COL866: Special Topics in Algorithms

Lecture 16

Manipulation in Voting

Oct 13, 2023 | Rohit Vaish

VOTING RULE

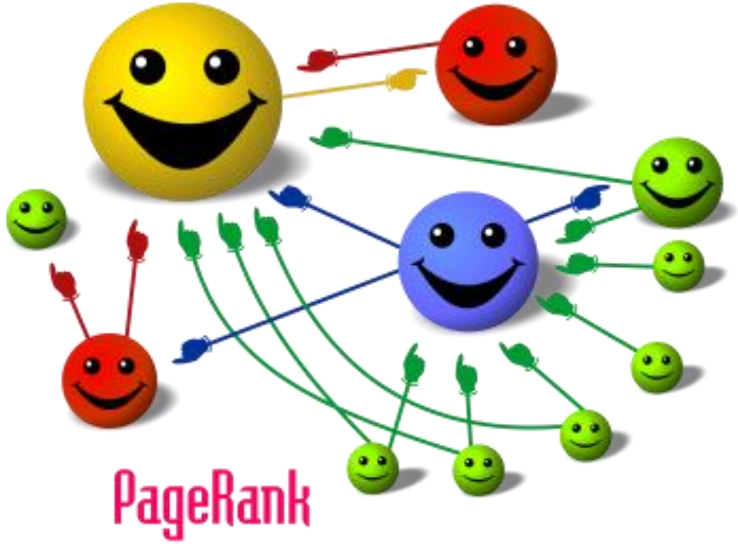
A mapping from preference profiles to candidates.

$$f: \begin{array}{cccc} v_1 & v_2 & & v_n \\ a & b & & c \\ b & c & \cdots & a \\ c & a & & b \end{array} \longrightarrow c$$

(also known as a *social choice function*)

Score-based	Runoff-based	Head-to-head election based
Plurality	Plurality with Runoff	Copeland
Borda Count	Single Transferable Vote	Schulze

Voting Rules are Everywhere!



IEEE

AMS
American Mathematical Society


Bucklin
Count
Voting
Random
Plurality
Minimax
Kemeny-Young
Winner
Copeland
Pairs
Ranked
Borda
Majority-Judgement
Range
Schulze
Approval

One Rule to Rule Them All?

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Axiomatic Approach

- Formulate a set of "reasonable" properties
- Check if there is a voting rule that satisfies them

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Axiomatic Approach

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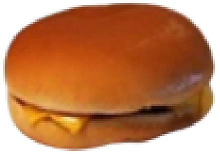
Expectation

Unique voting rule with all the desirable properties

Reality

No such voting rule



ONTO

For any candidate "a", there exists a profile where "a" wins.

$$f\left(\begin{array}{cccc} v_1 & v_2 & \cdots & v_n \\ \end{array}\right) = a$$

STRATEGYPROOF

No voter can improve by misreporting its preferences.

For any profile , any voter v_i, and any misreport , it must be that

An Obviously Bad Voting Rule

A voting rule is called a dictatorship if there exists a voter v_i such that for any preferences of the other voters, the voting outcome is the favorite candidate of voter v_i.

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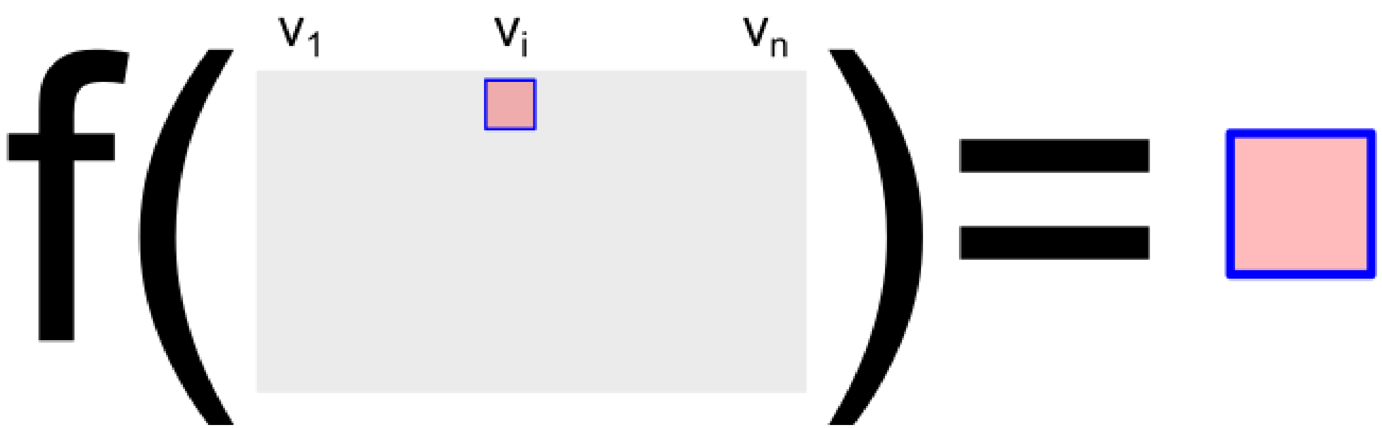

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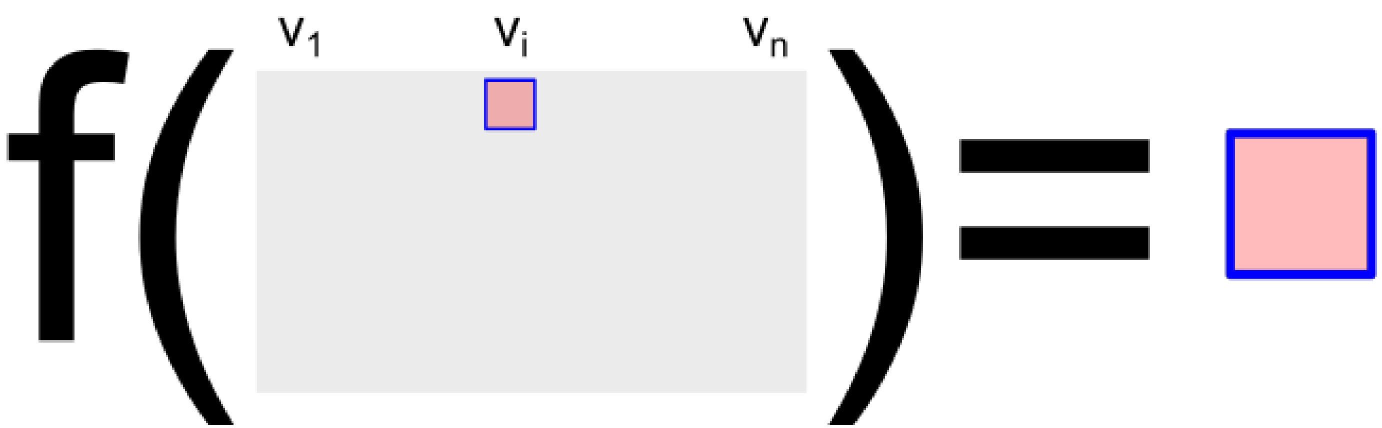

A dictatorship is onto and strategyproof.

[Gibbard'73; Satterthwaite'75]

With three or more candidates,
a voting rule is onto and strategyproof
if and only if it is a dictatorship.

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Quiz!

[Gibbard'73; Satterthwaite'75]

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Onto + strategyproof but two candidates:

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Onto + strategyproof but two candidates:

≥3 candidates + strategyproof but not onto:

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Onto + strategyproof but two candidates: Majority

≥3 candidates + strategyproof but not onto:

≥3 candidates + onto but not strategyproof:

[Gibbard'73; Satterthwaite'75]

With three or more candidates,
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Onto + strategyproof but two candidates:	Majority
≥3 candidates + strategyproof but not onto:	Constant/Restricted Majority
≥3 candidates + onto but not strategyproof:	

[Gibbard'73; Satterthwaite'75]

With three or more candidates,
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Onto + strategyproof but two candidates:	Majority
≥3 candidates + strategyproof but not onto:	Constant/Restricted Majority
≥3 candidates + onto but not strategyproof:	Plurality/Borda/...

With three or more candidates,
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↓

With three or more candidates,
a voting rule is unanimous and monotone
if and only if it is a dictatorship.

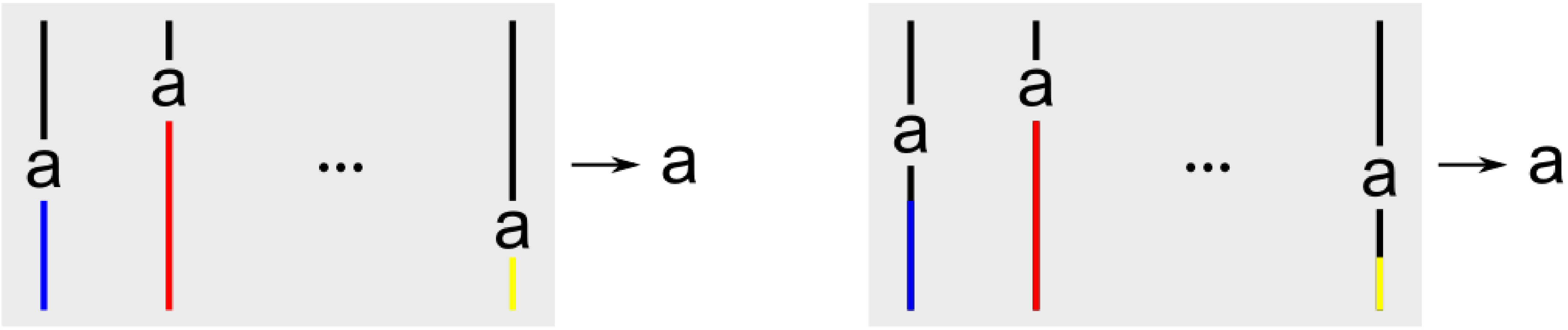
UNANIMOUS

If all voters agree on their top choice, then the voting rule picks it.

$$f\left(\begin{array}{cccc} v_1 & v_2 & & v_n \\ a & a & & a \\ \vdots & \vdots & \cdots & \vdots \end{array}\right)=a$$

MONOTONE

Suppose "a" is the current winner, and for each voter, any candidate that was ranked below "a" in its old vote is ranked below "a" in its new vote, then "a" continues to be the winner.



support of "a"

Strategyproofness = Monotonicity

STRATEGYPROOF ⇒ MON

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Suppose SP but not MON.

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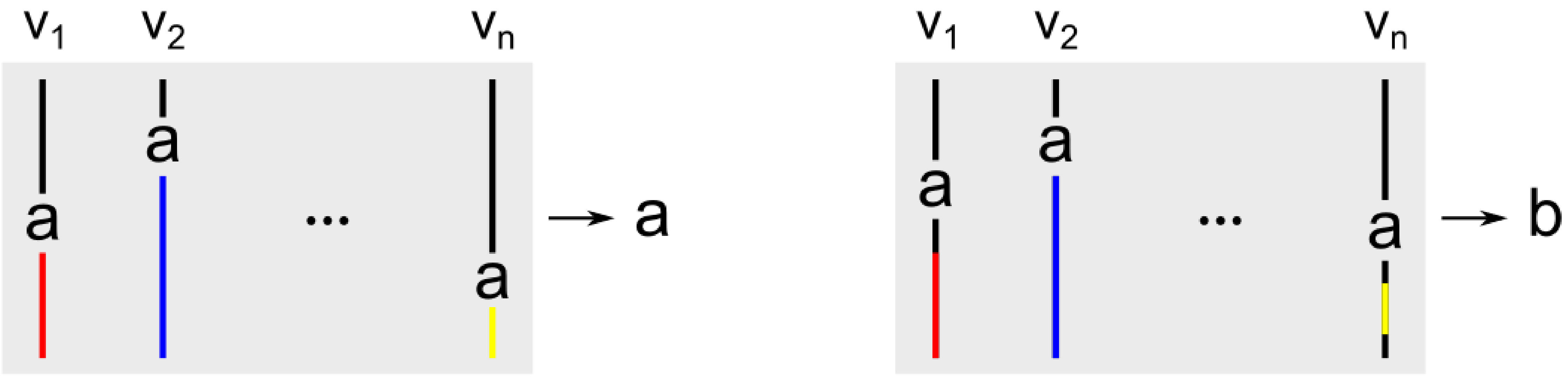

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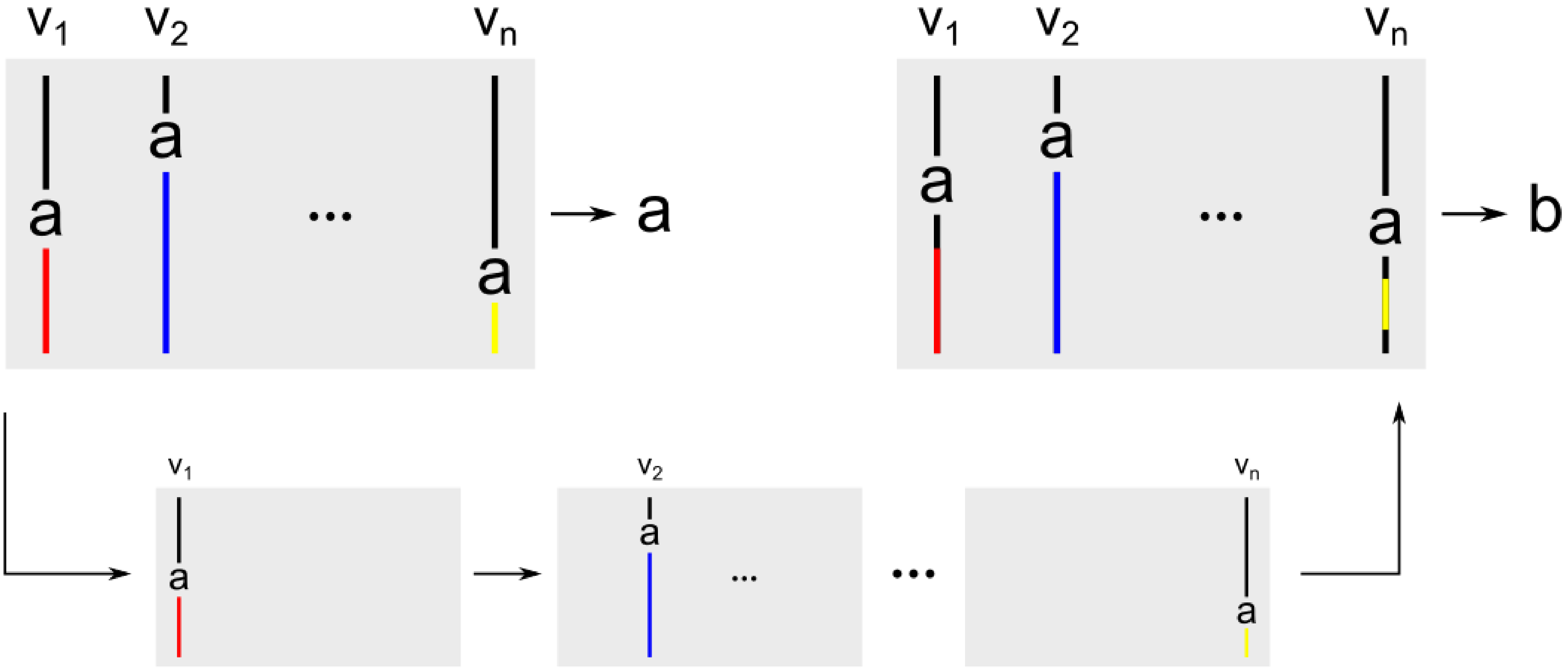

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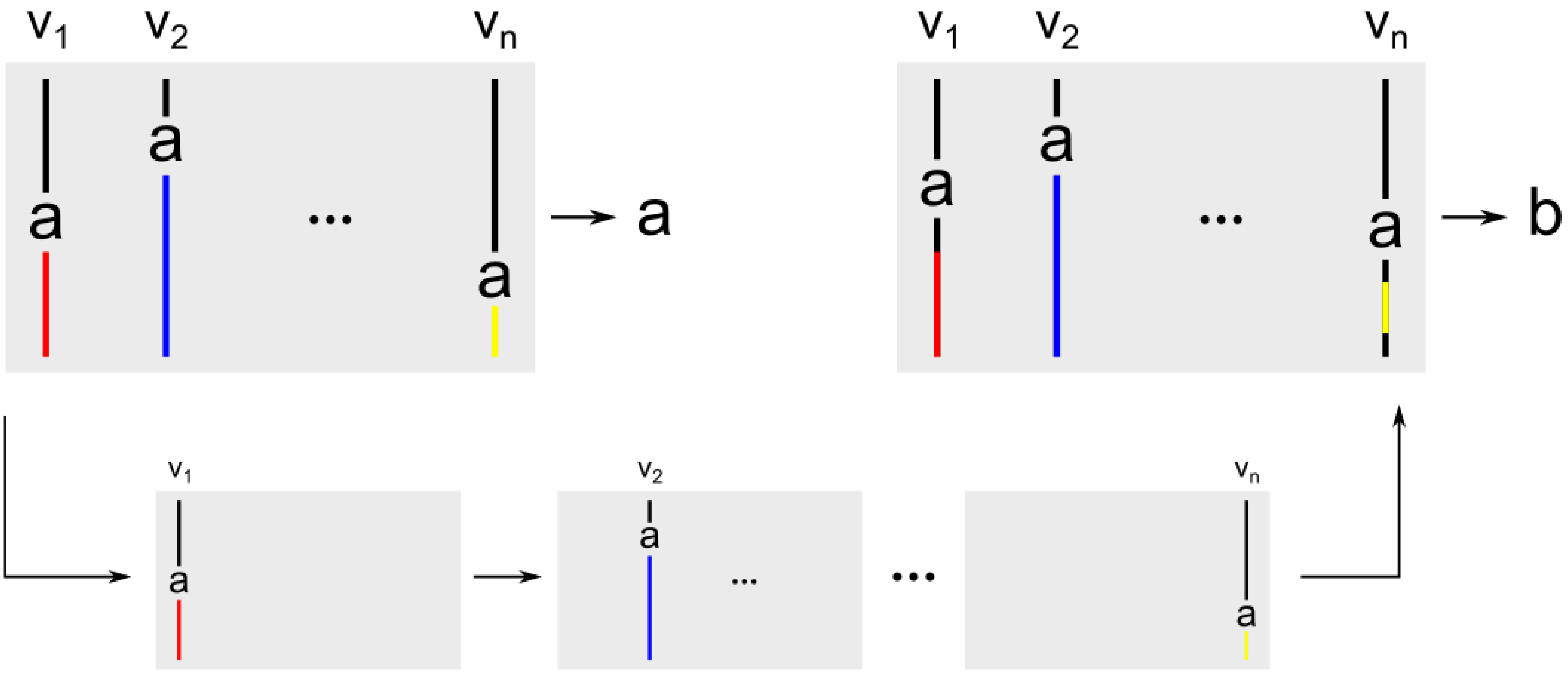
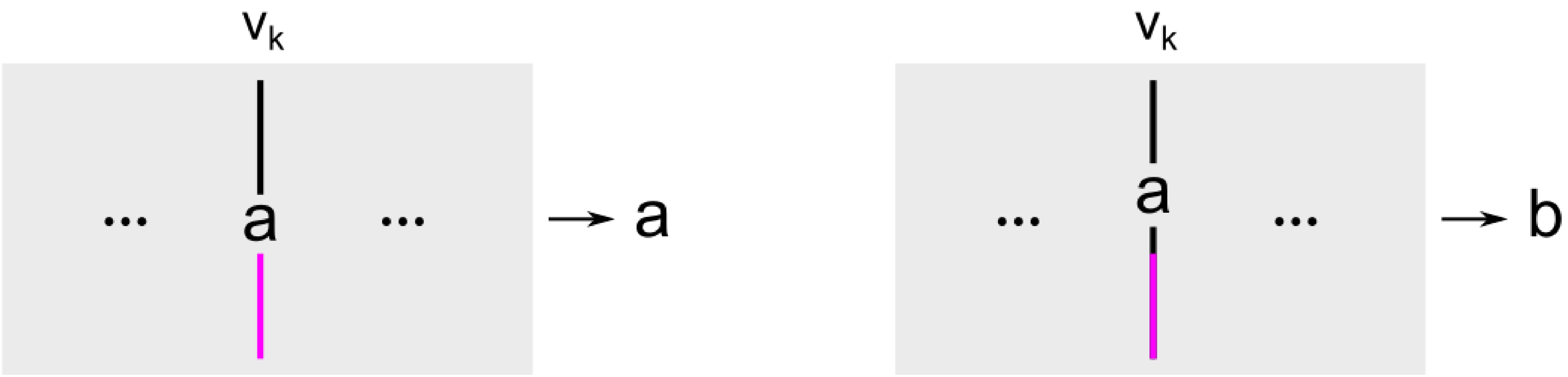


Suppose v_k is the first voter for which the outcome changes.

