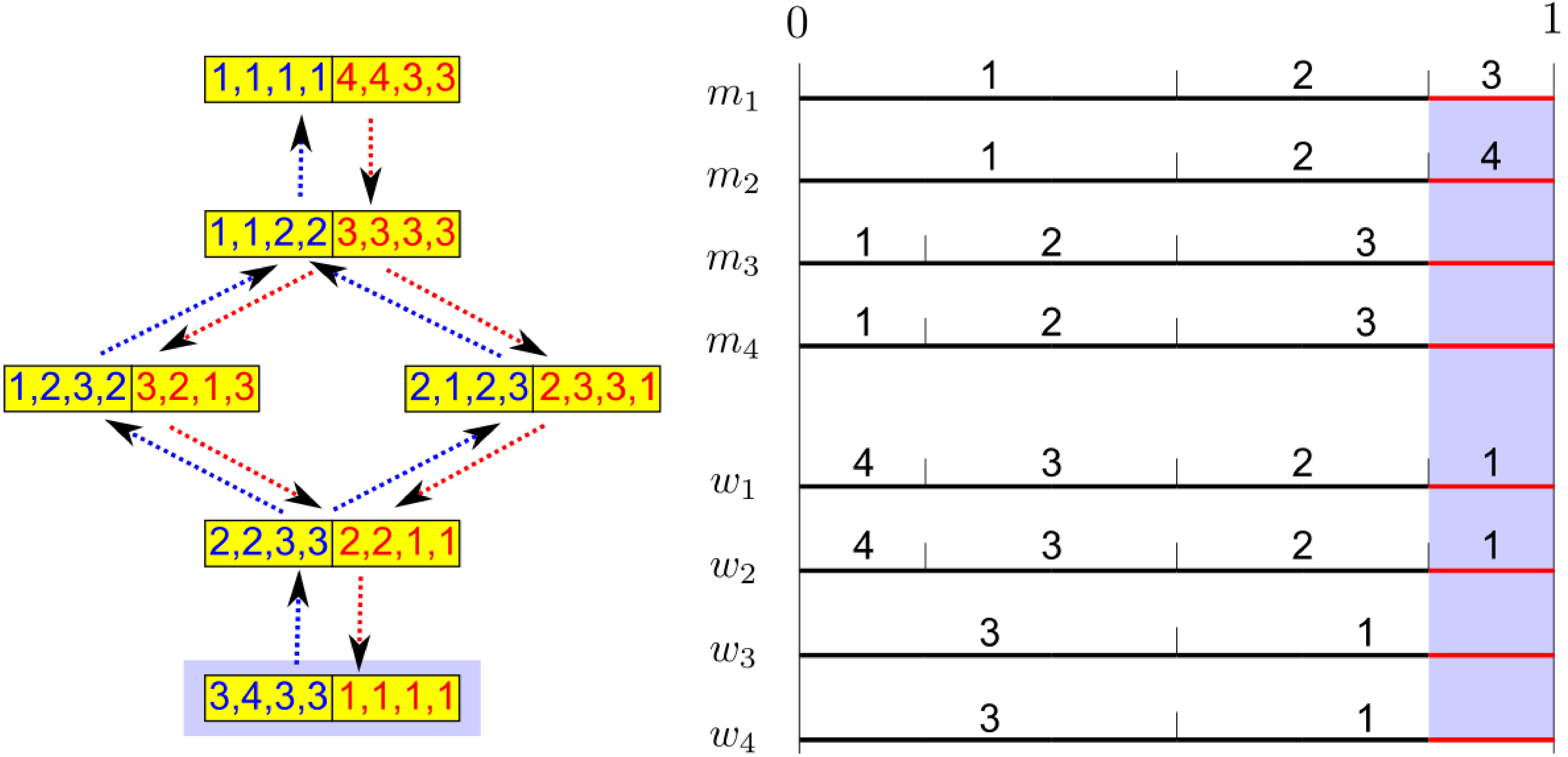

