| | |
|---|---|
| **COL866: Special Topics in Algorithms** | **Fall 2023** |

# Assignment 4

Total points: 100 — Deadline: Nov 17 (Friday)

1. [**20 points**] Recall that computing a ranking that minimizes the Kemeny score is NP-hard. Also recall that we saw a deterministic 2-approximation algorithm using the Footrule distance. In this exercise, we will design a *randomized* algorithm with the same approximation factor.

   For any natural number $m \in \mathbb{N}$, let $[m] := \{1, 2, \ldots, m\}$ and $\Pi([m])$ be the space of all permutations (or rankings) on the set $[m]$. Say we are given a set $S = \{\pi_1, \pi_2, \ldots, \pi_n\}$ consisting of $n$ rankings over a set of candidates $[m]$ and any ranking $\tau \in \Pi([m])$ not necessarily in $S$. The Kemeny score of $\tau$ with respect to $S$ is defined as $\mathtt{d}^{\mathtt{Kt}}(\tau, S) := \sum_{k \in [n]} \mathtt{d}^{\mathtt{Kt}}(\tau, \pi_k)$, where $\mathtt{d}^{\mathtt{Kt}}(\cdot)$ denotes the Kendall's tau distance between two rankings.

   Consider a randomized algorithm `ALG` whose output is ranking drawn uniformly at random from $S$. Show that `ALG` is a randomized 2-approximation algorithm. That is,

   $$\mathbb{E}_{\pi \sim \mathrm{Unif}(S)}[\mathtt{d}^{\mathtt{Kt}}(\pi, S)] \leqslant 2 \cdot \mathtt{d}^{\mathtt{Kt}}(\sigma^*, S),$$

   where $\sigma^*$ is a Kemeny optimal ranking for $S$, i.e.,

   $$\sigma^* \in \arg\min_{\sigma \in \Pi([m])} \mathtt{d}^{\mathtt{Kt}}(\sigma, S).$$

   Note that $\sigma^*$ may not belong to the set $S$.

2. [**20 points**] Consider the problem of determining whether, given a preference profile and a nonnegative integer $k$, there exists a ranking with Kemeny score at most $k$. Design an $\mathcal{O}(2^k \cdot \mathrm{poly}(n, m))$ algorithm for this problem, where $n$ is the number of the voters and $m$ is the number of candidates.

3. [**20 points**] Consider the following two-sided matching instance:

   | | | | | | | | |
   |---|---|---|---|---|---|---|---|
   | $m_1$: | $w_3$ | $w_2$ | $w_1$ | $w_1$: | $m_1$ | $m_3$ | $m_2$ |
   | $m_2$: | $w_1$ | $w_2$ | $w_3$ | $w_2$: | $m_3$ | $m_2$ | $m_1$ |
   | $m_3$: | $w_3$ | $w_1$ | $w_2$ | $w_3$: | $m_2$ | $m_1$ | $m_3$ |

   What is the distortion of the men-proposing and women-proposing deferred-acceptance algorithms on the above instance? Note that the optimal (i.e., utilitarian welfare maximizing) matching does not have to be stable.

4. [**20 points**] Consider the approval-based multiwinner voting problem. Here, we are given a set $C$ of $m$ candidates, a set $V$ of $n$ voters, and a positive integer $k$. Each voter $i \in V$ approves a subset $A_i \subseteq C$ of the candidates. The goal is to find a committee $W \subseteq C$ of $k$ candidates (i.e., $|W| = k$) that satisfies some notion of representation. We will focus on the notion called *justified representation* (JR) which says that no large, cohesive group of voters

should be unrepresented. Formally, a $k$-sized committee $W \subseteq C$ satisfies JR if there is no subset of voters $S \subseteq V$ with $|S| \geqslant n/k$ and $\cap_{i \in S} A_i \neq \emptyset$ such that $W$ contains no candidate from $\cup_{i \in S} A_i$.

Prove that if each candidate is approved by at least one voter (i.e., for every candidate $c \in C$, there exists some voter $i \in V$ such that $c \in A_i$), then there are at least $m - k + 1$ committees of size $k$ each that satisfy JR.

5. [**20 points**] In this exercise, we will think about the multiwinner voting problem where the voters' preferences are given not as *approvals* but as *rankings*. Let $C = \{c_1, c_2, \ldots, c_m\}$ denote the set of $m$ candidates, and $V = \{v_1, v_2, \ldots, v_n\}$ denote the set of $n$ voters. Each voter $v_i$ has a strict and complete ranking $R_i$ over the candidates in $C$.

   Consider the following voting rule $f$ for selecting a committee of $k$ candidates: Given any $k$-sized committee of candidates $W \subseteq C$, the score that voter $v_i$ assigns to the committee $W$ is equal to $m - j$ if voter $v_i$'s *favorite* candidate in $W$ is ranked at $j^{\text{th}}$ position in its ranking $R_i$. The score of the committee $W$ is the *minimum* of the scores it receives from all voters. The voting rule returns a committee with the highest score.

   For example, suppose there are three voters $v_1, v_2, v_3$ and five candidates $c_1, \ldots, c_5$. The rankings of the voters are $v_1 : c_1 \succ c_2 \succ c_3 \succ c_4 \succ c_5$, $v_2 : c_4 \succ c_2 \succ c_1 \succ c_5 \succ c_3$, and $v_3 : c_5 \succ c_4 \succ c_3 \succ c_2 \succ c_1$. Then, the committee $\{c_2, c_5\}$ gets a score of 3 from voter $v_1$ because its favorite candidate in the committee, namely $c_2$, is ranked second. Likewise, the committee gets scores of 3 and 4 from voters $v_2$ and $v_3$, respectively, resulting in an overall score of $\min\{3, 3, 4\} = 3$.

   Show that when the rankings $R_1, \ldots, R_n$ are single-peaked (with respect to a known axis $\sigma$), the output of the voting rule $f$ can be computed in polynomial time.