STRATEGYPROOF ⇒ MON



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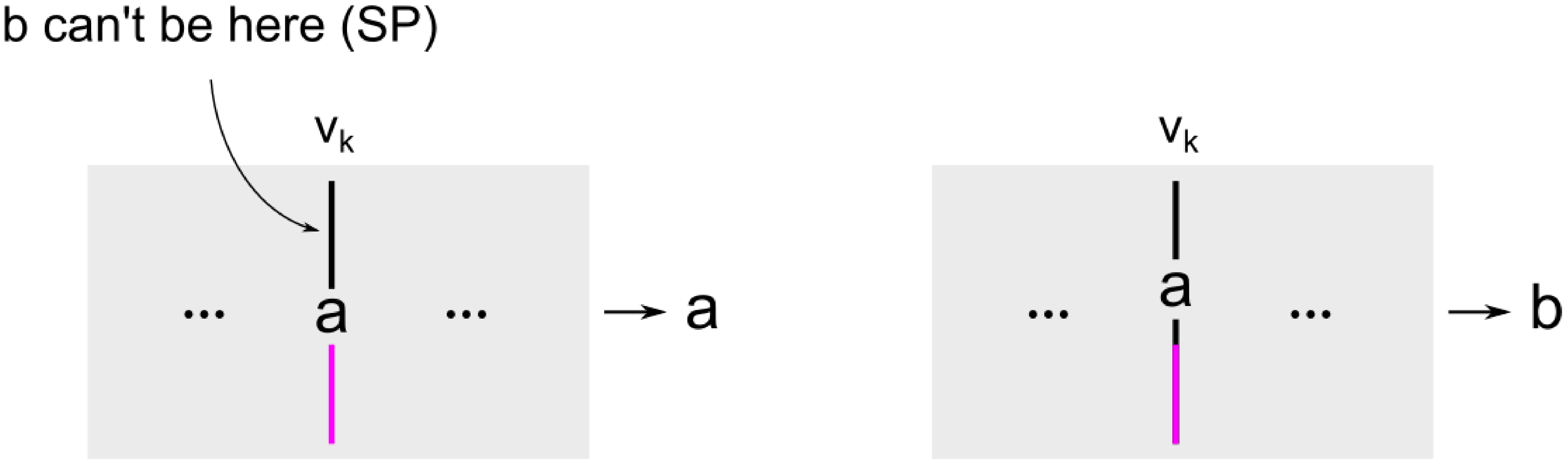

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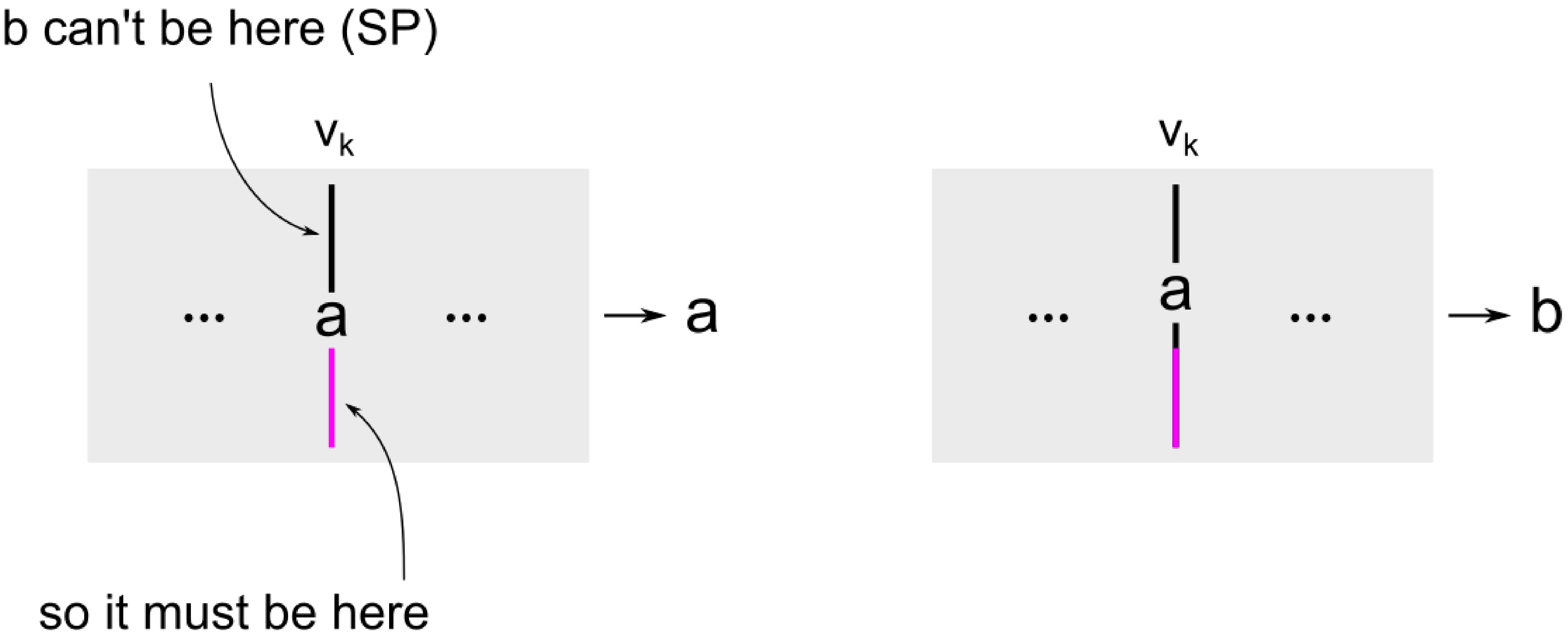

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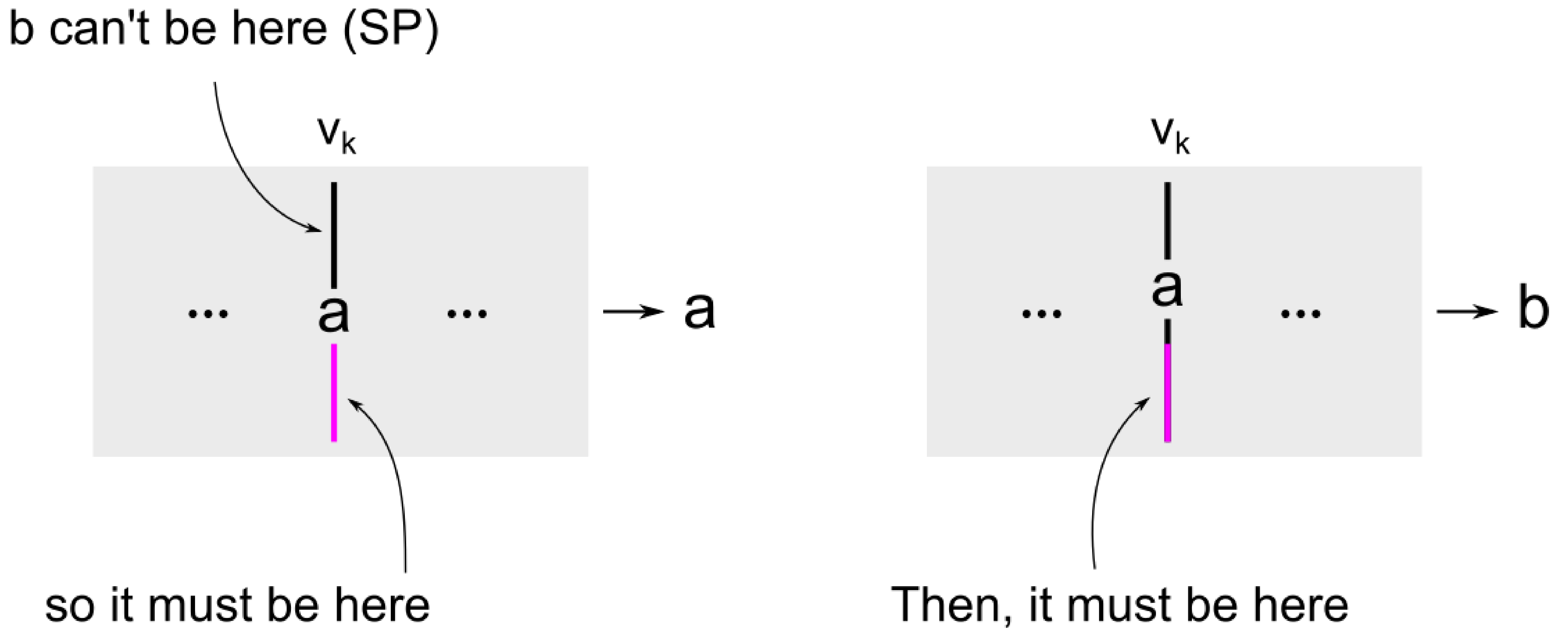

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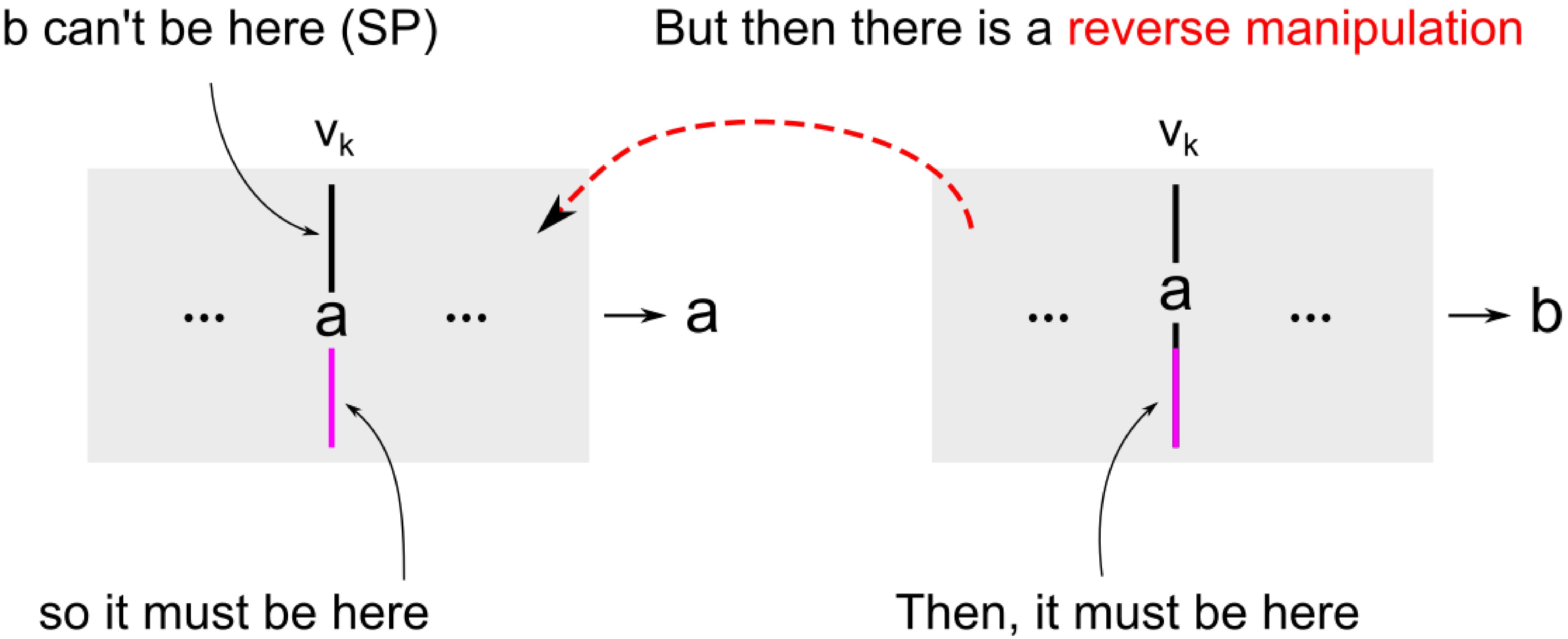
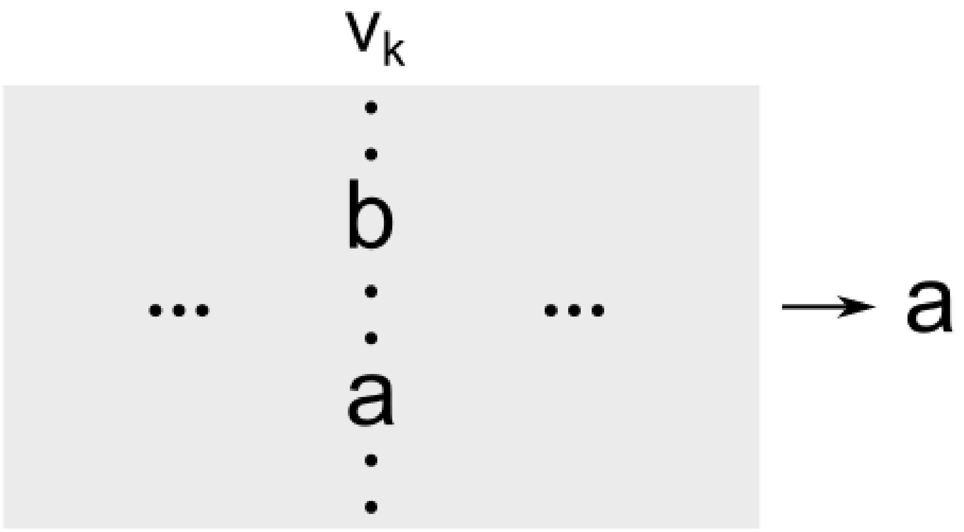
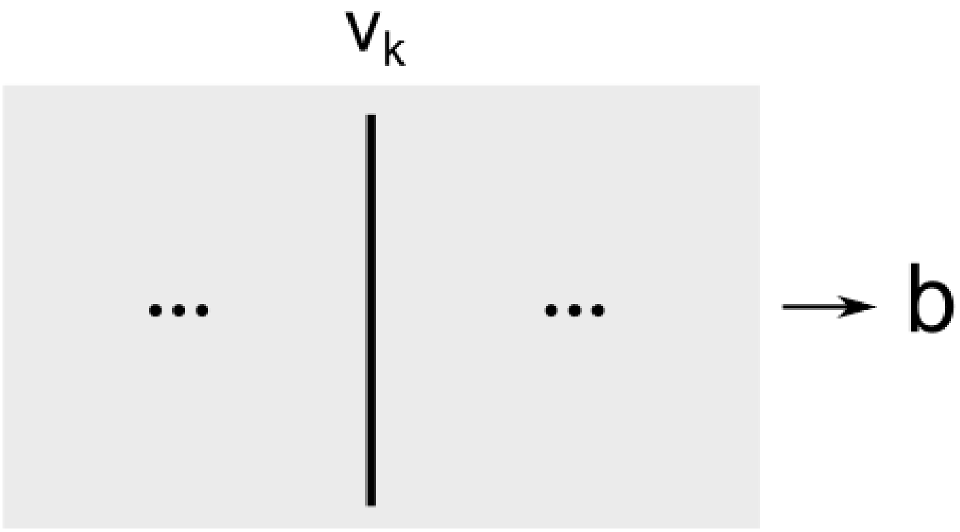


MON $\Rightarrow$ STRATEGYPROOF

MON ⇒ STRATEGYPROOF





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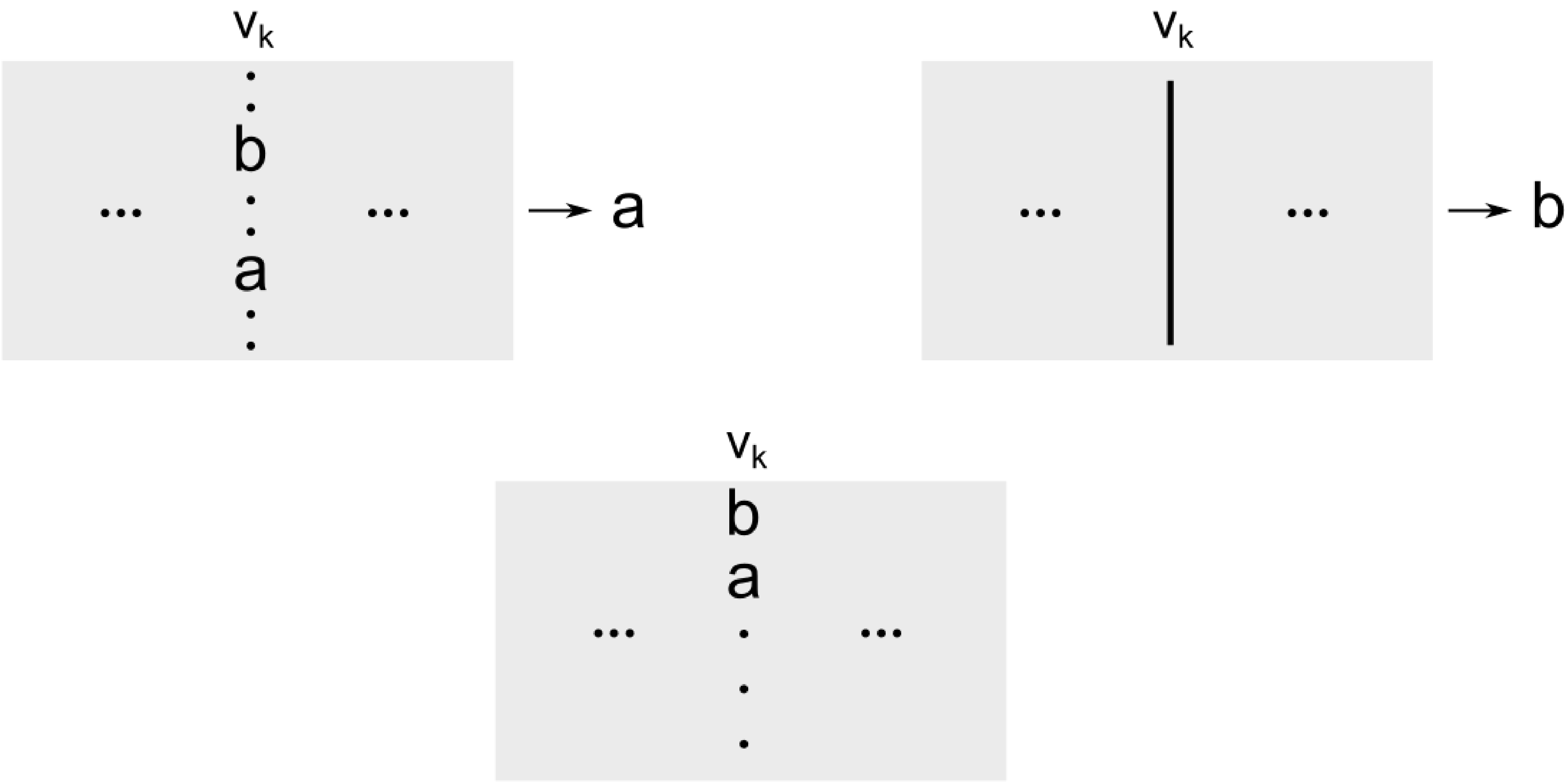

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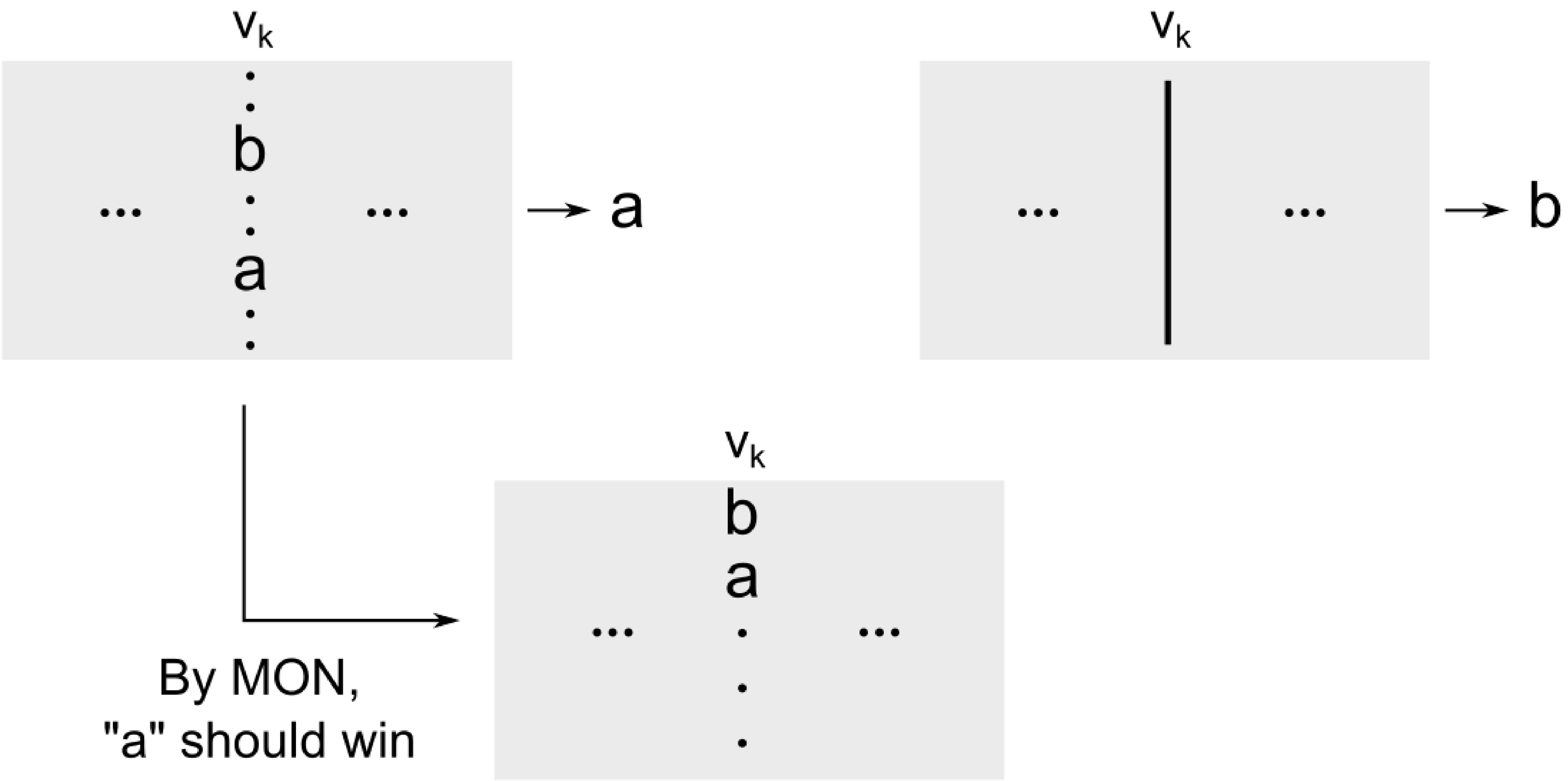

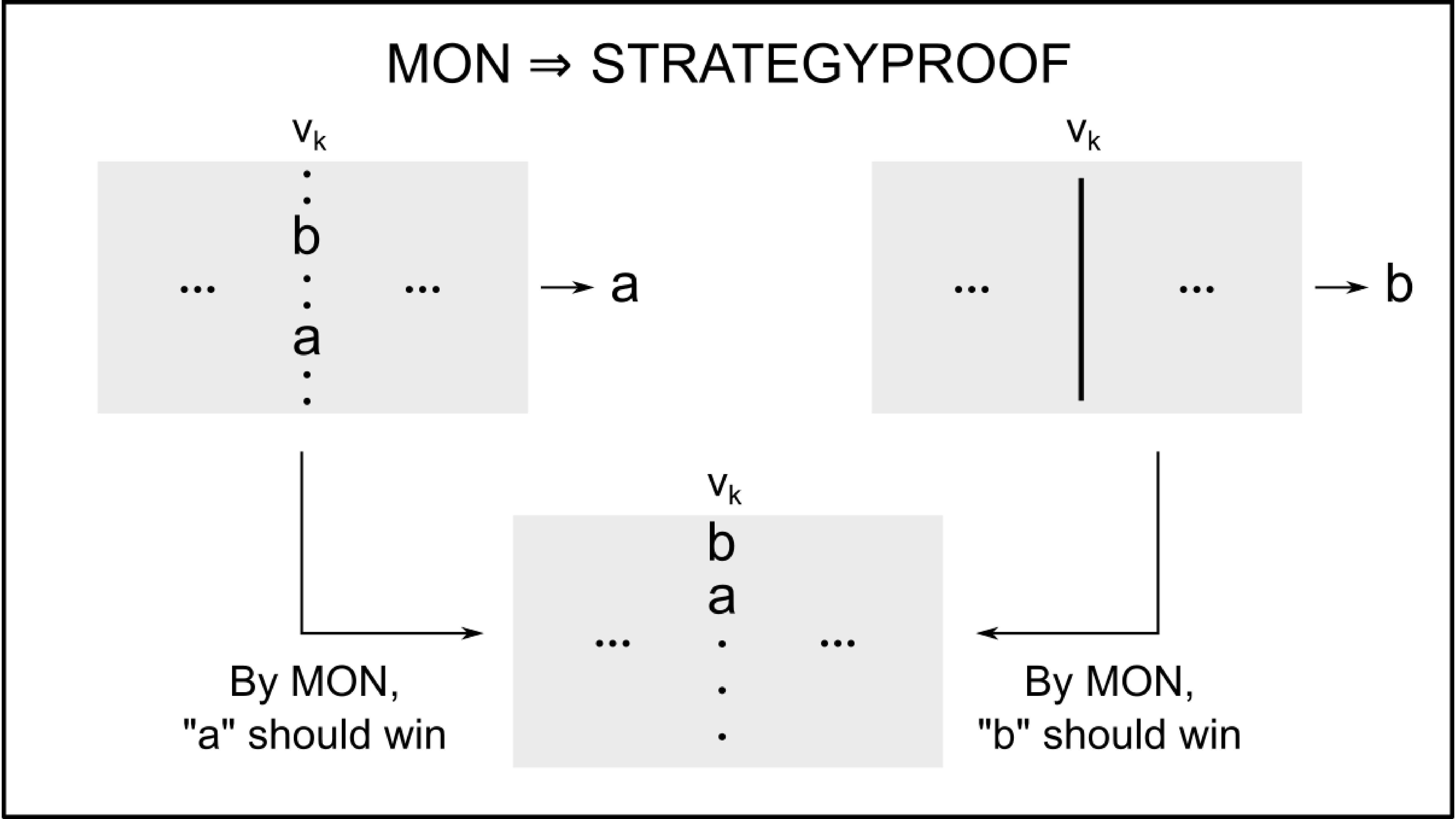

MON ⇒ STRATEGYPROOF
v_k
b
a
→ a
v_k
→ b
v_k
b
a
By MON,
"a" should win
By MON,
"b" should win