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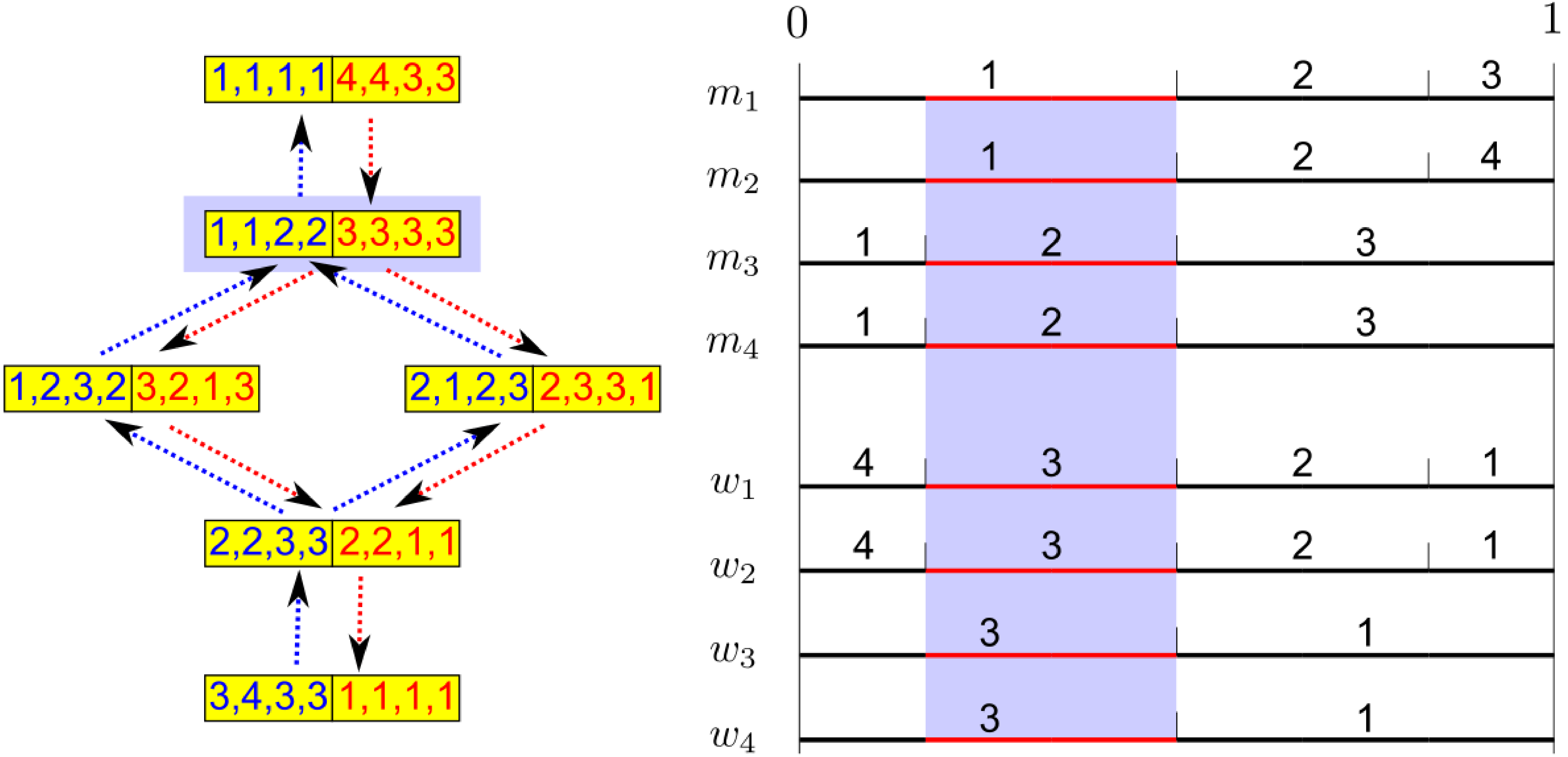