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a voting rule is onto and strategyproof
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With three or more candidates,
a voting rule is unanimous and monotone
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MON/SP + ONTO $\Rightarrow$ UNAN

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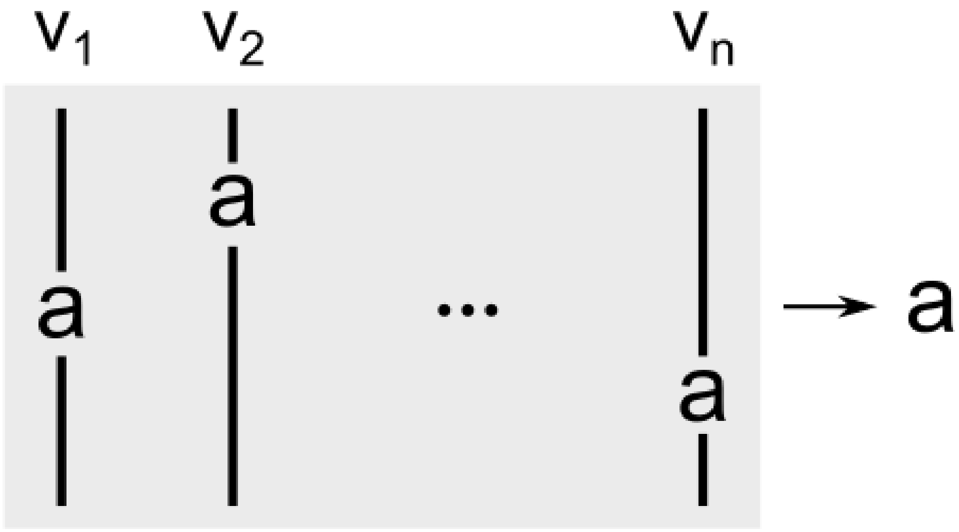
For any candidate "a", by ONTO, there is a profile where "a" wins



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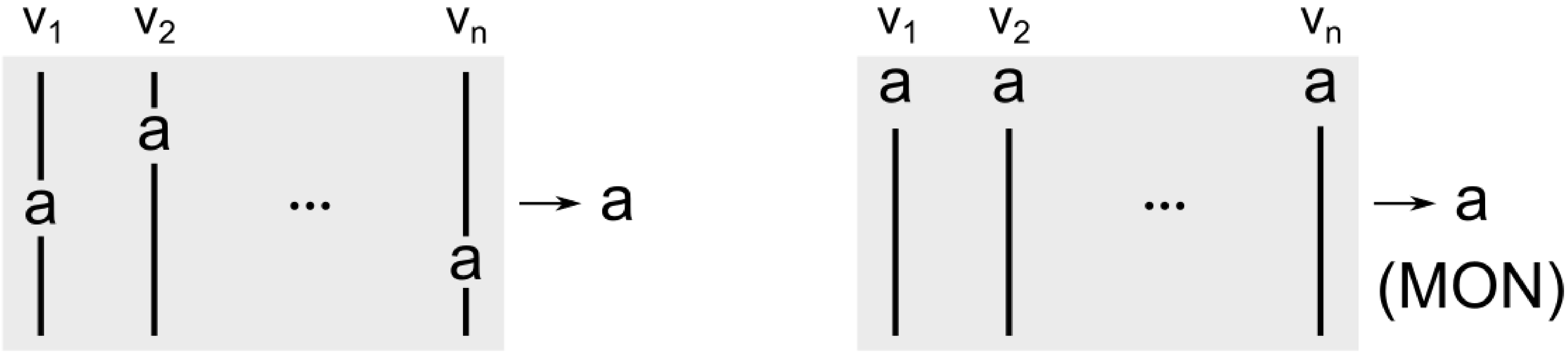

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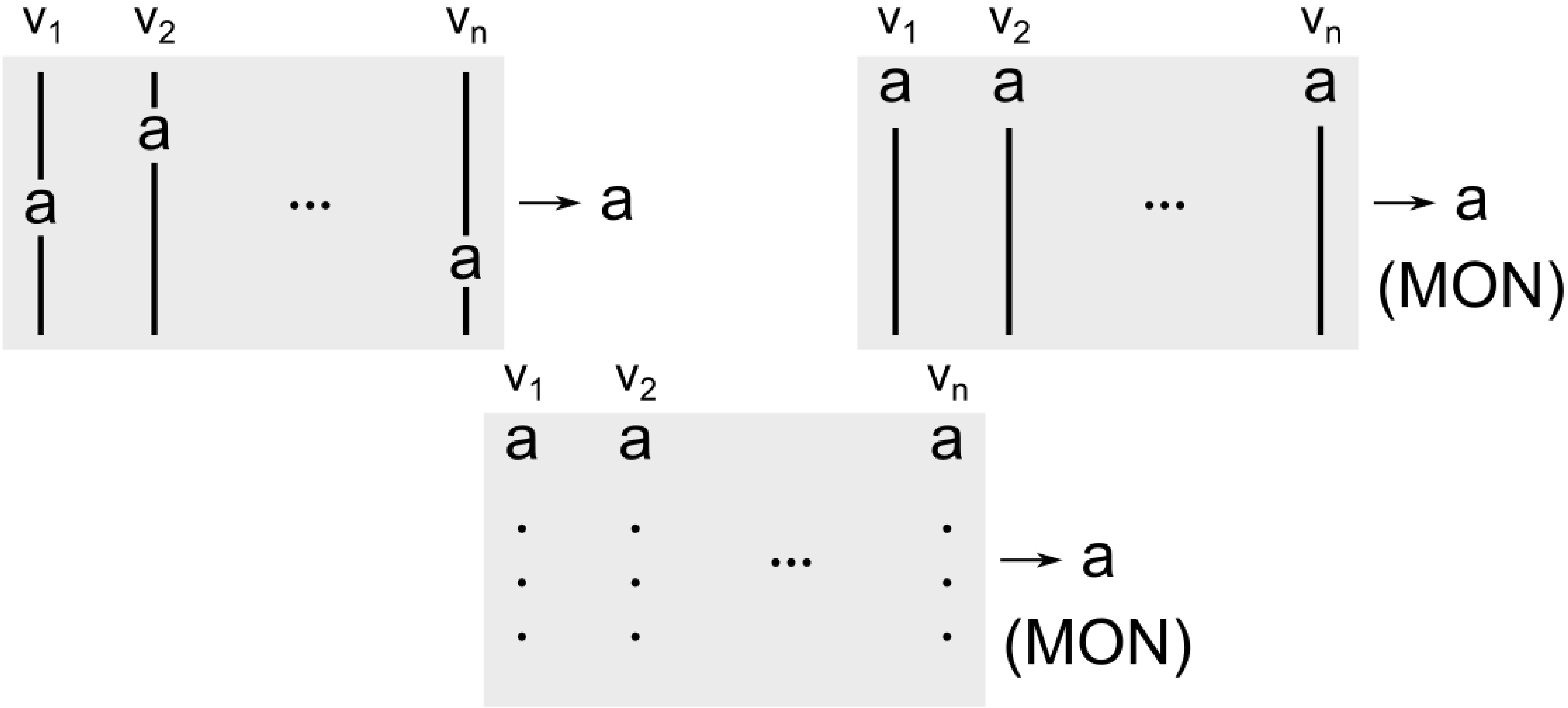

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UNAN + MON $\Rightarrow$ DICTATOR

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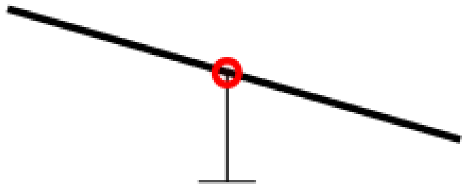
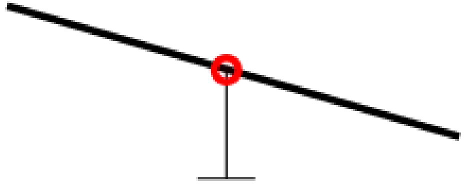
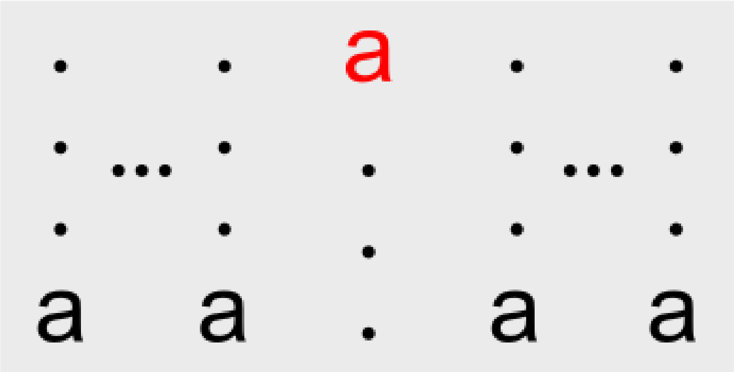


Identify a "pivotal voter" v_p for the pair {a,b}

UNAN + MON ⇒ DICTATOR





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Show that "a" wins if v_p likes it even if everyone else ranks it last

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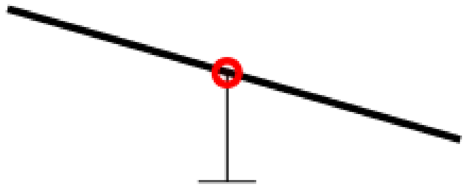

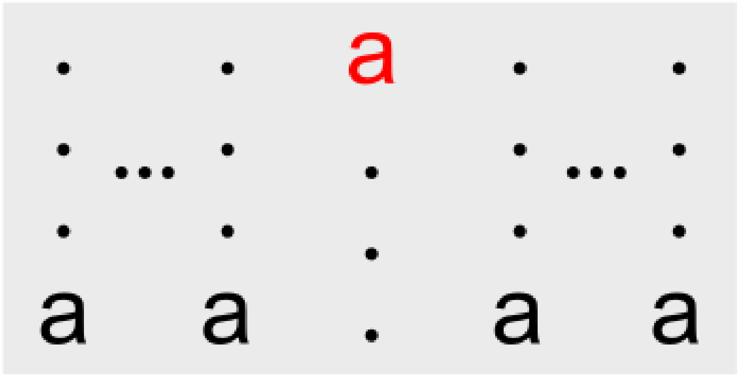

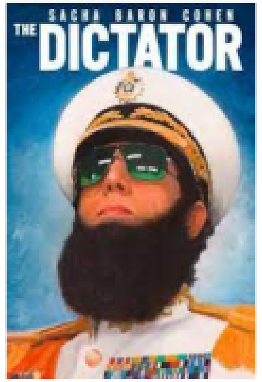

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v_p is the dictator for "a" (and every other candidate)

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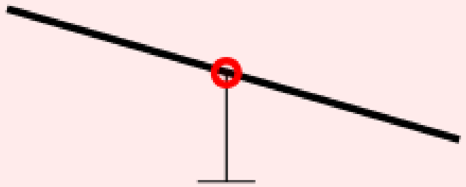

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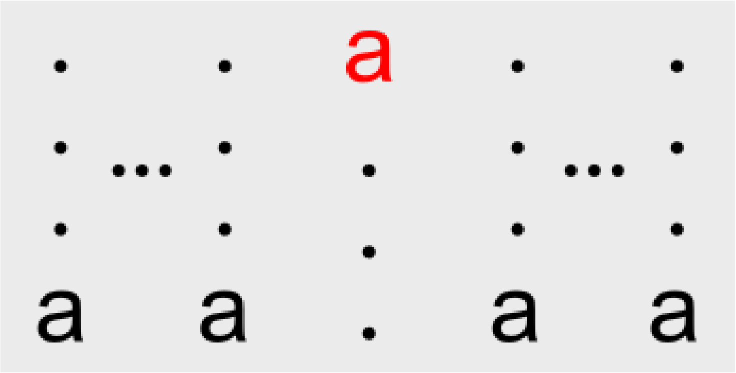

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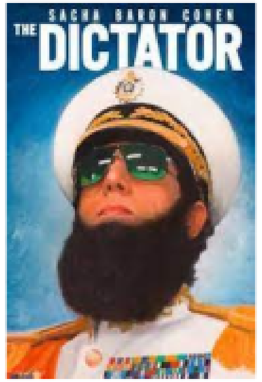

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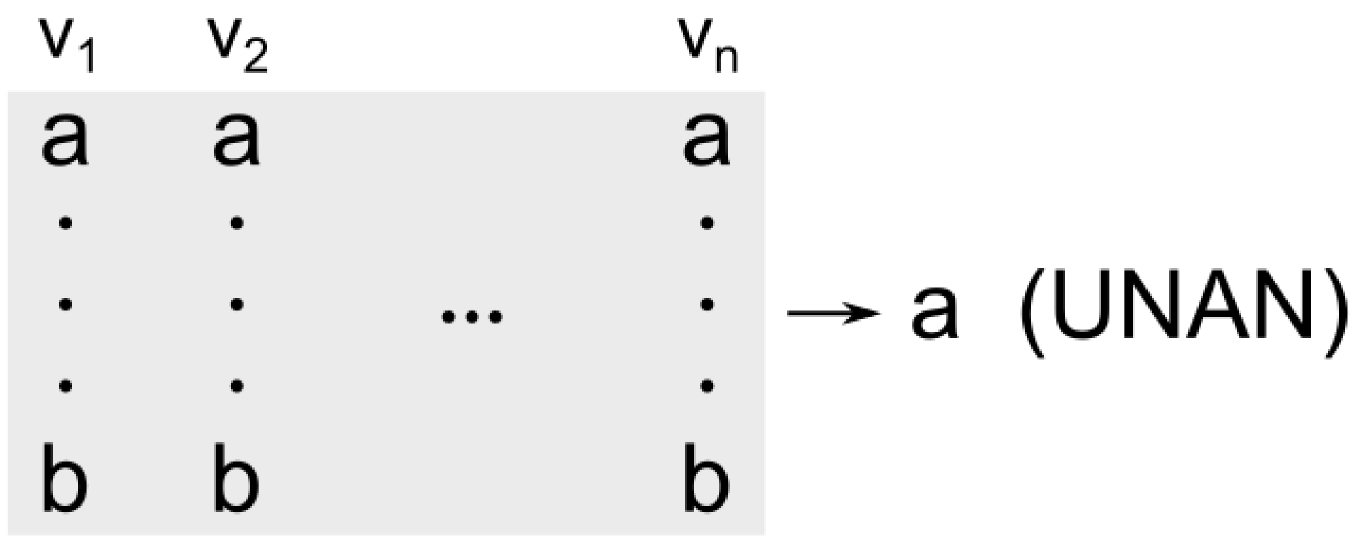
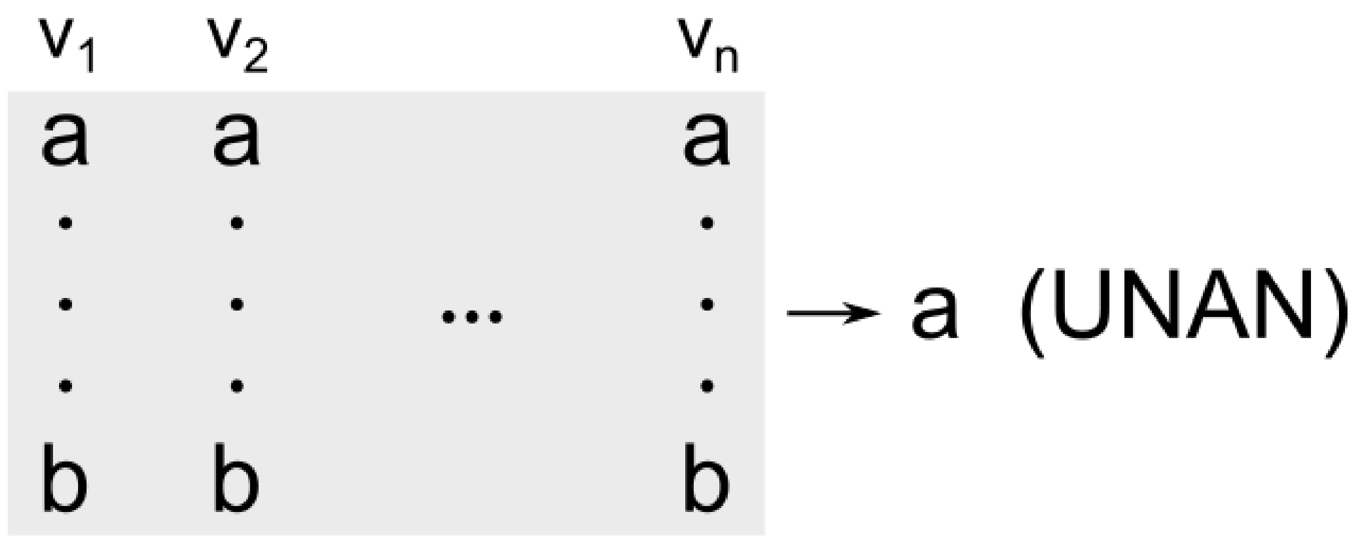
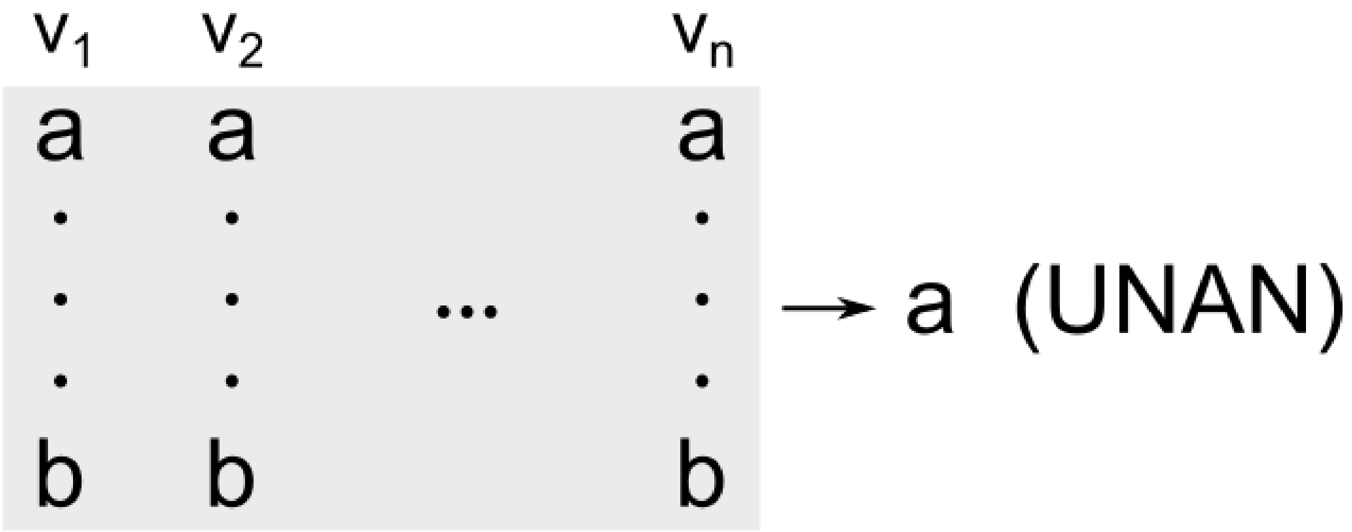
v_1	v_2		v_n
a	a		a
⋮	⋮	…	⋮
b	b		b


v1 v2 vn
a a a
b b b
→ a (UNAN)



Gradually promote "b" in each voter's list until it wins



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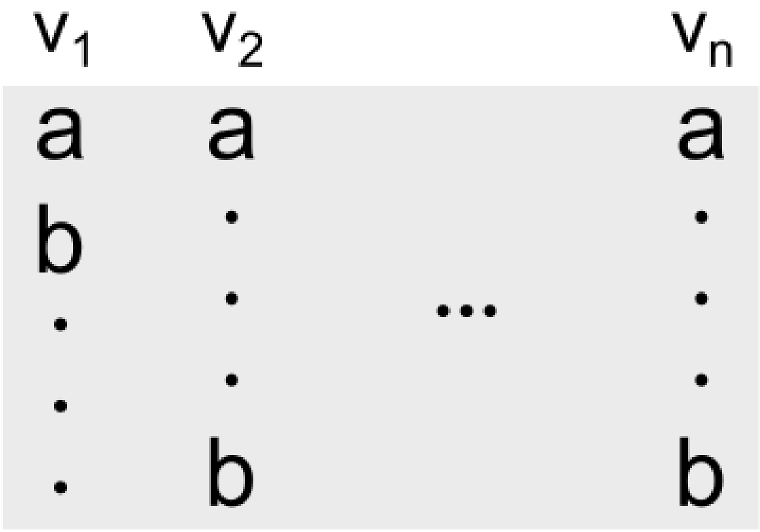

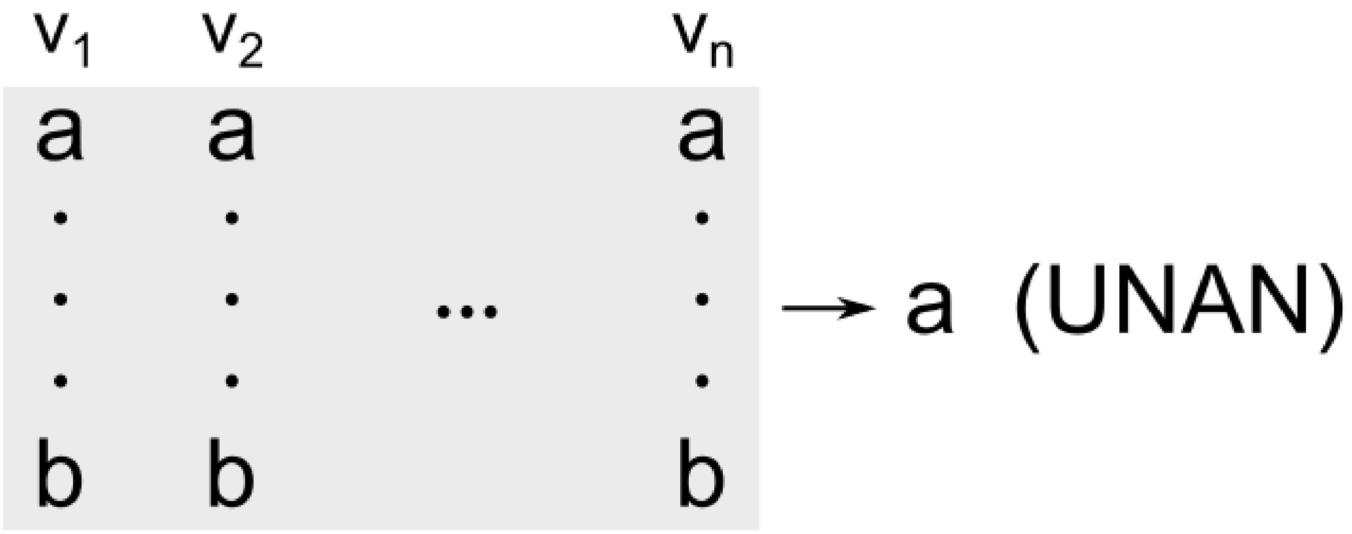

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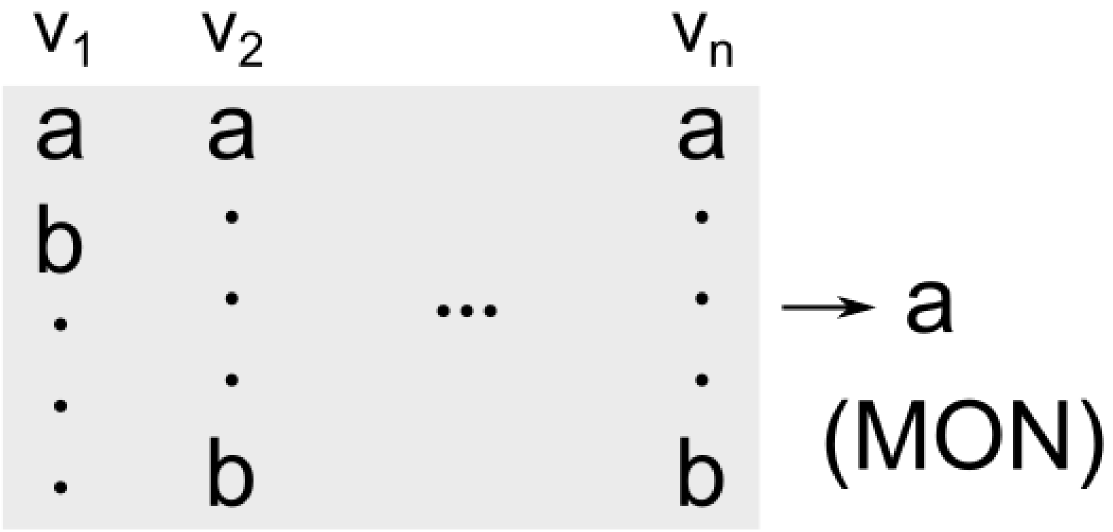

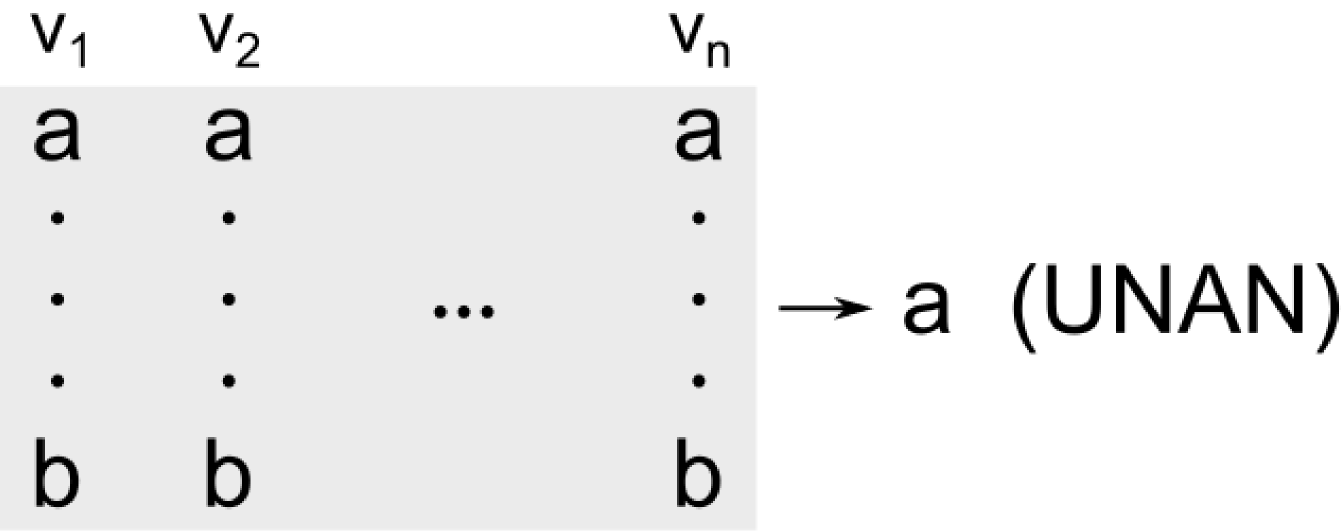

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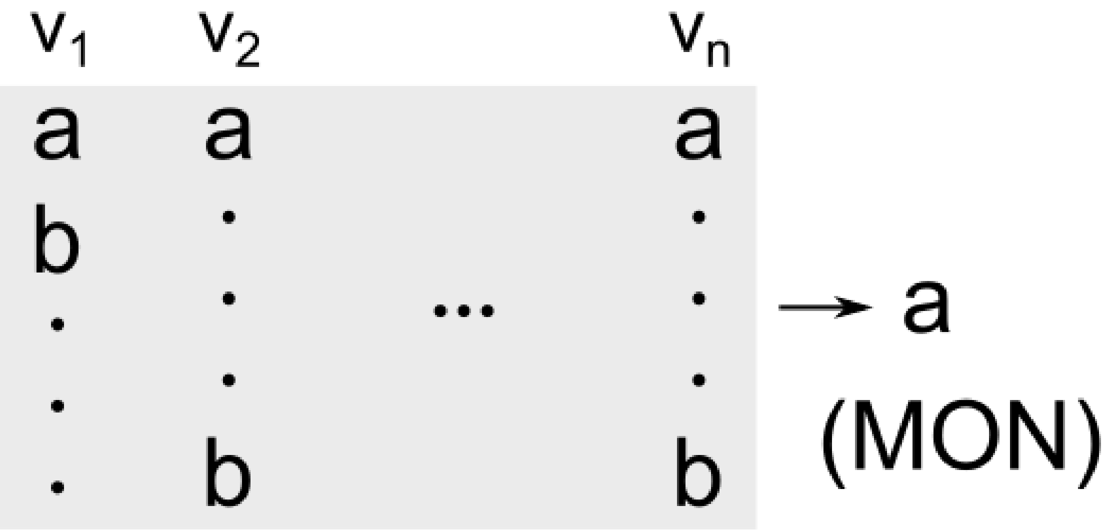

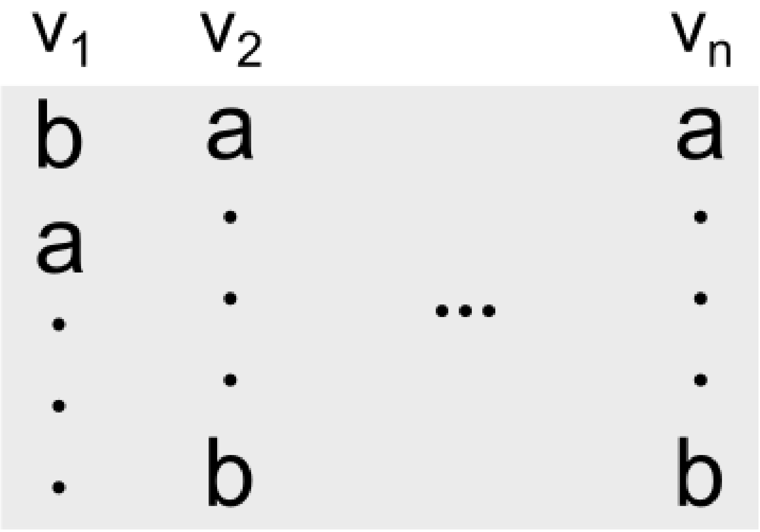

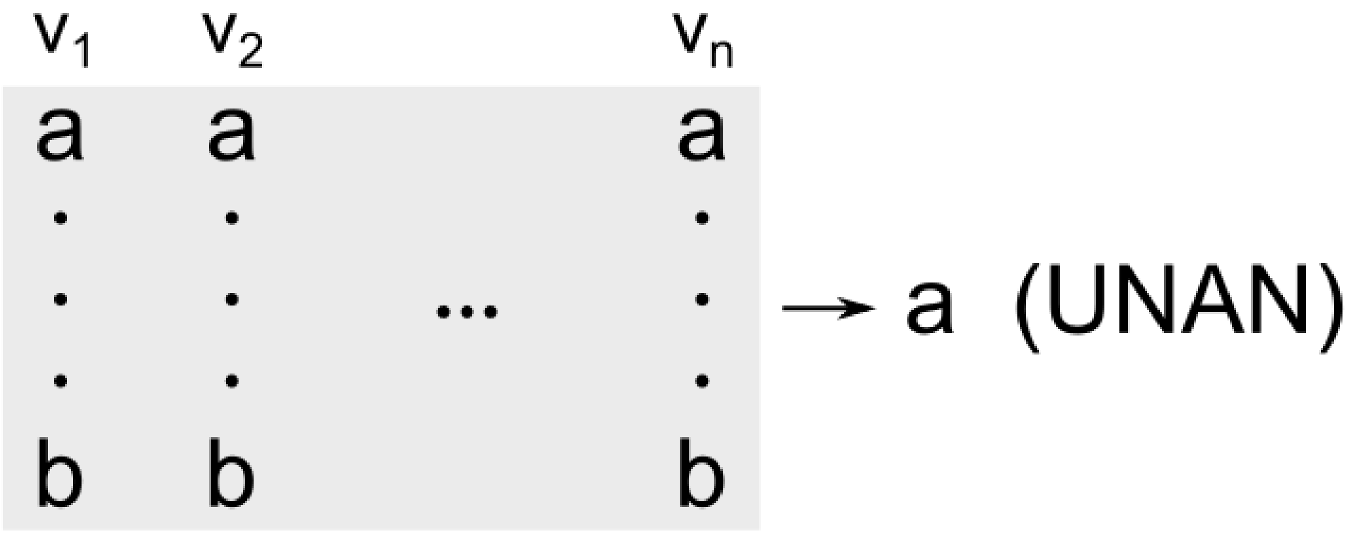

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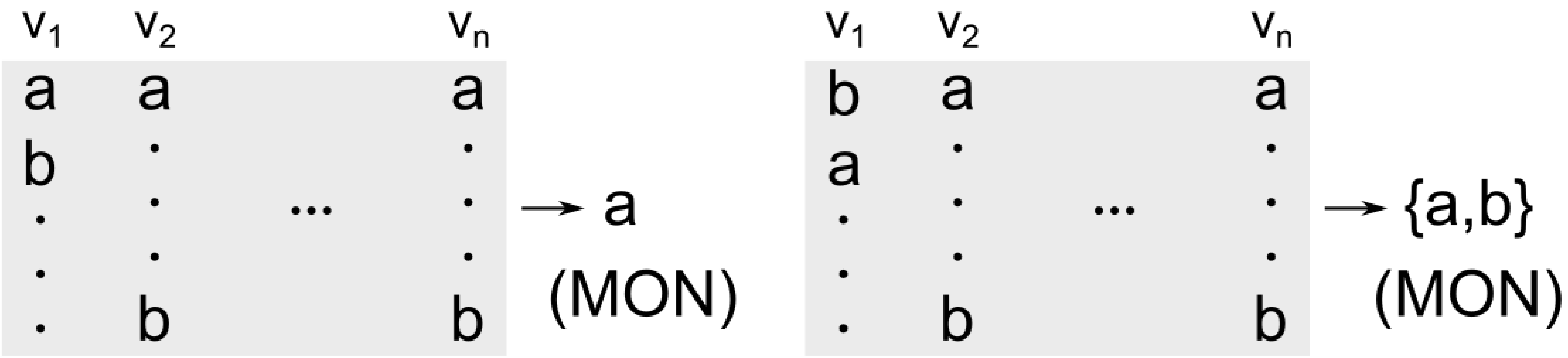
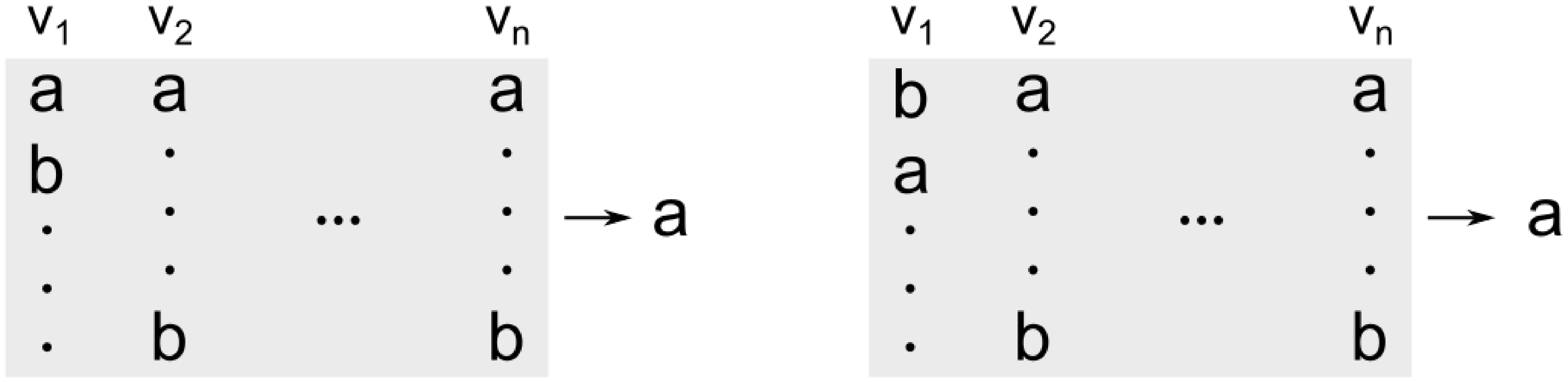
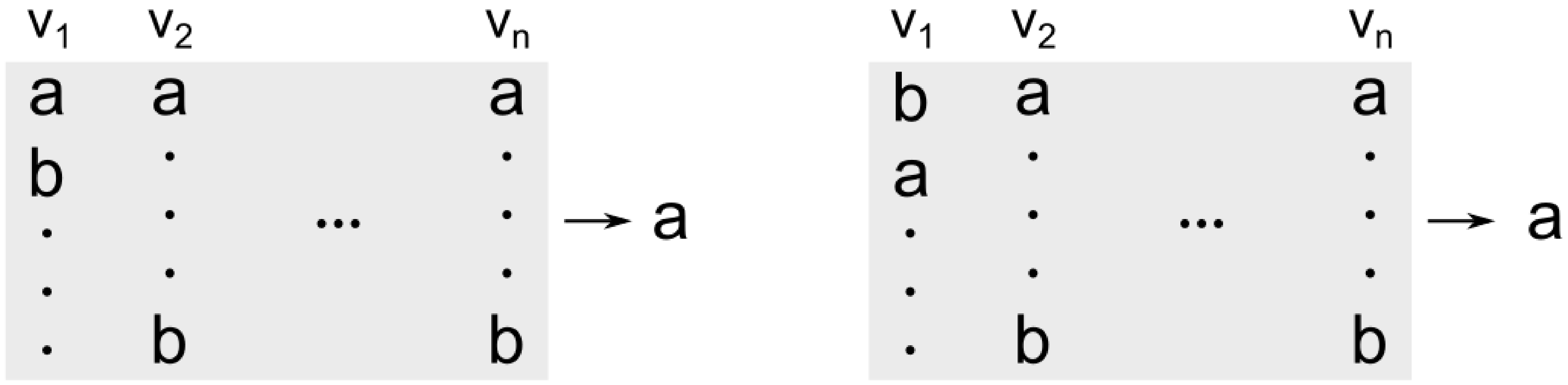
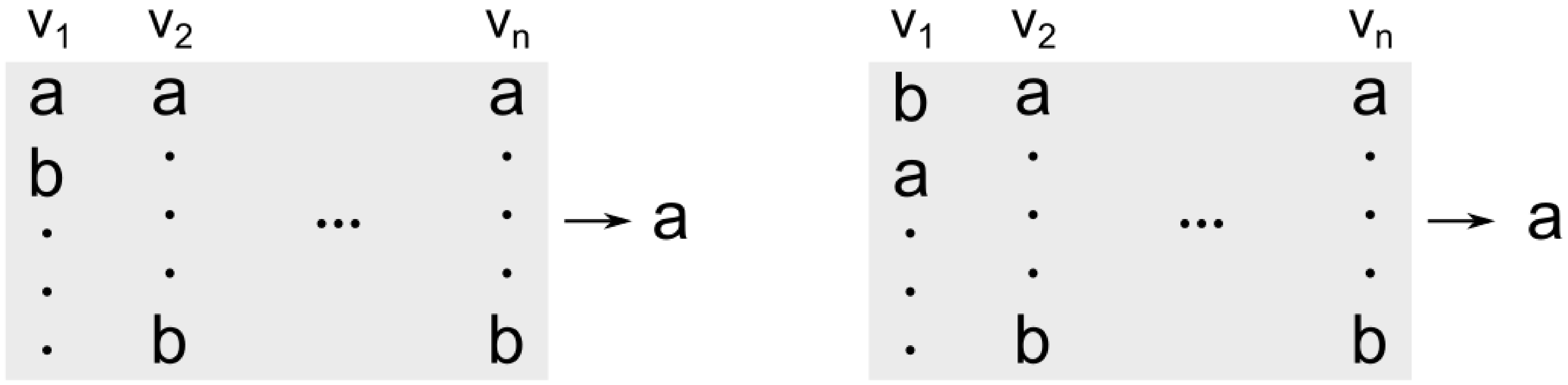




v_1 v_2 v_n
a a a
b
b b
→ a
v_1 v_2 v_n
b a a
a
b b
→ a



"b" must win *eventually* due to UNAN



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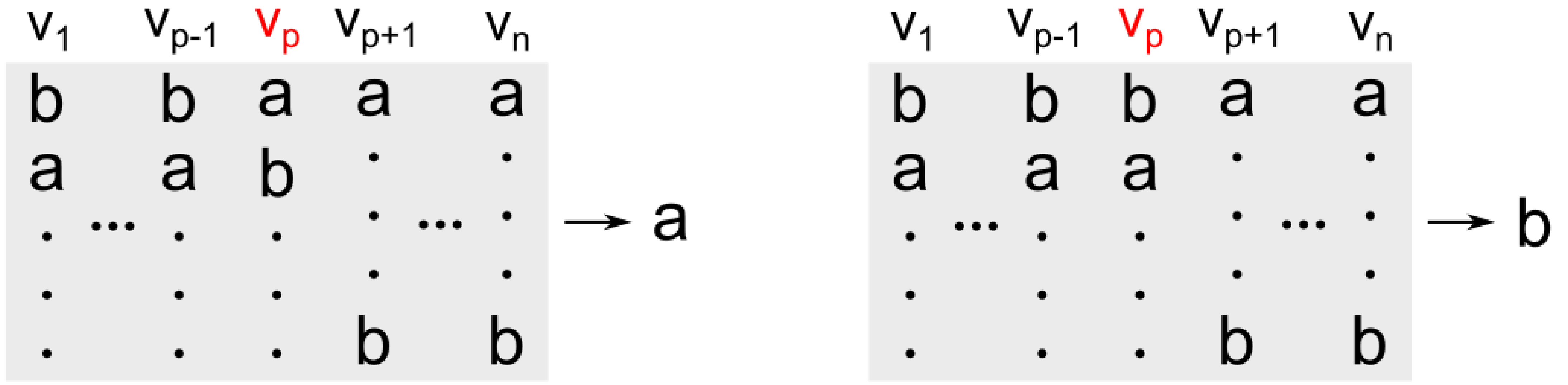
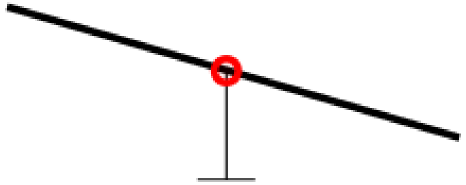
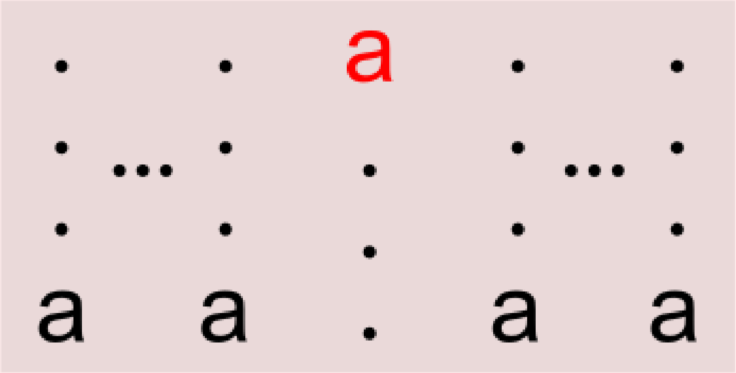


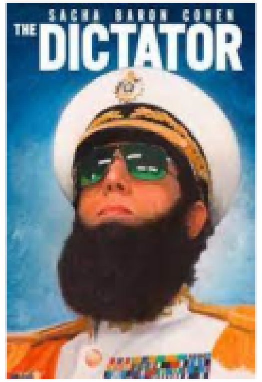
UNAN + MON ⇒ DICTATOR



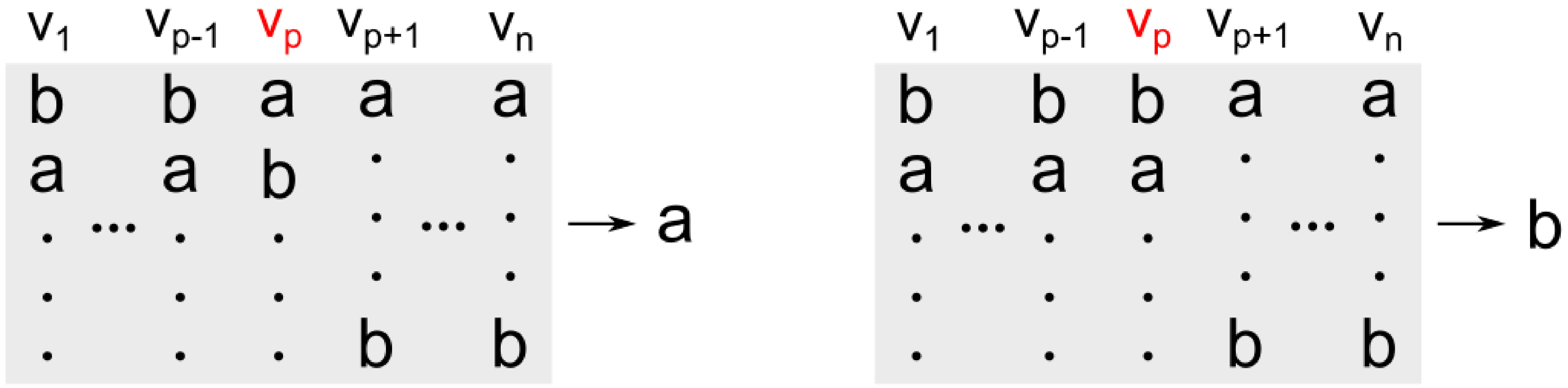
Identify a "pivotal voter" v_p for the pair {a,b}