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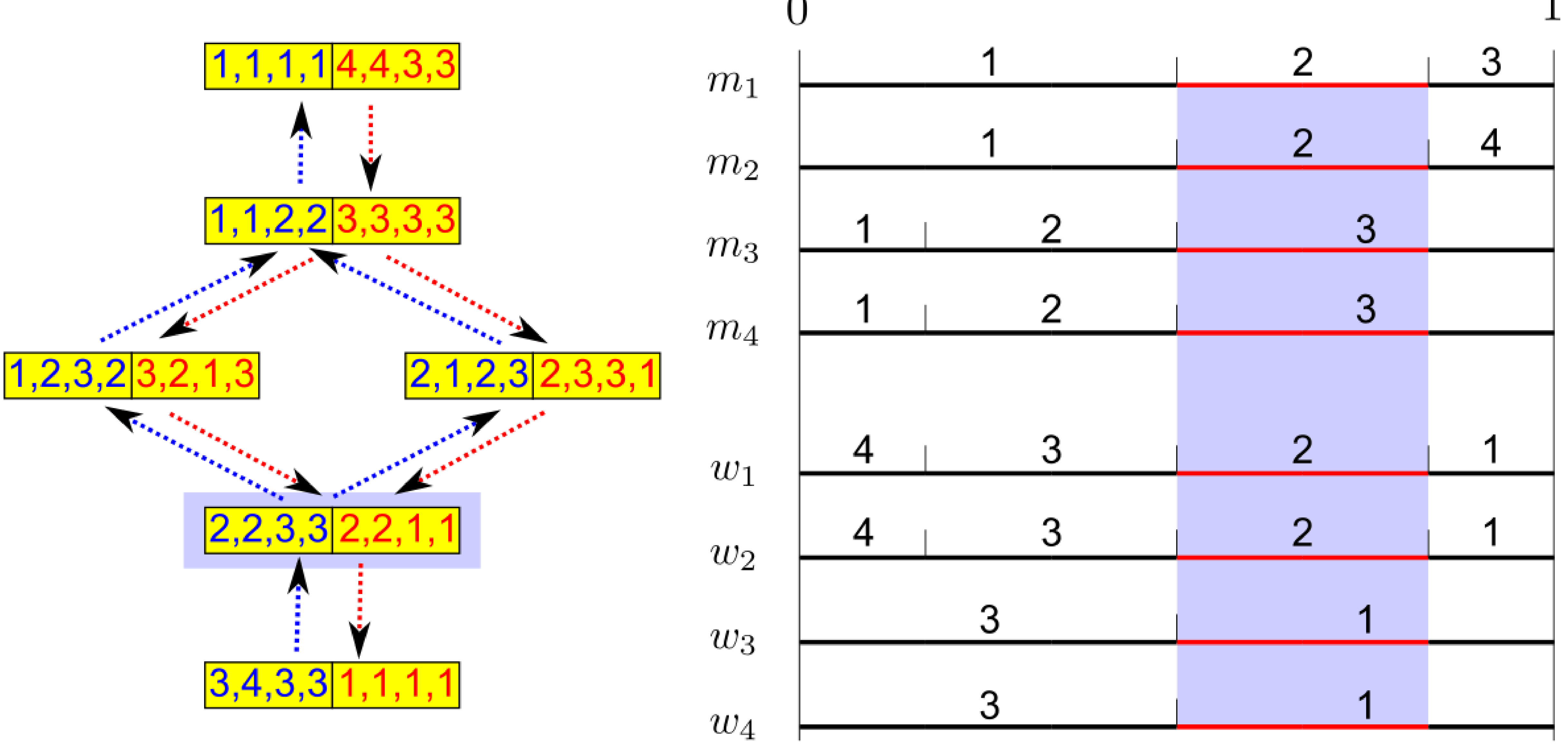

Bad News

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[Cheng, *Algorithmica* 2010]

Computing a median stable matching is #P-hard.

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as hard as...

- computing the permanent of a 0-1 matrix
- counting the number of perfect matchings in a bipartite graph
- counting the number of stable matchings for a given instance
- and many others...