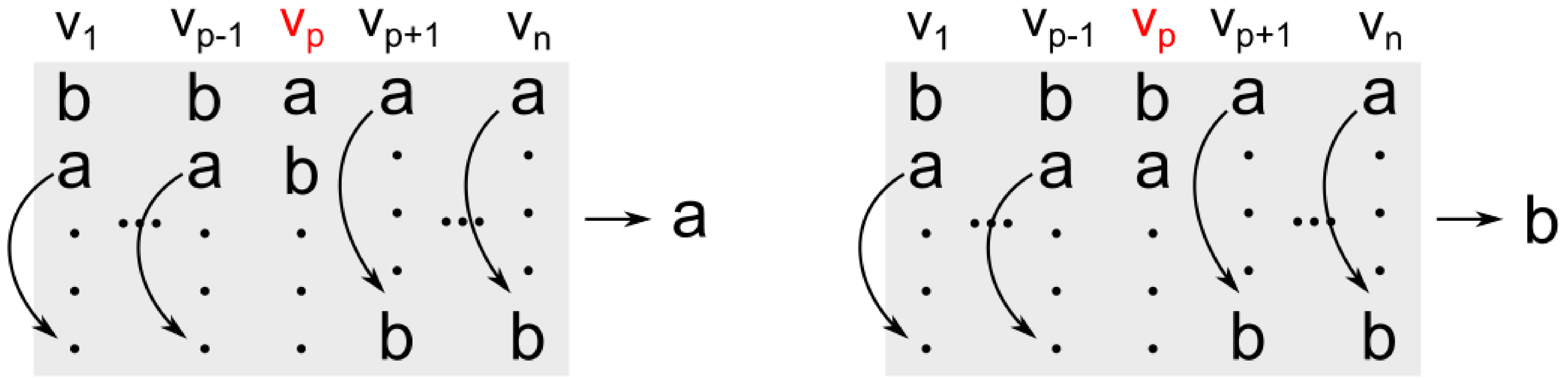


Show that "a" wins if v_p likes it even if everyone else ranks it last

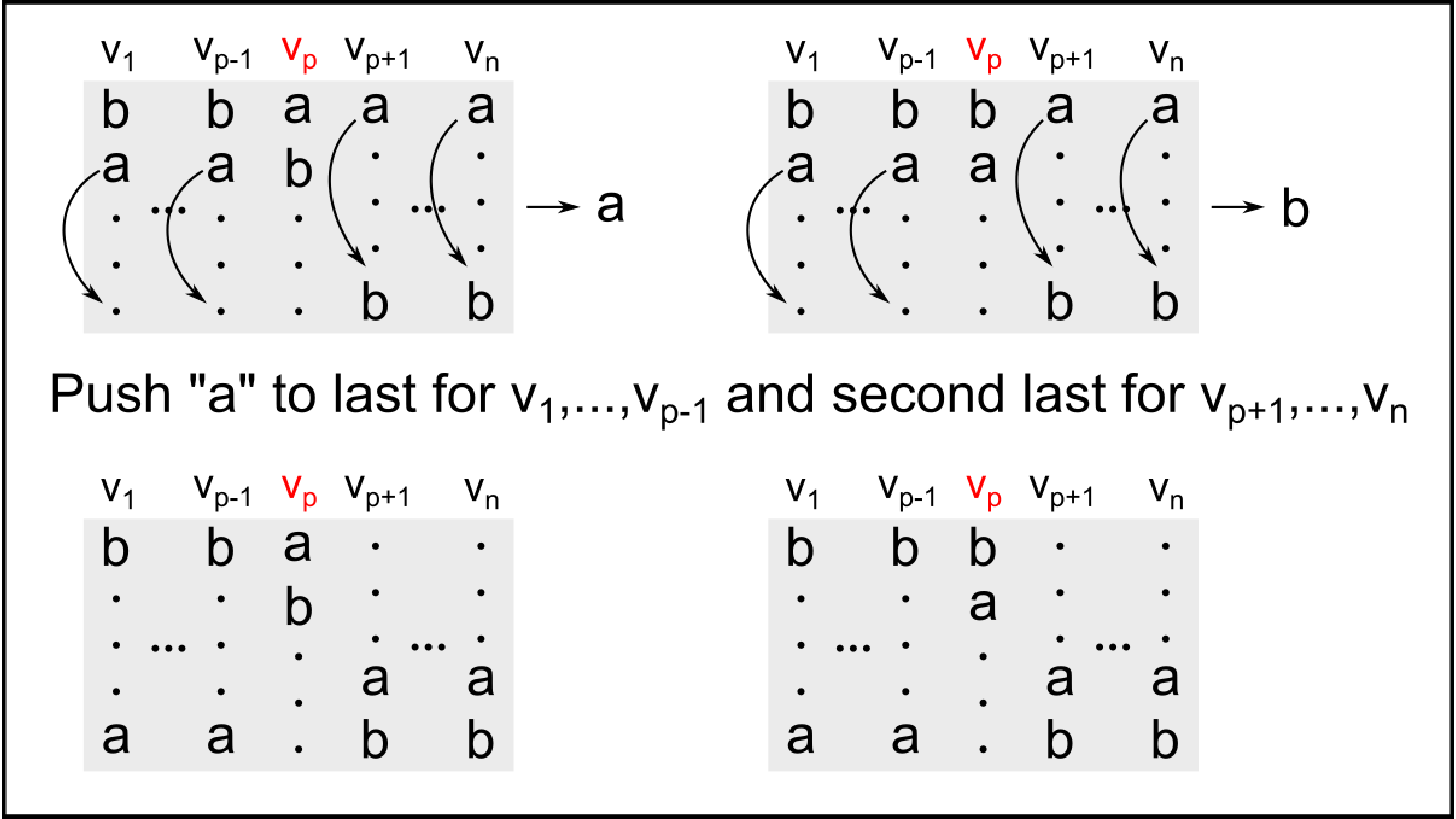


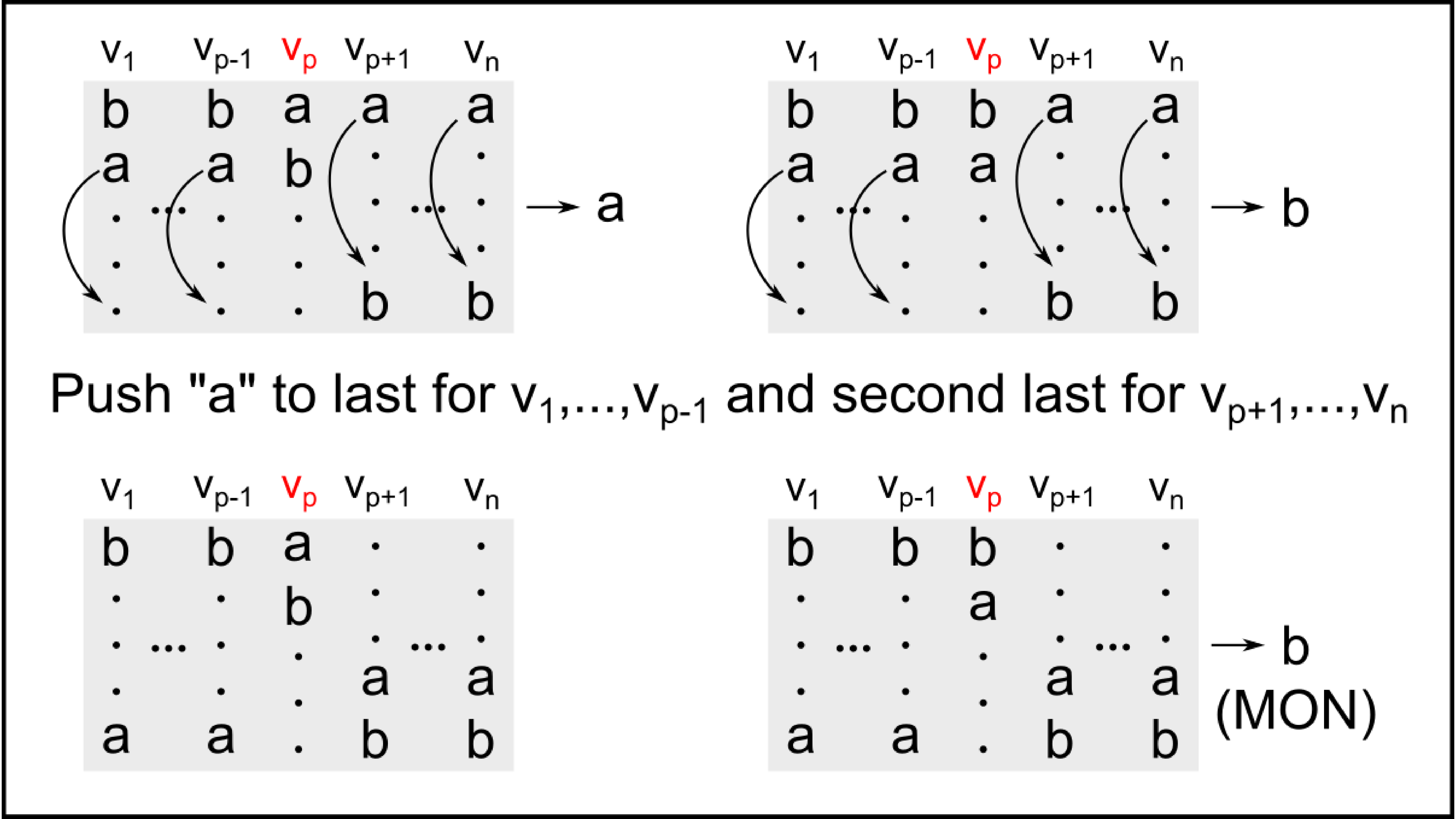
v_p is the dictator for "a" (and every other candidate)

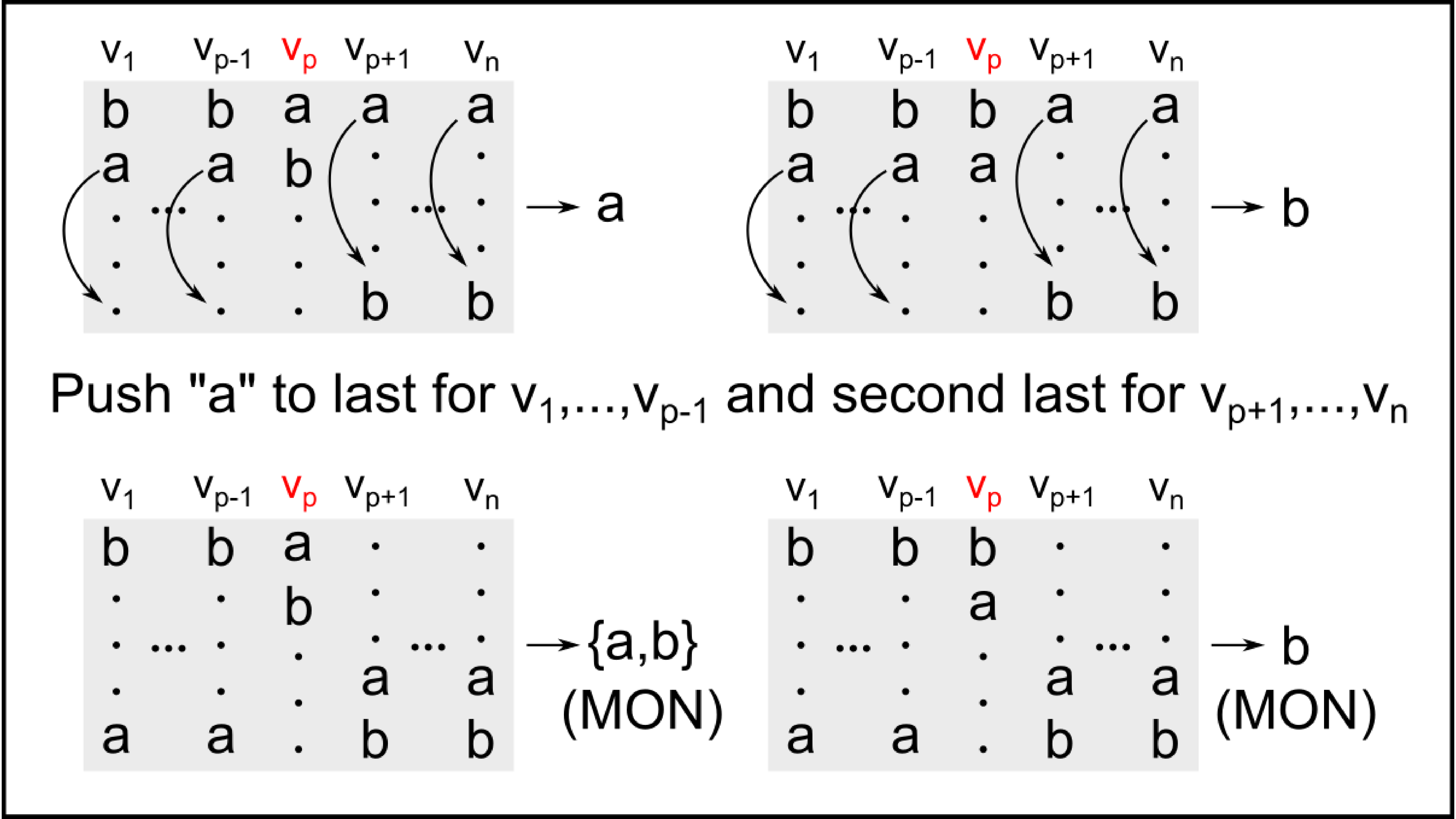

v1 vp-1 vp vp+1 vn
b b a a a
a a b
→ a
v1 vp-1 vp vp+1 vn
b b b a a
a a a
→ b

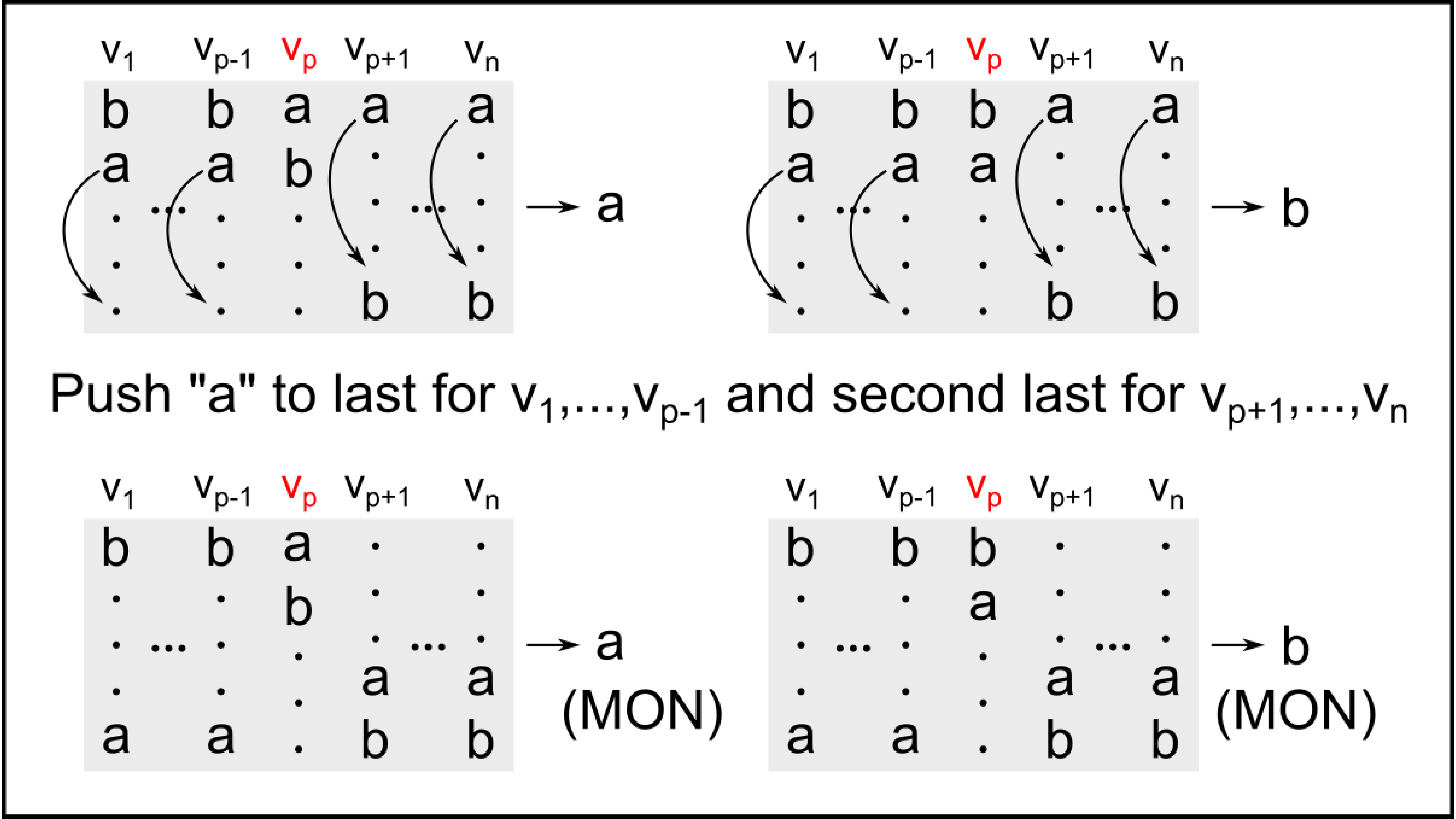


Push "a" to last for $v_1, \ldots, v_{p-1}$ and second last for $v_{p+1}, \ldots, v_n$

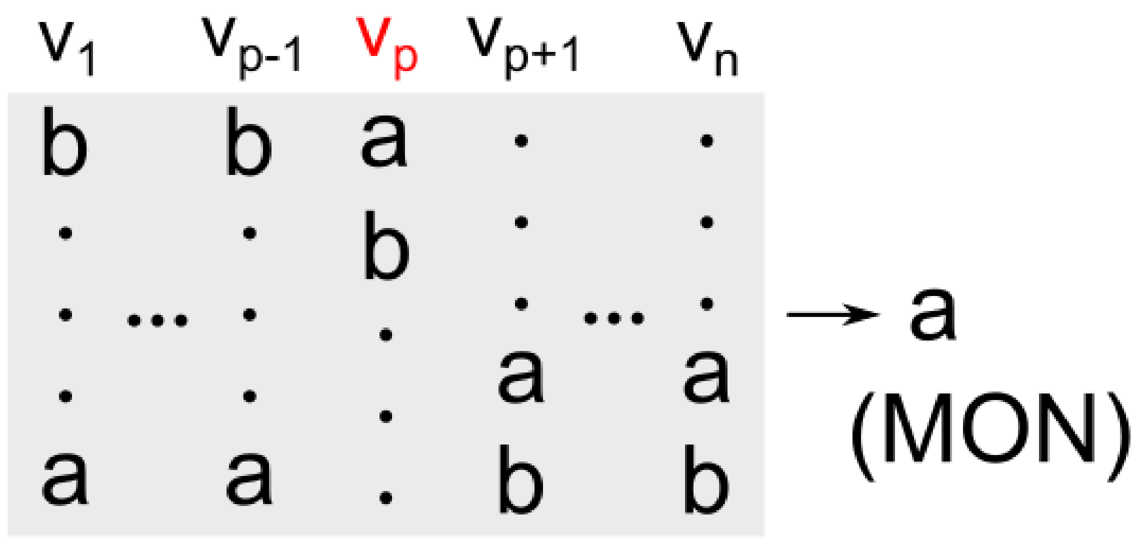

v1 vp-1 vp vp+1 vn
→ a
→ b
Push "a" to last for v1,...,vp-1 and second last for vp+1,...,vn
v1 vp-1 vp vp+1 vn

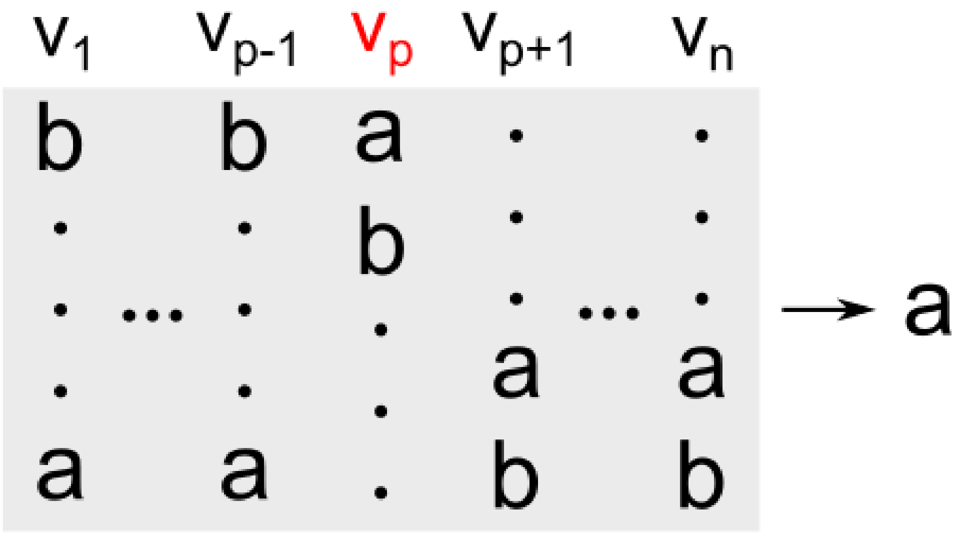

v1 vp-1 vp vp+1 vn
→ a
→ b
Push "a" to last for v1,...,vp-1 and second last for vp+1,...,vn
→ b
(MON)

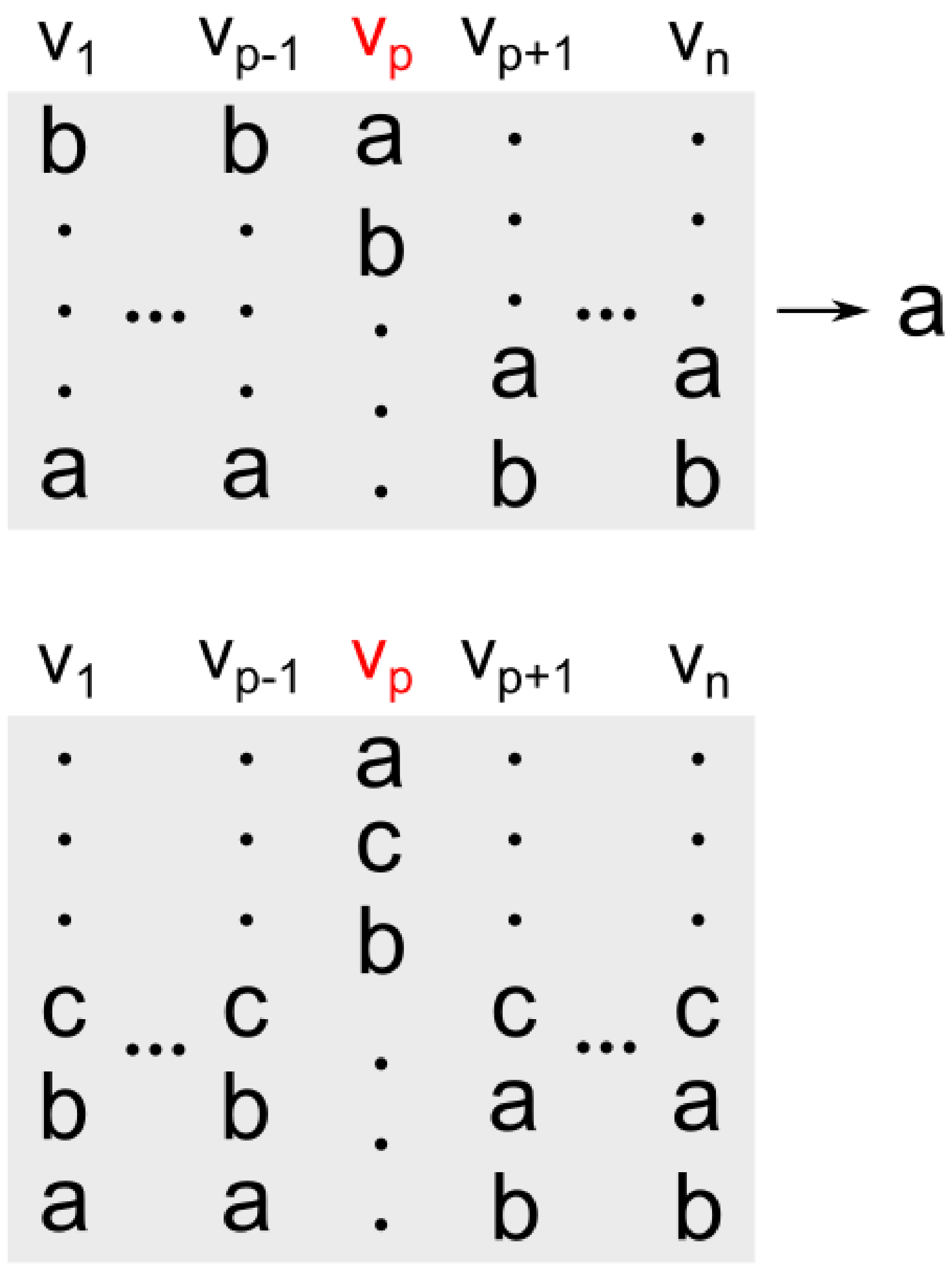

v1 vp-1 vp vp+1 vn
→ a
→ b
Push "a" to last for v1,...,vp-1 and second last for vp+1,...,vn
→ {a,b} (MON)
→ b (MON)

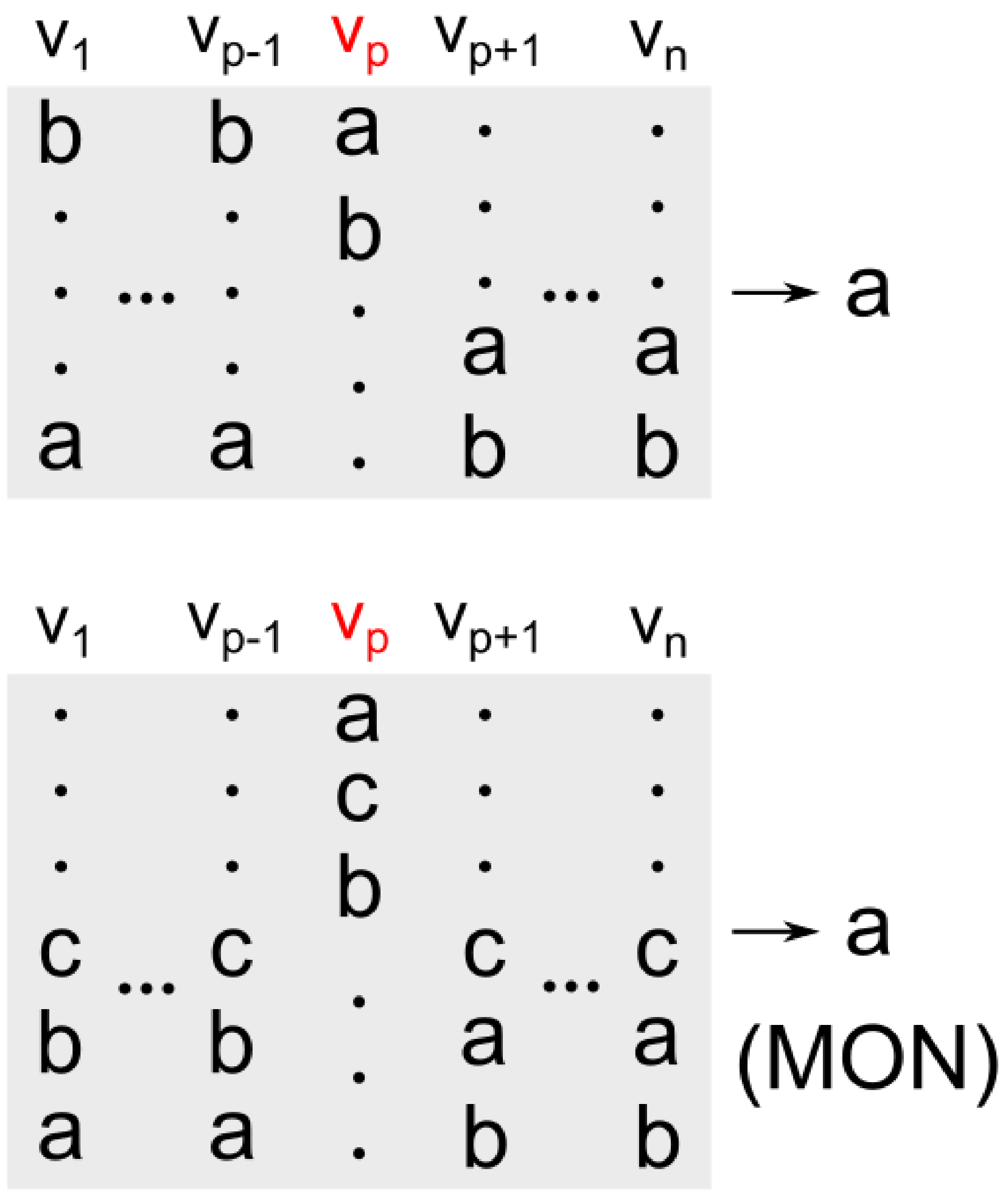


v_1 v_{p-1} v_p v_{p+1} v_n
→ a
→ b
Push "a" to last for $v_1,\ldots,v_{p-1}$ and second last for $v_{p+1},\ldots,v_n$
→ a (MON)
→ b (MON)

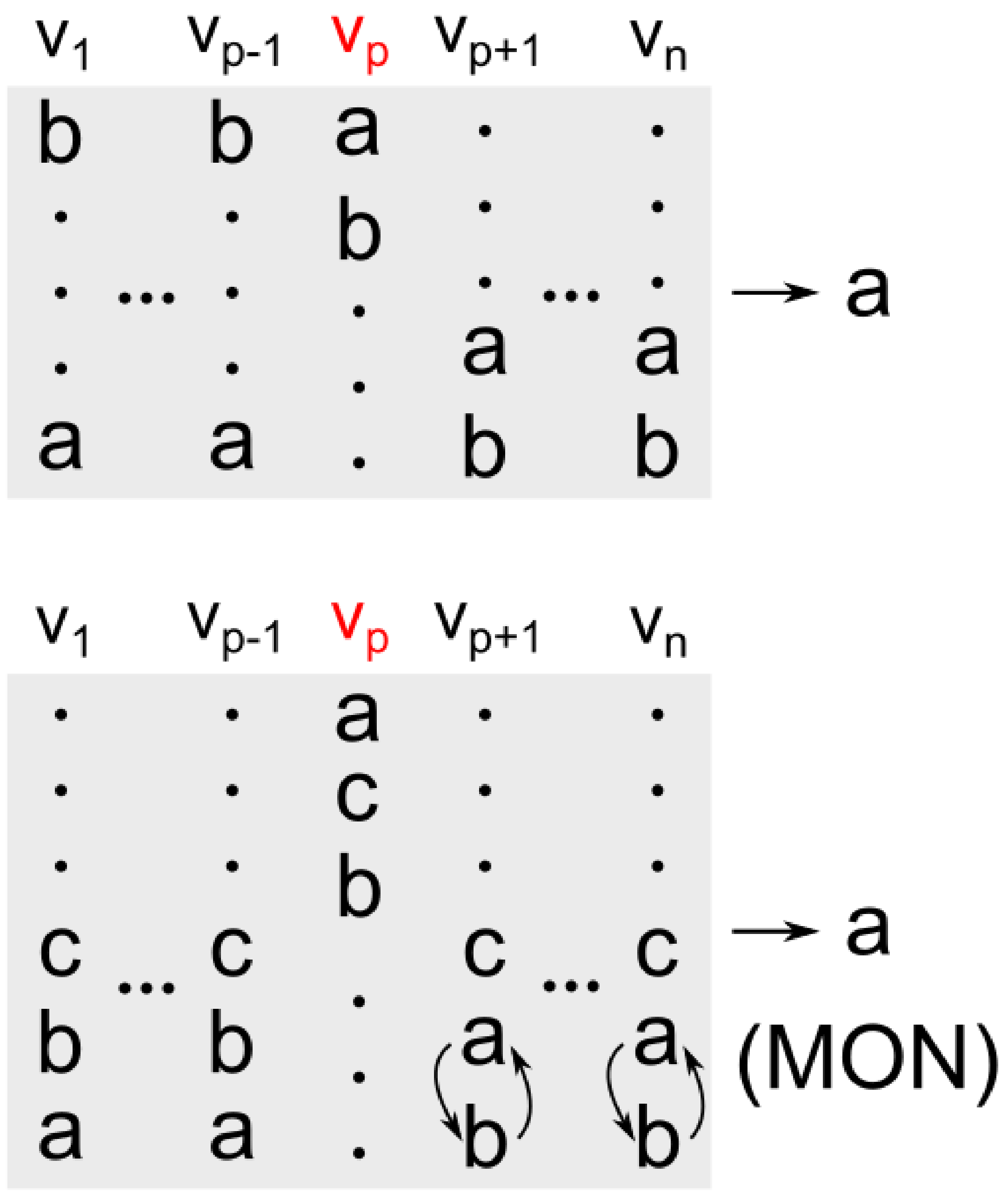

v1 vp-1 vp vp+1 vn
b b a
b
a a
a a b b
→ a
(MON)


v_1 v_{p-1} v_p v_{p+1} v_n
b b a
b
a a
a a b b
→ a

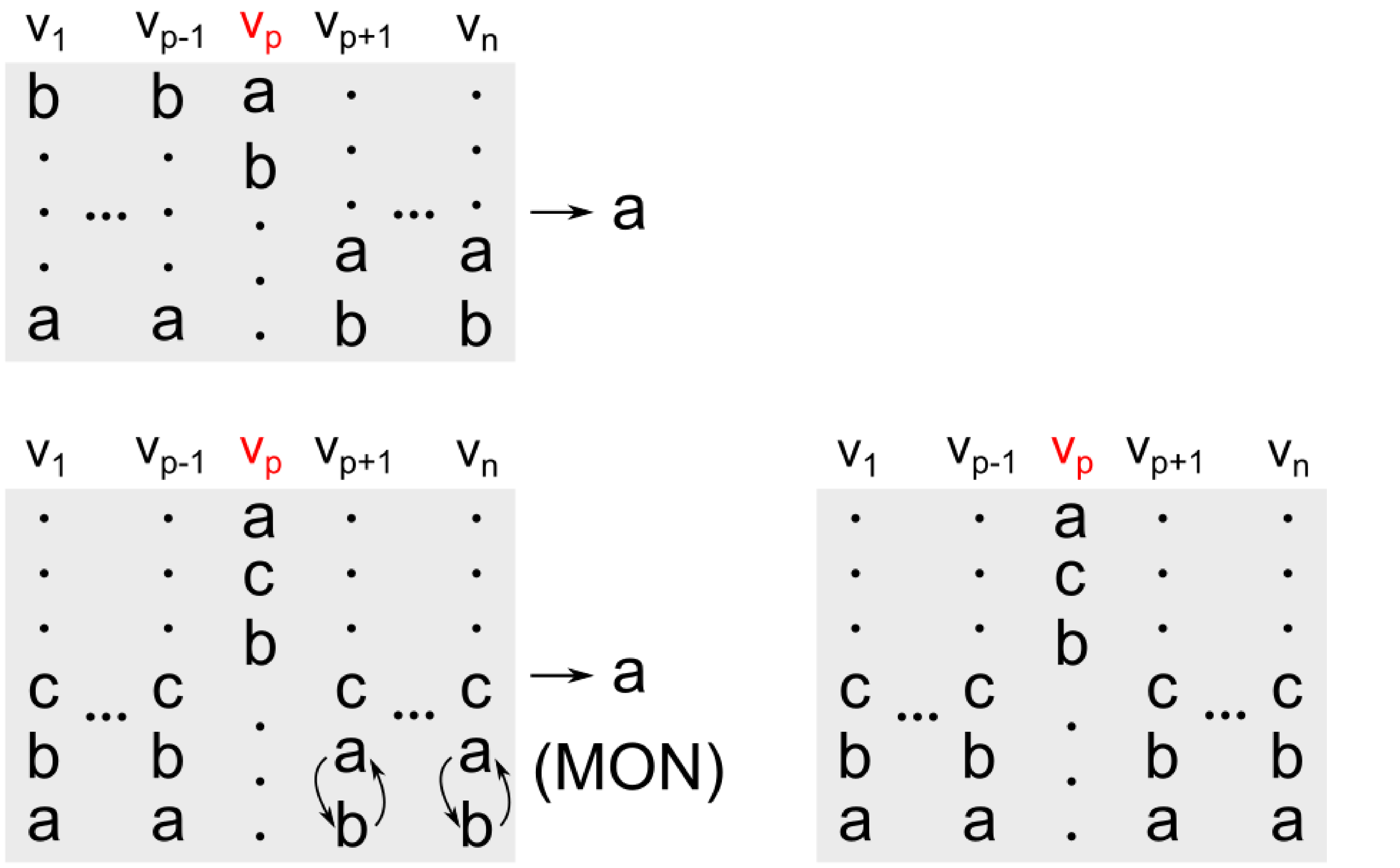

v1 vp-1 vp vp+1 vn
b b a
b
a a
a a b b
a
v1 vp-1 vp vp+1 vn
a
c
b
c c c c
b b a a
a a b b

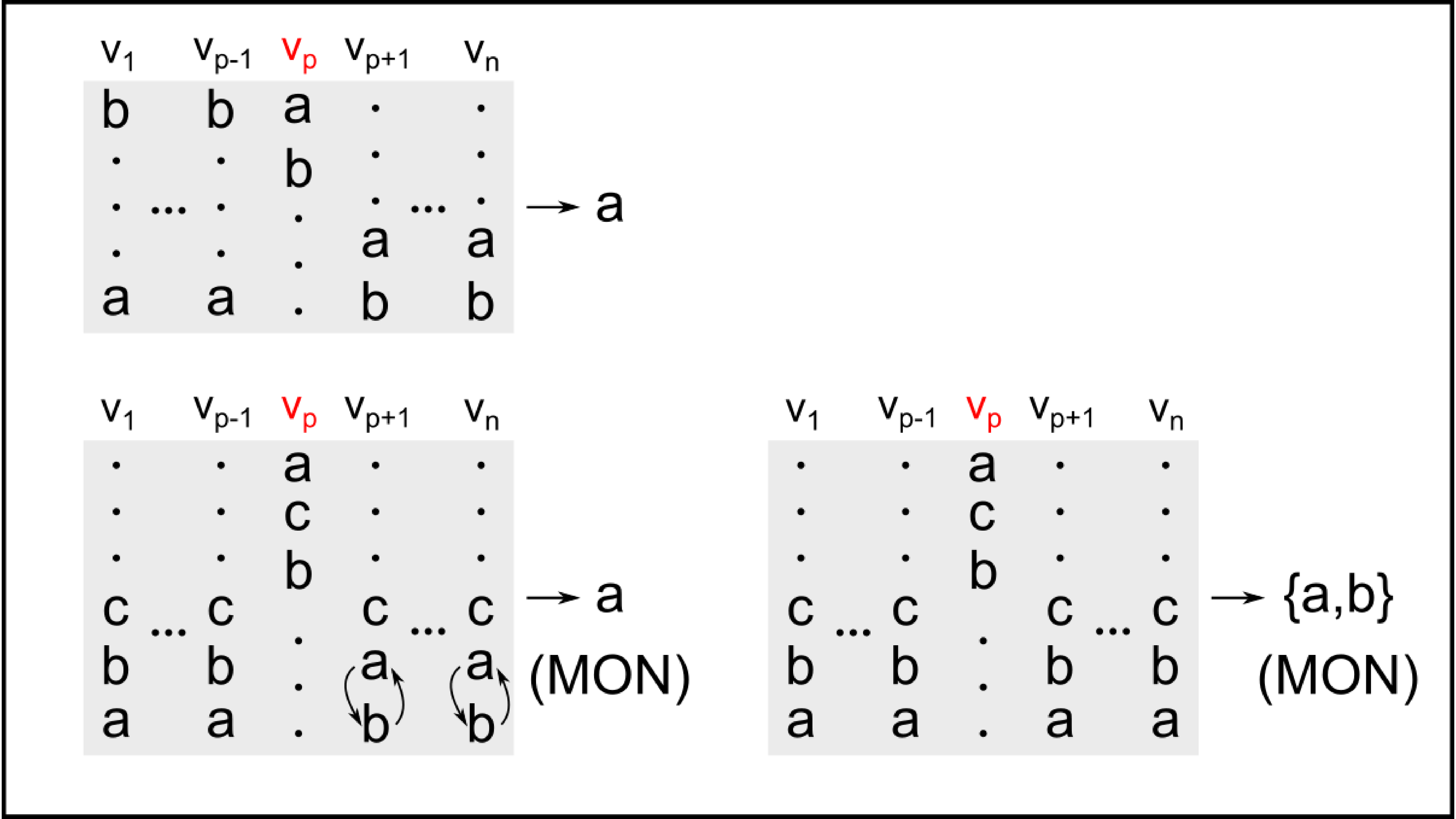


v1 ... vp-1 vp vp+1 ... vn
b b a
b
a a
a a b b
→ a
v1 ... vp-1 vp vp+1 ... vn
a
c
b
c c c c
b b a a
a a b b
→ a
(MON)

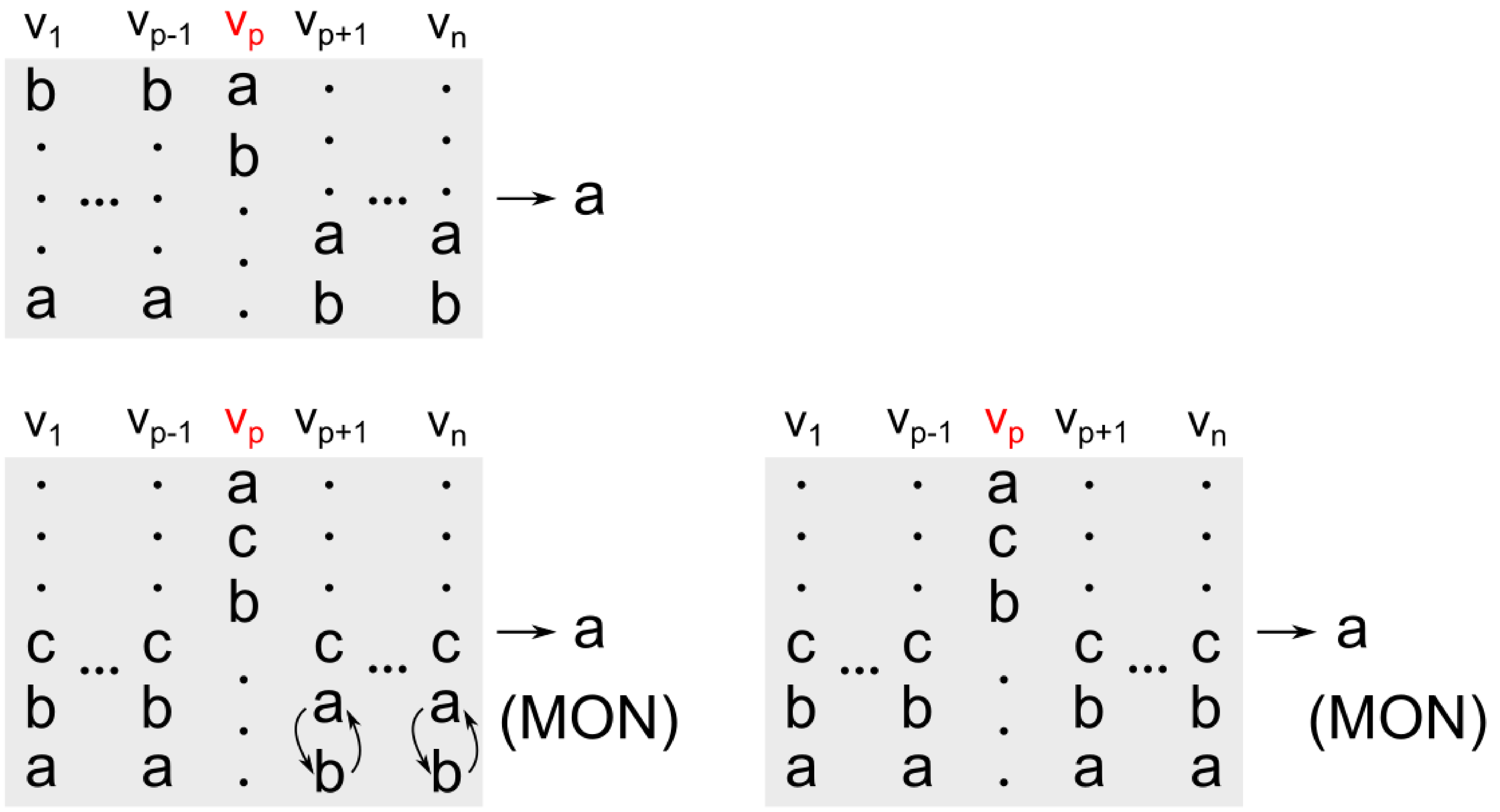


v1 vp-1 vp vp+1 vn
→ a
v1 vp-1 vp vp+1 vn
→ a
(MON)

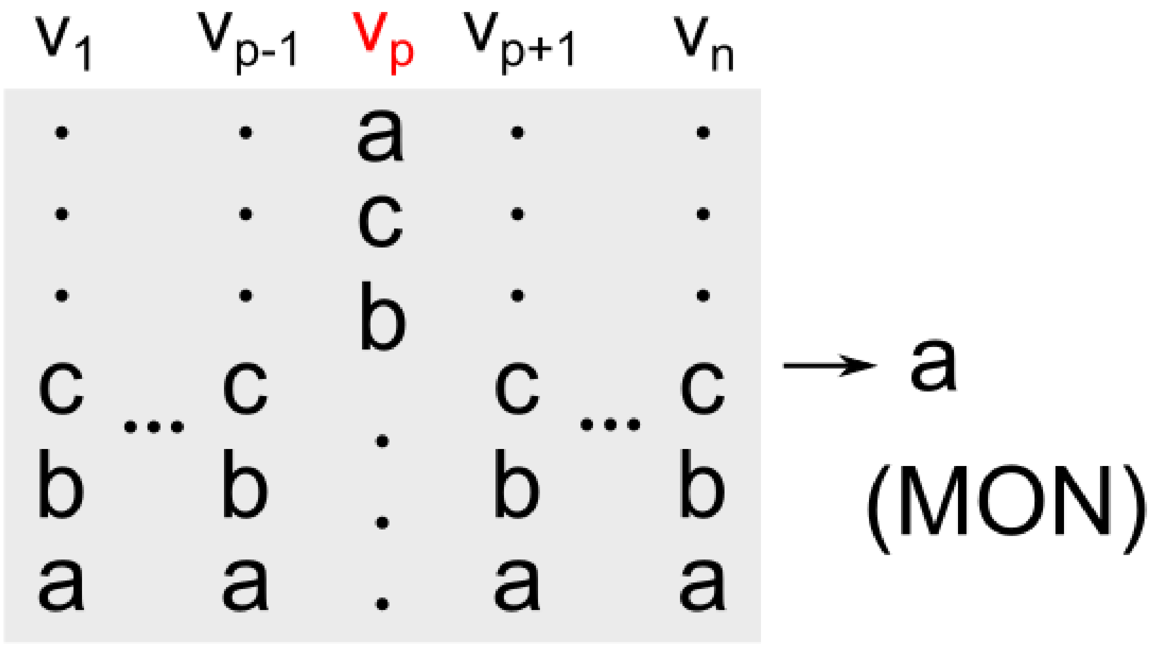

v1 vp-1 vp vp+1 vn
→ a
(MON)