Other Notions of Fairness

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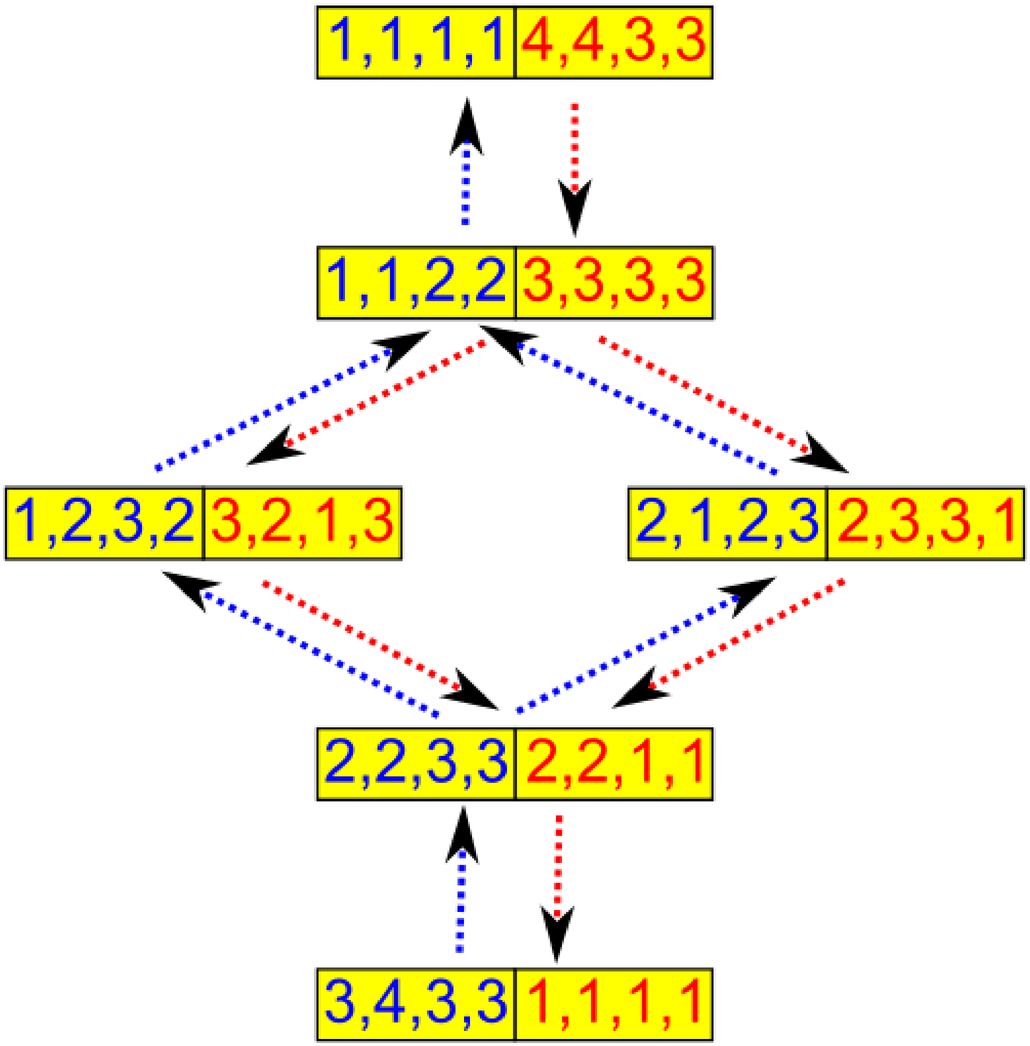