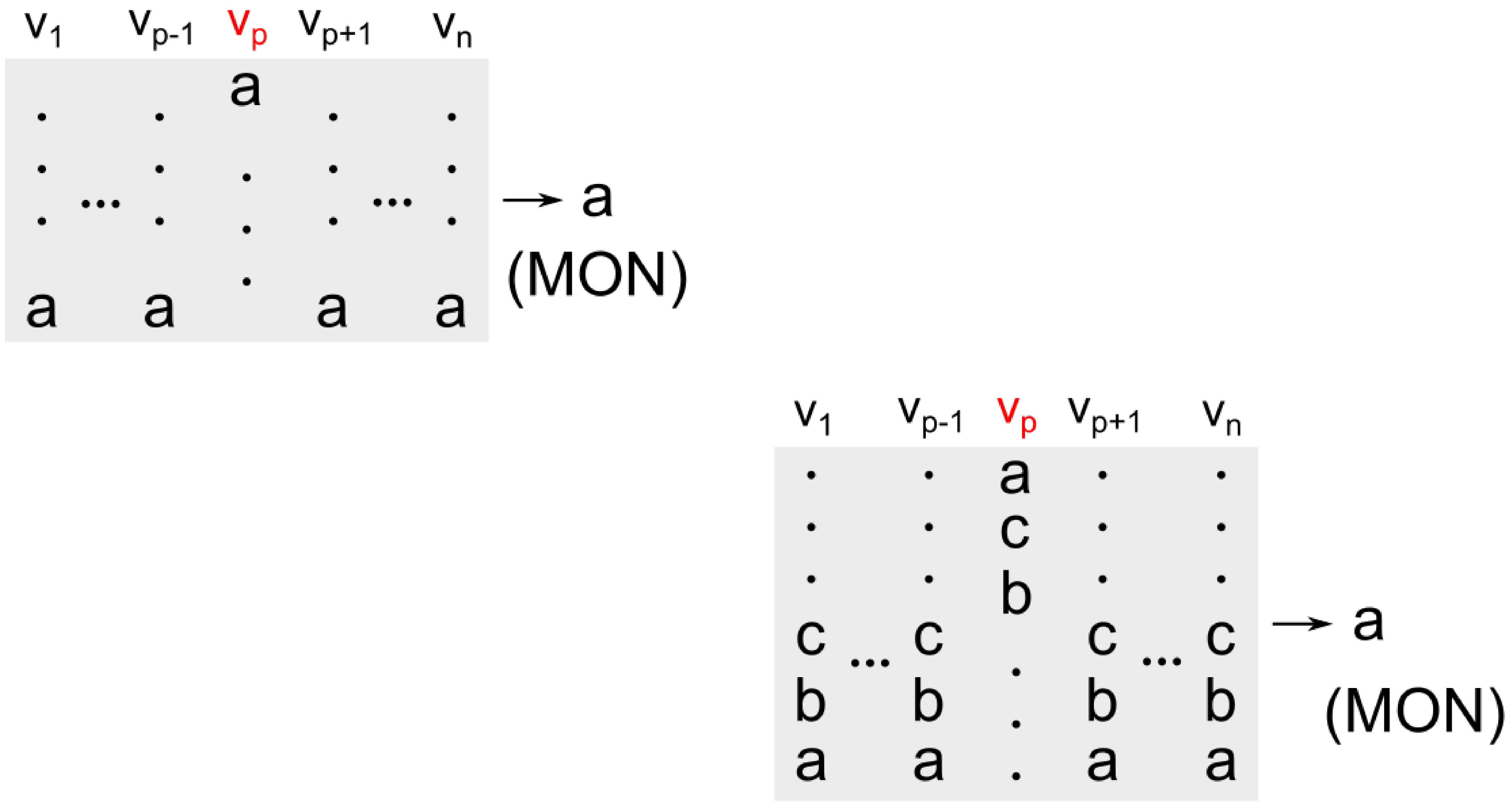


v1 vp-1 vp vp+1 vn
b b a
b
a a
a a b b
→ a
v1 vp-1 vp vp+1 vn
a
c
b
c c c c
b b a a
a a b b
→ a
(MON)
v1 vp-1 vp vp+1 vn
a
c
b
c c c c
b b b b
a a a a
→ {a,b}
(MON)

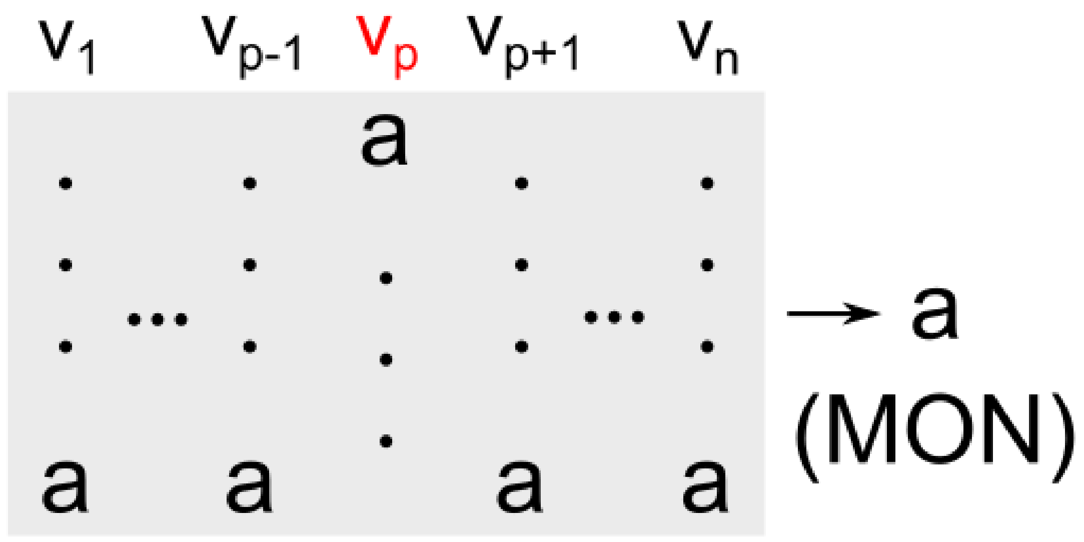


v1 vp-1 vp vp+1 vn
→ a
v1 vp-1 vp vp+1 vn
→ a
(MON)
v1 vp-1 vp vp+1 vn
→ a
(MON)


v_1 v_{p-1} v_p v_{p+1} v_n
a
c
b
c ... c c ... c
b b b b
a a a a
a
(MON)

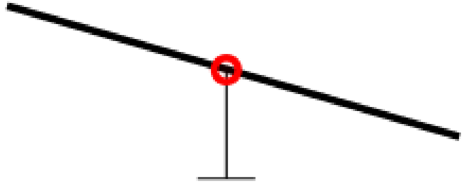


v1 vp-1 vp vp+1 vn
a
a a a a
→ a
(MON)
v1 vp-1 vp vp+1 vn
a
c
b
c c c c
b b b b
a a a a
→ a
(MON)

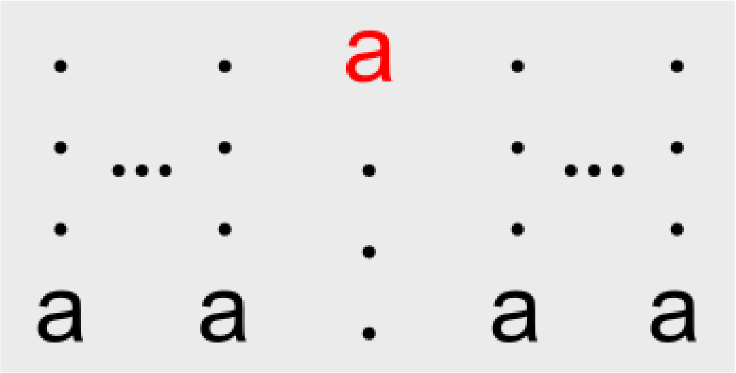


v1 vp-1 vp vp+1 vn
a
a a a a
→ a
(MON)

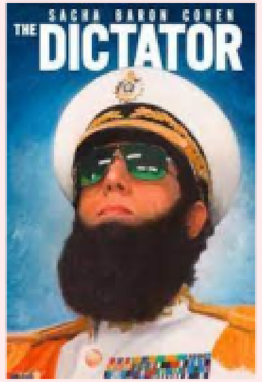
UNAN + MON ⇒ DICTATOR



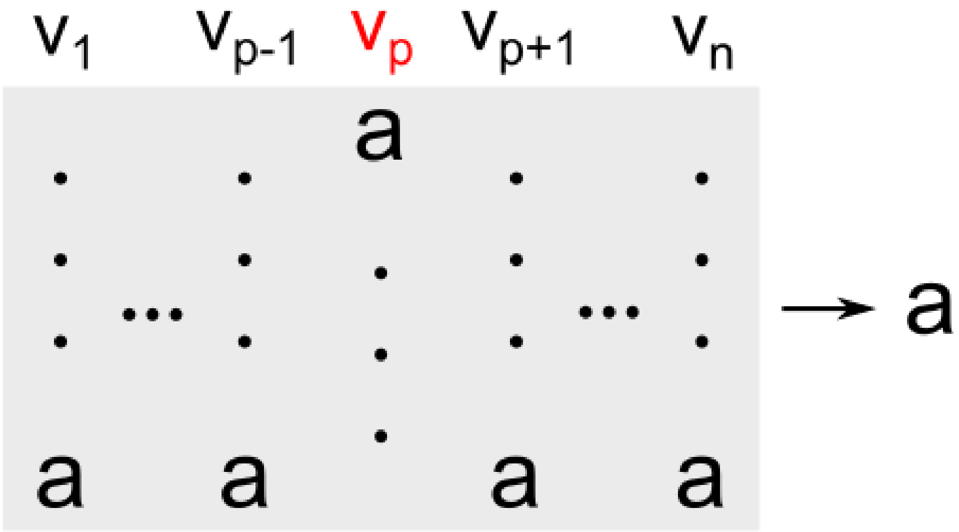
Identify a "pivotal voter" v_p for the pair {a,b}



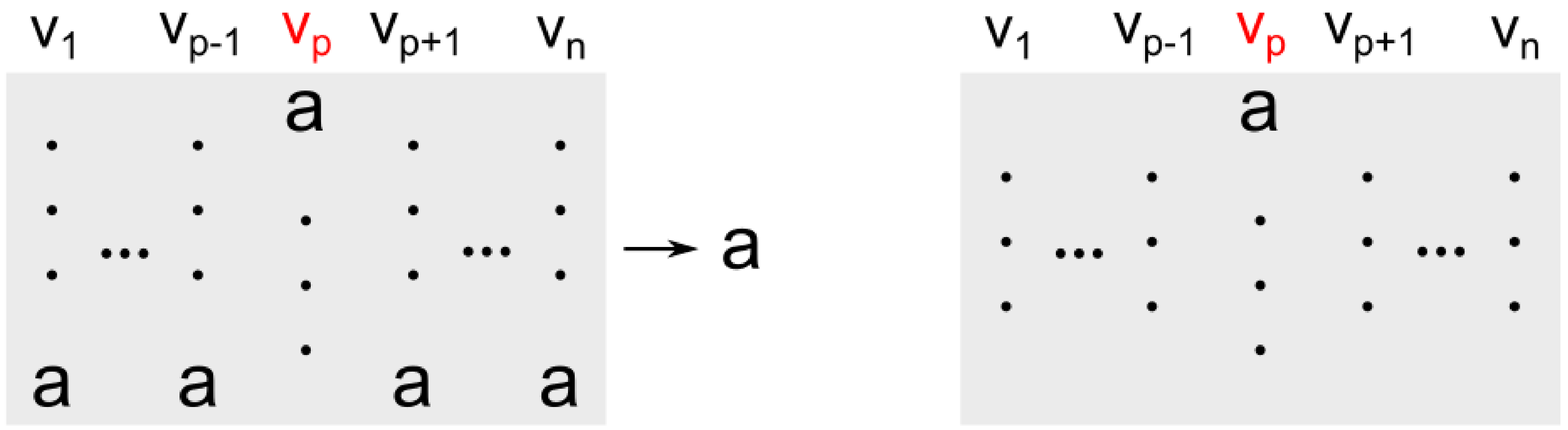
Show that "a" wins if v_p likes it even if everyone else ranks it last



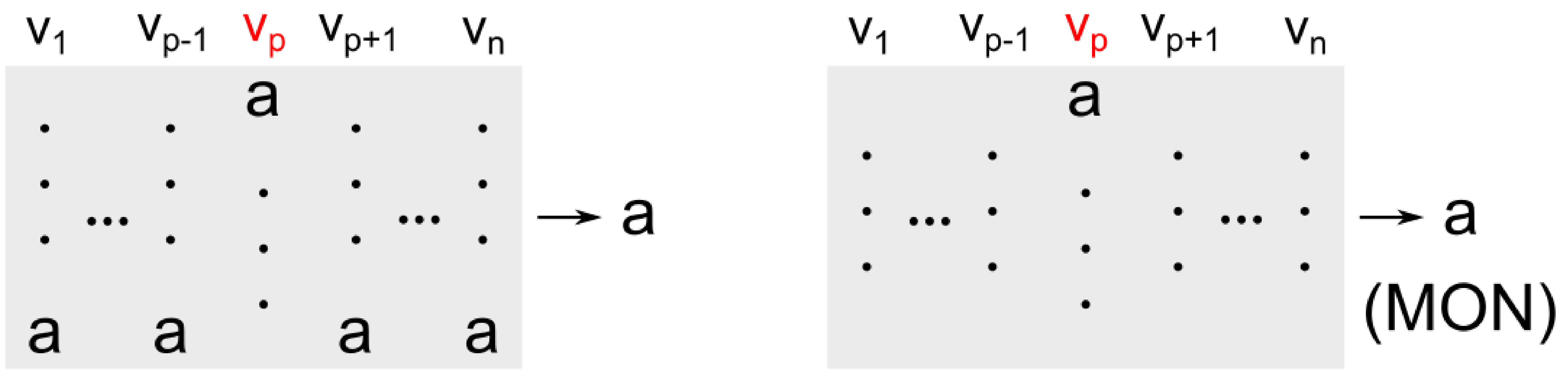
v_p is the dictator for "a" (and every other candidate)

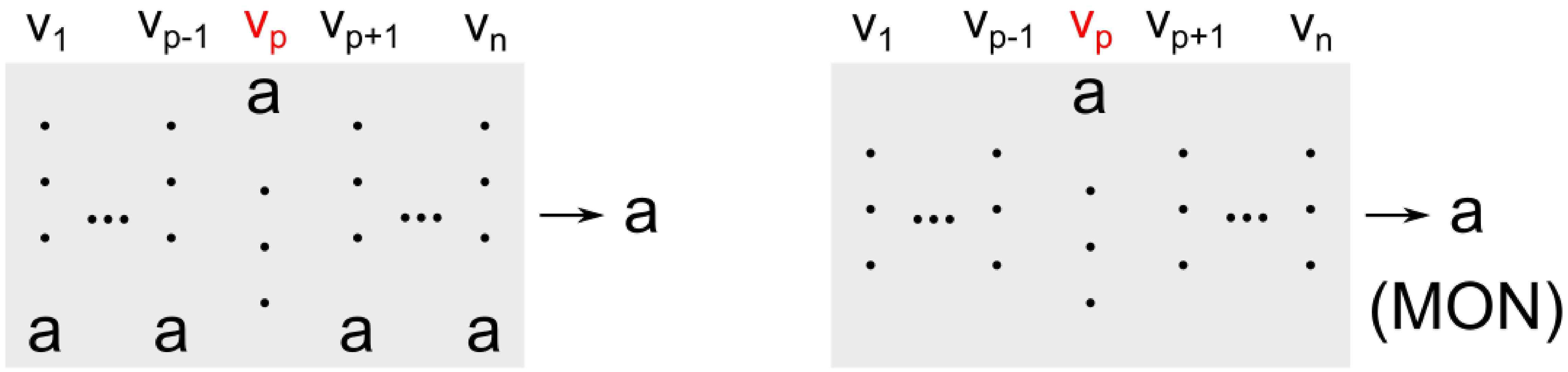


v_1 v_{p-1} v_p v_{p+1} v_n
a
a a a a
→ a

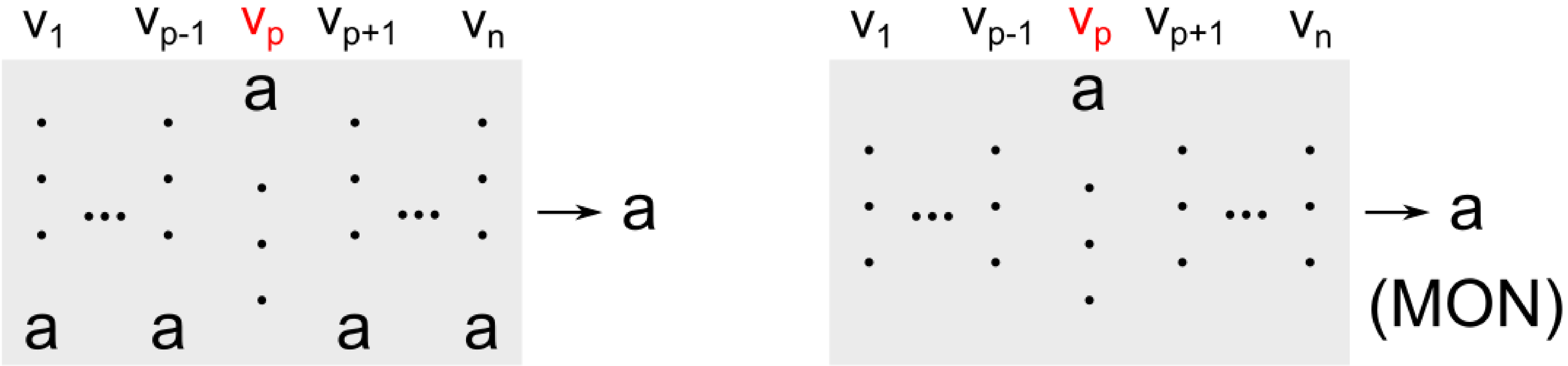


v_1 v_{p-1} v_p v_{p+1} v_n
a
a a a a
→ a
v_1 v_{p-1} v_p v_{p+1} v_n
a


v1 vp-1 vp vp+1 vn
a
a a a a
→ a
v1 vp-1 vp vp+1 vn
a
→ a
(MON)

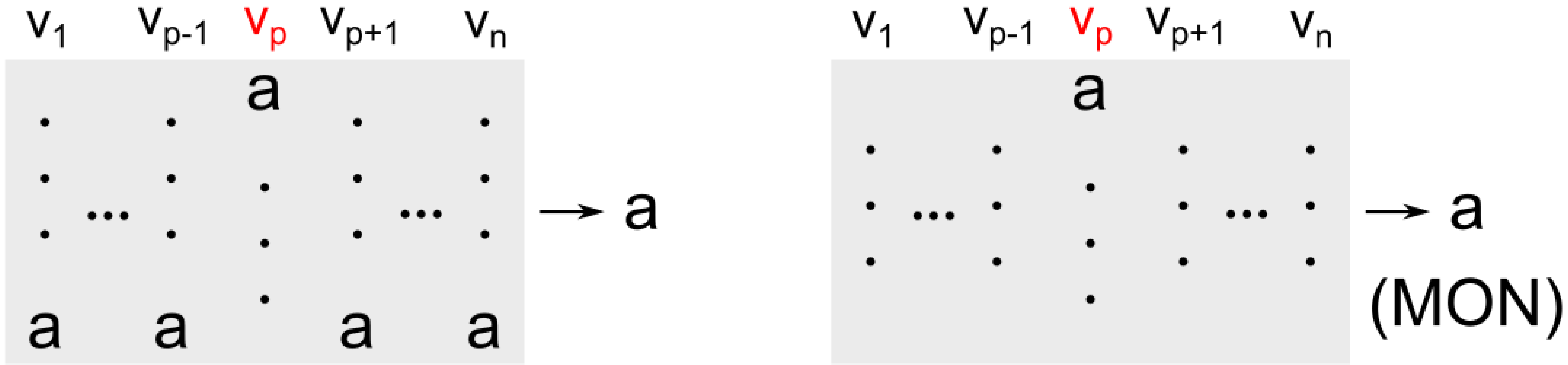


v_p is the dictator for "a"



v_p is the dictator for "a"

Since the choice of "a" was arbitrary, *every* candidate must have a dictatorial voter associated with it.



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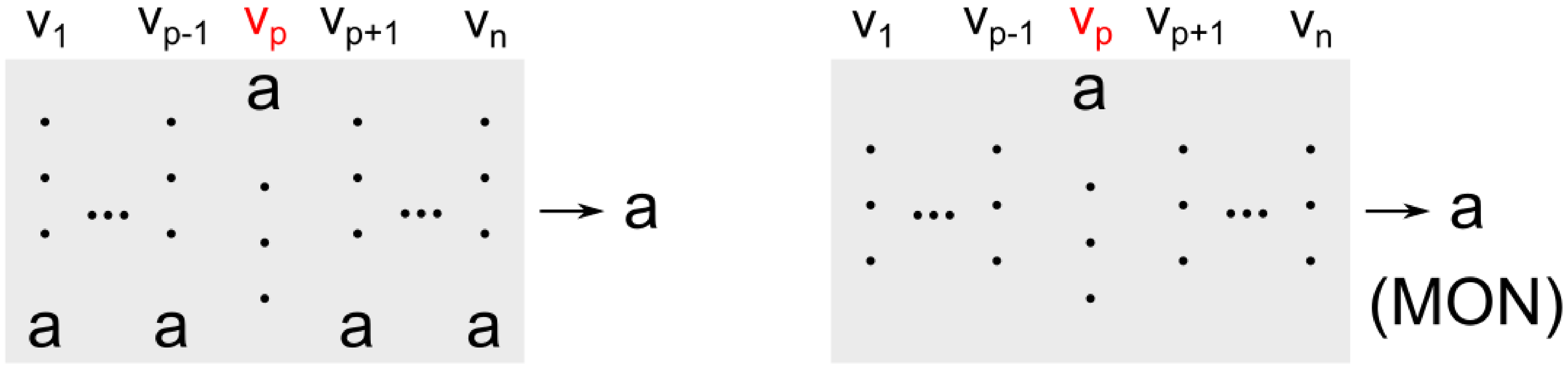
Since the choice of "a" was arbitrary, *every* candidate must have a dictatorial voter associated with it.

But there can't be distinct dictators for different candidates.