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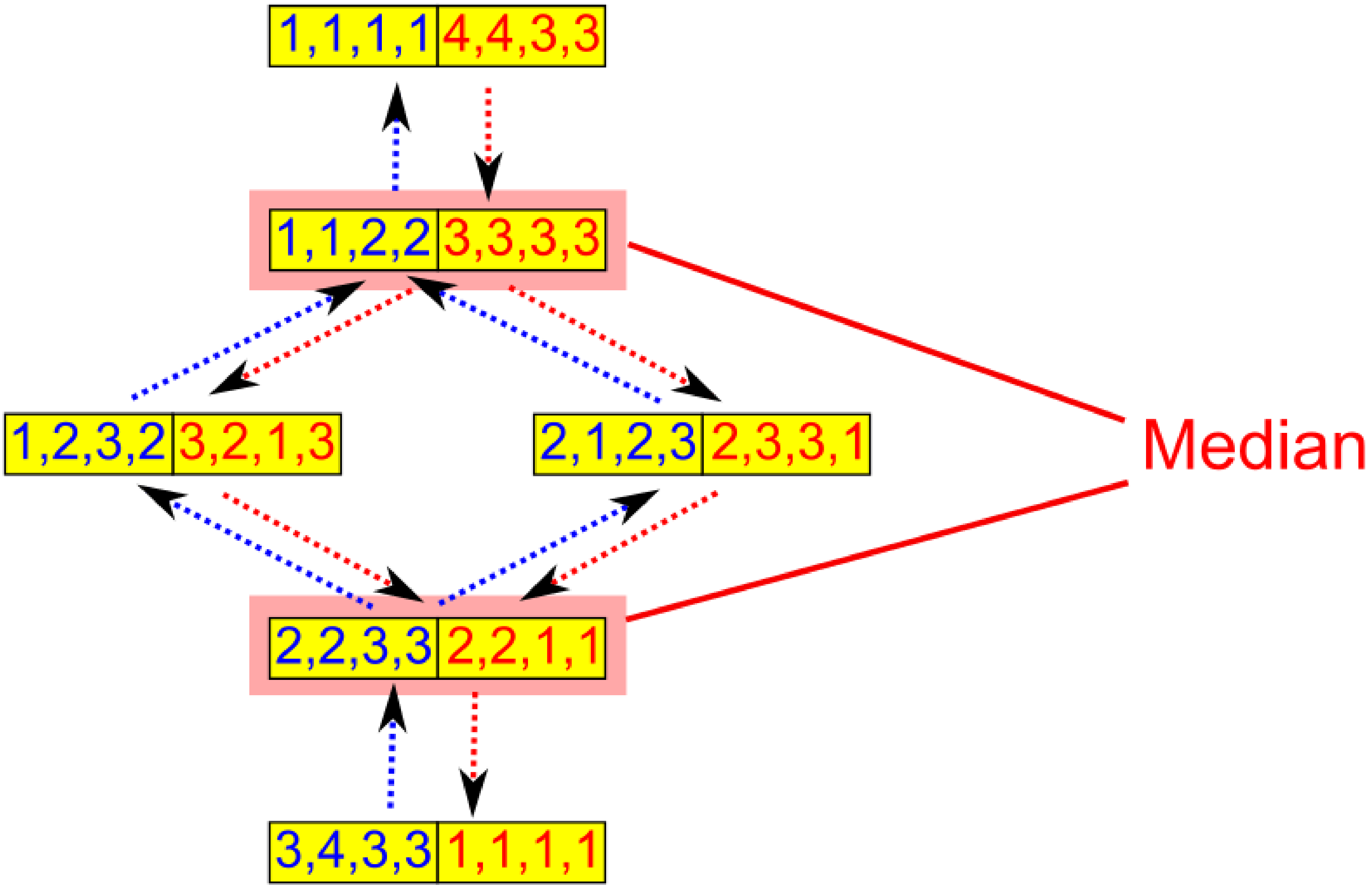
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Median
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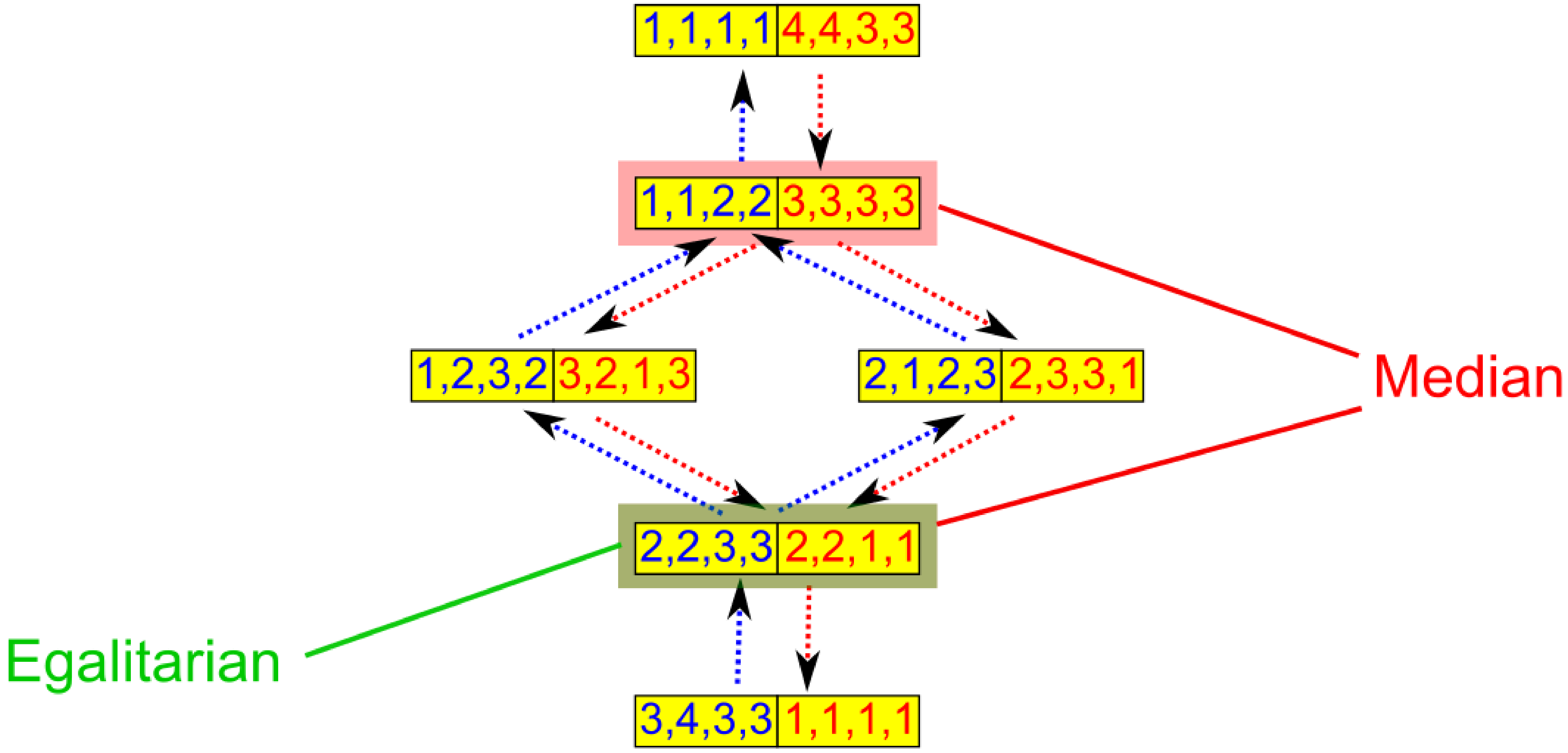
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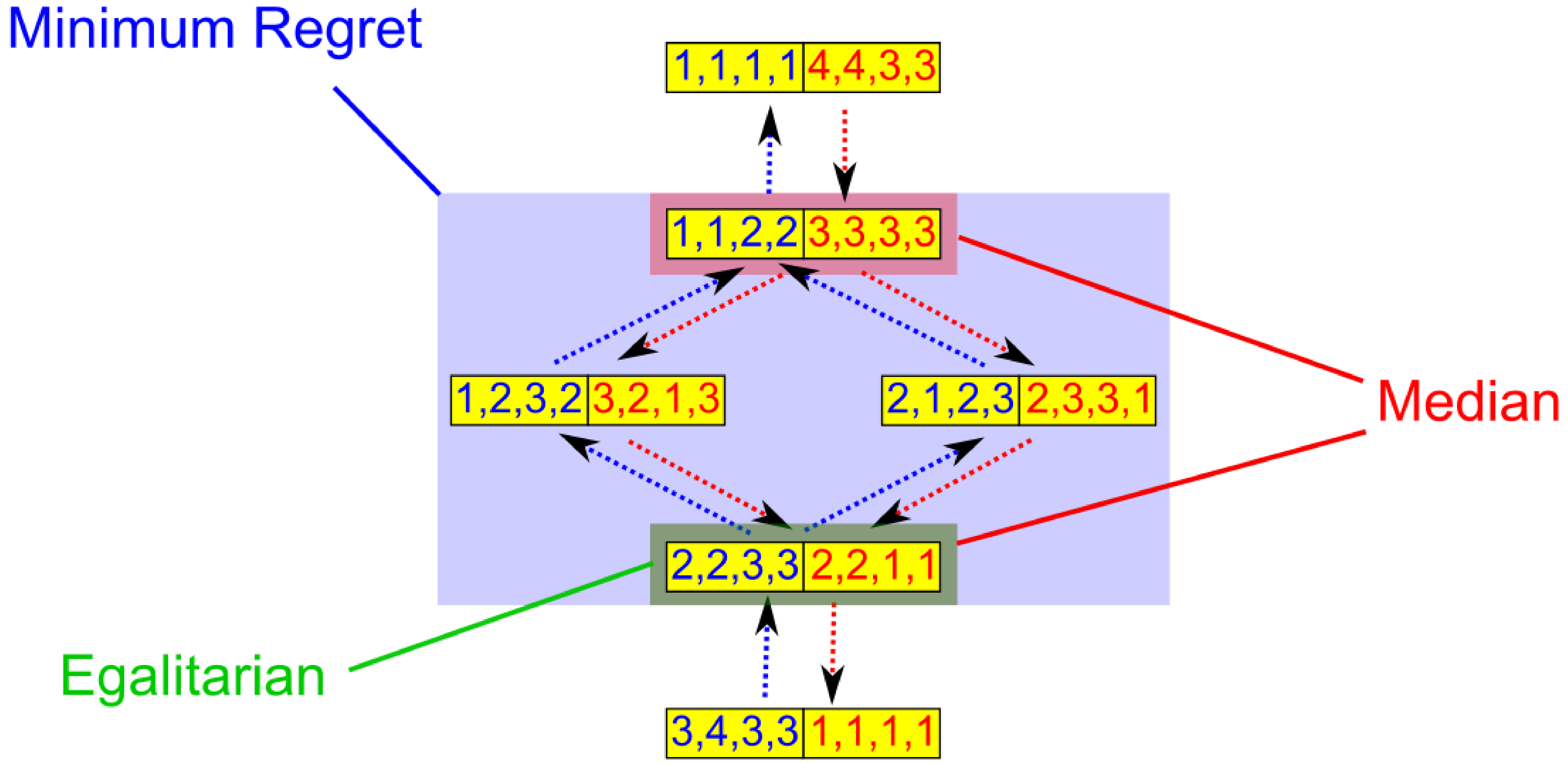