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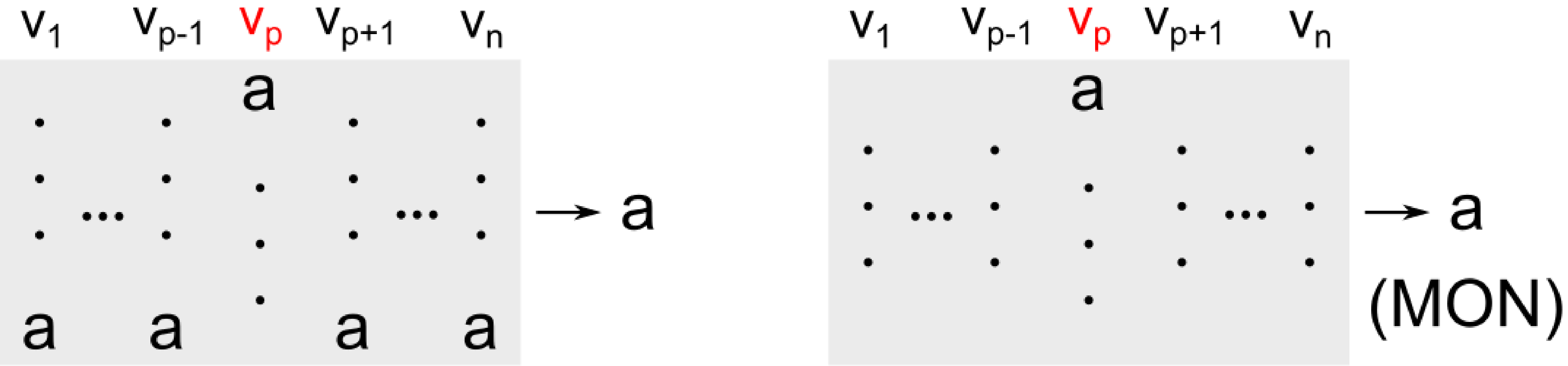
So, v_p is the dictator for all candidates.



v_p is the dictator for "a"

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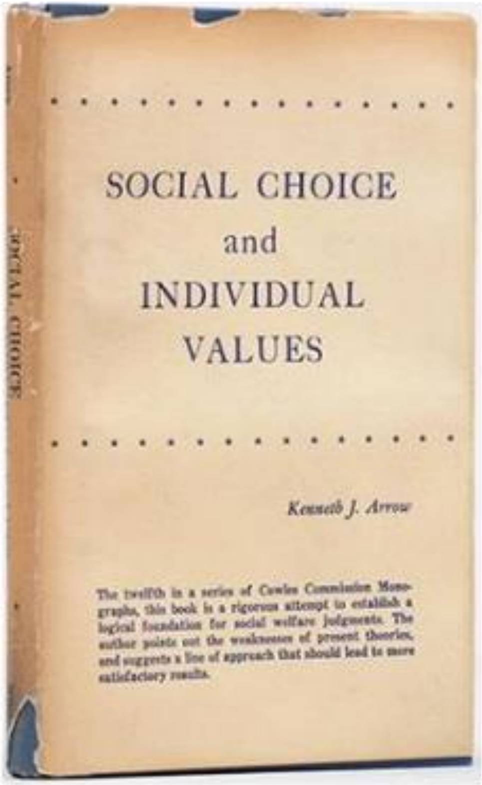
But there can't be distinct dictators for different candidates.

So, v_p is the dictator for all candidates. ∎

The use of axiomatic approach in voting goes back much further.



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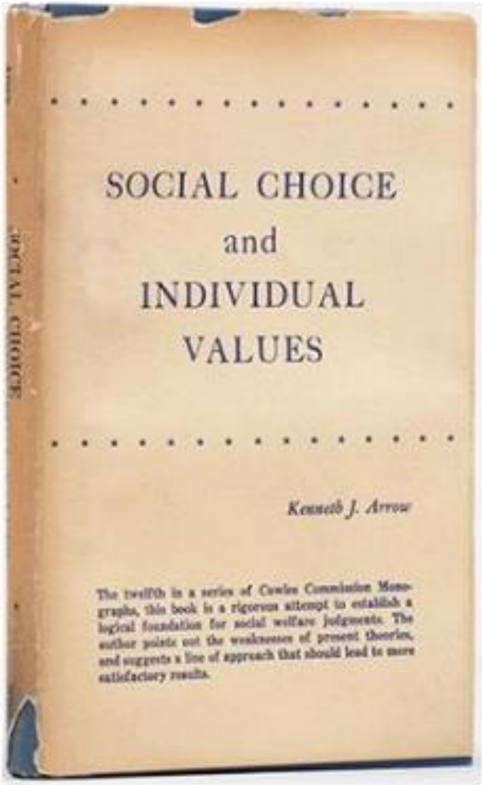

1951 PhD thesis of Kenneth Arrow

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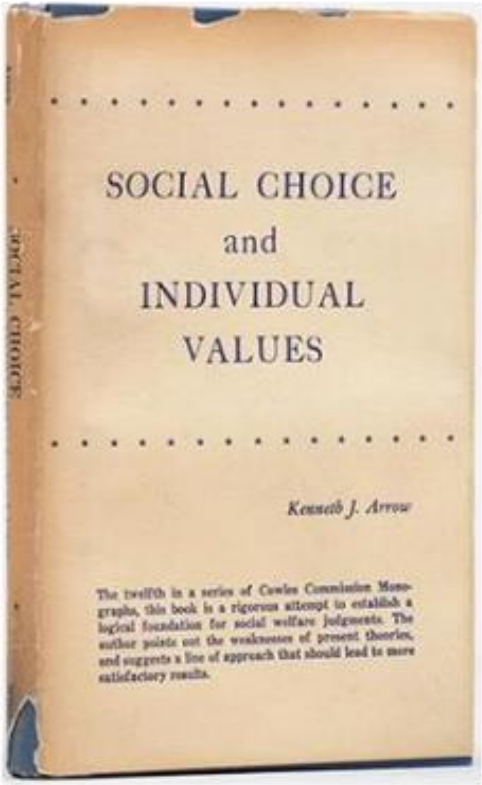

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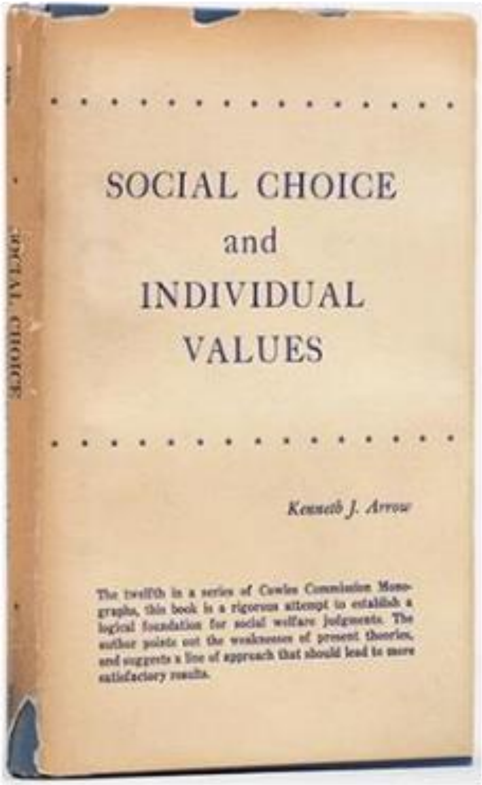
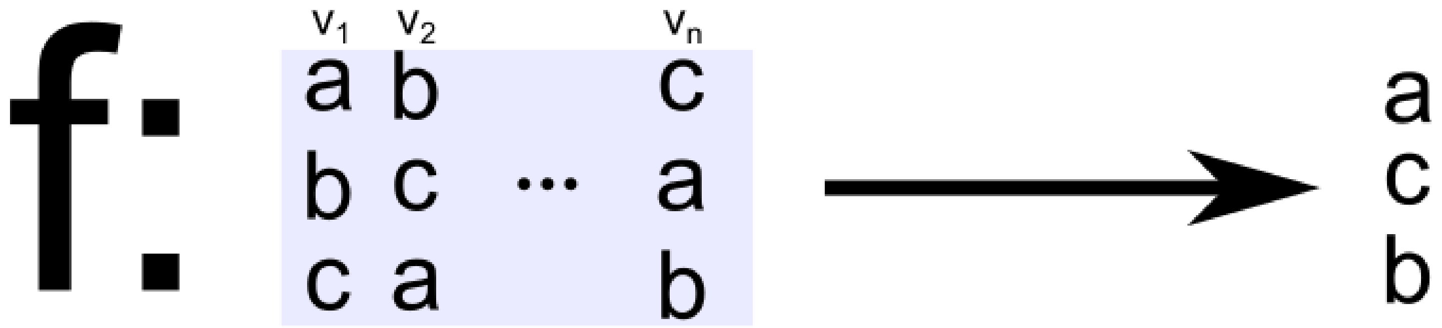


1951 PhD thesis of Kenneth Arrow

Voting Rules that Output Rankings

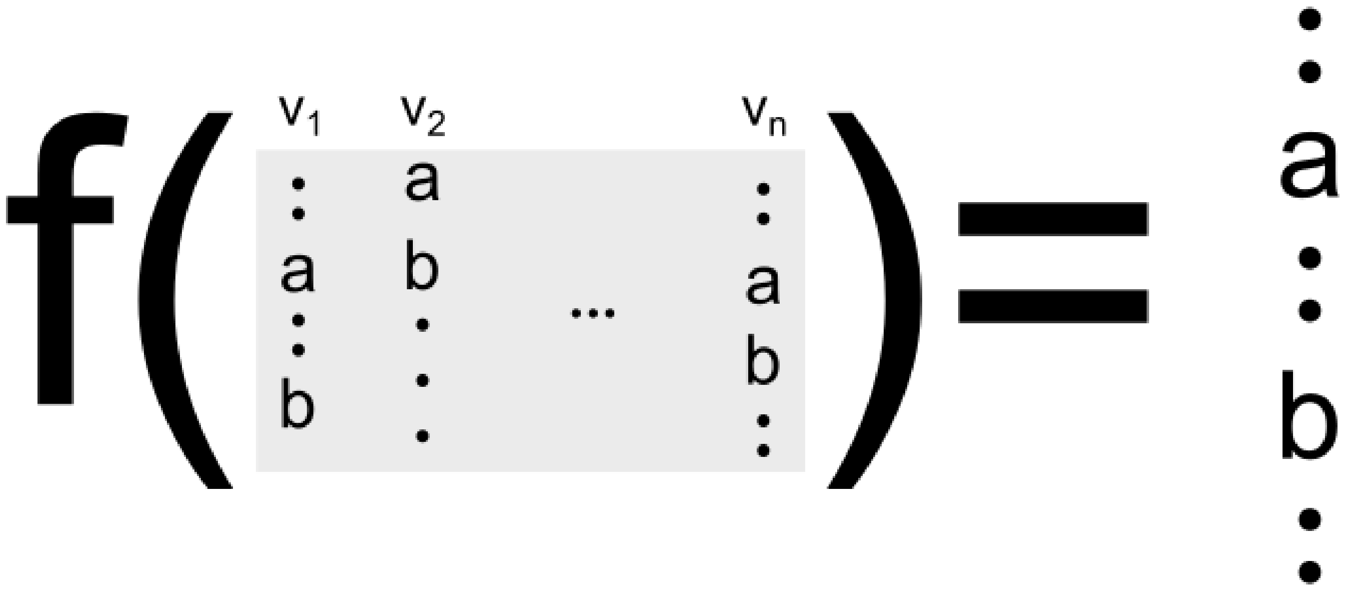
A mapping from preference profiles to rankings over candidates.



(also known as a *social welfare function* or SWF)

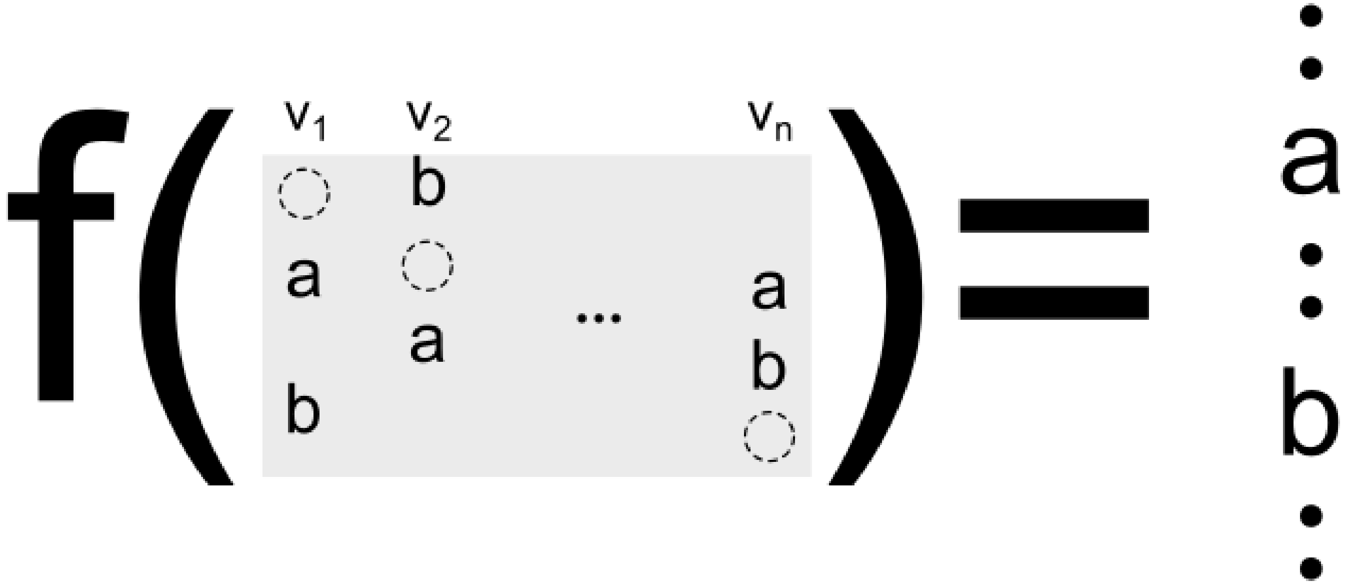
UNANIMOUS

If all voters prefer "a" over "b", then so does the SWF.



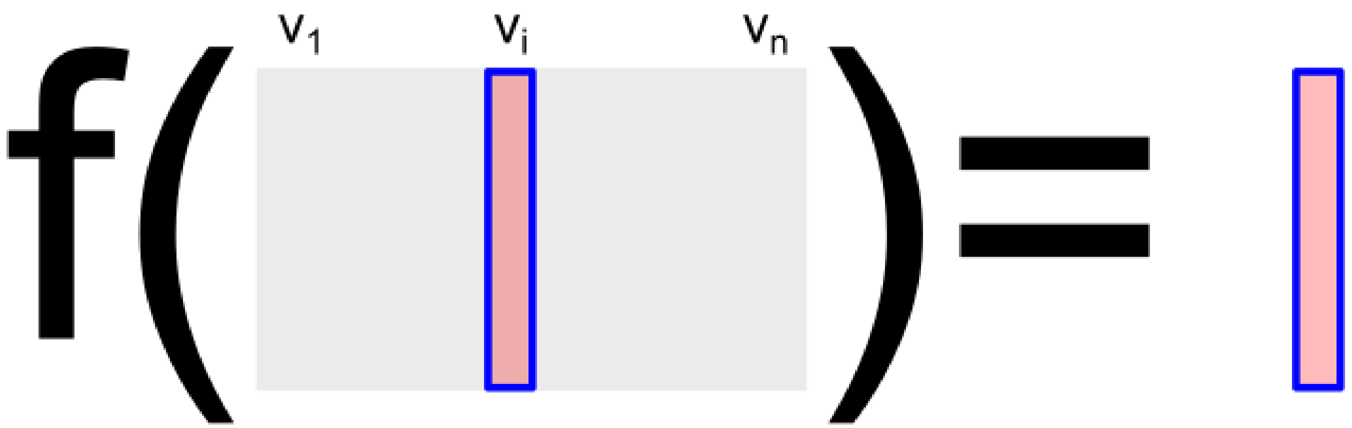
INDEPENDENCE OF IRRELEVANT ALTERNATIVES

If the relative ranking of "a" and "b" in each vote is unchanged, then their relative ranking in the SWF outcome is also unchanged.



DICTATORSHIP

An SWF that mimics the preferences of a fixed voter on all inputs.



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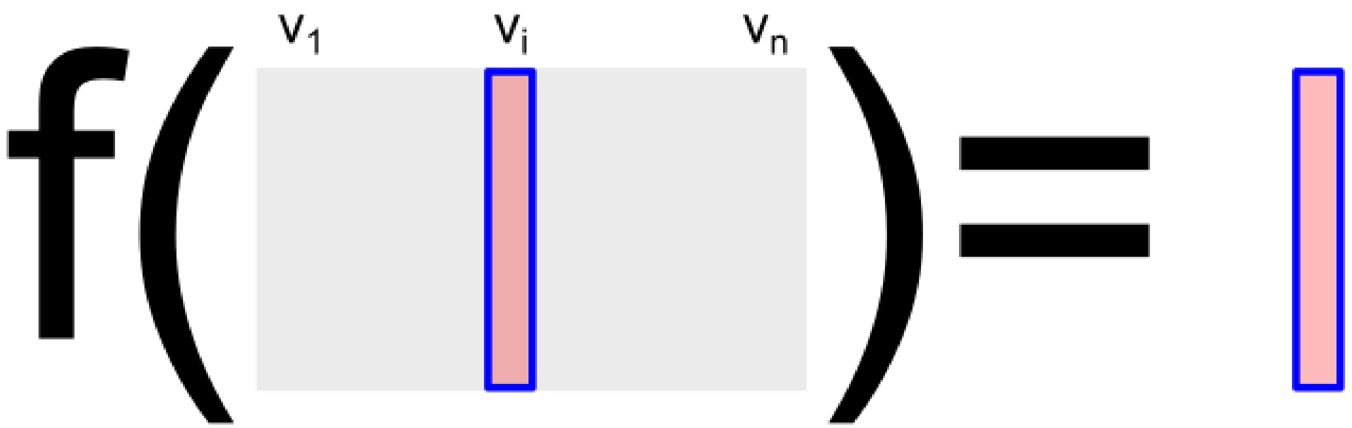

A dictatorship is unanimous and IIA.

[Arrow'51]

With three or more candidates,
an SWF is unanimous and IIA
if and only if it is a dictatorship.

[Arrow'51]

With three or more candidates,
an SWF is unanimous and IIA
if and only if it is a dictatorship.

Proof almost identical to the one we saw for Gibb-Satt thm.

Criterion / Method	Majority	Maj. loser	Mutual maj.	Condorcet	Cond. loser	Smith/ ISDA	LIIA	IIA	Cloneproof	Monotone	Consistency	Participation	Reversal symmetry	Polytime/ resolvable		Summable	Later-no-		No favorite betrayal	Ballot type	Ranks	
																	Harm	Help			=	>2
Approval	Rated[a]	No	No	No[b][c]	No	No[b]	Yes	Yes[d]	Yes[e]	Yes	Yes	Yes	Yes	O(N)	Yes	O(N)	No	Yes[f]	Yes	Approvals	Yes	No
Borda count	No	Yes	No	No[b]	Yes	No	No	No	Teams	Yes	Yes	Yes	Yes	O(N)	Yes	O(N)	No	Yes	No	Ranking	Yes	Yes
Bucklin	Yes	Yes	Yes	No	No	No	No	No	No	Yes	No	No	No	O(N)	Yes	O(N)	No	Yes	If equal preferences	Ranking	Yes	Yes
Copeland	Yes	Yes	Yes	Yes	Yes	Yes	No	No[b]	Teams, crowds	Yes	No[b]	No[b]	Yes	$O(N^2)$	No	$O(N^2)$	No[b]	No	No[b]	Ranking	Yes	Yes
IRV (AV)	Yes	Yes	Yes	No[b]	Yes	No[b]	No	No	Yes	No	No	No	No	$O(N^2)$	Yes[g]	O(N!)[h]	Yes	Yes	No	Ranking	No	Yes
Kemeny–Young	Yes	Yes	Yes	Yes	Yes	Yes	Yes	No[b]	Spoilers	Yes	No[b][i]	No[b]	Yes	O(N!)	Yes	$O(N^2)$[j]	No[b]	No	No[b]	Ranking	Yes	Yes
Highest median/Majority judgment[k]	Rated[l]	Yes[m]	No[n]	No[b][c]	No	No[b]	Yes	Yes[d]	Yes	Yes	No[o]	No[p]	Depends[q]	O(N)	Yes	O(N)[r]	No[s]	Yes	Yes	Scores[t]	Yes	Yes
Minimax	Yes	No	No	Yes[u]	No	No	No	No[b]	Spoilers	Yes	No[b]	No[b]	No	$O(N^2)$	Yes	$O(N^2)$	No[b][u]	No	No[b]	Ranking	Yes	Yes
Plurality/FPTP	Yes	No	No	No[b]	No	No[b]	No	No	Spoilers	Yes	Yes	Yes	No	O(N)	Yes	O(N)	N/A[v]	N/A[v]	No	Single mark	N/A	No
Score voting	No	No	No	No[b][c]	No	No[b]	Yes	Yes[d]	Yes	Yes	Yes	Yes	Yes	O(N)	Yes	O(N)	No	Yes	Yes	Scores	Yes	Yes
Ranked pairs	Yes	Yes	Yes	Yes	Yes	Yes	Yes	No[b]	Yes	Yes	No[b]	No[p][b]	Yes	$O(N^3)$	Yes	$O(N^2)$	No[b]	No	No[p][b]	Ranking	Yes	Yes
Runoff voting	Yes	Yes	No	No[b]	Yes	No[b]	No	No	Spoilers	No	No	No	No	O(N)[w]	Yes	O(N)[w]	Yes	Yes[x]	No	Single mark	N/A	No[y]
Schulze	Yes	Yes	Yes	Yes	Yes	Yes	No	No[b]	Yes	Yes	No[b]	No[p][b]	Yes	$O(N^3)$	Yes	$O(N^2)$	No[b]	No	No[p][b]	Ranking	Yes	Yes
STAR voting	No[z]	Yes	No[aa]	No[b][c]	Yes	No[b]	No	No	No	Yes	No	No	Depends[ab]	O(N)	Yes	$O(N^2)$	No	No	No[ac]	Scores	Yes	Yes
Sortition, arbitrary winner[ad]	No	No	No	No[b]	No	No[b]	Yes	Yes	No	Yes	Yes	Yes	Yes	O(1)	No	O(1)	Yes	Yes	Yes	None	N/A	N/A
Random ballot[ae]	No	No	No	No[b]	No	No[b]	Yes	Yes	Yes	Yes	Yes	Yes	Yes	O(N)	No	O(N)	Yes	Yes	Yes	Single mark	N/A	No

"Most systems are not going to work badly all of the time. All I proved is that all can work badly at times."

Next Time

Algorithms for Manipulating Voting Rules

Oct 14 (Sat)

References

- A shared proof of Arrow’s and Gibbard-Satterthwaite theorems: “Arrow’s Theorem and the Gibbard-Satterthwaite Theorem: A Unified Approach” by Philip J. Reny. https://www.sciencedirect.com/science/article/pii/S0165176500003323

- Another nice proof of Arrow’s theorem: https://www.youtube.com/watch?v=QLi_5LCwJ20

- The “big table” of voting rules is from the Wikipedia article “Comparison of Electoral Systems”.

- A tribute to Arrow (see the end of the article for a nice anecdote): https://www.nytimes.com/2017/02/21/business/economy/kenneth-arrow-dead-nobel-laureate-in-economics.html