Minimum Regret
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1,1,2,2 3,3,3,3
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2,1,2,3 2,3,3,1
Median
2,2,3,3 2,2,1,1
Egalitarian
3,4,3,3 1,1,1,1

Quiz

Quiz

Use the geometric method to write the following fractional stable matching as a convex combination of integral stable matchings:

$m_1: w_1 > w_2 > w_3$ $\quad$ $w_1: m_3 > m_2 > m_1$

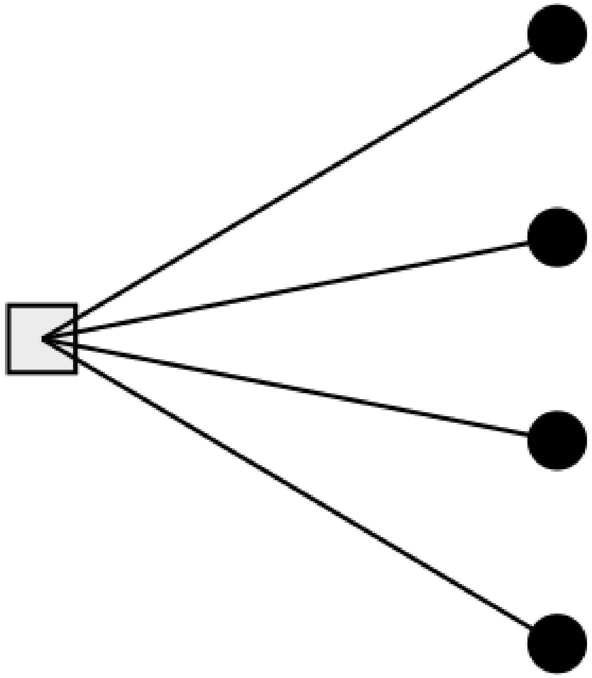
$m_2: w_2 > w_1 > w_3$ $\quad$ $w_2: m_3 > m_1 > m_2$

$m_3: w_3 > w_1 > w_2$ $\quad$ $w_3: m_1 > m_2 > m_3$

	w_1	w_2	w_3
m_1	1/2	1/2	0
m_2	1/6	1/2	1/3
m_3	1/3	0	2/3

Next Time

Many-To-One Matchings



References

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https://www.optimalmatching.com/MATCHUP2015/slides/ChristineCheng.pdf

- Linear programming-based formulation of the stable matching problem

John Vande Vate
“Linear Programming Brings Marital Bliss”
Operations Research Letters, 8(3), 1989 pg 147-153

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• A median stable matching always exists.

Chung-Piaw Teo and Jay Sethuraman
"The Geometry of Fractional Stable Matchings and Its Applications"
Mathematics of Operations Research, 23(4), 1998 pg 874-891

• Computing a median stable matching is #P-hard.

Christine Cheng
"Understanding the Generalized Median Stable Matchings"
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- An algorithm for computing an egalitarian stable matching.

Robert Irving, Paul Leather, and Dan Gusfield
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